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HOROCYCLE FLOWS ON CERTAIN SURFACES WITHOUT CONJUGATE POINTS[1]

BY

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ABSTRACT. We study the topological but not ergodic properties of the horocycle flow $\{h_t\}$ in the unit tangent bundle SM of a complete two dimensional Riemannian manifold M without conjugate points that satisfies the "uniform Visibility" axiom. This axiom is implied by the curvature condition $K \leq c < 0$ but is weaker so that regions of positive curvature may occur. Compactness is not assumed. The method is to relate the horocycle flow to the geodesic flow for which there exist useful techniques of study. The nonwandering set $\Omega_h \subseteq SM$ for $\{h_t\}$ is classified into four types depending upon the fundamental group of M. The extremes that Ω_h be a minimal set for $\{h_t\}$ and that Ω_h admit periodic orbits are related to the existence or nonexistence of compact "totally convex" sets in M. Periodic points are dense in Ω_h if they exist at all. The only compact minimal sets in Ω_h are periodic orbits if M is noncompact. The flow $\{h_t\}$ is minimal in SM if and only if M is compact. In general $\{h_t\}$ is topologically transitive in Ω_h and the vectors in Ω_h with dense orbits are classified. If the fundamental group of M is finitely generated and $\Omega_h = SM$ then $\{h_t\}$ is topologically mixing in SM.

Introduction. Horocycles have played an important role in noneuclidean geometry since its beginning, but horocycle flows on the unit tangent bundle of an orientable surface were evidently studied seriously for the first time by Hedlund and Hopf in the 1930's. The horocycle flow was defined for surfaces of constant negative curvature and was shown to be minimal if M is compact and ergodic if M has finite area. Apparently there was no study of the horocycle flow for the case of an arbitrary orientable, noncompact surface where the nonwandering set of the flow need not be the full unit tangent bundle, SM, of M.

In this paper we define and study the horocycle flow on the unit tangent bundle of a more general class of orientable surfaces, the *uniform Visibility surfaces*. We consider arbitrary surfaces of this type, both compact and noncompact, and we obtain basic information about the nonwandering set $\Omega_h \subseteq SM$. In particular we classify Ω_h into four possible types, find criteria for the existence and classification of dense orbits in Ω_h and the existence and

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density of periodic points in Ω_h, and describe the minimal subsets of the flow restricted to Ω_h. For example, we show that the horocycle flow is minimal in SM, M a uniform Visibility surface, if and only if M is compact. We do not consider ergodic problems. For recent work in this direction see [11], [18] and [19].

Uniform Visibility surfaces are surfaces that satisfy the "uniform Visibility" axiom, a certain condition on geodesics that is implied by the curvature condition $K \leqslant -c^2 < 0$. However, the geometric condition is much more general. One can show that any compact surface with Anosov geodesic flow is a uniform Visibility surface. Moreover, it is shown in [5] that any compact surface with negative Euler characteristic and without conjugate points along any geodesic is a uniform Visibility surface. One may create uniform Visibility surfaces whose curvature has both signs by starting with a complete surface M of constant negative curvature and modifying the metric in a neighborhood of a set of points $\{p_\alpha\}$ that lie at a distance $\geqslant \varepsilon > 0$ from each other. In fact, for a suitable modification one may obtain a complete metric with Anosov geodesic flow that agrees with the original metric outside the union of some neighborhoods $\{V_\alpha\}$ of $\{p_\alpha\}$ and that has prescribed constant curvature $k_\alpha > 0$ on a neighborhood U_α of p_α, $\overline{U}_\alpha \subseteq V_\alpha$. Details of this construction may be found in [13]. If M is compact, then any small C^2 perturbation of such a metric is also a uniform Visibility metric.

In the first section of the paper we list some basic background results that are needed for the exposition. Briefly, one studies the geometry of a uniform Visibility surface $M = H/D$ by extending the action of the deckgroup D on H, the universal Riemannian covering surface, to the *points at infinity*, denoted by $H(\infty)$. One defines a limit set $L(D) \subseteq H(\infty)$ that is invariant under D, and obtains information about the geometry of M by analyzing the action of D on $L(D)$.

In the second section we define and state the continuity of the horocycle flow in the unit tangent bundle of an orientable uniform Visibility surface M. In special cases this section is unnecessary. For example, if M is compact with $K < 0$ or with Anosov geodesic flow in SM, then the flow arises from a C^1 vector field on SM and is automatically continuous. In general the flow does not appear to arise from a C^1 vector field (see the discussion in §2), and consequently the flow maps must be defined explicitly. The construction is clear but technical and only statements of results are given in this section. The proofs are found in the Appendix.

In the third section we obtain basic dynamical information about the horocycle flow. The method, an old one, is to use information about the geodesic flow in SM to obtain information about the closely related horocycle flow. We characterize and classify the nonwandering set Ω_h of the horocycle flow. We also describe the periodic points of the flow in terms of the "parabolic" fixed points in the limit set $L(D)$, and we obtain existence theorems for the periodic points. For example, if $\pi_1(M)$ is finitely generated, then Ω_h contains no periodic points if and only if M admits a nonempty compact, totally convex set (Theorem 3.7). We also show that the periodic points are dense in Ω_h if they exist at all.

In the fourth section we apply the results of the third. Some of these results, particularly those regarding the existence and classification of dense orbits in SM, are obtained in the case $K \equiv -1$, $\Omega_h = SM$ by Hedlund in [15]. We show that the horocycle flow has a dense orbit in Ω_h if $\Omega_h = SM$, and in general Ω_h, if nonempty, has a dense orbit except possibly in an exceptional case that we believe does not occur and in the degenerate case that Ω_h contains only periodic vectors (see Theorem 4.1). Assuming that Ω_h does admit a dense horocycle orbit we characterize those vectors in Ω_h whose horocycle orbit is dense in Ω_h. As one consequence of this discussion we show that if M is a noncompact, finitely connected uniform Visibility surface of finite area (or more generally if $\Omega_h = SM$), then every horocycle orbit in SM is either dense in SM or periodic. In this section we also study the minimal sets of the flow. If the flow is minimal in Ω_h, then M admits a nonempty compact totally convex set, and the converse is true except possibly in the exceptional case referred to above. If M is finitely connected and Ω_h contains periodic orbits, then the periodic orbits are the only minimal sets in Ω_h. In the infinitely connected case we know little about the minimal sets in Ω_h except that they consist entirely of "almost minimizing" vectors (Proposition 4.6). However, the only compact minimal sets are periodic orbits. (By finitely connected we mean that the fundamental group is finitely generated. Equivalent conditions for surfaces without conjugate points are given in Theorem A of [3].)

We conclude §4 with a discussion of topological mixing. Our main result is that if $\Omega_h = SM$ and M is finitely connected but not necessarily compact, then the horocycle flow is topologically mixing in SM. In particular this result holds for all compact orientable surfaces with arbitrary curvature $K \leqslant 0$ and negative Euler characteristic. The basic technique is due to Brian Marcus [18] who used it to prove topological mixing in the case that M is compact with $K < 0$.

1. **Notation and preliminaries.** We begin with notation. M will always denote a complete Riemannian manifold, and $d(\ ,\ )$ will denote its Riemannian metric. All vectors tangent to M will be assumed to have length one, and SM will denote the bundle of unit tangent vectors of M with μ: $SM \to M$ the projection map. All geodesics of M will be assumed to have unit speed, and for any vector v in SM, γ_v will denote the unique geodesic of M whose velocity at $t = 0$ is v. The terms *maximal geodesic*, *geodesic ray* and *geodesic segment*

will denote a geodesic of M defined on $\mathbf{R}$, $[0,\infty]$ and a compact interval respectively. A geodesic ray γ is *divergent* if for any compact subset C of M there exists a positive number t_0 such that $\gamma(t) \in M - C$ for $t > t_0$. A geodesic ray is *minimizing* (on $[0,\infty]$) if $d(\gamma 0, \gamma t) = t$ for all $t \geqslant 0$; *ultimately minimizing* if there exists a positive number t_0 such that $d(\gamma t, \gamma t_0) = t - t_0$ for all $t \geqslant t_0$; and *almost minimizing* if $d(\gamma 0, \gamma t) - t \geqslant -A$ for some positive number A and all $t \geqslant 0$. A vector v in SM is minimizing, ultimately minimizing, almost minimizing or divergent if the geodesic γ_v has this property.

The rest of this section is a rapid sketch of basic definitions and facts. Details are omitted and may be found in §§1 through 5 of [10], §§1 and 2 of [5] or §§1 and 2 of [4].

Two points p and q on a geodesic γ are *conjugate* along γ if there exists a nonzero Jacobi vector field on γ that vanishes at p and q. M has *no conjugate points* if no geodesic of M has a pair of conjugate points. If M is simply connected and has no conjugate points, then any two points of M are joined by a unique geodesic. In the sequel, H will always denote a complete, simply connected manifold without conjugate points and M an arbitrary complete manifold without conjugate points. M may be represented as a quotient H/D, where D is a freely acting, properly discontinuous group of isometries of H.

Definition 1.1. If p and q are distinct points of H, then γ_{pq} denotes the unique geodesic of H such that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(a) = q$, where $a = d(p,q)$. Let $V(p,q)$ denote $\gamma'_{pq}(0)$.

If H is two dimensional, then for any maximal geodesic γ of H, $H - \gamma$ consists of two connected components, each of which is convex in the sense that it contains the unique geodesic segment joining any two of its points.

Definition 1.2. Let H be two dimensional, and let γ be a maximal geodesic of H. Relative to a fixed orientation of H a point p in $H - \gamma$ lies to the *right* (*left*) of γ if for some number t the pair of unit vectors $\{V(\gamma t, p), \gamma'(t)\}$ has the same (opposite) orientation. The *right* (*left*) *halfplane* determined by γ consists of those points lying to the right (left) of γ.

Assuming now that H has arbitrary dimension and that q, r are points of H distinct from a point p in H we define $\measuredangle_p(q,r)$ to be the angle subtended by $V(p,q)$ and $V(p,r)$, the value lying in $[0,\pi]$.

Definition 1.3. H satisfies the *Visibility axiom* if for every point p in H and every positive number ε there exists a positive number $R = R(p,\varepsilon)$ such that if $\gamma\colon [a,b] \to H$ is a geodesic segment satisfying the condition $d(p,\gamma) \geqslant R$, then $\measuredangle_p(\gamma a, \gamma b) \leqslant \varepsilon$. H satisfies the *uniform Visibility axiom* if the constant R may be chosen to depend only on ε.

H satisfies the uniform Visibility axiom if the sectional curvature K is everywhere $\leqslant -c^2 < 0$. Henceforth we shall assume that either H satisfies the uniform Visibility axiom or H has nonpositive sectional curvature and satisfies the Visibility axiom. $M = H/D$ will be called a (uniform) Visibility manifold (or surface in dimension two). Our arguments will always assume that H satisfies the uniform Visibility axiom, but they work equally well in the second case. We use extensively the results of [7] which are also true in the uniform Visibility case. Some proofs in [7] require modification, but this can be accomplished using the results of §2 of [4].

DEFINITION 1.4. Geodesics γ and σ in H are *asymptotes* if there exists a constant $c > 0$ such that $d(\gamma t, \sigma t) \leqslant c$ for all $t \geqslant 0$. Geodesics γ and σ in $M = H/D$ are asymptotic if they have lifts $\tilde{\gamma}$ and $\tilde{\sigma}$ to H that are asymptotic. Vectors v, w in SM or SH are asymptotic if the geodesics γ_v and γ_w are asymptotic.

Let $\gamma(\infty)$ denote the asymptote equivalence class of the geodesic γ, and let $\gamma(-\infty)$ denote the equivalence class of the geodesic $\gamma^{-1}: t \to \gamma(-t)$. A *point at infinity* for H is an equivalence class of geodesics of H, and $H(\infty)$ denotes the set of all points at infinity. A geodesic γ is said to *join* points x, y in $H(\infty)$ if $\{x,y\} = \{\gamma(\infty), \gamma(-\infty)\}$ as unordered sets. Let $\overline{H}$ denote $H \cup H(\infty)$.

PROPOSITION 1.5. *Let γ be a geodesic in H, and let p be any point of H. Then there exists a unique geodesic σ such that $\sigma(0) = p$ and σ is asymptotic to γ. If x and y are distinct points in $H(\infty)$, then there exists a geodesic γ joining x to y.*

The geodesic joining x to y need not be unique (Proposition 5.1 of [10]). Geodesics γ and σ of H are *equivalent* if they join the same points in $H(\infty)$. Geodesics γ and σ of $M = H/D$ are equivalent if they have lifts to H that are equivalent.

If p in H and x in $H(\infty)$ are arbitrary points let γ_{px} denote the unique geodesic γ such that $\gamma(0) = p$ and $\gamma(\infty) = x$. Let $V(p,x)$ denote $\gamma'_{px}(0)$. If q, r are points of $\overline{H} = H \cup H(\infty)$ distinct from a point p in H, then let $\measuredangle_p(q,r)$ denote the angle subtended by $V(p,q)$ and $V(p,r)$. The space $\overline{H}$ has a *cone topology* that makes it homeomorphic to the closed unit n-ball. Relative to this topology the functions $V(p,x)$ and $\measuredangle_p(q,r)$ are continuous in the variables p, x and p, q, r.

Isometries of H and limit sets. If φ is an isometry of H then φ extends to a homeomorphism of H by requiring that $\varphi[\gamma(\infty)] = (\varphi \circ \gamma)(\infty)$. Each isometry φ of H has a fixed point in $\overline{H}$ since $\overline{H}$ is an n-ball. If φ has a fixed point in H, then φ is *elliptic*, a case we do not consider.

DEFINITION 1.6. Let φ be an isometry of H that generates a freely acting, properly discontinuous (infinite) cyclic group of isometries of H. Then φ is *parabolic* if it has a single fixed point in $H(\infty)$, and φ is *axial* if it has exactly two fixed points in $H(\infty)$.

If φ is a nonidentity isometry of H that generates a freely acting, properly discontinuous isometry group, then φ has at most two fixed points in $H(\infty)$ by

Proposition 2.6 of [5] and hence must be either parabolic or axial. If φ is axial with fixed points x, y in $H(\infty)$, then there exists a geodesic γ joining x to y such that $(\varphi \circ \gamma)(t) = \gamma(t + c)$ for all t and some positive number c.

DEFINITION 1.7. Let D denote a freely acting, properly discontinuous group of isometries of H. Let $L(D)$ be the set of accumulation points in $H(\infty)$ of an orbit $D(p)$, where p is a point of H. $L(D)$ is called the *limit set of D* and its complement $O(D) = H(\infty) - L(D)$ is the set of *ordinary points of D*.

$L(D)$ is a closed, D-invariant subset of $H(\infty)$ that does not depend on the point p. It is precisely the set of points in $\overline{H}$ where D fails to act freely and properly discontinuously. $L(D)$ consists of one point, two points, a Cantor set or equals $H(\infty)$. If $M = H/D$ is two dimensional, then $L(D)$ consists of one point or two if and only if $D \cong \pi_1(M)$ is infinite cyclic. If $L(D)$ is an infinite set, then M admits infinitely many inequivalent periodic geodesics, and the orbit $D(x)$ is dense in $L(D)$ for each x in $L(D)$.

DEFINITION 1.8. Points x and y of $H(\infty)$, not necessarily distinct, are *dual relative to D* or simply *dual* if there exists a sequence $\varphi_n \subseteq D$ such that $\varphi_n(p) \to x$ and $\varphi_n^{-1}(p) \to y$ for every point p in H.

From Propositions 2.6 and 2.8 of [5] we obtain

PROPOSITION 1.9. *If $L(D)$ is a single point x, then x is dual to itself. If $L(D)$ consists of two points x, y then x and y are dual, but neither point is dual to itself. If $L(D)$ is an infinite set, then any two points of $L(D)$, not necessarily distinct, are dual.*

Horospheres. For details see §3 of [10], §2 of [7] and §2 of [4].

Define B: $SH \times H \to \mathbf{R}$ by $B(v,p) = \lim_{t \to +\infty} d(p, \gamma_v t) - t$. Define α: $H \times \overline{H} \times H \to \mathbf{R}$ by

(1) If $x \in H(\infty)$, then $\alpha(p,x,q) = B(V(p,x),q)$.

(2) If $x \in H$ then $\alpha(p,x,q) = d(q,x) - d(p,x)$.

The functions B and α are continuous relative to the product topologies.

DEFINITION 1.10. The *horosphere* determined by a unit vector $v = \{q \in H$: $B(v,q) = 0\}$. Let $L(p,x)$ denote the horosphere determined by $V(p,x)$; alternatively, $L(p,x) = \{q \in H$: $\alpha(p,x,q) = 0\}$. A *horosphere at x* is a horosphere $L(p,x)$ for some p in H.

DEFINITION 1.11. A *Busemann function f* at a point x in $H(\infty)$ is one of the functions f: $q \to \alpha(p,x,q)$.

PROPOSITION 1.12. *Busemann functions have the following properties*:

(1) *If f is a Busemann function at any point x in $H(\infty)$, then $|f(p) - f(q)| \leqslant d(p,q)$ for any points p, q of H.*

(2) *If f and g are any two Busemann functions at a point x in $H(\infty)$, then $f - g$ is constant in H. In particular $f(\gamma t) - f(\gamma s) = s - t$ for any geodesic γ with $\gamma(\infty) = x$ and for all numbers s, t.*

(3) *Any Busemann function f at x is C^1 in H and* $(\operatorname{grad} f)(q) = -V(q,x)$ *for all q in H. The level sets of f are the horospheres at x.*

DEFINITION 1.13. Let f be a Busemann function at x, L a horosphere at x and p a point of L. The *inside of* $L = \{q \in H: f(q) < f(p) = f(L)\}$, and the *outside of* $L = \{q \in H: f(q) > f(p)\}$.

The various minimizing conditions on geodesics of $M = H/D$ can be formulated usefully in terms of Busemann functions. Let $\tilde{\gamma}$ be a geodesic ray in H, and let $\gamma = \pi \circ \tilde{\gamma}$ be the corresponding geodesic ray in M. Let f be a Busemann function at $x = \tilde{\gamma}(\infty)$. Then γ is minimizing if and only if $f(\varphi\tilde{\gamma}0) \geqslant f(\tilde{\gamma}0)$ for all φ in D; ultimately minimizing if and only if $f(\varphi\tilde{\gamma}t_0) \geqslant f(\tilde{\gamma}t_0)$ for some $t_0 > 0$ and all φ; almost minimizing if and only if $f(\varphi\tilde{\gamma}0) \geqslant f(\tilde{\gamma}0) - A$ for all φ and some positive number A. See the argument of Lemma 7.3 of [10] for details.

2. Definition of the horocycle flow. We assume in this section that $M = H/D$ is a complete, orientable two dimensional manifold that either has nonpositive Gaussian curvature or has no conjugate points and satisfies the uniform Visibility axiom. Fix orientations of H and M so that the projection map $\pi: H \to M$ is orientation preserving. In defining the horocycle flow in SM we obtain a continuous map $h: SM \times \mathbf{R} \to SM$ with associated flow maps $h_t: v \to h(t,v)$. We apparently cannot easily obtain this flow from a C^1 vector field on SM in the general situation that we consider. There does exist a naturally defined vector field Z on SM, and it gives rise to the flow maps $\{h_t\}$ whenever it is C^1 (for example, M compact with $K < 0$). In general it is not clear that Z satisfies Lipschitz conditions strong enough to produce unique integral curves through each point of M. For this reason the construction of the flow depends upon elementary but technical results that are only stated here and proved in detail in the Appendix.

PROPOSITION 2.1. *Let $L(p,x)$ be an arbitrary horocycle in H. Then there exists a unique C^1 unit speed curve β: $\mathbf{R} \to L(p,x)$ that is a diffeomorphism of $\mathbf{R}$ onto $L(p,x)$ such that $\beta(0) = p$ and the pair $\{V(p,x), \beta'(0)\}$ is positively oriented.*

Assuming this fact we may define the horocycle flow in SM. We call the curve β the positively oriented parametrization of $L(p,x)$ starting at p.

DEFINITION 2.2. Let $t \in \mathbf{R}$ and $v \in SH$ be given. Define h_0 to be the identity map on SH. If $t \neq 0$ and v is written as $V(p,x)$, then define $h_t v = V(\beta t, x)$, where β is the positively oriented parametrization of $L(p,x)$ starting at p.

PROPOSITION 2.3. *For any numbers s, t we have $h_{s+t} = h_s \circ h_t$.*

PROPOSITION 2.4. *Define h: $SH \times \mathbf{R} \to SH$ by $h(v,t) = h_t(v)$. If $SH \times \mathbf{R}$ has* *the product topology, then the map h is continuous.*

We now define the horocycle flow in SM, $M = H/D$. If β is the positively oriented parametrization of a horocycle $L(p,x)$ that starts at p, then $\varphi \circ \beta$ is the positively oriented parametrization of $L(\varphi p, \varphi x)$ that starts at φp, for any orientation preserving isometry φ of H. It follows that $\varphi_* \circ h_t = h_t \circ \varphi_*$ in SH for any $t \in \mathbf{R}$ and any orientation preserving isometry φ of H. We may define maps $\tilde{h}_t$: $SM \to SM$ by setting $\tilde{h}_t(\pi_* w) = \pi_*(h_t w)$ for any vector $w \in SH$ and any $t \in \mathbf{R}$. The maps $\tilde{h}_t$ are well defined, and it follows from the corresponding assertions for the maps h_t that $\tilde{h}_t \circ \tilde{h}_s = \tilde{h}_{t+s}$ and $\tilde{h}$: $SM \times \mathbf{R} \to SM$ is continuous.

3. **Basic properties of the horocycle flow.** In this section we describe for the horocycle flow the nonwandering set, the periodic vectors and the α and ω-limit sets determined by a vector. Our method is to use information about the geodesic flow to obtain information about the closely related horocycle flow. This approach was used by Hedlund, E. Hopf and others [14], [15], [17]. If $\{g_t\}$ denotes the geodesic flow on SM, then each map g_t carries the horocycle orbit of v onto the horocycle orbit of $g_t(v)$. In fact, if s, t are numbers and $v \in SM$ is any vector, then

$$(g_t \circ h_s)(v) = (h_{s^*} \circ g_t)(v),$$

where $s^* = s^*(t,s,v)$ depends in general on three variables. However, if M has Gaussian curvature $K \equiv 0$, then $s^*(t,s,v) = s$, while if M has Gaussian curvature $K \equiv -1$ then $s^*(t,s,v) = se^{-t}$. One can show that if the horocycle flow arises from a C^1 vector field Z on SM, then Z is orthogonal to the C^∞ vector field V determined by the geodesic flow, relative to the inner product on SM arising from the Riemannian connection on M.

Our basic hypothesis is still that M is an orientable uniform Visibility surface or an orientable Visibility surface with nonpositive Gaussian curvature. Fix compatible orientations of H and M.

A *complete flow* on a second countable Hausdorff space X is a homomorphism φ of the additive real numbers into the group of homeomorphisms of X. Let φ_t denote $\varphi(t)$ and let $\{\varphi_t\}$ denote the entire flow. For each point x in X there are associated some closed sets that are invariant under each map φ_t.

(1) $\omega(x) = \{y \in X\colon \varphi_{t_n} x \to y$ for some sequence t_n diverging to $+\infty\}$.

(2) $\alpha(x) = \{y \in X\colon \varphi_{t_n} x \to y$ for some sequence t_n diverging to $-\infty\}$.

These are the ω and α-limit sets of x. For each $x \in X$ the sets $\omega(x)$ and $\alpha(x)$ are contained in the *nonwandering set* $\Omega_\varphi = \{x \in X\colon$ for any open set U containing x, $\varphi_t(U) \cap U$ is nonempty for arbitrarily large positive values of $t\}$. The set Ω_φ is also closed and invariant under each map φ_t.

In the case that we consider $X = SM$. We denote the horocycle flow in both

SH and SM by $\{h_t\}$ and the geodesic flow by $\{g_t\}$. The invariant sets for $\{h_t\}$ will be denoted by $\alpha_h(v)$, $\omega_h(v)$ and Ω_h respectively and those for $\{g_t\}$ by $\alpha_g(v)$, $\omega_g(v)$ and Ω_g. Let $h(v)$ denote the orbit of v under $\{h_t\}$. Vectors periodic relative to $\{h_t\}$ will be called *h-periodic* or simply *periodic* while vectors periodic relative to $\{g_t\}$ will always be called *g-periodic*.

We begin by proving our remark that the map g_t carries $h(v)$ onto $h(g_t v)$. It suffices to verify this assertion in SH. Given $v \in SH$ and $t \in \mathbf{R}$ let $x = \gamma_v(\infty)$ and let L, L' be the horocycles determined by $v, g_t v$. If f is the Busemann function at x such that $L = f^{-1}(0)$, then $L' = f^{-1}(-t)$. Then

$$h(v) = \{V(q,x) \colon q \in L\} \text{ and } h(g_t v) = \{V(q',x) \colon q' \in L'\}.$$

By Proposition 1.12, $f(\gamma_{qx}t) = -t$ for all $q \in L$ and $f(\gamma_{q'x}(-t)) = 0$ for all $q' \in L'$, which implies that $\gamma_{qx}t \in L'$ and $\gamma_{q'x}(-t) \in L$. Therefore $g_t h(v) = h(g_t v)$.

Since $\{g_t\}$ permutes the horocycle orbits it follows that for any $v \in SM$ and any numbers s, t we have $(g_t \circ h_s)(v) = (h_{s*} \circ g_t)(v)$, where $s^* = s^*(t,s,v)$. It is not difficult to show that s^* has the same sign as s.

PROPOSITION 3.1. *Let $t \in \mathbf{R}$ and $v \in SM$ be given. Then $g_t \alpha_h(v) = \alpha_h(g_t v)$, $g_t \omega_h(v) = \omega_h(g_t v)$ and $g_t \Omega_h = \Omega_h$.*

PROOF. If s_n diverges to $+\infty$ $(-\infty)$ then it is straightforward to show, using Lemmas 2.1c and 2.4d (Appendix), that $s_n^* = s_n^*(t, s_n, v)$ also diverges to $+\infty$ $(-\infty)$. This shows that g_t permutes the α and ω-limit sets. The invariance of Ω_h under g_t is a consequence of the next result, which should be compared with Lemma 3.5 of [5].

PROPOSITION 3.2. *Let $v \in SM$ be given and let $\tilde{v} \in SH$ be a lift of v. Then $v \in \Omega_h$ if and only if $\gamma_{\tilde{v}}(\infty)$ lies in $L(D)$ and is dual to itself.*

PROOF. Suppose first that $v \in \Omega_h$. Choose sequences $t_n \subseteq \mathbf{R}$ and $v_n \subseteq SM$ such that $t_n \to +\infty$, $v_n \to v$ and $h_{t_n} v_n \to v$. Let $\tilde{v}_n$ and $\tilde{v}$ be lifts to SH of v_n and v such that $\tilde{v}_n \to \tilde{v}$ and choose a sequence $\varphi_n \subseteq D$ such that $(\varphi_n)_* h_{t_n} \tilde{v}_n \to \tilde{v}$. We assert that $\varphi_n p$ and $\varphi_n^{-1} p$ converge to $x = \gamma_{\tilde{v}}(\infty)$ for any point p in H, which shows that x lies in $L(D)$ and is self-dual.

We show first that $\varphi_n p \to x$, where $p = \mu(\tilde{v})$. Let $x_n = \gamma_{\tilde{v}_n}(\infty)$. By the choice of φ_n the point $q_n = \mu(h_{t_n} \tilde{v}_n)$ can be written as $\varphi_n^{-1} \bar{p}_n$, where $\bar{p}_n \to p$. If $p_n = \mu(\tilde{v}_n)$ then the distance from q_n to the geodesic ray $\gamma_{\tilde{v}_n} = \gamma_{p_n x_n}$ is $\geqslant \frac{1}{2} d(p_n, q_n)$ by the argument used to prove fact 2) of Theorem 5.2 of [7]. Hence

$$d(\varphi_n^{-1} p, \gamma_{px_n}[0,\infty)) \to \infty \quad \text{and} \quad \sphericalangle_p (\varphi_n p, \varphi_n x_n) = \sphericalangle_{\varphi_n^{-1} p} (p, x_n) \to 0$$

by the uniform Visibility axiom. Therefore

$$\lim_{n\to\infty} \varphi_n p = \lim_{n\to\infty} \varphi_n x_n = \lim_{n\to\infty} \gamma_{w_n}(\infty) = \gamma_{\tilde{v}}(\infty) = x,$$

where $w_n = (\varphi_n)_* h_{t_n} \tilde{v}_n$. Since $\sphericalangle_p (\varphi_n p, \varphi_n q) \to 0$ by the uniform Visibility axiom it follows that $\varphi_n q \to x$ for any point q in H.

Next we show that $\varphi_n^{-1} p \to x$. Let p_n, q_n, x_n be as above and let σ_n be the geodesic ray $\gamma_{q_n x_n}$. It suffices to prove that $d(p_n, \sigma_n[0,\infty)) \to \infty$, for then $\sphericalangle_p (q_n, x_n) = \sphericalangle_p(\sigma_n(0), \sigma_n(\infty)) \to 0$ by the uniform Visibility axiom. Then $\lim_{n\to\infty} \varphi_n^{-1} p = \lim_{n\to\infty} q_n = \lim_{n\to\infty} x_n = x$. Suppose that by passing to a subsequence we can find numbers $s_n \geqslant 0$ such that $d(p_n, \sigma_n s_n) \leqslant K^*$ for all n and some positive number K^*. Now $s_n \to +\infty$ since $d(p_n, q_n) \to +\infty$ by Lemmas 2.1c and 2.4d (Appendix). If f_n is any Busemann function at x_n, then $f_n(\sigma_n s_n) = f_n(q_n) - s_n = f_n(p_n) - s_n \to -\infty$ by Proposition 1.12. However, Proposition 1.12 also shows that

$$|f_n(\sigma_n s_n) - f_n(p_n)| \leqslant d(p_n, \sigma_n s_n) \leqslant K^*,$$

a contradiction. Therefore $d(p_n, \sigma_n[0,\infty)) \to +\infty$.

Next suppose that $x \in L(D)$ is dual to itself, and let $v = (\pi \circ \tilde{\gamma})'(0)$, where $\tilde{\gamma}(\infty) = x$. We show that $v \in \Omega_h$. Let $p = \tilde{\gamma}(0)$. By hypothesis there exists a sequence $\varphi_n \subseteq D$ such that $\varphi_n p \to x$ and $\varphi_n^{-1} p \to x$. By Proposition 2.7 of [3] there exists a point $x_n \in H(\infty)$ such that $L(p, x_n) = L(\varphi_n^{-1} p, x_n)$ and $\varphi_n^{-1} p$ lies to the left of γ_{px_n}. The points x_n converge to x by the argument proving fact 1) of Theorem 5.2 of [7]. If β_n is the canonical unit speed parametrization of $L(p, x_n)$ starting at p, then $\varphi_n^{-1} p = \beta_n(t_n)$, where $t_n \to +\infty$. If $v_n = \pi_* V(p, x_n)$ then we assert that $h_{t_n} v_n \to v$ and $v_n \to v$, which will prove that $v \in \Omega_h$. Clearly $v_n \to v$ since $x_n \to x$. If $\tilde{v}_n = V(p, x_n)$ it suffices to prove that $w_n = (\varphi_n)_* h_{t_n} \tilde{v}_n \to \tilde{v} = V(p, x)$. Now $\mu(w_n) = p$ and $\gamma_{w_n}(\infty) = \varphi_n(x_n)$ so it suffices to prove that $\varphi_n(x_n) \to x$. But $\sphericalangle_{\varphi_n^{-1} p} (p, x_n) \to 0$ since

$$d(\varphi_n^{-1} p, \gamma_{px_n}[0,\infty)) \geqslant \tfrac{1}{2} d(p, \varphi_n^{-1} p)$$

by the proof of fact 2) of Theorem 5.2 of [7]. Hence

$$\sphericalangle_p (\varphi_n p, \varphi_n x_n) = \sphericalangle_{\varphi_n^{-1} p} (p, x_n) \to 0,$$

which implies that $\lim_{n\to\infty} \varphi_n x_n = \lim_{n\to\infty} \varphi_n p = x$. This completes the proof.

Remark. For each $v \in SM$ one can define prolongational limit sets $h^+(v)$ and $h^-(v)$ as in Definition 3.2 of [6]. Then $v \in \Omega_h$ if and only if $v \in h^+(v) \cap h^-(v)$. If v, w are vectors in SM with lifts $\tilde{v}$, $\tilde{w}$ in SH, then by arguing as in the previous result one may show that

$$h^+(v) = h^-(v) = \{w \in SM: \gamma_{\dot{v}}(\infty) \text{ and } \gamma_{\dot{w}}(\infty) \text{ lie in } L(D) \text{ and are dual}\}.$$

Compare Proposition 3.7 of [6].

The classification of Ω_h.

THEOREM 3.3. *Given $M = H/D$, one of the following possibilities must occur:*

(1) *$L(D)$ is a single point x. Then Ω_h is a connected set consisting of a single asymptote class of vectors in SM. Every vector in Ω_h is h-periodic, and D is an infinite cyclic group of parabolic isometries with fixed point x.*

(2) *$L(D)$ consists of two points x, y. Then Ω_h is empty, and D is an infinite cyclic group of axial isometries with fixed points x and y.*

(3) *$L(D)$ is a Cantor set or $L(D) = H(\infty)$. Either Ω_h is a connected nowhere dense subset of SM or $\Omega_h = SM$. Ω_h is the orbit $h(\Omega_g) = \bigcup_{t \in \mathbf{R}} h_t(\Omega_g)$ of the geodesic flow nonwandering set. If M is compact then it has negative Euler characteristic, and if M is noncompact, then D is a nonabelian free group.*

PROOF. This result should be compared with Theorem 3.9 of [6]. All of these cases do occur, and an example of each may be found in [6, p. 499]. In particular if M is compact then $\Omega_g = SM$ and this implies that $\Omega_h = SM$.

(1) The fact that D is an infinite cyclic group of parabolic isometries with fixed point x is proved in Theorem 2.18 of [4]. By Propositions 1.9 and 3.2, Ω_h consists of the vectors $\pi_* V(p, x)$, where $p \in H$ is arbitrary. Therefore Ω_h is an entire asymptote class of vectors in SM and is a connected set. The fact that every vector of Ω_h is h-periodic is proved in the next result, Proposition 3.4.

(2) If $L(D)$ consists of two points x, y, then D is an infinite cyclic group of axial isometries with fixed points x and y by Theorem 2.18 of [4]. Moreover that result implies that $\varphi^n p \to x$ and $\varphi^{-n} p \to y$ for any p in H and a suitable generator φ for D. Therefore x and y are not self-dual, and Ω_h is empty by Proposition 3.2.

(3) The assertions regarding D and $L(D)$ are proved in Theorem 2.18 of [4]. If M is compact, then D contains a free subgroup on an infinite number of generators by Theorem 1 of [8], which is also valid in the uniform Visibility case. From Lemma 3.5 of [5] and Propositions 1.9 and 3.2 it follows that $h(\Omega_g) \subseteq \Omega_h$. Conversely let $v \in \Omega_h$ be given, and let $v^* \in SH$ be a lift of v. By Proposition 3.2, $x = \gamma_{v^*}(\infty)$ lies in $L(D)$. Let z be a point in $L(D)$ distinct from x, and let σ be a geodesic such that $\sigma(-\infty) = z$ and $\sigma(\infty) = x$. Parametrize σ so that $\sigma(0)$ lies on the horocycle determined by v^*. Thus $v^* = h_t \sigma'(0)$ for some number t since σ and γ_{v^*} are asymptotes. The vector $w = \pi_* \sigma'(0)$ lies in Ω_g by Proposition 1.9 and Lemma 3.5 of [5], and therefore $\Omega_h \subseteq h(\Omega_g)$ since $v = h_t w$ and $v \in \Omega_h$ was arbitrary.

Suppose that Ω_h contains an open subset 0 of SM. Then $(\pi_*)^{-1}\Omega_h = \{V(p, x): p \in H, x \in L(D)\}$ contains an open subset U of SH. The fact that $V: H \times H(\infty) \to SH$ is a homeomorphism, where $H(\infty)$ has the topology induced from $\bar{H}$ and $H \times H(\infty)$ has the product topology, implies that there exist open sets $A \subseteq H$ and $B \subseteq H(\infty)$ such that $V(A \times B) \subseteq U \subseteq (\pi_*)^{-1}\Omega_h$. Therefore $L(D) \supseteq B$ and by Theorem 2.18 of [4] $L(D) = H(\infty)$. Hence $\Omega_h = SM$ by Proposition 3.2.

Finally we show that Ω_h is connected. Let vectors v and w of Ω_h be given, and let $V(p,x)$ and $V(q,y)$ be lifts of v and w respectively to SH. If $r \in H$ is arbitrary, then $\pi_* V(r,x) \in \Omega_h$ by Proposition 3.2. Any arc joining p to r induces an arc of asymptotic vectors in Ω_h from v to $\pi_* V(r,x)$. By the remark following Proposition 2.8 of [5] we can find a sequence $\varphi_n \subseteq D$ such that $\varphi_n x \to y$. Therefore

$$w_n = \pi_* V(\varphi_n^{-1} q, x) = \pi_* V(q, \varphi_n x) \to \pi_* V(q,y) = w.$$

The vector w lies in the connected component $C(v)$ containing v since w_n lies in $C(v)$ for each n. Therefore Ω_h is connected since v, w were arbitrary.

Periodic vectors. We characterize the h-periodic vectors and derive some existence theorems and a density theorem. A vector $v \in SM$ is *parabolic* if for any lift $\tilde{\gamma}$ of γ_v to H the asymptote class $\tilde{\gamma}(\infty)$ is fixed by some parabolic isometry in the deckgroup D for M.

PROPOSITION 3.4. *A vector $v \in SM$ is h-periodic if and only if v is parabolic.*

PROOF. Suppose that $v \in SM$ is h-periodic and choose $t \neq 0$ so that $h_t v = v$. If $v^* \in SH$ is any lift of v, then $v^* = (\varphi)_* h_t v^* = h_t(\varphi_* v^*)$ for some φ in D. The isometry φ fixes $x = \gamma_{v^*}(\infty)$ since v^* and $\varphi_* v^*$ are asymptotic. If $p = \mu(v^*)$ then φ leaves invariant the horocycle $L(p,x)$ by Proposition 1.12(3) since $L(p,x)$ and $L(\varphi p, x) = \varphi L(p,x)$ have the point $\varphi p = \mu(\varphi_* v^*)$ in common. By Proposition 2.15 of [4] φ is a parabolic isometry and thus v is a parabolic vector.

Conversely let $v \in SM$ be a parabolic vector. Let $v^* \in SH$ be a lift of v, and let $\varphi \in D$ be a parabolic isometry fixing $x = \gamma_{v^*}(\infty)$. It follows that $\varphi_* v^*$ is asymptotic to v^*, and moreover $\varphi_* v^* = h_t v^*$ for some nonzero number t since φ leaves invariant all horocycles at x by Proposition 2.15 of [4]. Therefore $h_t v = v$ and v is h-periodic.

As a corollary we obtain

PROPOSITION 3.5. *Suppose that Ω_h contains h-periodic vectors. Then the h-periodic vectors are dense in Ω_h.*

PROOF. Let $v \in \Omega_h$ be given, and let $V(p,x) \in SH$ be a lift of v. Let $w \in \Omega_h$ be h-periodic, and let $V(q,y) \in SH$ be a lift of w. By the preceding result y is fixed by some parabolic isometry φ in D. By Theorem 3.3 we may assume that $L(D)$ is an infinite set, and then since all orbits of D in $L(D)$ are dense we may choose a sequence $\varphi_n \subseteq D$ such that $\varphi_n y \to x$. For each n the point $\varphi_n y$ is fixed by the parabolic isometry $\varphi_n \varphi \varphi_n^{-1}$ and hence

$$v_n = \pi_* V(p, \varphi_n y)$$

is h-periodic. Since $v_n \to v$ and v was arbitrary the result follows.

Parabolic vectors are also significant for the geodesic flow. The next result is a combination of Propositions 3.1 and 4.2 of [4]. We remark that if M is not finitely connected, then by Proposition 4.3 of [4] there exists for each point p in M a unit vector v in $T_p(M)$ such that $v \in \Omega_h$, v is minimizing but v is not parabolic.

PROPOSITION 3.6. *If $v \in SM$ is parabolic then v is ultimately minimizing. If M is finitely connected, then any divergent vector $v \in \Omega_h$ is parabolic.*

If Ω_g is nonempty then one may show from Theorem 2.15 of [4] and Lemma 2.7 of [5] that Ω_g contains g-periodic vectors. The analogy for Ω_h is false.

THEOREM 3.7. *Suppose that $L(D)$ contains at least three points. Then the following conditions are equivalent.*

(1) *M is finitely connected and SM has no h-periodic vectors.*

(2) *M contains a nonempty compact totally convex set A.*

(3) *For every $v \in \Omega_g$ the maximal geodesic γ_v is contained in some compact subset of M.*

(4) *The set Ω_g is a compact subset of SM.*

In the terminology of §4 of [4], (1) is equivalent to the condition that M be finitely connected and admit only expanding ends. We remark that Ω_g is nonempty by Proposition 1.9 and Lemma 3.5 of [5]. By finitely connected we mean that $\pi_1(M)$ is finitely generated. A subset A of a complete Riemannian manifold N is totally convex if for any two points p, q of A, not necessarily distinct, the set A contains all geodesic segments joining p to q. If N has nonpositive sectional curvature then every closed totally convex subset A is a strong deformation retract of N. See [1] for a detailed discussion. We note that M also admits a compact totally convex set A if $L(D)$ contains exactly two points x and y. If $\tilde{A}$ is the union of those geodesics in H joining x to y, then $\pi(\tilde{A}) = A$ is compact and totally convex and in fact $A = \mu(\Omega_g)$.

PROOF OF THE THEOREM. We show that (1) implies (2). By Theorem 3.3 either $L(D) = H(\infty)$ or $L(D)$ is an infinite proper subset of $H(\infty)$. We consider these cases separately.

Suppose that $L(D) = H(\infty)$. Let p be any point in H, and let $R_p \subseteq H$ be the canonical fundamental domain for D with center

$$p = \{q \in H\colon d(p,q) \leqslant d(\varphi p, q) \text{ for all } \varphi \text{ in } D\}.$$

(See §2 of [3] for basic results about R_p.) If R_p were noncompact, then there would exist a point x in $H(\infty)$ that is an accumulation point of some sequence in R_p. The geodesic ray $\gamma_{px}[0,\infty)$ would therefore be contained in R_p since R_p is closed and starshaped relative to p. Since $x \in L(D)$ it follows from Proposition 3.2 that $(\pi \circ \gamma_{px})'(0)$ is a minimizing vector in Ω_h and hence v is parabolic by Proposition 3.6. This contradicts the hypothesis of (1). Therefore R_p is compact and $M = \pi(R_p)$ is compact. Set $A = M$ in this case.

Next suppose that $L(D)$ is an infinite proper subset of $H(\infty)$. Under this condition we showed in §6 of [4] that M admits a closed, totally convex subset M_0 that is contained in every closed, totally convex subset of M. Proposition 6.4 of [4] shows that M_0 is compact under the hypothesis of (1). Therefore (1) implies (2).

We prove that (2) implies (3). Let A be a nonempty compact, totally convex subset of M. Given $v \in \Omega_g$ there exists a vector $v^* \in \Omega_g$ that is equivalent to v and such that $\gamma_{v^*}t \in A$ for all t in $\mathbf{R}$ by Lemma 6.3b of [4]. We recall that v and v^* in SM are equivalent if there exist lifts $\tilde{\gamma}$ and $\tilde{\sigma}$ of γ_v and γ_{v^*} to H such that $\tilde{\gamma}$ and $\tilde{\sigma}$ join the same points in $H(\infty)$. Let $\tilde{\gamma}$ and $\tilde{\sigma}$ be such lifts. For any point $q = \tilde{\gamma}(t)$ we know that

$$\sphericalangle_q(\tilde{\gamma}(\infty), \tilde{\gamma}(-\infty)) = \sphericalangle_q(\tilde{\sigma}(\infty), \tilde{\sigma}(-\infty)) = \pi.$$

The uniform Visibility axiom implies that $d(q, \tilde{\sigma}) \leqslant R$ for some positive number R not depending on q. Therefore

$$\gamma_v t \in \overline{B_R(A)} = \{q \in M: d(q, A) \leqslant R\}$$

for all t in $\mathbf{R}$. The set $\overline{B_R(A)}$ is compact since A is compact. In fact, $\overline{B_R(A)}$ does not depend on the choice of $v \in \Omega_g$.

We prove that (3) implies (1). If M were infinitely connected, then Ω_h would contain a minimizing vector v by the remark preceding Proposition 3.6. There would exist a vector v^* in Ω_g that is asymptotic to v since $\Omega_h = h(\Omega_g)$ by Theorem 3.3. Therefore γ_{v^*} would be divergent, contradicting the hypothesis of (3). Therefore M is finitely connected. If Ω_h contained an h-periodic vector v, then v would be parabolic and ultimately minimizing by Propositions 3.4 and 3.6. By choosing a vector $v^* \in \Omega_g$ asymptotic to v we would obtain a divergent geodesic γ_{v^*}, contradicting the hypothesis of (3). This proves that (3) implies (1) and shows that (1), (2) and (3) are equivalent.

We prove that (2) implies (4). Since Ω_g consists of unit vectors it suffices to show that the image $\mu(\Omega_g)$ is compact in M. In the proof that (2) implies (3) we showed that there exists a compact subset A of M such that $\gamma_v t \in A$ for all t in $\mathbf{R}$ and all v in Ω_g. It is not difficult to show that $\mu(\Omega_g)$ is a closed subset of M since Ω_g is a closed subset of SM. Therefore $\mu(\Omega_g)$ is compact since it is a closed subset of A. We have shown that (2) implies (4) and since (4) obviously implies (3) the proof of the theorem is complete.

As a corollary we obtain

PROPOSITION 3.8. *Let M be finitely connected and noncompact, and let $\Omega_h = SM$. Then Ω_h contains h-periodic vectors.*

PROOF. Proposition 3.2 implies that $L(D) = H(\infty)$ since $\Omega_h = SM$. Therefore $\Omega_g = SM$ by Proposition 1.9 and Lemma 3.5 of [5]. If Ω_h contained no h-periodic vectors, then $\Omega_g = SM$ would be compact by the preceding result. This can happen only if M is compact, contrary to our assumption.

We conclude this section by characterizing the surfaces described in the preceding result in a more classical way. The proof is not difficult, but we omit it since the result is not used.

PROPOSITION 3.9. *The following statements are equivalent.*

(1) *M is noncompact and finitely connected and $\Omega_h = SM$.*

(2) *For some point p in H the fundamental domain R_p for D is noncompact and its boundary points in $H(\infty)$ are fixed points of parabolic isometries and finite in number.*

(3) *For every point p in H the fundamental domain R_p has the properties of* (2).

By a boundary point of R_p in $H(\infty)$ we mean a point x in $H(\infty)$ that is a limit of a sequence of points in R_p. In the terminology of §4 of [4] these surfaces have finitely many ends, all of them parabolic. If M has Gaussian curvature $K \leqslant -c^2 < 0$, then M has finite area if it satisfies any of the conditions above.

4. Applications. We begin by investigating the existence of dense orbits of the horocycle flow $\{h_t\}$ in Ω_h. Clearly we must assume that $\pi_1(M)$ is not infinite cyclic for in that case Ω_h is empty or consists entirely of periodic vectors.

THEOREM 4.1. *If $\Omega_h = SM$ then $\{h_t\}$ has a dense orbit in Ω_h. In general suppose that Ω_h contains nonperiodic vectors and $\{h_t\}$ has no dense orbit in Ω_h. Then there exists a positive number c such that the period of every g-periodic vector is an integer multiple of c.*

As a consequence we obtain

COROLLARY 4.2. *Let SM contain g-periodic vectors v_1, v_2 with periods c_1, c_2 such that c_1/c_2 is irrational. Then Ω_h is nonempty and $\{h_t\}$ has a dense orbit in Ω_h.*

We believe that the exceptional case in Theorem 4.1 does not occur and that $\{h_t\}$ has a dense orbit in Ω_h whenever $L(D)$ is an infinite set. By Proposition 8.9F of [10] there exist infinitely many inequivalent periodic geodesics in this case.

To prove the corollary it suffices to show that $L(D)$ must have at least three points, for then $\Omega_g \subseteq \Omega_h$ admits g-periodic vectors (which are not h-periodic) by Proposition 2.7 of [5] and Proposition 2.15 of [4]. If $L(D)$ were a single point, then there would be no periodic geodesics by Proposition 2.15 of [4], for example. If $L(D)$ consisted of two points, then all g-periods would be integer multiples of a smallest period since D is infinite cyclic (Theorem 2.18 of [4]).

We now prove the theorem. If $\Omega_h = SM$ then $L(D) = H(\infty)$ by Proposition 3.2 and hence $\Omega_g = SM$ by Proposition 1.9 and Lemma 3.5 of [5]. The result is now a reformulation of Theorem 5.2 of [7]. The terminology of [7] is different from that used here; for a uniform Visibility manifold M of arbitrary dimension and a vector $v \in SM$ we constructed in [7] a *strong stable set* $W^{ss}(v)$, which is precisely the horocycle orbit $h(v)$ if M is two dimensional.

In the case that $\Omega_h = SM$ we shall need the following result, which is contained in the proof of Theorem 5.5 of [7] beginning with the second paragraph.

LEMMA 4.1. *Let $v \in \Omega_h$ be not almost minimizing and let $c > 0$ be the period of some g-periodic vector. Then for every vector $w \in \Omega_h$ there exists a number d with $0 \leqslant d \leqslant c$ such that $g_d w \in \overline{h(v)}$.*

We now complete the proof of Theorem 4.1. Let A_0 denote the additive subgroup of $\mathbf{R}$ generated by the periods of all g-periodic vectors. The closure of A_0 in $\mathbf{R}$, denoted by A, is also an additive subgroup of $\mathbf{R}$, and it is easy to see that either $A = \mathbf{R}$ or A consists of integer multiples of some positive number c. In the latter case all g-periods are integer multiples of c, so it suffices to prove the theorem by showing that if $A = \mathbf{R}$, then the horocycle flow has a dense orbit in Ω_h.

Let $v \in \Omega_h$ be a vector that is not almost minimizing; that is, $d(\gamma_v 0, \gamma_v t) - t \to -\infty$ as $t \to +\infty$. For example, any vector v that is g-periodic is not almost minimizing. We shall show that regardless of the nature of A, if $c' > 0$ is any element of A, then for any vector $v^* \in \Omega_h$ there exists a number d with $0 \leqslant d \leqslant c'$ such that $g_d v^*$ lies in the closure of the horocycle orbit of v, $\overline{h(v)}$. If $A = \mathbf{R}$ then A contains arbitrarily small positive numbers c', and it follows that any vector $v \in \Omega_h$ that is not almost minimizing has a dense orbit in Ω_h.

We may prove the assertion above in the case that $c' > 0$ lies in A_0 since A_0 is dense in A. A_0 consists of finite sums $\sum_{i=1}^{k} m_i w_i$, where m_i is an arbitrary integer and $w_i > 0$ is the period of some vector v_i that is g-periodic. We note that v_i is not almost minimizing. Replacing v_i by a suitable translate $g_{t_i} v_i$ we may further assume that $v_1 \in \overline{h(v)}$ and $v_{i+1} \in \overline{h(v_i)}$ for every $i \geqslant 1$ by Lemma 4.1. Let $v^* \in \Omega_h$ be arbitrary. Then $g_t v^* \in \overline{h(v_k)}$ for some number t by Lemma 4.1. Choose an integer n such that $nc' \leqslant t \leqslant (n+1)c'$. Since $g_t v^* \in \overline{h(v_k)}$ it follows that

$$g_{t-nm_k w_k} v^* \in g_{-nm_k w_k} \overline{h(v_k)} = \overline{h(g_{-nm_k w_k} v_k)} = \overline{h(v_k)}.$$

Now $\overline{h(v_k)} \subseteq \overline{h(v_{k-1})}$ since $v_k \in \overline{h(v_{k-1})}$. Therefore

$$g_{t-nm_k w_k - nm_{k-1} w_{k-1}} v^* \in g_{-nm_{k-1} w_{k-1}} \overline{h(v_{k-1})} = \overline{h(g_{-nm_{k-1} w_{k-1}} v_{k-1})} = \overline{h(v_{k-1})}.$$

Continuing in this fashion we see that $g_{t-nc'} v^* = g_{t-n(\sum m_i w_i)} v^* \in \overline{h(v)}$. We have proved the desired result since $0 \leqslant t - nc' \leqslant c'$ and $v^* \in \Omega_h$ was arbitrary.

Classification of vectors with dense orbits in Ω_h.

THEOREM 4.3. *Suppose that $\{h_t\}$ has a dense orbit in Ω_h. Then $v \in \Omega_h$ has a dense h-orbit if and only if v is not almost minimizing.*

PROOF. This is Theorem 5.5 of [7].

In Theorem 2.3 of [15] Hedlund shows that if M has Gaussian curvature -1, if $\Omega_h = SM$ and if $v \in SM$ is not almost minimizing, then v has a dense horocycle orbit in SM. He does not remark that the condition that v be not almost minimizing is also necessary. The necessity of this condition was also recently observed in [20]. One can show that $v \in SM$ is not almost minimizing if and only if for any lift $v^* \in SH$ of v the horocycles at $x = \gamma_{v^*}(\infty)$ have the following property: given a point p in H, a positive number R and a horocycle L at x, there exists an isometry φ in D such that the open disc $B_R(p)$ lies inside $\varphi(L)$. Hedlund's result is stated in terms of this formulation of the almost minimizing property.

COROLLARY 4.3. *Let M be noncompact and finitely connected, and let $\Omega_h = SM$. Then every orbit of $\{h_t\}$ is either dense in SM or periodic. Moreover, periodic orbits exist.*

Hedlund proved an equivalent formulation of this result in Theorem 2.6 of [15] for the case that M has Gaussian curvature -1. The fact that his formulation is equivalent follows from Proposition 3.9.

PROOF OF THE COROLLARY. Every almost minimizing vector v determines a divergent geodesic γ_v. Consequently if v is not h-periodic, then v is nondivergent by Proposition 3.6, and thus v is not almost minimizing. There exists a dense h-orbit in SM by Theorem 4.1, and therefore every h-orbit in SM is either dense in SM or periodic by the previous result. The existence of periodic orbits is a consequence of Proposition 3.8.

Minimal sets. If $\{\varphi_t\}$ is any complete flow on a space X, then a closed subset A of X is *minimal* if it is invariant under $\{\varphi_t\}$ and if the $\{\varphi_t\}$ orbit of every point a in A is dense in A. A periodic orbit is the simplest example of a minimal set.

PROPOSITION 4.4. *If $A \subseteq SM$ is a minimal set for the horocycle flow, then for any number t the set $g_t(A)$ is another minimal set.*

PROOF. The orbit $h(g_t v) = g_t h(v)$ is dense in $g_t(A)$ if and only if $h(v)$ is dense in A. Therefore A is a minimal set if and only if $g_t(A)$ is a minimal set.

THEOREM 4.5. *The horocycle flow $\{h_t\}$ is minimal in SM if and only if M is compact. If $\{h_t\}$ is minimal in Ω_h, then M admits a nonempty, compact, totally convex set. If M admits a nonempty, compact, totally convex set and if Ω_h is nonempty, then the orbit closure $\overline{h(v)}$ is a minimal subset of Ω_h for every $v \in \Omega_h$. Moreover there exists a positive number c such that if A, B are any two minimal subsets of Ω_h, not necessarily distinct, then $g_d(A) = B$ for some positive number $d \leqslant c$. In addition $\{g_t\}$ is a suspension flow in Ω_g over $\overline{h(v)} \cap \Omega_g$ for any $v \in \Omega_g$.*

It follows immediately that if M admits a nonempty, compact, totally convex set and $\{h_t\}$ has a dense orbit in Ω_h, then $\{h_t\}$ is minimal in Ω_h.

PROOF. The first statement of the theorem is Theorem 6.1 of [7]. The proof of that result must be modified; the proof given here is simpler although identical in outline. If M is compact, then $\Omega_g = SM$ since SM has finite measure relative to the natural Riemannian measure that is invariant under the geodesic flow. Therefore $\Omega_h = SM$ by Theorem 3.3, and there is a dense h-orbit in SM by Theorem 4.1. Every vector $v \in SM$ is not almost minimizing since M is compact, and hence $\overline{h(v)} = SM$ for all $v \in SM$ by Theorem 4.3. Conversely suppose that $\{h_t\}$ is minimal in SM. Note that Ω_h is nonempty; since $SM = \omega_h(v) \cup \alpha_h(v) \cup h(v)$ for every $v \in SM$, either $\omega_h(v)$ or $\alpha_h(v)$ is nonempty. In fact, $\Omega_h = SM$ for if $v \in \Omega_h$ then $SM = \overline{h(v)} \subseteq \Omega_h$. If M were noncompact then for any point p in M there would exist a minimizing geodesic ray starting at p. The h-orbit of $v = \gamma'(0)$ would not be dense in SM by Theorem 4.3, which contradicts the minimality assumption. Therefore M is compact.

Suppose now that Ω_h is nonempty and that $\{h_t\}$ is minimal in Ω_h. Then Ω_h contains no h-periodic vectors since it never consists of a single periodic orbit (Theorem 3.3). We assert that M is finitely connected. If this were false, then Ω_h would contain a minimizing vector that is not parabolic by Proposition 4.3 of [4]. Therefore $h(v)$ would not be dense in Ω_h by Theorem 4.3, a contradiction. Thus, M is finitely connected, and by Theorem 3.7, M admits a nonempty, compact, totally convex set.

Next, assume that M admits a nonempty, compact, totally convex set. Proposition 3.2 and Lemma 6.3b of [4] show that the geodesic ray γ_v is nondivergent on $[0, \infty)$ for every $v \in \Omega_h$. Therefore any vector $v \in \Omega_h$ is not almost minimizing. Hence if v and w are any two vectors in Ω_h, then $g_d w \in \overline{h(v)}$ for some number d by Lemma 4.1. Since Ω_g is compact an application of Zorn's lemma shows that $\Omega_h = h(\Omega_g)$ contains some minimal set A. Let v be a vector in A, and let w be any vector in Ω_h. Then $g_d w \in \overline{h(v)}$ for some number d, which implies that $w \in g_{-d}(A)$. The set $g_{-d}(A)$ is minimal by Proposition 4.4 and since w was arbitrary this proves that every vector in Ω_h lies in a minimal subset. Hence $\overline{h(w)}$ is a minimal subset for every $w \in \Omega_h$. Now let A and B be any two minimal sets, possibly the same, and let $w \in A$ and $v \in B$ be given. Choose a number d such that $g_d w \in \overline{h(v)} = B$. The minimal sets $g_d(A)$ and B must be equal since they intersect.

Define a number c to be a *period* if $g_c(A) = A$ for some minimal set A. It follows that $g_c(B) = B$ for any minimal set B since $B = g_d(A)$ for some number d. If c is a positive period, then ones sees immediately that for any two minimal subsets A and B in Ω_h, $g_d(A) = B$ for some number d with $0 < d \leqslant c$. The set A^* of periods of the minimal subsets of Ω_h forms a closed additive subgroup of $\mathbf{R}$. If $A^* = \mathbf{R}$, then clearly the horocycle flow is minimal in Ω_h, while if A^* is an additive cyclic group generated by some positive number c, then c is the smallest positive period for the minimal subsets of Ω_h. We remark that if $c' > 0$ is the period of a vector v that is g-periodic, then $c' \in A^*$. If $B = \overline{h(v)}$, then the minimal sets B and $g_{c'}(B)$ both contain v and hence must be equal. Finally it is clear that $\{g_t\}$ is a suspension flow in Ω_g over $\overline{h(v)} \cap \Omega_g$ for any $v \in \Omega_g$ if $\{h_t\}$ has no dense orbit.

If M does not admit a compact, totally convex set, then the minimal sets are entirely different.

PROPOSITION 4.6. *Suppose that Ω_h contains an almost minimizing vector. Then every minimal subset A of Ω_h consists entirely of almost minimizing vectors.*

COROLLARY 4.7. *Let M be finitely connected, and suppose that Ω_h contains vectors periodic relative to the horocycle flow. Then the only minimal subsets of Ω_h are the periodic orbits.*

PROOF OF THE COROLLARY. This result says that the minimal subsets of Ω_h are of the simplest possible type. Note that if Ω_h contains no periodic vectors, then M admits a compact, totally convex set by Theorem 3.7. A vector $v \in \Omega_h$ is almost minimizing if and only if it is parabolic, hence periodic, by Propositions 3.4 and 3.6. The corollary now follows from Proposition 4.6.

REMARK. In Corollary 4.3 we actually proved the stronger result that if M is noncompact and finitely connected and if $\Omega_h = SM$, then the only closed sets invariant under the horocycle flow are SM and unions of periodic orbits.

Before proving Proposition 4.6 we shall need the following result.

LEMMA 4.6. *Let $v \in \Omega_h$ be almost minimizing. Then $\overline{h(v)}$ contains only almost minimizing vectors.*

PROOF. Let $v \in \Omega_h$ be almost minimizing. Then clearly $h_s(v)$ is almost minimizing for any number s since the geodesics with initial velocities $h_s(v)$ are asymptotic to γ_v. It suffices to show that $\omega_h(v)$ and $\alpha_h(v)$ contain only almost minimizing vectors since $\overline{h(v)} = \omega_h(v) \cup \alpha_h(v) \cup h(v)$. We shall consider only the ω-limit case.

Suppose that $v^* \in \omega_h(v)$, and let $s_n \to +\infty$ be a sequence such that $v_n = h_{s_n} v \to v^*$. Let $V(p,x)$ and $V(q,y)$ be lifts to SH of v and v^* respectively. Then there exists a sequence $\varphi_n \subseteq D$ such that $(\varphi_n)_* V(\beta s_n, x) \to V(q,y)$, where β is the positively oriented unit speed parametrization of $L(p,x)$ that starts at p. Note, $v_n = \pi_* V(\beta s_n, x)$. There exists a number $A > 0$ such that $d(\gamma_v t, \gamma_v 0) - t \geqslant -A$ for all $t \geqslant 0$ since v is almost minimizing. It suffices to prove that given a number $\varepsilon > 0$ there exists an integer $N > 0$ such that

$$d(\gamma_{v_n} 0, \gamma_{v_n} t) - t \geqslant -(A + d(p,q) + \varepsilon)$$

for all $n \geqslant N$ and all $t \geqslant 0$. If this is established, then it will follow by continuity that $d(\gamma_{v^*} 0, \gamma_{v^*} t) - t \geqslant -(A + d(p,q))$ for all $t \geqslant 0$ since $v_n \to v^*$. This will prove that v^* is almost minimizing.

Let $\varepsilon > 0$ be given. It follows by continuity of the vector function V that

$$d(\varphi_n \beta s_n, q) = d(\beta s_n, \varphi_n^{-1} q) \to 0 \quad \text{and} \quad \varphi_n x \to y$$

as $n \to \infty$ since $(\varphi_n)_* V(\beta s_n, x) = V(\varphi_n \beta s_n, \varphi_n x) \to V(q,y)$. Choose $N > 0$ so large that $d(\beta s_n, \varphi_n^{-1} q) < \varepsilon$ for $n \geqslant N$, and let f be the Busemann function at x such that $L(p,x) = f^{-1}(0)$. We note that $L(\beta s_n, x) = L(p,x)$ for every n since $\beta s_n \in L(p,x)$. It follows by the discussion at the end of §1 that $f(\varphi p) \geqslant -A$ for all $\varphi \in D$ since $d(\gamma_v t, \gamma_v 0) - t \geqslant -A$ for all $t \geqslant 0$. Given an element $\varphi \in D$ we observe that for $n \geqslant N$,

$$\begin{aligned} f(\varphi\beta s_n) &= f(\varphi\beta s_n) - f(\varphi\varphi_n^{-1} p) + f(\varphi\varphi_n^{-1} p) \\ &\geqslant -|f(\varphi\beta s_n) - f(\varphi\varphi_n^{-1} p)| - A \geqslant -d(\varphi\beta s_n, \varphi\varphi_n^{-1} p) - A \\ &= -d(\beta s_n, \varphi_n^{-1} p) - A \geqslant -(A + d(p,q) + \varepsilon) \end{aligned}$$

since $d(\beta s_n, \varphi_n^{-1} p) \leqslant d(\beta s_n, \varphi_n^{-1} q) + d(\varphi_n^{-1} q, \varphi_n^{-1} p) < d(p,q) + \varepsilon$. It follows that $d(\gamma_{v_n} t, \gamma_{v_n} 0) - t \geqslant -(A + d(p,q) + \varepsilon)$ for all $n \geqslant N$ and all $t \geqslant 0$ since we have shown that $f(\varphi\beta s_n) \geqslant -(A + d(p,q) + \varepsilon)$ for all $n \geqslant N$ and all $\varphi \in D$. This completes the proof of the lemma.

Proof of Proposition 4.6. Let $A \subseteq \Omega_h$ be a minimal subset for the horocycle flow. Suppose that A contains a vector v that is not almost minimizing. By hypothesis Ω_h contains a vector w that is almost minimizing. For some number d, $g_d w \in \overline{h(v)} = A$ by Lemma 4.1. We know that $\overline{h(g_d w)} = A$ since the horocycle flow is minimal in A, but the lemma above says that v cannot be in $\overline{h(g_d w)}$ since $g_d w$ is almost minimizing. This contradiction shows that A contains only almost minimizing vectors.

If M is infinitely connected we do not know what the minimal sets of Ω_h actually look like aside from the fact that they contain only almost minimizing vectors. We believe that in the infinitely connected case there exist vectors $v \in \Omega_h$ such that $\alpha_h(v)$ and $\omega_h(v)$ are both empty, and in this case the orbit $h(v)$ would be a minimal set.

The next result shows that the compact minimal sets of Ω_h are particularly simple.

PROPOSITION 4.8. *Let $A \subseteq \Omega_h$ be a nonempty compact minimal set for the horocycle flow. Then either M is compact and $A = SM$ or M is noncompact and A is a periodic orbit.*

PROOF. We first dispose of the case that $L(D)$ contains one point or two. If $L(D)$ is a single point, then every orbit in Ω_h is periodic by Theorem 3.3. That result also excludes the possibility that $L(D)$ has exactly two points, for Ω_h would be empty in this case. We may therefore assume that $L(D)$ is an infinite set. If M is compact, then $\{h_t\}$ is minimal in SM by Theorem 4.5, and $A = SM$.

We now suppose that M is noncompact, and we show first that Ω_h is noncompact. If $L(D) = H(\infty)$, then let γ be any geodesic in M that is minimizing on $[0, \infty)$. By Propositions 1.9 and 3.2 the velocity vectors $\gamma'(t)$ lie in Ω_h for all $t \geqslant 0$, which shows that Ω_h is noncompact in this case. If $L(D)$ is an infinite proper subset of $H(\infty)$ let γ be a geodesic of H such that $\gamma(\infty) \in L(D)$ and $\gamma(-\infty) \in O(D)$. If $w = (\pi \circ \gamma)'(0)$ then the vectors $g_{-n}w$ lie in Ω_h for all positive integers n. If some subsequence of these vectors converged to a vector w^* in Ω_h, then we could find a sequence $\varphi_n \subseteq D$ and a number $R > 0$ such that $d(\varphi_n p, \gamma(-n)) \leqslant R$ for all n, where $p = \gamma(0)$. Hence $\varphi_n p$ would converge to $\gamma(-\infty)$, contradicting the fact that $\gamma(-\infty) \in O(D)$. The vectors $g_{-n}w$ therefore have no cluster point and Ω_h is noncompact.

We show next that any vector v in A is almost minimizing. Suppose that this is false for some v in A and let a number $c > 0$ be chosen as in the statement of Lemma 4.1. Since Ω_h is noncompact we can find a vector w in Ω_h such that $g_t w \in \Omega_h - A$ for all $|t| \leqslant c$. This contradicts the conclusion of Lemma 4.1 and the fact that $\overline{h(v)} \subseteq A$. Hence v is almost minimizing.

We now show that any $v \in A$ is h-periodic, which will complete the proof. Let $v \in A$ be given, and let $V(p,x) \in SH$ be a lift of v. Let β be the canonical unit speed parametrization of $L(p,x)$. Since the orbit $h(v)$ is contained in the compact set A there exists a number $R > 0$ and a sequence $\varphi_n \subseteq D$ such that $d(\varphi_n p, \beta(n)) \leqslant R$ for each positive integer n. Let $\psi_1, \ldots, \psi_k$ be those elements in D such that $d(p, \psi_i p) \leqslant 2R + 1$, and let G be the subgroup of D generated by $\psi_1, \ldots, \psi_k$. We show inductively that each element φ_n lies in G. Clearly $\varphi_1 \in G$ since $d(p, \varphi_1 p) \leqslant d(p, \beta(1)) + d(\beta(1), \varphi_1 p) \leqslant R + 1$. Suppose that

$\varphi_n \in D$. Then

$$d(p, \varphi_n^{-1}\varphi_{n+1}p) = d(\varphi_{n+1}p, \varphi_n p) \leqslant d(\varphi_{n+1}p, \beta(n+1)) + d(\beta(n+1), \beta(n)) + d(\beta(n), \varphi_n p) \leqslant 2R + 1$$

since β has unit speed. Hence $\varphi_n^{-1}\varphi_{n+1} = \psi_j$ for some $1 \leqslant j \leqslant k$, and therefore $\varphi_{n+1} = \varphi_n \psi_j \in G$.

We now complete the proof. The surface $M^* = H/G$ is finitely connected since G is finitely generated. Since v is almost minimizing in SM the discussion in §1 shows that there exists a number $c > 0$ such that $f(\varphi p) \geqslant f(p) - c$ for all $\varphi \in D$, where f is a fixed Busemann function at x. In particular $f(\varphi p) \geqslant f(p) - c$ for all $\varphi \in G \subseteq D$. This implies that $v^* = (k)_* V(p, x)$ is almost minimizing in SM^*, where $k: H \to M^*$ is the projection map. By hypothesis $d(\varphi_n p, \beta(n)) \leqslant R$ for each n and hence $\lim_{n\to\infty} \varphi_n p = \lim_{n\to\infty} \beta(n) = \beta(\infty) = x$ by Proposition 2.13 of [4]. Therefore $x \in L(G)$ since each φ_n lies in G. The vector v^* consequently lies in $\Omega_h \subseteq SM^*$, and by Proposition 3.6 there exists a parabolic element $\varphi \in G$ such that $\varphi x = x$. By Proposition 3.4 the vector $v \in SM$ is h-periodic. This completes the proof.

Topological mixing. A complete flow $\{\varphi_t\}$ on a topological space X is *topologically mixing* if for any two open sets O, U of X there exists a number $T = T(O, U) > 0$ such that $\varphi_s(O) \cap U \neq \varnothing$ for $|s| \geqslant T$. In the discussion below we assume that M is a Visibility surface with $K \leqslant 0$. The basic result of this section is

THEOREM 4.9. *Let M be a complete Visibility surface with $K \leqslant 0$ such that $\pi_1(M)$ is not infinite cyclic. Let $A \subseteq \Omega_h$ be an orbit closure $\overline{h(z)}$, $z \in \Omega_h$. Suppose that Ω_h contains a compact minimal set. Then for any two open sets O, U of Ω_h that intersect A there exists a number $T = T(O, U) > 0$ such that $h_s(O) \cap U \neq \varnothing$ for $|s| \geqslant T$.*

The result above is not an assertion of topological mixing since the sets O, U are open in Ω_h, not in A. This restriction is necessary, however, for the set A might be a periodic orbit, for example, a case in which $\{h_t\}$ restricted to A is not topologically mixing. As corollaries we obtain the following two results.

THEOREM 4.10. *Let M be a complete Visibility surface with $K \leqslant 0$ such that $\pi_1(M)$ is not infinite cyclic. Let $\{h_t\}$ admit both a dense orbit in Ω_h and a periodic orbit in Ω_h. Then $\{h_t\}$ is topologically mixing in Ω_h.*

THEOREM 4.11. *Let M be a complete Visibility surface with $K \leqslant 0$ such that $K \not\equiv 0$ and $\Omega_h = SM$. If $\pi_1(M)$ is finitely generated, then $\{h_t\}$ is topologically mixing in SM. In particular if M is a compact orientable surface with negative Euler characteristic and curvature $K \leqslant 0$, then $\{h_t\}$ is topologically mixing in SM.*

Before proving Theorem 4.9 we establish the two corollaries. Theorem 4.10 is an immediate consequence of Theorem 4.9. Consider Theorem 4.11, and let M be as given there. (In fact, the condition that $\Omega_h = SM$ actually implies that $K \not\equiv 0$, but we omit the details.) If M is compact, then the flow $\{h_t\}$ is minimal on the compact set SM by Theorem 4.5 above. The topological mixing of $\{h_t\}$ in SM now follows from Theorem 4.9. If M is noncompact, then SM admits h-periodic vectors by Proposition 3.8. It follows by Theorem 4.9 that $\{h_t\}$ is topologically mixing in SM since each periodic orbit is a compact minimal set and $\{h_t\}$ has a dense orbit in SM by Theorem 4.1. Finally let M be compact and orientable with curvature $K \leqslant 0$ and negative Euler characteristic. M is a Visibility surface by Theorem 5.1 of [5], and $K \not\equiv 0$ by the Gauss-Bonnet theorem. Now $\Omega_g = SM$ since SM is compact and the geodesic flow preserves a natural measure arising from a differential form. By Theorem 3.3, $\Omega_h = SM$, which reduces us to a case already considered. This completes the proof of Theorem 4.11.

The proof of Theorem 4.9 uses some rather technical preliminary results. We merely state them here and give the proofs in Appendix II. In each case we assume that $\pi_1(M)$ is not infinite cyclic.

LEMMA 4.9a. *Suppose that $\{h_t\}$ admits a compact minimal set $B \subseteq \Omega_h$. Then $\{v \in \Omega_h : \overline{h(v)}$ is a compact minimal set$\}$ is a dense subset of Ω_h.*

LEMMA 4.9b. *Let $B \subseteq \Omega_h$ be a compact minimal set, and let O be an open subset of Ω_h that meets B. Then there exists a number $s_0 > 0$ such that if $J \subseteq \mathbf{R}$ is any open interval of length $\geqslant s_0$, then for any point $x \in B$, $h_J(x) = \{h_t(x) : t \in J\}$ intersects O.*

An *arc* of an orbit $h(x)$ is a set $\sigma = h_J(x) = \{h_t(x) : t \in J\}$, where $J \subseteq \mathbf{R}$ is an interval, either bounded or unbounded. The *parametrized length* of σ is defined to be the length of J and is denoted by $L(\sigma)$.

LEMMA 4.9c *Let $B \subseteq \Omega_h$ be a compact minimal set, and let numbers $\varepsilon > 0$ and $s_0 > 0$ be given. Then there exists a number $T = T(\varepsilon, s_0, B) > 0$ such that for any $x \in B$ and any arc $\sigma \subseteq h(x)$ of parametrized length $\geqslant T$ we have $L(g_{-\varepsilon}\sigma) - L(\sigma) \geqslant s_0$.*

We are now ready to prove Theorem 4.9. Let $A = \overline{h(z)} \subseteq \Omega_h$ be given. Let O, U be open subsets of Ω_h that intersect A. Choose $x_1 = h_{t_1} z \in O \cap A$ and $x_2 = h_{t_2} z \in U \cap A$. Choose a number $\varepsilon > 0$ and open sets $O^* \subseteq O$ and $U^* \subseteq U$ such that $x_1 \in O^*$, $x_2 \in U^*$ and $g_t(O^*) \subseteq O$, $g_t(U^*) \subseteq U$ for all $|t| \leqslant \varepsilon$. Since $h_r x_1 = x_2$, where $r = t_2 - t_1$, we can choose O^* to be still smaller if necessary so that $h_r(O^*) \subseteq U^*$. Since $\{h_t\}$ admits a compact minimal set in Ω_h, Lemma 4.9a allows us to choose $v \in O^*$ so that $B = \overline{h(v)}$ is a compact minimal set in Ω_h. Choose $s_0 > 0$ as in Lemma 4.9b to correspond to B and U^*. The set $g_\varepsilon(B)$ is a compact minimal set by Proposition 4.4. Choose $T = T(\varepsilon, s_0, g_\varepsilon(B)) > 0$ as in Lemma 4.9c to correspond to ε, s_0 and $g_\varepsilon(B)$.

We assert that for all $|s| \geqslant T$, $h_s(O) \cap U$ is nonempty. Let C be the curve $\{g_t(v)\colon 0 \leqslant t \leqslant \varepsilon\}$. Then $C \subseteq O$ and it suffices to show that $h_s(C) \cap U$ is nonempty. This idea and the technique for proving it are due to Brian Marcus, who used it to prove topological mixing in the case that M is compact with negative Gaussian curvature [18]. Let a number s with $|s| \geqslant T$ be given. For simplicity we consider only the case that $s \geqslant T$. Let $v^* = g_\varepsilon v$, and let $\sigma \subseteq h(v^*)$ be the arc $\{h_t(v^*)\colon 0 \leqslant t \leqslant s\}$. The arc $g_{-\varepsilon}\sigma \subseteq h(v)$ consists of $\{h_t(v)\colon 0 \leqslant t \leqslant s^*\}$, where $h_{s^*}(v) = g_\varepsilon^{-1} h_s(v^*)$. Now $s^* \geqslant s + s_0$ by Lemma 4.9c and the choice of T. The interval $\sigma^* = \{h_t(v)\colon s \leqslant t \leqslant s^*\}$ is a subarc of $g_{-\varepsilon}\sigma$ of parametrized length $\geqslant s_0$, and by Lemma 4.9b, σ^* meets U^*. The curve $h_s(C)$ joins $h_s(v)$ to $h_s(v^*)$. To prove that $h_s(C) \cap U$ is nonempty it suffices by choice of U^* to show that each point q of σ^* is of the form $g_u(q^*)$ for some $u \in [-\varepsilon, 0]$ and some point $q^* \in h_s(C)$.

Define a map $\eta\colon [0,\varepsilon] \to h(v)$ by $\eta(t) = (g_t^{-1} h_s g_t)(v)$. The set $\eta[0,\varepsilon]$ is contained in $h(v)$ since g_t carries h-orbits into h-orbits. Now $\eta(0) = h_s(v)$ and $\eta(\varepsilon) = h_{s^*}(v)$, and hence $\eta[0,\varepsilon]$ contains $\sigma^* \subseteq h(v)$. Let $q \in \sigma^*$ be given and choose $t_0 \in [0,\varepsilon]$ such that $q = \eta(t_0)$. Then $q = g_{-t_0}(q^*)$, where

$$q^* = (h_s g_{t_0})(v) \in h_s(C).$$

This completes the proof of Theorem 4.9.

5. **Appendix I.** In this section we prove the results stated in §2. The first result defines the distance along a horocycle relative to a fixed point on it.

PROPOSITION 2.1. *Let $L(p,x)$ be an arbitrary horocycle in H. Then there exists a unique C^1 unit speed curve $\beta\colon \mathbf{R} \to L(p,x)$ which is a diffeomorphism of $\mathbf{R}$ onto $L(p,x)$ such that $\beta(0) = p$ and the pair $\{V(p,x), \beta'(0)\}$ is positively oriented.*

REMARK. $L(p,x) = f^{-1}(0)$ where f is the C^1 Busemann function $q \to \alpha(p,x,q)$, and hence $L(p,x)$ is a closed C^1 submanifold of H. Recall that the horocycle $L(q,x) = \{r \in H\colon f(r) = f(q)\}$. Since $(\operatorname{grad} f)(q) = -V(q,x)$ by Proposition 1.12, it follows that $V(q,x)$ is a perpendicular unit vector field on $L(q,x)$ for any q in H.

We prove the proposition in a series of lemmas. The parametrization β of $L(p,x)$ defined above will be referred to as the *positively oriented unit speed parametrization of $L(p,x)$ starting at p.*

LEMMA 2.1a. *Let $\beta_1\colon I \to L(p,x)$ and $\beta_2\colon I \to L(p,x)$ be two C^1 unit speed curves defined on an open interval I. If $\beta_1'(t_0) = \beta_2'(t_0)$ for some number t_0 in H, then $\beta_1 = \beta_2$ in I.*

PROOF. Let $I_0 = \{t \in I: \beta_1'(t) = \beta_2'(t)\}$. By assumption, I_0 is nonempty, and clearly I_0 is a closed subset of I. Since I is connected it suffices to prove that I_0 is open in I.

Let $s \in I_0$. Since β_1 and β_2 are nonsingular at s, there exists a neighborhood U of $\beta_1(s) = \beta_2(s)$ in $L(p,x)$ and intervals J_1, J_2 in I containing s such that $\beta_1: J_1 \to U$ and $\beta_2: J_2 \to U$ are C^1 diffeomorphisms. The map

$$\rho = \beta_2^{-1} \circ \beta_1 : J_1 \to J_2$$

is therefore a C^1 diffeomorphism, and hence $\beta_1(t) = \beta_2(\rho t)$ for all t in J_1. Differentiating we obtain the relation $\beta_1'(t) = \rho'(t)\beta_2'(\rho t)$ in J_1. Since β_1 and β_2 are both unit speed curves $|\rho'(t)| \equiv 1$ in J_1, and since $\rho(s) = s$ and $\rho'(s) = 1$, it follows that $\rho(t) = t$ in J_1. Therefore $\beta_1 = \beta_2$ in J_1, and this proves that I_0 is open in I.

LEMMA 2.1b. *There exists a one-one C^1 unit speed curve β: $\mathbf{R} \to L(p,x)$ such that $\beta(0) = p$ and $\{V(p,x), \beta'(0)\}$ is positively oriented.*

Note that the lemma does not assert that $\beta(\mathbf{R}) = L(p,x)$.

PROOF. Since $L(p,x)$ is a closed C^1 submanifold of H of dimension one, there exist an $\varepsilon > 0$ and a C^1 map β: $(-\varepsilon, \varepsilon) \to L(p,x)$ such that $\beta(0) = p$ and β is a diffeomorphism of $(-\varepsilon,\varepsilon)$ onto its image, an open subset of $L(p,x)$. By reparametrizing β we may assume that β has unit speed and that $\{V(p,x), \beta'(0)\}$ is positively oriented. Now let J_0 be the union of all open intervals J containing zero for which there exists a C^1 unit speed curve $\beta_J: J \to L(p,x)$ such that $\beta_J(0) = p$ and $\{V(p,x), \beta_J'(0)\}$ is positively oriented. The interval J_0 is nonempty and equals $(-A, B)$ for some positive extended real numbers A, B. Since $\beta_J'(0)$ is the same for all intervals J, the previous lemma implies that any two of the maps β_J agree on the intersection of their domains. We therefore obtain a well-defined C^1 unit speed curve β: $J_0 \to L(p,x)$ such that $\beta(0) = p$ and $\{V(p,x), \beta'(0)\}$ is positively oriented. We assert that $J_0 = \mathbf{R}$. If this were false, then either A or B, say B for convenience, would be finite. Let t_n be a sequence in J_0 converging to B. The points $\beta(t_n)$ have distance $\leqslant t_n$ from $p = \beta(0)$ since β is a unit speed curve, and therefore we may assume that $\beta'(t_n)$ converges to a unit vector v at a point q by passing to a subsequence. The vector v is tangent to $L(p,x)$ at q since the vectors $\beta'(t_n)$ are tangent to $L(p,x)$ and $L(p,x)$ is a closed subset of H. The pair $\{V(q,x), v\}$ is positively oriented since the pairs $\{V(\beta t_n, x), \beta'(t_n)\}$ are positively oriented for each n. There exists an open interval I containing B and a C^1 unit speed diffeomorphism $\tilde{\beta}$: $I \to L(p,x)$ such that $\tilde{\beta}(I)$ is an open set of $L(p,x)$ containing $q = \tilde{\beta}(B)$ and $\tilde{\beta}'(B) = v$. Let I^* be an open subinterval of I containing B whose closure is also contained in I. Since $\tilde{\beta}(I^*)$ is a neighborhood of q in $L(p,x)$, $\beta(t_n) \in \tilde{\beta}(I^*)$ for sufficiently large n. Choose numbers t_n^* in I^* such that $\beta(t_n) = \tilde{\beta}(t_n^*)$. The numbers t_n^* converge to B since $\tilde{\beta}$ is one-one on I. Note that $\beta'(t_n) = \tilde{\beta}'(t_n^*)$ since these are both unit vectors tangent to $L(p,x)$ at the same point and having the property that the orthogonal pairs

$$\{V(\beta t_n, x), \beta'(t_n)\} \quad \text{and} \quad \{V(\tilde{\beta} t_n^*, x), \tilde{\beta}'(t_n^*)\}$$

are positively oriented. The previous lemma implies that $\tilde{\beta}(t) = \beta(t + t_n - t_n^*)$ in $[t_n^*, B)$. Because this is true for all n the difference $t_n - t_n^*$ must be constant, and this difference must be zero since both sequences t_n and t_n^* converge to B. Therefore $\tilde{\beta}$ agrees with β on some interval (t_0, B), and this implies that β may be extended to $(-A, B + \varepsilon)$ for a small number $\varepsilon > 0$. This contradicts the maximality of $J_0 = (-A, B)$ and shows that J_0 must be $\mathbf{R}$. The fact that β is one-one follows from the next result and the fact that $\beta(t)$ and $\beta(-s)$ lie on opposite sides of γ_{px} for any positive numbers s, t.

LEMMA 2.1c. *Let I be an open interval containing zero, and let $\beta: I \to L(p,x)$ be a C^1 nonsingular map with $\beta(0) = p$. Then the derivative of the function $t \to d(\beta 0, \beta t)$ is defined and positive for any $t > 0$ in I. Moreover if I contains* $[0, \infty)$, *then*

$$d(\beta 0, \beta t) \to \infty \quad \text{as } t \to \infty.$$

PROOF. Let $f(t) = d(\beta 0, \beta t)$. By Lemma 2.3 of [4] it follows that $f'(t) = -\langle \beta'(t), V(\beta t, p)\rangle$ for all $t > 0$, where $p = \beta(0)$. If $f'(t_0)$ were zero for some $t_0 > 0$, then $V(\beta t_0, p)$ and $V(\beta t_0, x)$ would be collinear because Proposition 1.12 implies that

$$\langle \beta'(t_0), V(\beta t_0, x) = -(f \circ \beta)'(t_0) = 0,$$

where f is the Busemann function $q \to \alpha(p, x, q)$ whose zero level set is $L(p,x)$. Therefore $\beta(t_0)$ must lie on both $L(p,x)$ and γ_{px}, which implies that $\beta(t_0) = p$. Since $\beta'(0) \neq 0$, $\beta(t) \neq p$ for small numbers $t > 0$. Therefore there is a smallest positive t for which $f'(t) = 0$, and we may assume that t_0 is this value. Since $f(0) = 0$ and $f(t) > 0$ for $0 < t < t_0$, it follows that $f'(t) > 0$ for $0 < t < t_0$. This contradicts the fact that $\beta(t_0) = p$, and therefore we conclude that $f'(t) > 0$ for all $t > 0$ in I.

Suppose now that I contains $[0, \infty)$. For any $r > 0$ let $\overline{B_r(p)}$ denote the closed ball of radius r and center $p = \beta(0)$. Since $f'(t) = -\langle \beta'(t), V(\beta t, p)\rangle > 0$, an argument similar to that of the previous paragraph implies that there exist numbers δ_1 and δ_2 such that

$$0 < \delta_1 \leqslant \measuredangle\, (V(\beta t, p), V(\beta t, x)) \leqslant \delta_2 < \pi$$

for any $t > 0$ such that $\beta(t) \in B_r(p)$. Therefore $f'(t) \geqslant \delta > 0$ for some

$\delta > 0$ and any $t > 0$ such that $\beta(t) \in B_r(p)$. It follows that $\beta(t)$ leaves $\overline{B_r(p)}$ forever after a time interval of length at most r/δ. Since $r > 0$ is arbitrary this shows that $f(t) \to \infty$ as $t \to +\infty$.

LEMMA 2.1d. *Let β: $\mathbf{R} \to L(p,x)$ be any C^1 unit speed curve. Then $\beta(\mathbf{R}) = L(p,x)$.*

PROOF. The previous result implies that $\beta(\mathbf{R})$ is a closed subset of $L(p,x)$. Also $\beta(\mathbf{R})$ is an open subset of $L(p,x)$ since β is a nonsingular map of one dimensional manifolds. It suffices therefore to prove that $L(p,x)$ is connected. By Proposition 3.4 of [10] there exists a homeomorphism ψ: $H \to L(p,x) \times \mathbf{R}$, where $L(p,x) \times \mathbf{R}$ has the product topology. Since H and $\mathbf{R}$ are connected and $L(p,x) \times \mathbf{R}$ has the product topology, it follows that $L(p,x)$ is connected.

We now complete the proof of Proposition 2.1. The lemmas have shown that there exists a C^1 bijective unit speed map β: $\mathbf{R} \to L(p,x)$ such that $\beta(0) = p$ and $\{V(p,x), \beta'(0)\}$ is positively oriented. Clearly β is a diffeomorphism. The uniqueness of β follows from Lemma 2.1a.

We now define the horocycle flow in SH.

DEFINITION 2.2. Let $t \in \mathbf{R}$ and $v \in SH$ be given. Define h_0 to be the identity map on SH. If $t \neq 0$ and $x = \gamma_v(\infty)$ define $h_t(v) = V(\beta t, x)$, where β is the positively oriented unit speed parametrization starting at $p = \mu(v)$ of the horocycle in H determined by v, namely $\{q \in H\colon B(v,q) = 0\}$.

The map B: $SH \times H \to \mathbf{R}$ is defined in §1.

PROPOSITION 2.3. *For any numbers s, t in $\mathbf{R}$, $h_{t+s} = h_t \circ h_s$.*

PROOF. Let $v \in SH$ be given, and let $w = h_s(v)$. Let p and q be the points of tangency of v and w respectively, and let $x = \gamma_v(\infty) = \gamma_w(\infty)$. Then $L(p,x) = L(q,x)$ by Proposition 1.12 since both horocycles contain the point q. Let β: $\mathbf{R} \to L(p,x)$ denote the positively oriented unit speed parametrization of $L(p,x)$ starting at p. If we define $\alpha(u) = \beta(s+u)$ for all u in $\mathbf{R}$, then it is easy to see that α is the positively oriented unit speed parametrization of $L(p,x) = L(q,x)$ starting at q. Finally,

$$h_t(h_s v) = h_t(w) = V(\alpha t, x) = V(\beta(s+t), x) = h_{t+s}(v).$$

Define the usual flow map h: $SH \times \mathbf{R} \to SH$ given by $h(v,t) = h_t(v)$.

PROPOSITION 2.4. *If $SH \times \mathbf{R}$ is given the product topology, then the map h: $SH \times \mathbf{R} \to SH$ is continuous.*

It is convenient for the proof to introduce the "canonical" parametrization α of a horocycle $L(p,x)$ that starts at p. Let $\alpha(0) = p$. For $t > 0$ let $\alpha(t)$ be the unique point of $L(p,x)$ that lies to the left of γ_{px} at a distance t from p. If $t < 0$, let $\alpha(t)$ be the unique point of $L(p,x)$ that lies to the right of γ_{px} at a distance $|t|$ from p. The map α: $\mathbf{R} \to L(p,x)$ exists and is onto by Lemma 2.1c. The proof of Proposition 2.4 consists of a series of lemmas.

LEMMA 2.4a. *The map* α: $\mathbf{R} \to L(p,x)$ *starting at* p *is* C^1 *and nonsingular at any value* $t \neq 0$.

PROOF. It suffices to consider the case where $t > 0$ since the definition of α depends upon the orientation of H. Given $t > 0$, we may choose a C^1 diffeomorphism ρ: $(-\varepsilon,\varepsilon) \to L(p,x)$, where $\rho(-\varepsilon,\varepsilon)$ is a neighborhood of $\alpha(t) = \rho(0)$ in $L(p,x)$. Let $k(s) = d(p,\rho s)$. Since $\rho'(s)$ is orthogonal to $V(\rho s, x)$ for all s, $k'(0) = -\langle \rho'(0), V(\rho(0),p)\rangle \neq 0$ by Lemma 2.3 of [4] and the fact that p, $\alpha(t)$ and x are not collinear. Therefore, k is nonsingular on some neighborhood of $s = 0$, and for some $\delta > 0$, k: $(-\delta,\delta) \to (a,b)$ is a diffeomorphism where $t = k(0)$ is a point in (a,b). If g: $(a,b) \to (-\delta,\delta)$ is the inverse of k, then $\tilde{\alpha} = \rho \circ g$ is a C^1 nonsingular curve defined on (a,b). By definition, $\tilde{\alpha}(t) = \rho(0) = \alpha(t)$ and $d(p,\tilde{\alpha}s) = s$ for all s in (a,b). Therefore $\tilde{\alpha} = \alpha$ in (a,b) and this proves that α is C^1 and nonsingular at t.

LEMMA 2.4b. *Let* v_n *be a sequence of unit vectors in* H *that converge to a unit vector* v. *Let* p_n *and* p *be the points of tangency of* v_n *and* v, *and let* α_n *and* α *be the canonical parametrizations starting at* p_n *and* p *of the horocycles determined by* v_n *and* v *respectively. Let* t_n *be a sequence of numbers converging to a number* $t \neq 0$. *Then*

$$\alpha_n'(t_n) \to \alpha'(t) \quad \text{as } n \to \infty.$$

PROOF. We again consider only the case where $t > 0$. We show first that $\alpha_n(t_n) \to \alpha(t)$ as $n \to \infty$. Since $d(p_n, \alpha_n t_n) = t_n$, the sequence $\alpha_n(t_n)$ is bounded in H. If q is a cluster point, then $\alpha_n(t_n) \to q$ by passing to a subsequence. The hypothesis implies that $B(v_n, \alpha_n t_n) = 0$ for all n. Therefore $B(v,q) = 0$ and $d(p,q) = t$ by continuity. Now q lies to the left of γ_v since the orientation of the pair $\{v, V(p,q)\}$ equals the positive orientation of the pair $\{v_n, V(p_n, \alpha_n t_n)\}$ for sufficiently large n. Thus $q = \alpha(t)$, and since q was an arbitrary cluster point of $\alpha_n(t_n)$ it follows that $\alpha_n(t_n) \to \alpha(t)$.

To show that $\alpha_n'(t_n) \to \alpha'(t)$ we shall need to show first that $\|\alpha_n'(t_n)\|$ is a bounded sequence of real numbers. Let $w_n = \alpha_n'(t_n)/\|\alpha_n'(t_n)\|$, and let w_n converge to a unit vector w at $\alpha(t)$ by passing to a subsequence. Since $s = d(p_n, \alpha_n s)$ for all $s > 0$ and all n, by differentiating both sides we obtain the equation

$$1 = -\langle \alpha_n'(s), V(\alpha_n s, p_n)\rangle.$$

In particular

$$1 = -\langle \alpha_n'(t_n), V(\alpha_n t_n, p_n)\rangle = -\|\alpha_n'(t_n)\| \langle w_n, V(\alpha_n t_n, p_n)\rangle.$$

If $\|\alpha_n'(t_n)\|$ were unbounded, then $\langle w, V(\alpha t, p)\rangle$ would equal zero for reasons of continuity. However, $\langle w, V(\alpha t, x)\rangle = 0$ since $\langle w_n, V(\alpha_n t_n, x_n)\rangle = 0$ for every n, where $x = \gamma_v(\infty)$ and $x_n = \gamma_{v_n}(\infty)$. Since $\alpha(t)$ is distinct from p it cannot lie on both $L(p,x)$ and γ_{px}, which yields a contradiction. Therefore $\|\alpha_n'(t_n)\|$ is bounded and $\alpha_n'(t_n)$ has a cluster point w^* at $\alpha(t)$. By continuity it follows that $\langle w^*, V(\alpha t, x)\rangle = 0$ and $1 = -\langle w^*, V(\alpha t, p)\rangle$ since $\langle \alpha_n'(t_n), V(\alpha_n t_n, x_n)\rangle = 0$ and $1 = -\langle \alpha_n'(t_n), V(\alpha_n t_n, p_n)\rangle$ for every n. Now $\langle \alpha'(t), V(\alpha t, x)\rangle = 0$, and $1 = -\langle \alpha'(t), V(\alpha t, p)\rangle$ as one sees by differentiating the equation $t = d(p, \alpha t)$. Therefore it follows that $w^* = \alpha'(t)$, and since w^* was an arbitrary cluster point of $\alpha_n'(t_n)$ we conclude that $\alpha_n'(t_n) \to \alpha'(t)$ as $n \to \infty$.

LEMMA 2.4c. *Let v_n, v, p_n, p, α_n and α be as in the statement of the previous lemma. If s_n is any sequence of nonzero numbers that converges to zero, then $\|\alpha_n'(s_n)\| \to 1$ as $n \to \infty$.*

PROOF. Let s_n be a sequence of nonzero numbers that converges to zero. We may assume without loss of generality that every s_n is positive since the definition of the canonical parametrization α_n depends upon the orientation of H. Differentiating the equation $t = d(p_n, \alpha_n t)$, which holds for $t > 0$ and all n, we obtain the equation

$$1 = -\langle \alpha_n'(t), V(\alpha_n t, p_n)\rangle = -\|\alpha_n'(t)\| \cos\theta_n(t),$$

where $\theta_n(t)$, measured between 0 and π, is the angle subtended by $\alpha_n'(t)$ and $V(\alpha_n t, p_n)$. Thus $\|\alpha_n'(t)\| \geq 1$ and $\theta_n(t) > \pi/2$ for all n and all $t > 0$. Suppose that $\|\alpha_n'(s_n)\| \geq \eta > 1$ for some number η and all n by passing to a subsequence. Now

$$V(\alpha_n s_n, x_n) = h_{s_n^*} V(p_n, x_n)$$

for some $s_n^* > 0$, where $x_n = \gamma_{v_n}(\infty)$, and therefore

$$V(p_n, x_n) = h_{-s_n^*} V(\alpha_n s_n, x_n),$$

which implies that p_n lies to the right of $\gamma_{\alpha_n s_n, x}$. Therefore

$$\begin{aligned}\theta_n^* &= \measuredangle\, (V(\alpha_n s_n, p_n),\ V(\alpha_n s_n, x_n))\\ &= \theta_n(s_n) - \measuredangle\, (\alpha_n'(s_n),\ V(\alpha_n s_n, x_n)) = \theta_n(s_n) - \pi/2.\end{aligned}$$

Since $\|\alpha_n'(s_n)\| \geq \eta > 1$, there exists a number $\delta > 0$ such that $\pi/2 < \theta_n(s_n) < \pi - \delta$, and consequently $0 < \theta_n^* < \pi/2 - \delta$ for every n.

Consider the circle C_n of radius 1 with center $p_n' = \gamma_{\alpha_n s_n, x_n}(1)$ that passes through $\alpha_n s_n$. Let σ_n be the unit speed geodesic such that $\sigma_n(0) = \alpha_n(s_n)$ and $\sigma_n(s_n) = p_n$. The fact that $\theta_n^* < \pi/2 - \delta$ implies that an initial segment of σ_n of length ε_n lies within the circle C_n. Since the centers p_n' of C_n are a bounded sequence in H and the radii equal 1, one may choose an $\varepsilon > 0$ such that $\varepsilon_n \geqslant \varepsilon > 0$ for all n. The interior of C_n is contained in the interior of the horocycle $L(\alpha_n s_n, x_n) = L(p_n, x_n)$ for all n, and consequently $\sigma_n(0, \varepsilon]$ lies inside $L(\alpha_n s_n, x_n)$ for all n. This contradicts the fact that $p_n = \sigma_n(s_n)$ lies on $L(\alpha_n s_n, x_n)$ and $s_n \to 0$ as $n \to \infty$. This contradiction proves that $\|\alpha_n'(s_n)\| \to 1$ as $n \to \infty$.

One could end the proof of the lemma here, but the assertion that the numbers ε_n are bounded below by some $\varepsilon > 0$ requires more justification. If $g_n(t) = d(\sigma_n t, p_n')$, then

$$g_n''(t) = -\langle \sigma_n'(t), \nabla_{\sigma_n'(t)} W_n \rangle = \pm k_n(\sigma_n t),$$

where $W_n = V(\ , p_n')$ is the inward normal vector field for all circles with center p_n', and $k_n(q)$ denotes the geodesic curvature at q of the circle with center p_n' that passes through q. Let the Gaussian curvature be bounded below by $-c^2$ on a compact set C containing all circles with center p_n' and radius $\leqslant 2$. A standard comparison technique shows that the geodesic curvatures of any circle with center p_n' and radius $\leqslant 2$ are not greater than the geodesic curvatures of a circle of equal radius in the hyperbolic plane with curvature $-c^2$. Since $d(p_n', \alpha_n s_n) = 1$, it follows that $|k_n(\sigma_n t)| \leqslant c \cdot \coth(c/2) = B$ for $0 \leqslant t \leqslant \frac{1}{2}$ and all n. Thus $|g_n''(t)| \leqslant B$ for $0 \leqslant t \leqslant \frac{1}{2}$ and all n. Now

$$\begin{aligned} g_n'(0) &= -\langle \sigma_n'(0), W_n(\sigma_n 0) \rangle = -\langle V(\alpha_n s_n, p_n), V(\alpha_n s_n, p_n') \rangle \\ &= -\cos \theta_n^* \leqslant -\cos(\pi/2 - \delta) = -\sigma < 0. \end{aligned}$$

Therefore $g_n'(t) < 0$ for $0 \leqslant t \leqslant \varepsilon = \sigma/B$, and this implies that $\sigma_n(t)$ lies inside C_n for $0 < t \leqslant \varepsilon$.

LEMMA 2.4d. *Let v_n be a sequence of unit vectors in H that converges to a unit vector v, and let t_n be a sequence of numbers that converges to a number t. Then $\beta_n(t_n) \to \beta(t)$, where β_n and β are the positively oriented unit speed parametrizations starting at $p_n = \mu(v_n)$ and $p = \mu(v)$ of the horocycles determined by v_n and v respectively.*

PROOF. Since $d(\beta_n t_n, \beta_n t) \leqslant |t_n - t|$ it suffices to prove that $\beta_n(t) \to \beta(t)$ as $n \to \infty$. This result is trivial if $t = 0$, and as usual it suffices to consider only the case where $t > 0$. Let α_n and α be the canonical parametrizations starting at $p_n = \mu(v_n)$ and $p = \mu(v)$ of the horocycles determined by v_n and v respectively. Choose numbers $t_n^* > 0$ and $t^* > 0$ such that $\beta_n(t) = \alpha_n(t_n^*)$ and $\beta(t) = \alpha(t^*)$. By Lemma 2.4b it suffices to show that $t_n^* \to t^*$ as $n \to \infty$.

Since $t_n^* = d(p_n, \beta_n t) \leqslant t$ there exists a cluster point s^* of the sequence $\{t_n^*\}$ and by passing to a subsequence we may further assume that $t_n^* \to s^*$. As $\varepsilon > 0$ tends to zero, the length of α_n between ε and t_n^* tends to the length of β_n between 0 and t, namely t, for any fixed n. Therefore

$$t = \lim_{\varepsilon \to 0^+} \int_\varepsilon^{t_n} \|\alpha_n'(u)\| \, du$$

for each fixed n, and similarly

$$t = \lim_{\varepsilon \to 0^+} \int_\varepsilon^{t^*} \|\alpha'(u)\| \, du.$$

Although $\alpha_n'(0)$ and $\alpha'(0)$ do not exist we may define $\|\alpha_n'(0)\| = \|\alpha'(0)\| = 1$ and the resulting functions $u \to \|\alpha_n'(u)\|$ and $u \to \|\alpha'(u)\|$ are continuous at $u = 0$ by virtue of the preceding lemma. Fixing n, the bounded convergence theorem implies that

$$t = \int_0^{t_n^*} \|\alpha_n'(u)\| \, du,$$

where the integrand is defined everywhere in $[0, t_n^*]$. Similarly $t = \int_0^{t^*} \|\alpha'(u)\| \, du$. We will conclude the proof by showing that $\int_0^{t_n^*} \|\alpha_n'(u)\| \, du$ converges to $\int_0^{s^*} \|\alpha'(u)\| \, du$. This will show that $s^* = t^*$, and since s^* is an arbitrary cluster point of t_n^* it will show that $t_n^* \to t^*$ as $n \to \infty$. Let $f_n(u) = \|\alpha_n'(u)\| \mathfrak{X}_{[0, t_n^*]}(u)$ be defined on $[0, t]$, where $\mathfrak{X}$ denotes a characteristic function, and let $f(u) = \|\alpha'(u)\| \mathfrak{X}_{[0, s^*]}(u)$ be defined on the same interval. Now $f_n(u) \to f(u)$ for $u \in [0, t]$ since $t_n^* \to s^*$ and $\alpha_n'(u) \to \alpha'(u)$ for $u \neq 0$ by Lemma 2.4b. If $R \geqslant 1$ is an upper bound for $\|\alpha'(u)\|$ on $[0, t]$, then $\|\alpha_n'(u)\| \leqslant R + 1$ for $u \in [0, t]$ and all sufficiently large n; one observes that for any sequence s_n in $[0, t]$ that converges to a number s, either $\|\alpha_n'(s_n)\| \to \|\alpha'(s)\|$ by Lemma 2.4b if $s > 0$ or $\|\alpha_n'(s_n)\| \to 1$ by Lemma 2.4c if $s = 0$. Therefore

$$\int_0^{t_n^*} \|\alpha_n'(u)\| \, du = \int_0^t f_n(u) \, du \to \int_0^t f(u) \, du = \int_0^{s^*} \|\alpha'(u)\| \, du$$

by the bounded convergence theorem. This completes the proof of the lemma.

We now complete the proof of Proposition 2.4. Let (v_n, t_n) be a sequence in $SH \times \mathbf{R}$ that converges to a point (v, t). We show that $h(v_n, t_n) \to h(v, t)$. Let β_n and β be the positively oriented unit speed parametrizations starting at $p_n = \mu(v_n)$ and $p = \mu(v)$ of the horocycles determined by v_n and v respectively. Then $h(v_n, t_n) = V(\beta_n t_n, x_n)$, where $x_n = \gamma_{v_n}(\infty)$, and $h(v, t) = V(\beta t, x)$, where $x = \gamma_v(\infty)$. Now $x_n \to x$ since $v_n \to v$ and $\beta_n t_n \to \beta t$ by the previous lemma. Therefore $h(v_n, t_n) \to h(v, t)$ by the continuity of the vector function V.

6. **Appendix II.** In this section we prove the supporting results of Theorem 4.9.

PROOF OF LEMMA 4.9a. Let $B \subseteq \Omega_h$ be a compact minimal set. If B is a periodic orbit, then the result follows from Proposition 3.5. If $B = \Omega_h = SM$ and M is compact, then the result follows from Theorem 4.5. By Proposition 4.8 these are the only cases that arise for B. One can also construct a direct proof analogous to Proposition 3.5.

PROOF OF LEMMA 4.9b. Suppose that the lemma is false. Then for some compact minimal set $B \subseteq \Omega_h$ and some open subset O of Ω_h that meets B, there exist a divergent sequence of positive numbers t_n and a sequence of vectors v_n in B such that $O \cap \{h_t(v_n): -t_n \leqslant t \leqslant t_n\}$ is empty for each integer n. Passing to a subsequence we let v_n converge to a vector v in B. By continuity the entire orbit $h(v)$ is disjoint from O, but this contradicts the fact that $h(v)$ is dense in B since B is minimal.

PROOF OF LEMMA 4.9c. Suppose that the lemma is false for some compact minimal set $B \subseteq \Omega_h$ and some numbers $\varepsilon > 0$ and $s_0 > 0$. Using Sublemmas A and B (stated and proved below) we shall show that there exists v^* in B such that $L(g_{-\varepsilon}\sigma) = L(\sigma)$ for any segment $\sigma \subseteq h(v^*)$. Assume for the moment that this has been established. Let $\tilde{v} = V(p,x)$ be a lift of v^* for suitable $p \in H$, $x \in H(\infty)$.

We assert that the curvature at every point inside the horocycle $L(p,x)$ is zero. Every point inside $L(p,x)$ is of the form $\mu(g_{u_0}h_{t_0}v)$ for a suitable $u_0 > 0$ and $t_0 \in \mathbf{R}$. Let $\tilde{\sigma}$ be any segment of $h(\tilde{v})$ that contains $h_{t_0}\tilde{v}$, and let $L(s) = L(g_s\tilde{\sigma})$. From the discussion above we see that $L(-\varepsilon) = L(0)$. By Sublemma B it follows that $K = 0$ for all points $\mu(g_u w)$, $u > 0$, $w \in \tilde{\sigma}$, and in particular for $\mu(g_{u_0}h_{t_0}\tilde{v})$. Therefore $K \equiv 0$ inside $L(p,x)$.

By Sublemma A, $L(p,x)$ is a geodesic σ of H. Therefore if $y \in H(\infty)$ is a point such that $\measuredangle_p(x,y) \leqslant \pi/2$, then the curvature is zero on $\gamma_{py}[0,\infty)$. In particular, the point $x \in H(\infty)$ has a neighborhood U in the cone topology (see [10]) such that $K \equiv 0$ in $U \cap H$. However, $x \in L(D)$ by Proposition 3.2 since $v^* \in \Omega_h$. Since K is not identically zero in H and $x \in L(D)$ we may choose a sequence $\varphi_n \subseteq D$ and a point $q \in H$ such that $K(q) < 0$ and $\varphi_n q \to x$ in the cone topology. For large n, $\varphi_n q \in U$ and $K(\varphi_n q) = K(q) < 0$, a contradiction. This will complete the proof of Lemma 4.9c.

Assuming that Lemma 4.9c is false for some compact minimal set B and some numbers $\varepsilon > 0$ and $s_0 > 0$ we now establish the existence of v^* in B such that $L(g_{-\varepsilon}\sigma) = L(\sigma)$ for any segment $\sigma \subseteq h(v^*)$. By hypothesis we can find a sequence of horocycle segments $\sigma_n \subseteq h(v_n)$, $v_n \in B$, such that $L(\sigma_n) \to +\infty$ and $L(g_{-\varepsilon}\sigma_n) \leqslant L(\sigma_n) + s_0$ for every n. By changing v_n if necessary we may further assume that $\sigma_n = \{h_t(v_n): -t_n \leqslant t \leqslant t_n\}$, where $2t_n = L(\sigma_n)$. From Sublemma B it follows that for any positive number t and all $t_n \geqslant t$ we have

$$L(g_{-\varepsilon}\sigma_n[-t_n,-t]) \geqslant L(\sigma_n[-t_n,-t]) \quad \text{and} \quad L(g_{-\varepsilon}\sigma_n[t,t_n]) \geqslant L(\sigma_n[t,t_n]).$$

We conclude that

$$L(g_{-\varepsilon}\sigma_n[-t,t]) \leqslant L(\sigma_n[-t,t]) + s_0$$

for all $t > 0$ and all sufficiently large n.

Since B is compact we may let v_n converge to v in B by passing to a subsequence. Let $\bar{\sigma}(t)$ denote $h_t(v)$ for all t in $\mathbf{R}$. The inequality displayed above and continuity imply that for every $t > 0$ we have

$$L(g_{-\varepsilon}\bar{\sigma}[-t,t]) \leqslant L(\bar{\sigma}[-t,t]) + s_0.$$

By Sublemma B the function $\varphi(t) = L(g_{-\varepsilon}\bar{\sigma}[-t,t]) - L(\bar{\sigma}[-t,t])$ is nonnegative and nondecreasing in t for $t > 0$. Therefore there exists $\lim_{t\to\infty}\varphi(t) = A \leqslant s_0$. Choose a divergent sequence of numbers $T_n > 0$ such that for all $s \geqslant t \geqslant T_n$ we have $0 \leqslant \varphi(s) - \varphi(t) \leqslant 1/n$. It follows that for any $s \geqslant T_n$ we have

$$1/n \geqslant \varphi(s) - \varphi(T_n) \geqslant L(g_{-\varepsilon}\sigma[T_n,s]) - L(\sigma[T_n,s]).$$

Now let $\sigma_n^*(t) = \bar{\sigma}(t + 2T_n)$. It follows from the discussion above that

$$0 \leqslant L(g_{-\varepsilon}\sigma_n^*[-T_n,T_n]) - L(\sigma_n^*[-T_n,T_n]) \leqslant 1/n.$$

We see from the inequality above and Sublemma B that for any $t > 0$ we have

$$0 \leqslant L(g_{-\varepsilon}\sigma_n^*[-t,t]) - L(\sigma_n^*[-t,t]) \leqslant 1/n$$

for all n so large that $t \leqslant T_n$. Now let $v_n^* = \sigma_n^*(0)$ converge to v^* in B by passing to a subsequence. It follows by continuity that for every $t > 0$

$$L(g_{-\varepsilon}\sigma^*[-t,t]) = L(\sigma^*[-t,t]),$$

where $\sigma^*(t) = h_t(v^*)$. If σ is any segment in $h(v^*)$ then by Sublemma B we have $L(g_{-\varepsilon}\sigma) \geqslant L(\sigma)$. It now follows from the equality above that $L(g_{-\varepsilon}\sigma) = L(\sigma)$.

We conclude Lemma 4.9c with the statements and proofs of Sublemmas A and B.

Sublemma A. *Let $x \in H(\infty)$ and $p \in H$ be given, and suppose that each point inside $L(p,x)$ has zero Gaussian curvature. Then $L(p,x)$ is a geodesic of H.*

Proof. Let $\bar{\gamma}$ be the unit speed geodesic, unique up to orientation, that is perpendicular to γ_{px} at p. By Lemma 2.1d in Appendix I, it suffices to prove that $\bar{\gamma}(\mathbf{R}) \subseteq L(p,x)$ to conclude that $\bar{\gamma}(\mathbf{R}) = L(p,x)$. For any number $t \neq 0$, $\measuredangle_p(\bar{\gamma}t, x) = \pi/2$ and this implies that $d(\bar{\gamma}t, \gamma_{px}s) - s \geqslant 0$ for all $s > 0$ by the law of cosines [10, p. 47]. Hence $f(\bar{\gamma}t) \geqslant 0$ for all $t \in \mathbf{R}$, where f is the Busemann function at x such that $f(p) = 0$. To show that $f(\bar{\gamma}t) \leqslant 0$ for all $t \in \mathbf{R}$ it suffices to show that $(f \circ \sigma)(s) < 0$ for any $s > 0$ and any geodesic σ such that $\measuredangle(\sigma'(0), V(p,x)) < \pi/2$. Suppose that this is false for some such geodesic σ. Now $(f \circ \sigma)(s) < 0$ for small positive values of s since

$$(f \circ \sigma)'(0) = \langle \sigma'(0), -V(p,x) \rangle < 0$$

by Proposition 1.12. Let $s_0 > 0$ be the first positive value for which $f \circ \sigma$ is zero. The segment $\sigma[0, s_0]$ lies on or inside $L(p,x)$, and therefore for any $t > 0$ the points on or inside the geodesic triangle with vertices p, $\sigma(s_0)$ and $\gamma_{px}t$ have Gaussian curvature zero. Therefore the sum of the interior angles equals π. It follows that

$$\measuredangle_{\sigma s_0}(p,x) = \lim_{t\to\infty} \measuredangle_{\sigma s_0}(p, \gamma_{px}t) = \pi - \measuredangle_p(\sigma s_0, x) > \pi/2$$

since $\measuredangle_p(\sigma s_0, x) < \pi/2$ and $\measuredangle_{\gamma_{px}t}(p, \sigma s_0) \to 0$ as $t \to \infty$. On the other hand, $(f \circ \sigma)(s) < 0$ for $0 < s < s_0$ and therefore

$$0 \leqslant (f \circ \sigma)'(s_0) = \langle \sigma'(s_0), -V(\sigma s_0, x) \rangle.$$

This implies that $\measuredangle_{\sigma s_0}(p,x) \leqslant \pi/2$, a contradiction that completes the proof.

SUBLEMMA B. *Let M be a complete surface with $K \leqslant 0$. Let $v \in SM$ be given and let $\sigma \subseteq h(v)$ be an arc of parametrized length $a > 0$. For each number s let $L(s)$ be the parametrized length of the arc $g_s(\sigma) \subseteq g_s h(v) = h(g_s v)$. Then $L(s)$ is nonincreasing in s. Moreover if K is negative at some point $\mu(g_u w)$, $u \in \mathbf{R}$, $w \in \sigma$, then $L(s)$ is strictly decreasing on any interval $[s', u]$, where $s' < u$.*

PROOF. Recall that μ: $SM \to M$ is the projection map. Since the horocycle flow in SM is induced from that in SH it suffices to prove the result in SH. Note that the parametrized length of the arc $\sigma \subseteq h(v)$ is just the length of $\mu(\sigma)$ in M or H. Let p, x be those points such that $v = V(p,x)$, and let $\alpha(t)$, $0 \leqslant t \leqslant a$, be a nonsingular parametrization of σ. Define r: $\mathbf{R} \times [0,a] \to H$ by $r(s,t) = \gamma_t(s)$, where $\gamma_t = \gamma_{\alpha t, x}$. Let f be the Busemann function at x that is zero on $\mu(\sigma)$. The s-parameter curves are unit speed geodesics belonging to x, and the t-parameter curves are parametrizations of arcs of horocycles at x. Hence the s and t parameter curves are orthogonal. Let $r_s(s,t)$ and $r_t(s,t)$ denote $r_*(\partial/\partial s)(s,t)$ and $r_*(\partial/\partial t)(s,t)$ respectively. The vector field $Y_t(s,t) = r_t(s,t)$ is a Jacobi vector field on the unit speed geodesic γ_t: $s \to r(s,t)$ since all s-parameter curves are geodesics. By [16] or our own unpublished work f is a C^2 convex function. This means that $(f \circ \gamma)''(t) \geqslant 0$ for any number t and any geodesic γ, or alternatively that $\langle\nabla_w \operatorname{grad} f, w\rangle \geqslant 0$ for any vector w. Note that $(\operatorname{grad} f)(r(s,t)) = -r_s(s,t)$ since $\operatorname{grad} f(q) = -V(q,x)$ for any $q \in H$ by Proposition 1.12. Hence $\langle\nabla_{r_t} r_s, r_t\rangle(s,t) \leqslant 0$ for all s, t. In particular,

$$\partial/\partial s\langle Y_t(s), Y_t(s)\rangle = 2\langle\nabla_{r_s} r_t, r_t\rangle(s,t) = 2\langle\nabla_{r_t} r_s, r_t\rangle(s,t) \leqslant 0.$$

This implies that $y(s,t) = \|Y_t(s)\|$ is nonincreasing in s for each $t \in [0,a]$. The parametrized length of $g_s\sigma$ equals the length of $\mu(g_s\sigma)$: $t \to r(s,t)$. Hence

$$L(s) = \int_0^a \|r_t(s,t)\|\, dt = \int_0^a y(s,t)\, dt$$

since $y(s,t)$ is never zero. It is now clear that $L(s)$ is nonincreasing in s.

Suppose now that K is negative at $\mu(g_u w)$ for some $u \in R$ and some $w \in \sigma$. We may write $g_u w = r_s(u,t_0)$ for some $t_0 \in [0,a]$. We show that $L(s)$ is strictly monotone decreasing on $[s',u]$ for any number $s' < u$. Assume that this is false and choose numbers $s' \leqslant s_1 < s_2 \leqslant u$ such that $L(s_1) = L(s_2)$. It is well known that $y(s,t)$ satisfies the Jacobi equation

$$(\partial^2 y/\partial s^2)(s,t) + K(s,t)y(s,t) = 0$$

where $K(s,t)$ is the Gaussian curvature at $r(s,t)$. Since $L(s)$ is nonincreasing it follows that $L(s) = L(s_1)$ for all $s \in [s_1,s_2]$, which implies that $y(s,t) \equiv y(s_1,t)$ for all $s \in [s_1,s_2]$ and all $t \in [0,a]$. In particular for $s \geqslant s_1$, $(\partial y/\partial s)(s,t_0) \geqslant 0$ since

$$(\partial y/\partial s)(s_1,t_0) = 0 \quad \text{and} \quad (\partial^2 y/\partial s^2)(s,t_0) = -K(s,t_0)y(s,t_0) \geqslant 0.$$

Therefore $y(s,t_0) \equiv y(s_1,t_0)$ for all $s \geqslant s_1$ since $(\partial y/\partial s)(s,t_0) \leqslant 0$ for all s by the discussion above. From the Jacobi equation above and the fact that $y(s,t_0)$ is never zero we conclude that $K(s,t_0) \equiv 0$ for $s \geqslant s_1$. This contradicts the fact that $K(u,t_0) < 0$ and proves that L is strictly monotone decreasing on $[s',u]$.

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