JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 78B, No. 3, July–September 1974

How Bad Is the Hadamard Determinantal Bound?*

Charles R. Johnson and Morris Newman**

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(July 22, 1974)

The Hadamard bound for the determinant of an n by n matrix is a good one in that equality may be attained in a rich class of cases. However, the bound generally gives up a good deal, and we answer the title question "on the average." Assuming the entries of $A=(a_{ij})$ are uniformly distributed over some interval symmetric about the origin, the expected value of the ratio of $(\det A)^2$ to the square of the Hadamard bound is found to be $\frac{n!}{n^n}$. The expectations of the square of the Hadamard bound and of $(\det A)^2$ are also computed individually, and their ratio turns out also to be $\frac{n!}{n^n}$.

Key words: Determinant; expected value; Hadamard determinantal bound; uniform distribution.

The Hadamard bound is often used as an upper estimate for the determinant of an n by n matrix in computational algorithms as well as other numerical estimates. It states that for an n by n matrix $A=(a_{ij})$,

$$|\det A| \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \equiv H(A).$$

Since equality may be attained (either when the rows of A are orthogonal or when one row is zero), the bound is both simple and theoretically sound. However, computational experience suggests that the bound generally exaggerates the determinant rather heavily. This is unfortunate when the computation time of an algorithm is proportional to a determinantal estimate. One might well ask what would be a good estimate "on the average."

In order to provide a partial answer, we assume that $A=(a_{ij})$ is a real n by n matrix whose entries are chosen independently from a uniform distribution on $[-1, 1]$. Then $d \equiv \det A$, $H \equiv H(A)$ and d/H may be considered as random variables. If $f=f(A)$ is any scalar-valued function of the entries of the matrix A, we denote the expected value (if it exists) of the random variable f by

$$E(f) \equiv 2^{-n^2} \int_{-1}^{1} f da_{ij}$$

where the notation means that each of the n^2 variables of integration a_{ij} runs over the interval $[-1, 1]$. We shall prove:

THEOREM 1: *We have*

$$\mathrm{E}(\mathrm{d}^2) = \mathrm{n}!/3^{\mathrm{n}}, \tag{i}$$

$$\mathrm{E}(\mathrm{H}^2) = \mathrm{n}^{\mathrm{n}}/3^{\mathrm{n}}, \tag{ii}$$

AMS Subject Classification: 15A15, 65F99.

*This work was done while the first author was a National Academy of Sciences – National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

**Present Address: Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Md. 20742

554-917 OL - 74 - 5

$$E(d^2/H^2) = E(d^2)/E(H^2) = n!/n^n. \tag{iii}$$

PROOF: We have

$$\det(A) = \sum_{\sigma} \chi(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)},$$

where σ runs over all elements of the symmetric group S_n, and $\chi(\sigma)$ is the alternating character. Thus

$$(\det(A))^2 = \sum_{\sigma,\tau} \chi(\sigma)\chi(\tau) a_{1\sigma(1)} a_{1\tau(1)} \ldots a_{n\sigma(n)} a_{n\tau(n)}.$$

Now let $f = f(A)$ be any scalar-valued function of the variables a_{ij} which is *even* in each variable. Then

$$E(d^2 f) = 2^{-n^2} \sum_{\sigma,\tau} \chi(\sigma)\chi(\tau) \int_{-1}^{1} f \cdot a_{1\sigma(1)} a_{1\tau(1)} \ldots a_{n\sigma(n)} a_{n\tau(n)} da_{ij}.$$

Consider the contribution of the term

$$f \cdot a_{1\sigma(1)} a_{1\tau(1)} \ldots a_{n\sigma(n)} a_{n\tau(n)}$$

to the integral. If $\sigma(k) \neq \tau(k)$ for some k such that $1 \leqslant k \leqslant n$, then this term is an odd function of $a_{k\sigma(k)}$, and so the integral of this term vanishes, since the range of each variable of integration a_{ij} is $-1 \leqslant a_{ij} \leqslant 1$. It follows that

$$E(d^2 f) = 2^{-n^2} \sum_{\sigma} \int_{-1}^{1} f a^2_{1\sigma(1)} \ldots a^2_{n\sigma(n)} da_{ij}.$$

We first make the choice $f = 1$. Then

$$\begin{aligned} E(d^2) &= 2^{-n^2} \sum_{\sigma} \int_{-1}^{1} a^2_{1\sigma(1)} \ldots a^2_{n\sigma(n)} da_{ij} \\ &= 2^{-n^2} \sum_{\sigma} \left(\frac{2}{3}\right)^n \cdot 2^{n^2-n} \\ &= n!/3^n. \end{aligned}$$

Hence (i) is proved.

We now choose $f = H^{-2}$. Then

$$E(d^2/H^2) = 2^{-n^2} \sum_{\sigma} \int_{-1}^{1} a^2_{1\sigma(1)} \ldots a^2_{n\sigma(n)} \Big/ \prod_{i=1}^{n} (a^2_{i1} + \ldots + a^2_{in}) da_{ij}.$$

A moment's reflection shows that the value of the integral is the same for each σ, so that

$$\begin{aligned} E(d^2/H^2) &= 2^{-n^2} n! \int_{-1}^{1} \prod_{i=1}^{n} a^2_{ii}/(a^2_{i1} + \ldots + a^2_{in}) da_{ij} \\ &= 2^{-n^2} n! \left\{ \int_{-1}^{1} x_1^2/(x_1^2 + \ldots + x_n^2) dx_1 \ldots dx_n \right\}^n \end{aligned}$$

$$=2^{-n^2}n!\left\{\frac{1}{n}\int_1^1 (x_1^2+\ .\ .\ .\ +x_n^2)/(x_1^2+\ .\ .\ .\ +x_n^2)dx_1\ .\ .\ .\ dx_n\right\}^n$$

$$=2^{-n^2}n!n^{-n}2^{n^2}$$

$$=n!/n^n.$$

Hence (iii) is proved.

The proof of (ii) is similar and we omit it.

REMARK 1: An interpretation may be given to part (iii) of theorem 1 by the observation that the quotient $d(A)^2/H(A)^2$ is just the ratio of det AA^T to the product of the diagonal entries of AA^T. (That this ratio is less than or equal to 1 is another version of Hadamard's inequality.)

REMARK 2: If alternatively the a_{ij} are independently and uniformly distributed over $[-M, M]$, we have:

$$E(d^2)=M^{2n}\frac{n!}{3^n} \tag{i'}$$

and

$$E(H^2)=M^{2n}\frac{n^n}{3^n}. \tag{ii'}$$

The fact that

$$E(d^2/H^2)=E(d^2)/E(H^2)$$

could certainly not have been predicted beforehand, and strikes us as a rather remarkable occurrence. Unfortunately, it seems quite difficult to derive similar formulae for

$$E(|d|),\ E(H),\qquad E(|d|/H).$$

The previous discussion may be generalized directly to any generalized matrix function. Thus if

$$d(\chi, G)=\sum_{\sigma\in G}\chi(\sigma)a_{1\sigma(1)}\ .\ .\ .\ a_{n\sigma(n)},$$

where G is any subgroup of S_n and χ any irreducible character on G, then we have

THEOREM 2: *The expected value of* $|\mathrm{d}(\chi, \mathrm{G})|^2$ *is given by*

$$\mathrm{E}(|\mathrm{d}(\chi, \mathrm{G})|^2)=\mathrm{o(G)}/3^{\mathrm{n}},$$

where $\mathrm{o(G)}$ *is the order of* G.

The proof is just as before, except that the character relationship

$$\sum_{\sigma\in G}|\chi(\sigma)|^2=o(G)$$

comes into play.

(Paper 78B3–411)