HOROCYCLE FLOW ON FLAT PROJECTIVE BUNDLES: TOPOLOGICAL REMARKS AND APPLICATIONS

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Abstract. In this paper we study topological aspects of the dynamics of the foliated horocycle flow on flat projective bundles over hyperbolic surfaces and we derive ergodic consequences. If $\rho : \Gamma \to \mathrm{PSL}(n+1, \mathbb{R})$ is a representation of a non-elementary Fuchsian group Γ, the unit tangent bundle Y associated to the flat projective bundle defined by ρ admits a natural action of the affine group B obtained by combining the foliated geodesic and horocycle flows. If the image $\rho(\Gamma)$ satisfies *Conze-Guivarc'h conditions*, namely strong irreducibility and proximality, the dynamics of the B-action is captured by the proximal dynamics of $\rho(\Gamma)$ on $\mathbb{R}\mathrm{P}^n$ (Theorem A). In fact, the dynamics of the foliated horocycle flow on the unique B-minimal subset of Y can be described in terms of dynamics of the horocycle flow on the non-wandering set in the unit tangent bundle X of the surface $S = \Gamma \backslash \mathbb{H}$ (Theorem B). Assuming the existence of a continuous limit map, we prove that the B-minimal set is an attractor for the foliated horocycle flow restricted to the proximal part of the non-wandering set in Y (Theorem C). As a corollary, we deduce that the restricted flow admits a unique conservative ergodic U-invariant Radon measure (defined up to a multiplicative constant) if and only if Γ is convex-cocompact. For example, the foliated horocycle flow on the projective bundle defined by the Cannon-Thurston map is uniquely ergodic.

1. Introduction

In the 1930s G.A. Hedlund [13] proved the minimality of the right action of the unipotent subgroup

$$U = \{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \}$$

of $\mathrm{PSL}(2, \mathbb{R}) = \{\pm Id\} \backslash \mathrm{SL}(2, \mathbb{R})$ on the quotient $X = \Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$ by a cocompact Fuchsian group Γ. Later H. Furstenberg [11] obtained a stronger result, namely the U-action is uniquely ergodic. Identifying $\mathrm{PSL}(2, \mathbb{R})$ and the unit tangent bundle of the hyperbolic plane $\mathbb{H}$ with the Poincaré metric, when Γ is torsion-free, the quotient X becomes the unit tangent bundle of the hyperbolic surface $S = \Gamma \backslash \mathbb{H}$. In this geometric setting, the U-action on X identifies with the *horocycle flow* and we write

$$h_s(\Gamma u) = \Gamma u \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

for all $u \in \mathrm{PSL}(2, \mathbb{R})$ and all $s \in \mathbb{R}$. Hedlund's and Furstenberg's results have been extended to the case where Γ is finitely generated, but replacing X by the non-wandering set Ω_X of the U-action. Notice that Ω_X is also the unique non-empty minimal invariant closed set for the action of the affine group

$$B = \{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid b \in \mathbb{R}, a \in \mathbb{R}_*^+ \}$$

on X. In that case, the dynamic properties of the U-action from a double topological and measurable perspective can be gathered in the following statement:

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Theorem. *Let Γ be a finitely generated Fuchsian group.*

(1) For any $x \in \Omega_X$, either xU is periodic or $\overline{xU} = \Omega_X$ [8, 10, 13].

(2) For any ergodic U-invariant Radon measure μ supported by Ω_X, either μ is supported by a periodic orbit or μ is the Burger-Roblin measure up to a multiplicative constant [5, 19, 20, 21, 22].

As explained in [8], it turns out that property (1) is true if and only if Γ is finitely generated. However, the topological dynamics of the U-action on Ω_X is not well understood otherwise. On the other hand, it follows from Ratner's work that the measure μ in property (2) is finite if and only if μ supported by a periodic orbit or μ is the Haar measure (up to a constant) and in this case the surface S has finite volume.

In this paper we study the foliated horocycle flow on flat projective bundles over hyperbolic surfaces. Given a non-elementary Fuchsian group Γ, we consider a representation

$$\rho : \Gamma \to \mathrm{PSL}(n+1, \mathbb{R})$$

with $n \geq 1$. The subgroup $\Gamma_\rho = \{ (\gamma, \rho(\gamma) \,|\, \gamma \in \Gamma \}$ of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(n+1,\mathbb{R})$ acts properly discontinuously on $\tilde{Y} = \mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}\mathrm{P}^n$. As this action preserves the product structure of $\tilde{Y}$, the projective bundle $Y = \Gamma_\rho \backslash \tilde{Y}$ over $X = \Gamma \backslash \mathrm{PSL}(2,\mathbb{R})$ admit a foliation transverse to the fibration $\pi : Y \to X$ (which sends $\Gamma_\rho(u,x) \in Y$ onto $\Gamma_u \in X$). The leaves are 3-manifolds endowed with a natural $\mathrm{PSL}(2,\mathbb{R})$-geometric structure. Clearly the U-action on $\tilde{Y}$ defined by right translation on the first factor induces an U-action on Y preserving each leaf. This action defines the *foliated horocycle flow* on Y [16]. In the same way, the affine group B acts on Y preserving each leaf.

Our goal is to prove topological properties of the actions of B and U on Y when ρ satisfies two conditions, which we call *Conze-Guivarc'h conditions*:

(CG1) $\rho(\Gamma)$ is strongly irreducible,

(CG2) $\rho(\Gamma)$ contains a proximal element.

Both conditions guarantee the existence of a unique non-empty minimal $\rho(\Gamma)$-invariant closed set $L(\rho(\Gamma))$ in $\mathbb{R}\mathrm{P}^n$ [7]. It is the closure of the dominant directions of the proximal elements of $\rho(\Gamma)$.

The following results extend well known properties of the actions of B and U on X to the projective bundle Y:

Theorem A. *Let Γ be a non-elementary Fuchsian group and $\rho : \Gamma \to \mathrm{PSL}(n+1,\mathbb{R})$ be a representation satisfying conditions* (CG1) *and* (CG2)*. Then there is a unique B-minimal set $\mathcal{M}_B \subset Y$, i.e. $\mathcal{M}_B$ is a non-empty B-invariant closed set such that $\overline{yB} = \mathcal{M}_B$ for all $y \in \mathcal{M}_B$.*

Theorem B. *Under the same assumptions of Theorem A, for each point $y \in \mathcal{M}_B$, we have:*

$$\overline{yU} = \mathcal{M}_B \;\Leftrightarrow\; \overline{\pi(y)U} = \Omega_X.$$

Corollary 1. *Let Γ be a non-elementary Fuchsian group and $\rho : \Gamma \to \mathrm{PSL}(n+1,\mathbb{R})$ be a representation satisfying conditions* (CG1) *and* (CG2)*. Then $\mathcal{M}_B$ is U-minimal (and therefore the unique U-minimal subset of Y) if and only if Γ is convex-cocompact.*

In the last part of the paper, we will add a strong condition on ρ, called *Nielsen's condition*, implying the existence of a continuous section for π:

(N) ρ induces a continuous map $\varphi : L(\Gamma) \to L(\rho(\Gamma))$, called *limit map*, such that $\varphi \circ \gamma = \rho(\gamma) \circ \varphi$ for all $\gamma \in \Gamma$.

As $L(\rho(\Gamma))$ is minimal, the map φ is always surjetive. If we denote

$$Y_{prox} = \Gamma_\rho \backslash \mathrm{PSL}(2, \mathbb{R}) \times L(\rho(\Gamma)),$$

the B-invariant closed set $\Omega_{prox} = Y_{prox} \cap \pi^{-1}(\Omega_X)$ inherits from Y a natural structure of $L(\rho(\Gamma))$-fibre bundle over Ω_X. By construction, it always contains the B-minimal set $\mathcal{M}_B$. Condition (N) gives arise to a continuous section $\Phi : X \to Y$ for the fibration $\pi : Y \to X$.

Theorem C. *Let* Γ *be a non-elementary Fuchsian group and* $\rho : \Gamma \to \mathrm{PSL}(n+1, \mathbb{R})$ *be a representation satisfying conditions* (CG1), (CG2) *and* (N)*. Then*

(1) $\mathcal{M}_B = \Phi(\Omega_X)$*,*

(2) $\mathcal{M}_B$ *is a* U*-attractor relative to* Ω_{prox}*, i.e. for any point* $y \in \Omega_{prox}$ *and for any sequence* $s_k \to +\infty$*, we have:*

$$h_{s_k}(y) \to y' \;\; \Rightarrow \;\; y' \in \mathcal{M}_B.$$

Corollary 2. *Under the conditions of Theorem C, if* m *is a conservative ergodic* U*-invariant Radon measure on* Y *supported by* Ω_{prox}*, then the support of* m *is equal to* $\mathcal{M}_B$ *and there exist a conservative ergodic* U*-invariant Radon measure* μ *on* X *supported by* Ω_X *such that* $m = \Phi_* \mu$*.*

Corollary 3. *Under the conditions of Theorem C, assume* Γ *is finitely generated. Then there is a unique conservative ergodic* U*-invariant Radon measure* m *on* Ω_{prox} *(defined up to a multiplicative constant and supported by the unique* U*-minimal set* $\mathcal{M}_B$ *in* Ω_{prox}*) if and only if* Γ *is convex-cocompact. In particular, there is a unique* U*-invariant probability measure* m *on* Y_{prox} *if and only if* Γ *is cocompact.*

The uniqueness of U-invariant probability measures on Y projecting to Haar measure on X has been proved by C. Bonatti, A. Eskin and A. Wilkinson [4] when Γ has finite covolume. Here we use Nielsen's condition (N) to deduce a similar property, both for finite and infinite measures, from the existence of a topological attractor. However, a strictly ergodic approach can be applied to prove Corollary 3 for Y under conditions (CG1) and (CG2). Details will be discussed elsewhere.

2. Preliminaries

A matrix $A \in SL(n+1, \mathbb{R})$ is said to be *proximal* if A admits a simple dominant real eigenvalue λ_A. Let $w_A \in \mathbb{R}^{n+1}$ be an eigenvector associated to λ_A and $\chi_A \in \mathbb{R}\mathrm{P}^n$ its direction, also called *dominant* for A. Further, we have the decomposition

$$\mathbb{R}^{n+1} = \mathbb{R} w_A \oplus W_A$$

where

$$W_A = \{\, w \in \mathbb{R}^{n+1} \mid \lambda_A^{-k} A^k w \to 0 \text{ as } k \to +\infty \,\}.$$

Definition 1. Let G be a subgroup of $\mathrm{SL}(n+1, \mathbb{R})$. We say G is:

(CG1) *strongly irreducible* if there does no exist any proper non-trivial subspace of $\mathbb{R}^{n+1}$ invariant by the action of a subgroup of finite index of G;

(CG2) *proximal* if G contains a proximal element A.

Both conditions will be called *Conze-Guivarc'h conditions.*

Conditions (CG1) and (CG2) are satisfied by G if and only if they are satisfied by its Zariski closure in $\mathrm{SL}(n+1,\mathbb{R})$ [7]. But these conditions do not imply that G is Zariski dense in $\mathrm{SL}(n+1,\mathbb{R})$ (since $\mathrm{SO}(n,1)$ satisfies (CG1) and (CG2)).

Proposition 1 ([7]). *Let G be a subgroup of $\mathrm{SL}(n+1,\mathbb{R})$ satisfying* (CG1) *and* (CG2). *Then*

$$L(G) = \overline{\{\, \chi_A \in \mathbb{R}\mathrm{P}^n \,|\, A \in G \text{ proximal}\}}.$$

is the unique G-minimal set in $\mathbb{R}\mathrm{P}^n$. □

Remark 1. Assume $G \subset \mathrm{PSL}(n+1,\mathbb{R})$ is discrete and consider the G-action induced on $L^c(G) = \mathbb{R}\mathrm{P}^n - L(G)$. For $n = 1$, as this action is properly discontinuous, the set $L(G)$ is a *G-attractor* (i.e. for any point $\xi \in \mathbb{R}\mathrm{P}^1$ and for any non stationary sequence g_k in G, the condition $g_k.\xi \to \xi'$ implies $\xi' \in L(G)$) which captures the proximal dynamics of G. However, for $n \geq 2$, these properties are not true in general as the following example proves.

Example 1. Consider $\mathbb{R}^3$ equipped with the Lorentz quadratic form

$$q(x) = x_1^2 + x_2^2 - x_3^2.$$

For $i = -1, 0, 1$, denote $\mathcal{H}_i = \{\, x \in \mathbb{R}^3 \,|\, q(x) = i\}$ and let $p : \mathbb{R}^3 - \{0\} \to \mathbb{R}\mathrm{P}^2$ be the canonical projection. Let $SO^+(2,1)$ be the connected component of the identity in the group $SO(2,1)$ of orientation-preserving linear isometries of q and take a discrete subgroup G of $SO^+(2,1)$. If G is non-elementary and contains no elliptic elements, then $G\backslash\mathcal{H}^+_{-1}$ is isometric to a hyperbolic surface S, where $\mathcal{H}^+_{-1} = \mathcal{H}_{-1} \cap \{\, x \in \mathbb{R}^3 \,|\, x_3 \geq 0\,\}$ (see [8]). The limit set $L(G)$ is contained into $p(\mathcal{H}^+_0 - \{0\})$. For any vector $x \in \mathcal{H}_1$, the orthogonal plane (with respect to q) intersects $\mathcal{H}_0$ along two lines $D_1(x)$ and $D_2(x)$. Let $\mathcal{H}_1(G)$ be the set of vectors $x \in \mathcal{H}_1$ such that the directions of $D_1(x)$ and $D_2(x)$ belong to $L(G)$. This is a G-invariant closed subset of $\mathbb{R}^3 - \{0\}$. According to [8, Proposition VI.2.5], the dynamics of the G-action on $\mathcal{H}_1(G)$ is dual to that of the geodesic flow on the non-wandering set of T^1S. In particular, the G-action on $\mathcal{H}_1(G)$ has dense orbits (see [8, Property VI.2.12]), as well many non-empty proprer minimal sets, and hence the G-action on the closure $\mathcal{F}(G)$ of $p(\mathcal{H}_1(G))$ in $\mathbb{R}\mathrm{P}^2$ also has dense orbits, as well many non-empty invariant closed sets $F \subset \mathcal{F}(G)$ such that $L(G) \subset F$. In conclusion, the G-action on $L^c(G)$ is not discontinuous and $L(G)$ is far from being a G-attractor.

Let Γ be a non-elementary Fuchsian group and $\rho : \Gamma \to \mathrm{SL}(n+1,\mathbb{R})$ a representation satisfying conditions (CG1) and (CG2). Theorems A and B will be proved using a dual approach. Namely, as the linear action of Γ on $E = \{\pm Id\}\backslash\mathbb{R}^2 - \{0\}$ and the projective action of Γ on $\mathbb{R}\mathrm{P}^1$ are conjugated to the Γ-actions on $\mathrm{PSL}(2,\mathbb{R})/U$ and $\mathrm{PSL}(2,\mathbb{R})/B$ respectively, both actions are dual to the U-action and the B-action on $X = \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$.

In our case, the linear and projective actions extend to actions of

$$\Gamma_\rho = \{\, (\gamma, \rho(\gamma) \,|\, \gamma \in \Gamma\,\}$$

on $E \times \mathbb{R}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^n$. As before, they are dual to the U-action and the B-action on the flat projective bundle $Y = \Gamma_\rho\backslash\tilde{Y}$ over X where $\tilde{Y} = \mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}\mathrm{P}^n$. From a geometrical point of view, Y is the unitary tangent bundle to the foliation by hyperbolic surfaces on $\Gamma_\rho\backslash\mathbb{H} \times \mathbb{R}\mathrm{P}^n$ which is induced by the horizontal foliation on $\mathbb{H} \times \mathbb{R}\mathrm{P}^n$.

Theorem A*. *Under the assumptions of Theorem A, there is a unique non-empty minimal Γ_ρ-invariant closed set $\mathcal{M} \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^n$. Moreover $\mathcal{M} \subset L(\Gamma) \times L(\rho(\Gamma))$.*

The relation between the sets $\mathcal{M}_B$ and $\mathcal{M}$ considered in Theorems A and A* is given by
$$\mathcal{M}_B = \{\, \Gamma_\rho(u,\chi) \in Y \mid (u(+\infty),\chi) \in \mathcal{M} \,\},$$
where $u(+\infty)$ is the endpoint of the geodesic ray associated to $u \in T^1\mathbb{H}$.

For each vector $v \in E$, denote $\bar{v} \in \mathbb{R}\mathrm{P}^1$ its direction. Clearly the set
$$E(\Gamma) = \{\, v \in E \mid \bar{v} \in L(\Gamma) \,\}$$
is dual to Ω_X and the set
$$E(\mathcal{M}) = \{(v,\chi) \in E \times \mathbb{R}\mathrm{P}^n \mid (\bar{v},\chi) \in \mathcal{M}\,\}.$$
is dual to $\mathcal{M}_B$.

Theorem B*. *Under the assumptions of Theorem A, for each pair $(v,\chi) \in E(\mathcal{M})$, we have:*
$$\overline{\Gamma_\rho(v,\chi)} = E(\mathcal{M}) \;\Leftrightarrow\; \overline{\Gamma v} = E(\Gamma).$$

3. Proof of Theorems A* and B*

Let Γ be a non-elementary Fuchsian group and $\rho : \Gamma \to \mathrm{PSL}(n+1,\mathbb{R})$ be a representation satisfying (CG1) and (CG2). Take $(\gamma, A) \in \Gamma_\rho$ with A proximal and denote $\chi_A \in \mathbb{R}\mathrm{P}^n$ the dominant direction of A. Since A has infinite order, γ is hyperbolic or parabolic. Consider $\gamma^+ = \lim_{k\to+\infty} \gamma^k(z)$ for any $z \in \mathbb{H}$.

Lemma 1. *For any non-empty Γ_ρ-invariant closed set $F \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^n$, we have $(\gamma^+,\chi_A) \in F$.*

Proof. Since $\mathbb{R}\mathrm{P}^1$ is compact, F projects on a $\rho(\Gamma)$-invariant closed subset of $\mathbb{R}\mathrm{P}^n$ containing $L(\rho(\Gamma))$. It follows that there exists $\xi \in \mathbb{R}\mathrm{P}^1$ such that $(\xi,\chi_A) \in F$. If $\xi \neq \lim_{k\to+\infty} \gamma^{-k}(z)$, then $\lim_{k\to+\infty} \gamma^k(\xi) = \gamma^+$ and hence
$$\lim_{k\to+\infty} \big(\gamma^k(\xi), \rho(\gamma^k)\chi_A\big) = \lim_{k\to+\infty} \big(\gamma^k(\xi), A^k\chi_A\big) = (\gamma^+,\chi_A) \in F.$$
Otherwise, by the irreducibility condition (CG1), there exists $\gamma' \in \Gamma - <\gamma>$ such that $\rho(\gamma')\chi_A$ does not belong to the projection $\overline{W}_A$ of W_A into $\mathbb{R}\mathrm{P}^n$. Since Γ is discret, we have $\gamma'(\xi) \neq \xi$. As a consequence, we have:
$$\lim_{k\to+\infty} \big(\gamma^k(\gamma'(\xi)), A^k\rho(\gamma')\chi_A\big) = (\gamma^+,\chi_A) \in F. \quad \square$$

Proof of the Theorem A.* By Lemma 1, the intersection of all non-empty closed Γ_ρ sets contains
$$\mathcal{M} = \overline{\{\, (\gamma^+,\chi_A) \mid \gamma \in \Gamma, A = \rho(\gamma) \text{ proximal} \,\}} \subset L(\Gamma) \times L(\rho(\Gamma)).$$
Thus $\mathcal{M}$ becomes the unique minimal set for the Γ_ρ-action on $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^n$. $\square$

Remark 2 (on the shape of $\mathcal{M}$). (1) If ρ is not injective, then $\mathcal{M} = L(\Gamma) \times L(\rho(\Gamma))$ because $N = Ker\,\rho$ is a normal subgroup of Γ and then $L(\Gamma) = L(N)$.

(2) A similar conclusion holds if ρ is indiscrete (in the sense that $\rho(\Gamma)$ is not discrete). Indeed, let γ_k be a non-stationary sequence of elements of Γ such that $\rho(\gamma_k) \to Id$. Passing to a subsequence if necessary, there exist two points ξ^- and ξ^+ in $\mathbb{R}\mathrm{P}^1$ such that
$$\lim_{k\to+\infty} \gamma_k(\xi) = \xi^+$$
for any $\xi \neq \xi^-$ (see for example [2, Lemma 2.2]). For any $\chi \in L(\rho(\Gamma))$, take an element $(\xi,\chi) \in \mathcal{M}$ such that $\xi \notin \Gamma\xi^-$. Since Γ is non elementary, such element always exists. For any $\gamma \in \Gamma$, we have:
$$\lim_{k\to+\infty} \big(\gamma\gamma_k\gamma^{-1}(\xi), \rho(\gamma\gamma_k\gamma^{-1})\chi\big) = (\gamma(\xi^+),\chi).$$

Therefore $\Gamma_\rho\big(\gamma(\xi^+),\chi\big) \subset \mathcal{M}$ and hence $L(\Gamma) \times L(\rho(\Gamma)) = \mathcal{M}$.

(3) In the opposite side, if $n = 1$ and ρ is the natural inclusion of Γ into $\mathrm{PSL}(2,\mathbb{R})$, then $\mathcal{M} = \{\, (\xi,\xi) \,|\, \xi \in L(\Gamma) \,\}$.

Two lemmas are needed to prove Theorem B*:

Lemma 2. *Let Γ be a non-elementary Fuchsian group and $\rho : \Gamma \to \mathrm{PSL}(n+1,\mathbb{R})$ be a representation satisfying* (CG1) *and* (CG2)*. There are two hyperbolic elements γ_1 and γ_2 of Γ such that*

(1) the dominant eigenvalues λ_1 and λ_2 generate a dense subgroup of the positive multiplicative group $\mathbb{R}^*_+$,

(2) $A_1 = \rho(\gamma_1)$ and $A_2 = \rho(\gamma_2)$ are proximal.

Proof. Under conditions (CG1) and (CG2), the group $\rho(\Gamma)$ contains two elements A_1 and A_2 which generate a non-abelian free group containing only proximal elements (see [3, Lemma 3.9] and [12, Lemma 3]). Let γ_1 and γ_2 be two elements of Γ such that $\rho(\gamma_1) = A_1$ and $\rho(\gamma_2) = A_2$. Reasoning as in [9], we can replace γ_1 and γ_2 with two hyperbolic elements of Γ whose dominant eigenvalues λ_1 and λ_2 generate a dense subgroup of $\mathbb{R}^*_+$. □

For each hyperbolic element γ of Γ, we denote v_γ the unit eigenvector in E associated to dominant eigenvalue λ_γ. Clearly $v_\gamma \in E(\Gamma)$ since its direction $\bar{v}_\gamma = \gamma^+ \in L(\Gamma)$. From Theorem A*, it follows that

$$E(\mathcal{M}) \subset E(\Gamma) \times L(\rho(\Gamma)).$$

Lemma 3. *Let $(v,\chi) \in E(\mathcal{M})$ such that $\overline{\Gamma v} = E(\Gamma)$. For any hyperbolic element $\gamma \in \Gamma$ such that $A = \rho(\gamma)$ is proximal, there exists $\alpha \in \mathbb{R}^*$ such that*

$$(\alpha v_\gamma, \chi_A) \in \overline{\Gamma_\rho(v,\chi)}.$$

Proof. Assuming $\overline{\Gamma v} = E(\Gamma)$, there exists a sequence of elements $\gamma_k \in \Gamma$ such that the norms $\|\, \gamma_k v \,\|$ converge to 0 as $k \to +\infty$. Since Γ is non elementary and $\rho(\Gamma)$ is irreducible, replacing γ_k by $\gamma' \gamma_k$ for some $\gamma' \in \Gamma$, up to take a subsequence, we can suppose:

(1) $\gamma_k v = a_k v_\gamma + b_k v_{\gamma^{-1}}$ where $a_k \neq 0$ for any k,

(2) $\rho(\gamma_k)\chi \to \chi_0 \notin \overline{W}_A$.

Let p_k an increasing sequence of integers converging to $+\infty$ such that $\lambda_\gamma^{p_k} a_k$ converges to some $\alpha \neq 0$. Then we have $\gamma^{p_k}\gamma_k v \to \alpha v_\gamma$. Let us prove that

$$A^{p_k}\rho(\gamma_k)\chi \to \chi_A. \tag{3.1}$$

Since $\chi_0 \notin \overline{W}_A$, there exist an open neighbourhood $V(\chi_A)$ of χ_A containing χ_0, an integer $N \gg 0$ and a constant $0 < c < 1$ satisfying [12, Lemma 3]:

i) $A^{Nk}(V(\chi_A)) \subset V(\chi_A)$ for all $k \geq 0$,

iii) $\delta(A^{Nk}\chi_1, A^{Nk}\chi_2) \leq c^k \delta(\chi_1,\chi_2)$ for all $\chi_1, \chi_2 \in V(\chi_A)$ and for all $k \geq 0$,

For $k \geq 0$ large enough, we have $\rho(\gamma_k)\chi \in V(\chi_A)$. Assuming $p_k = Nq_k + r_k$ with $0 \leq r_k < N$, the inequality

$$\delta(A^{Nq_k}\rho(\gamma_k)\chi, \chi_A) \leq c^{q_k}\delta(\rho(\gamma_k)\chi, \chi_A)$$

implies

$$\lim_{k\to+\infty} \delta(A^{Nq_k}\rho(\gamma_k)\chi, \chi_A) = 0$$

and hence

$$\lim_{k\to+\infty} \delta(A^{p_k}\rho(\gamma_k)\chi, \chi_A) = 0.$$

This proves (3.1). Finally, we deduce:

$$\lim_{k\to+\infty}\left(\gamma^{p_k}\gamma_k v, A^{p_k}\rho(\gamma_k)\chi\right) = (\alpha v_\gamma, \chi_A) \in \overline{\Gamma_\rho(v,\chi)}. \quad \square$$

Proof of the Theorem B.* Let $(v,\chi) \in E(\mathcal{M})$ with $\overline{\Gamma v} = E(\Gamma)$. Take $\gamma_1, \gamma_2 \in \Gamma$ given by Lemma 2 and its images $A_1 = \rho(\gamma_1)$ and $A_2 = \rho(\gamma_2)$. Applying Lemma 3, there exists a real number $\alpha_1 \neq 0$ such that $(\alpha_1 v_{\gamma_1}, \chi_{A_1}) \in \overline{\Gamma_\rho(v,\chi)}$ and hence

$$(\alpha_1 \lambda_1^p v_{\gamma_1}, \chi_{A_1}) \in \overline{\Gamma_\rho(v,\chi)} \tag{3.2}$$

for any $p \in \mathbb{Z}$. Since $\overline{\Gamma v_{\gamma_1}} = E(\Gamma)$ [8, Theorem V.3.1], by the same argument, we obtain another real number $\alpha_2 \neq 0$ such that

$$(\alpha_1\alpha_2 \lambda_1^p \lambda_2^q v_{\gamma_2}, \chi_{A_2}) \in \overline{\Gamma_\rho(v,\chi)} \tag{3.3}$$

for any pair $p, q \in \mathbb{Z}$. As λ_1 and λ_2 generate a dense subgroup of $\mathbb{R}_+^*$ by Lemma 2, we deduce from (3.2) and (3.3) that

$$(\lambda v_{\gamma_2}, \chi_{A_2}) \in \overline{\Gamma_\rho(v,\chi)}$$

for any $\lambda > 0$.

For any $(v',\chi') \in E(\mathcal{M})$, since $(\bar{v}',\chi')$ and (γ_2^+, χ_{A_2}) belong to the minimal set $\mathcal{M}$, there exists a sequence $\gamma_k \in \Gamma$ such that

$$\gamma_k \gamma_2^+ \to \bar{v}' \quad \text{and} \quad \rho(\gamma_k)\chi_{A_2} \to \chi'.$$

It follows there exists a sequence $\lambda_k \in \mathbb{R}$ such that

$$\lambda_k \gamma_k v_{\gamma_2} \to \alpha v' \tag{3.4}$$

for some $\alpha \neq 0$. As $(\lambda_k v_{\gamma_2}, \chi_{A_2}) \in \overline{\Gamma_\rho(v,\chi)}$, we deduce $(\alpha v', \chi') \in \overline{\Gamma_\rho(v,\chi)}$. The same argument applies when multiply the two terms of (3.4) by a real number $\lambda > 0$. $\square$

We deduce from Theorem B* that $E(\mathcal{M})$ is a non-empty minimal Γ_ρ-invariant closed set if and only if $E(\Gamma)$ is a minimal Γ-invariant closed set. Since this condition is satisfied if and only if Γ is convex-compact [8, Proposition V.4.3], we retrieve Corollary 1:

Corollary 4. *The set $\mathcal{M}_B$ is U-minimal if and only if Γ is convex-cocompact*

More generally, since $\mathbb{R}\mathrm{P}^n$ is compact, any non-empty minimal Γ_ρ-invariant closed subset $F \subset E \times \mathbb{R}\mathrm{P}^n$ projets onto a non-empy minimal Γ-invariant closed subset $p_1(F) \subset E$. If Γ is finitely generated, then either F projets onto a closed Γ-orbit or $F = E(\mathcal{M})$ and Γ is convex-compact [8, Theorem V.4.1]. On the contrary, if Γ is not finitely generated, there exist examples where $E(\Gamma)$ does not admit any non-empty minimal Γ-invariant closed subset [14, 17].

Corollary 5. *There exist infinitely generated Fuchsian groups Γ such that for any representations $\rho : \Gamma \to \mathrm{PSL}(n+1,\mathbb{R})$ satisfying conditions* (CG1) *and* (CG2)*, the projective bundle Y does not admit any non-empty U-minimal subset of $\pi^{-1}(\Omega_X)$.*

4. Proof of Theorem C

In this section, we restrict our attention to the space

$$Y_{prox} = \Gamma_\rho \backslash \mathrm{PSL}(2,\mathbb{R}) \times L(\rho(\Gamma)).$$

This space is a $\mathrm{PSL}(2,\mathbb{R})$-invariant closed subset of Y for which the induced $\mathrm{PSL}(2,\mathbb{R})$-action is minimal. From a geometrical point of view, Y_{prox} is the unit tangent bundle to a minimal lamination by hyperbolic surfaces. Intersecting with $\pi^{-1}(\Omega_X)$, we obtain a B-invariant closed set

$$\Omega_{prox} = Y_{prox} \cap \pi^{-1}(\Omega_X)$$

such that:

(i) Ω_{prox} is included in the non-wandering set for the U-action on Y_{prox},

(ii) Ω_{prox} inherits from Y a natural structure of $L(\rho(\Gamma))$-fibre bundle over Ω_X with projection $\pi : \Omega_{prox} \to \Omega_X$.

By duality, U-orbits in Ω_{prox} are in one-to-one correspondance with Γ_ρ-orbits in $E(\Gamma) \times L(\rho(\Gamma)$. Note that $\mathcal{M}_B \subset \Omega_{prox}$ is the unique non-empty minimal B-invariant closed subset of Ω_{prox}.

We also add a new condition on the representation $\rho : \Gamma \to \mathrm{PSL}(n+1, \mathbb{R})$, which we call *Nielsen's condition*:

(N) there exists a continuous map

$$\varphi : L(\Gamma) \to L(\rho(\Gamma)),$$

called *limit map*, such that $\varphi \circ \gamma = \rho(\gamma) \circ \varphi$ for all $\gamma \in \Gamma$.

Conditions (CG1), (CG2) and (N) imply ρ is discrete injective and φ is surjective.

A wide familiy of representations ρ satisfying conditions (CG1), (CG2) and (N) can be find in the litterature: for $\rho(\Gamma) \subset SO(n, 1)$ see [24] and for $\rho(\Gamma) \subset SL(n + 1, R)$ Anosov see [15]. In general, even if Γ is finitely generated, φ is not necessarily injective. This is the case for example if γ is hyperbolique and $\rho(\gamma)$ is parabolic [23]. One of the most surprising examples is a discrete faithful representation $\rho : \Gamma \to SO(3, 1)$ of a torsion-free cocompact Fuchsian group Γ that gives raise to sphere-filling map $\varphi : S^1 \to S^2$ called the *Cannon-Thurston map* [6].

Proof of the Theorem C. Assume Γ is non-elementary and $\rho : \Gamma \to \mathrm{PSL}(n + 1, \mathbb{R})$ satisfies conditions (CG1), (CG2) and (N). Under condition (N), we can immediately deduce the two following facts:

(i) the graph of map φ is a non-empty Γ_ρ-invariant closed subset of $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^n$,

(ii) the map φ define a continuous section $\Phi : \Omega_X \to \Omega_{prox}$ given by

$$\Phi(\Gamma u) = \Gamma_\rho(u, \varphi(u(+\infty))).$$

(1) The unique B-minimal set $\mathcal{M}_B$ is given by

$$\mathcal{M}_B = \Phi(\Omega_X) = \{\, \Gamma_\rho(u, \varphi(u(+\infty))) \in Y \mid u \in \mathrm{PSL}(2, \mathbb{R}), u(+\infty) \in L(\Gamma) \,\}.$$

Indeed, by Theorem A*, we know that the unique Γ-minimal set $\mathcal{M} \subset \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^n$ coincides with the graph of φ. Now our statement follows by duality.

(2) The unique B-minimal set $\mathcal{M}_B$ is a U-attractor relative to Ω_{prox}. Indeed, take $y = \Gamma_\rho(u, \chi) \in \Omega_{prox}$ and assume $h_{s_k}(y) \to y'$ for some sequence $s_k \to +\infty$. Since Ω_{prox} is U-invariant, $y' = \Gamma_\rho(u', \chi')$ with $u'(+\infty) \in L(\Gamma)$ and $\chi' \in L(\rho(\Gamma))$. As $y \in \mathcal{M}_B$ implies $y' \in \mathcal{M}_B$, the proof reduces to the case where $\chi = \varphi(\xi) \neq \varphi(u(+\infty))$. By construction, there exists a sequence $\gamma_k \in \Gamma$ such that

$$\gamma_k u \begin{pmatrix} 1 & s_k \\ 0 & 1 \end{pmatrix} \to u' \quad \text{and} \quad \rho(\gamma_k)\chi \to \chi'$$

Let us return to the hyperbolic point of view, identifying $\mathrm{PSL}(2, \mathbb{R})$ with the unit tangent bundle $T^1\mathbb{H}$. In this model, each element $u \in \mathrm{PSL}(2, \mathbb{R})$ identifies with $u = (u(0), \vec{u}) \in T^1\mathbb{H}$ where $u(0)$ is a point of $\mathbb{H}$ and $\vec{u}$ is a unit tangent vector to $\mathbb{H}$ at $u(0)$. Denoting $B_{u(+\infty)}(i, u(0))$ the Busemann cocycle centred at $u(+\infty)$ and calculated at i and $u(0)$, we have the following conditions [8, Chapter V]:

(a) $\gamma_k(u(+\infty)) \to u'(+\infty)$,

(b) $B_{\gamma_k(u(+\infty))}\big(i, \gamma_k(u(0))\big) \to B_{u'(+\infty)}\big(i, u'(0)\big)$,

(c) $\rho(\gamma_k)\chi \to \chi'$.

Properties (a) and (b) imply

$$\lim_{k\to+\infty} \gamma_k(u(0)) = u'(+\infty).$$

Since

$$B_{\gamma_k(u(+\infty))}\big(i, \gamma_k(u(0)\big) = B_{u(+\infty)}\big(\gamma_k^{-1}(i), u(0)\big),$$

applying again Property (b), we deduce:

$$\lim_{k\to+\infty} \gamma_k^{-1}(i) = u(+\infty).$$

As a consequence, since $\xi \neq u(+\infty)$, we have (see [2, Lemma 2.2]):

$$\gamma_k(\xi) \to u'(+\infty).$$

By continuity of φ, it follows:

$$\rho(\gamma_k)\chi = \varphi(\gamma_k(\xi)) \to \varphi(u'(+\infty)).$$

Property (c) implies $\chi' = \varphi(u'(+\infty))$ and hence $y' \in \mathcal{M}_B$. □

Remark 3. Example 1 shows that we cannot expect $\mathcal{M}_B$ to be a global U-attractor in general. This is why we introduced the laminated space Y_{prox}.

Proof of the Corollary 2. Let m be a U-invariant (non necessarily finite) Radon measure on Ω_{prox}. If m is conservative, then Poincaré's Recurrence Theorem (see [1, Theorem 1.1.5] for the discrete version) implies that the set of U-recurrent points

$$\mathcal{R}_{prox} = \{\, y \in \Omega_{prox} \,|\, \exists\, s_n \to +\infty \,:\, h_{s_n}(y) \to y \,\}$$

has full-measure, that is, $m(\Omega_{prox} - \mathcal{R}_{prox}) = 0$. Since $\mathcal{M}_B$ is a U-attractor relative to $\mathcal{R}_{prox}$, then $\mathcal{R}_{prox} \subset \mathcal{M}_B$ and therefore $m(\Omega_{prox} - \mathcal{M}_B) = 0$. As the continuous section Φ sends homeomorphically Ω_X onto $\mathcal{M}_B$ and m is supported by $\mathcal{M}_B$, the push-forward $\mu = \pi_* m$ is a U-invariant measure μ on Ω_X. It is also conservative and verifies $\Phi_*\mu = m$. Finally m is ergodic if and only if μ is ergodic. □

If Γ is finitely generated, as we recall in the introduction, any ergodic U-invariant measure μ either is supported by a closed orbit or is equal to the Burger-Roblin measure [5, 22] up to a multiplicative constant. In the last case, μ is conservative, so Corollary 3 follows from Corollary 2. Namely, under the conditions of Theorem C and assuming that Γ is finitely generated, there is a unique conservative ergodic U-invariant Radon measure m on Ω_{prox} (defined up to a multiplicative constant and supported by the unique U-minimal set $\mathcal{M}_B$ in Ω_{prox}) if and only if Γ is convex-cocompact. In particular, there is a unique U-invariant probability measure m on Y_{prox} if and only if Γ is cocompact. Notice that the unique U-invariant probability measure on Y (which is obtained by lifting the Haar measure) in [4, Corollary 2.4] is supported by Y_{prox}. In the counterexample constructed by S. Matsumoto [18] on a 3-dimensional compact solvmanifold, there is a unique B-invariant probability measure m supported by the unique B-minimal set $\mathcal{M}_B$, but there are uncountable many U-invariant probability measures (specifically, the ergodic components of m) supported by uncountable many U-minimal sets.

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