How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?

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Abstract

The chemotaxis-Navier-Stokes system

$$\left\{ \begin{array}{rcl} n_t + u \cdot \nabla n &=& \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + u \cdot \nabla c &=& \Delta c - nf(c), \\ u_t + (u \cdot \nabla) u &=& \Delta u + \nabla P + n\nabla\Phi, \\ \nabla \cdot u &=& 0, \end{array} \right. \qquad (\star) \qquad (0.1)$$

is considered under boundary conditions of homogeneous Neumann type for n and c, and Dirichlet type for u, in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, where $\Phi \in W^{1,\infty}(\Omega)$ and χ and f are sufficiently smooth given functions generalizing the prototypes $\chi \equiv const.$ and $f(s) = s$ for $s \geq 0$.

It is known that for all suitably regular initial data n_0, c_0 and u_0 satisfying $0 \not\equiv n_0 \geq 0$, $c_0 \geq 0$ and $\nabla \cdot u_0 = 0$, a corresponding initial-boundary value problem admits at least one global weak solution which can be obtained as the pointwise limit of a sequence of solutions to appropriately regularized problems. The present paper shows that after some relaxation time, this solution enjoys further regularity properties and thereby complies with the concept of *eventual energy solutions* which is newly introduced here, and which inter alia requires that two quasi-dissipative inequalities are ultimately satisfied.

Moreover, it is shown that actually for any such eventual energy solution (n, c, u) there exists a waiting time $T_0 \in (0, \infty)$ with the property that (n, c, u) is smooth in $\bar{\Omega} \times [T_0, \infty)$, and that

$$n(x,t) \to \overline{n_0}, \qquad c(x,t) \to 0 \qquad \text{and} \qquad u(x,t) \to 0$$

hold as $t \to \infty$, uniformly with respect to $x \in \Omega$.

This resembles a classical result on the three-dimensional Navier-Stokes system, asserting eventual smoothness of arbitrary weak solutions thereof which additionally fulfill the associated natural energy inequality. In consequence, our results inter alia indicate that under the considered boundary conditions, the possibly destabilizing action of chemotactic cross-diffusion in $(\star)$ does not substantially affect the regularity properties of the fluid flow at least on large time scales.

Key words: chemotaxis, Navier-Stokes, global existence, boundedness, eventual regularity, stabilization, entropy dissipation
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1 Introduction

Chemotaxis and blow-up. When primitive microorganisms interact with their environment, their individually unstructured behavior may switch to quite complicated dynamics at macroscopic levels. Prototypical situations include spontaneous formation of aggregates such as in slime mold formation processes e.g. of *Dictyostelium discoideum*, organization of cell positioning during embryonic development, or also the invasion of tumors into healthy tissue. An important role in numerous structure-generating processes is known to be played by various types of biased cell movement in response to external cues such as chemical signal substances, mechanical stimuli, or gradients in voltage or acidic concentration, for instance. Such *taxis* mechanisms have been thoroughly studied in various contexts, also at a theoretical level, with the celebrated Keller-Segel system of chemotaxis consituting the apparently most paradigmatic representative in the field of macroscopic mathematical models ([19]; see also [15] for a comprehensive survey on modeling aspects). Indeed, intense analysis on the latter has confirmed the conjecture that spontaneous formation of aggregates, in the extreme mathematical sense of finite-time blow-up of solutions, may arise even in the simple two-component framework containing a population of cells moving chemotactically upward gradients of a signal substance, provided that the system reinforces itself in that cells produce the chemical in question ([16], [18], [36], [25]).

Chemotaxis-fluid interaction. In the case of even more primitive organisms, chemotactically moving toward a nutrient which they consume rather than produce, the correspondingly modified chemotaxis system possesses global bounded smooth solutions in the spatially two-dimensional setting, whereas in the three-dimensional counterpart at least global weak solutions can be constructed which eventually become smooth and bounded ([29]). On the other hand, more recent findings indicate that also populations of such simple individuals may exhibit quite colorful collective behavior: As suggested by striking experiments revealing spontaneous formation of plume-like aggregates in populations of *Bacillus subtilis* suspended in sessile water drops, in such situations it may be necessary to take into account the mutual interaction of cells and their movement on the one hand, and of the surrounding medium on the other. Accordingly, the modeling approach in [8] and [32] in particular assumes that besides chemotactic movement, signal consumption and transport of both cells and signal through the fluid, there is a significant buoyancy-driven effect of cells on the fluid dynamics. One is thereby led to considering the coupled chemotaxis-Navier-Stokes system

$$\left\{ \begin{array}{rcll} n_t + u\cdot\nabla n &=& \Delta n - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &=& \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &=& \Delta u + \nabla P + n\nabla\Phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u &=& 0, & x\in\Omega,\ t>0, \end{array} \right. \tag{1.2}$$

for the unknown (n, c, u, P) in the physical domain $\Omega \subset \mathbb{R}^N$, where the chemotactic sensitivity χ, the signal consumption rate f and the gravitational potential Φ are given parameter functions.

From a viewpoint of mathematical analysis, this system couples the well-known obstacles from the theory of the Navier-Stokes equations to the typical difficulties arising in the study of chemotaxis systems. Up to now, in the case $N = 3$ it is not only unknown whether the incompressible Navier-Stokes equations possess global smooth solutions for arbitrarily large smooth initial data (e.g. under Dirichlet boundary conditions in bounded domains, cf. [23] or also [34] and [26]); also the chemotaxis-only subsystem of (1.2) obtained on letting $u \equiv 0$ is far from understood in this framework, with no answer available yet e.g. to the question whether the global weak solutions, known to exist for any reasonably regular initial data in a corresponding Neumann-type initial-boundary value problem in bounded convex domains, may blow up in finite time before becoming ultimately smooth ([29]).

Accordingly, the knowledge on such coupled chemotaxis-fluid systems is at a rather early stage yet, with most previous works focusing on the basic issues of global solvability in various functional frameworks. For instance, global existence of uniquely determined smooth solutions is known for an initial-boundary value problem associated with the two-dimensional version of (1.2), under structural assumptions on the parameter functions χ and f which are mild enough so as to include the prototypical choices

$$\chi \equiv const. \qquad \text{and} \qquad f(s) = s, \quad s \geq 0, \tag{1.3}$$

and for all reasonably smooth initial data ([35]).

In the context of three-dimensional frameworks, large bodies of the existing literature address variants of (1.2) involving diverse types of regularizing modifications. For the chemotaxis-Stokes system obtained from (1.2) on neglecting the convective term $(u \cdot \nabla)u$ in the fluid evolution, global existence results have been derived for the Cauchy problem in $\mathbb{R}^3$ under certain additional requirements on χ and f and a smallness assumption e.g. on c ([9]), and also for a corresponding boundary-value problem without any such further restrictions ([35]). Even in this simplified setting, all these solutions constructed so far are merely weak solutions, with widely unknown boundedness and regularity properties which in fact might be poor so as to be consistent with several conceivable types of blow-up phenomena, for example of finite-time blow up of n with respect to the norm in $L^\infty(\Omega)$ (cf. also [4] for a detailed discussion of extensibility criteria for local-in-time smooth solutions).

As an additional regularizing mechanism, numerous works study the enhancement of cell diffusion at large densities, modeled by replacing Δn with the porous medium-type diffusion term Δn^m for $m > 1$. Under mild assumptions on χ and f, the corresponding three-dimensional chemotaxis-Stokes system then again admits global weak solutions whenever $m > 1$ ([10]; cf. also [24] for a precedent); in the case when moreover $m > \frac{8}{7}$ and Ω is a bounded convex domain in $\mathbb{R}^3$, the first component n of such a solution is in fact locally bounded in $\bar{\Omega} \times [0, \infty)$ ([31]), and if even $m > \frac{7}{6}$, then in the latter situation n actually remains bounded in all of $\Omega \times (0, \infty)$ ([38]). In the case $m > \frac{4}{3}$, global existence of – possibly unbounded – weak solutions has been established in [33] even for the associated full chemotaxis-Navier-Stokes system; results on global existence and boundedness in two-dimensional chemotaxis-fluid systems with nonlinear cell diffusion can be found in [6], [30] and [17]. Examples of further regularizations, as discussed in the literature with regard to global weak solvability, consist in considering saturation effects in the cross-diffusive term at large cell densities ([3]), or also including logistic-type cell proliferation and death ([33]).

Solvability and asymptotics in the three-dimensional chemotaxis-Navier-Stokes system. Concerning the full three-dimensional chemotaxis-Navier-Stokes system (1.2) with linear cell diffusion, the question of global solvability is apparently more delicate, and accordingly the first result in this direction resorted to the construction of global solutions to a corresponding Cauchy problem in $\mathbb{R}^3$ which emenate from initial data suitably close to one of the constant equilibria $(a, 0, 0)$ for $a > 0$ ([9]). As for general – and, in particular, large – initial data, in view of the limited knowledge on global regularity in the Navier-Stokes subsystem of (1.2) only weak solutions can currently be expected. A result optimal in this respect has recently been achieved in [39], where it has been shown that under the assumptions (1.7) and (1.8) below, for any initial data fulfilling (1.6) the problem (1.2) possesses at least one global weak solution.

Concerning the large time behavior of solutions, only very few seems known even in simplified situations: Except for some results on convergence of solutions satisfying certain smallness conditions ([9], [4]), the only statememts we are aware of which cover arbitrarily large initial data in (1.2) address its two-dimensional version in bounded convex domains in which any solution approaches the constant state $(\overline{n_0}, 0, 0)$ in the large time limit at an exponential rate, where $\overline{n_0} := \fint_\Omega n_0 > 0$ ([37], [40]). In the three-dimensional counterpart, a similar stabilization result so far could be obtained only for the chemotaxis-Stokes variant in which additional regularity is enforced by the presence of porous medium-type cell diffusion ([38]).

Main results: Eventual smoothness and stabilization. The objective of the present work is to undertake a further step toward a qualitative understanding of the chemotaxis-fluid interaction modeled in (1.2), especially with regard to possible effects of the chemotaxis-driven forcing on the fluid motion, and of the latter on the distribution of cells. Our main results in this direction will reveal that any such mutual influence will in fact disappear asymptotically in that the large time behavior of solutions is essentially governed by the decoupled chemotaxis-only and Navier-Stokes subsystems obtained on neglecting the components u and (n, c), respectively. In particular, as in the unforced Navier-Stokes equations ([34]), the fluid velocity u will become smooth eventually and decay uniformly in the large time limit; likewise, the couple (n, c) enjoys enjoys a similar ultimate smoothness property and approaches the spatially homogeneous limit $(\overline{n_0}, 0)$ associated with the respective mass level, thus resembling the behavior in the associated fluid-free chemotaxis system ([29]).

In order to make this more precise, let us consider (1.2) in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, along with the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.4}$$

and under the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{on } \partial\Omega. \tag{1.5}$$

Here we shall require that

$$\begin{cases} n_0 \in L\log L(\Omega) & \text{is nonnegative with } n_0 \not\equiv 0, \quad \text{that} \\ c_0 \in L^\infty(\Omega) & \text{is nonnegative and such that } \sqrt{c_0} \in W^{1,2}(\Omega), \quad \text{and that} \\ u_0 \in L^2_\sigma(\Omega), \end{cases} \tag{1.6}$$

where as usual, $L\log L(\Omega)$ denotes the Orlicz space corresponding to the Young function $[0,\infty) \ni z \mapsto z\ln(1+z)$, and where for $p > 1$, by $L^p_\sigma(\Omega) := \{\varphi \in L^p(\Omega;\mathbb{R}^3) \mid \nabla\cdot\varphi = 0\}$ we abbreviate the space of all solenoidal vector fields in $L^p(\Omega)$.

Throughout this paper, the chemotactic sensitivity χ, the signal consumption rate f in (1.2) and the gravitational potential Φ are assumed to be such that

$$\begin{cases} \chi \in C^2([0,\infty)) & \text{is positive on } [0,\infty), \\ f \in C^1([0,\infty)) & \text{is positive on } (0,\infty) \text{ with } f(0) = 0, \text{ that} \\ \Phi \in W^{1,\infty}(\Omega), \end{cases} \tag{1.7}$$

and that the structural requirements

$$\Big(\frac{f}{\chi}\Big)' > 0, \qquad \Big(\frac{f}{\chi}\Big)'' \le 0 \quad \text{and} \quad (\chi\cdot f)'' \ge 0 \qquad \text{on } [0,\infty) \tag{1.8}$$

are fulfilled, noting that the latter hypotheses are mild enough so as to allow e.g. for the choices in (1.3).

Within this framework, in view of the unsolved uniqueness problem for the Navier-Stokes equations we cannot expect weak solutions of (1.2) to be unique; accordingly, it seems desirable to derive results on qualitative behavior which are independent of a particular construction of solutions. Inspired by analogues from the analysis of the Navier-Stokes system, we shall thus consider rather arbitrary weak solutions enjoying certain additional properties which are essentially linked to strutural features of (1.2). Here besides the natural energy inequality (1.11) associated with the Navier-Stokes system, singling out the so-called turbulent solutions among all weak solutions of the latter, a central role will be played by a second energy-like functional $\mathcal{F}_\kappa$ which for given $\kappa > 0$ is defined by

$$\mathcal{F}_\kappa[n,c,u] := \int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{\chi(c)}{f(c)}|\nabla c|^2 + \kappa\int_\Omega |u|^2 \tag{1.9}$$

whenever $n \in L\log L(\Omega)$ and $c \in W^{1,2}(\Omega)$ are nonnegative and such that $\frac{\chi(c)}{f(c)}|\nabla c|^2 \in L^1(\Omega)$, and $u \in L^2(\Omega;\mathbb{R}^3)$ ([39]).

We now select a subclass of weak solutions to (1.2) as follows.

Definition 1.1 *Suppose that (n,c,u) is a global weak solution of (1.2) in the sense of Definition 2.1 below. Then we call (n,c,u) an* eventual energy solution *of (1.2) if there exists $T>0$ such that*

$$\begin{aligned} & n \in L^4_{loc}(\bar\Omega\times[T,\infty)) \cap L^2_{loc}([T,\infty);W^{1,2}(\Omega)) \quad \textit{with} \quad n^{\frac{1}{2}} \in L^2_{loc}([T,\infty);W^{1,2}(\Omega)), \\ & c \in L^\infty_{loc}(\Omega\times[T,\infty)) \quad \textit{with} \quad c^{\frac{1}{4}} \in L^4_{loc}([T,\infty);W^{1,4}(\Omega)) \quad \textit{and} \\ & u \in L^\infty_{loc}([T,\infty);L^2_\sigma(\Omega)) \cap L^2_{loc}([T,\infty);W^{1,2}_0(\Omega)), \end{aligned} \tag{1.10}$$

if

$$\frac{1}{2}\int_\Omega |u(\cdot,t)|^2 + \int_{t_0}^t\int_\Omega |\nabla u|^2 \le \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 + \int_{t_0}^t\int_\Omega nu\cdot\nabla\Phi \qquad \textit{for a.e. } t_0 > T \textit{ and all } t > t_0, \tag{1.11}$$

and if there exist $\kappa>0$ and $K>0$ such that

$$\frac{d}{dt}\mathcal{F}_\kappa[n,c,u](t) + \frac{1}{K}\int_\Omega \Big\{\frac{|\nabla n|^2}{n} + \frac{|\nabla c|^4}{c^3} + |\nabla u|^2\Big\} \le K \qquad \textit{in } \mathcal{D}'((T,\infty)). \tag{1.12}$$

Remark. The regularity assumptions in (1.10) warrant that $\frac{|\nabla n|^2}{n}$, $\frac{|\nabla c|^4}{c^3}$ and $|\nabla u|^2$ belong to $L^1_{loc}(\bar\Omega\times(T,\infty))$, and that moreover $(T,\infty) \ni t \mapsto \mathcal{F}_\kappa[n,c,u](t) \in L^1_{loc}((T,\infty))$ (cf. Lemma 7.1), implying that (1.12) indeed is meaningful.

It has been shown in [39] that under the assumptions (1.7) and (1.8), for any initial data fulfilling (1.6) the problem (1.2) possesses at least one global weak solution in the natural sense specified in Definition 2.1 below. The first of our results asserts that this solution actually enjoys all the above properties of an eventual energy solution; we shall thereby prove the following.

Theorem 1.2 *Let (1.7) and (1.8) hold, and assume that n_0, c_0 and u_0 comply with (1.6). Then there exists at least one eventual energy solution of (1.2).*

We can secondly prove that in fact any such eventual energy solution becomes smooth ultimately, and that it approaches the unique spatially homogeneous steady state compatible with the preserved total mass $\int_\Omega n_0 > 0$.

Theorem 1.3 *Let (1.7) and (1.8) hold, and suppose that (n, c, u) is an eventual energy solution of (1.2) with some initial data n_0, c_0 and u_0 satisfying (1.6). Then there exist $T > 0$ and $P \in C^{1,0}(\bar{\Omega} \times [T, \infty))$ such that*

$$\begin{cases} n \in C^{2,1}(\bar{\Omega} \times [T, \infty)), \\ c \in C^{2,1}(\bar{\Omega} \times [T, \infty)) \qquad \textit{and} \\ u \in C^{2,1}(\bar{\Omega} \times [T, \infty); \mathbb{R}^3), \end{cases} \tag{1.13}$$

and such that (n, c, u, P) solves the boundary value problem in (1.2) classically in $\bar{\Omega} \times [T, \infty)$. Furthermore,

$$n(\cdot, t) \to \overline{n_0} \quad \textit{in } L^\infty(\Omega), \qquad c(\cdot, t) \to 0 \quad \textit{in } L^\infty(\Omega) \qquad \textit{and} \qquad u(\cdot, t) \to 0 \quad \textit{in } L^\infty(\Omega) \tag{1.14}$$

as $t \to \infty$, where $\overline{n_0} := \fint_\Omega n_0$.

Theorem 1.3 may be interpreted as rigorously reflecting that despite the possibly disordering influence of the fluid, the signal consumption process in (1.2) occurs in such a regular manner that ultimately even the gradients of the chemical become irrelevant with regard to their chemoattractive impact, and that in consequence the cell population homogenizes efficiently enough so as to let any substantial destabilizing effect on the fluid vanish asymptotically. This may become substantially different in situations when different types of boundary conditions are considered, possibly accounting for signal influx, or when signal absorption is replaced with mechanisms of signal production by cells, as recently proposed and studied in contexts involving fluid interaction in [20], [21], [7] and [28].

Main ideas. Organization of the paper. The overall strategy pursued in the course of our reasoning consists in showing that firstly the solution component c must decay with respect to the norm in $L^\infty(\Omega)$ as $t \to \infty$, and that secondly appropriate smallness of this component in $L^\infty(\Omega \times (T_0, \infty))$ for some $T_0 > 0$ entails smoothness of (n, c, u) in $\bar{\Omega} \times (T_1, \infty)$ for some $T_1 > T_0$.

The first of these properties will be a consequence of some basic dissipative features of (1.2) combined with suitable uniform-in-time regularity estimates implied by the energy inequality (1.12) (Sections 3 and 4). In accomplishing the second of the mentioned steps, we will generalize a related procedure pursued in [37] for smooth solutions of the two-dimensional version of (1.2), where a similar conclusion was derived on the basis of the observation that for any given $p \geq 2$ the functional $\int_\Omega \frac{n^p}{\delta - c}$ acts as an entropy, provided that $\delta = \delta(p)$ is suitably small and c remains below the threshold δ throughout evolution. Since in the present case we intend to address arbitrary eventual energy solutions, the lack of a priori knowledge on regularity properties beyond those listed in Definition 1.1 will require the use of a substantially more subtle testing technique to track the time evolution of functionals of the above type, simultaneously involving the first two equations in (1.2). Moreover, this limited information on regularity will force us to firstly restrict our key statement in this direction, presented in the extensive Lemma 5.1, to functionals of the form $\int_\Omega \psi(n)\rho(c)$ with convex ψ and ρ subject to technical assumptions which inter alia require that $\psi(n)$ does not increase faster than $n^{\frac{9}{5}}$ as $n \to \infty$; only in a second step we will see in Section 6 by means of an approximation argument that actually any algebraic growth of ψ can be achieved.

In Section 7 we shall infer from the correspondingly gained entropy-dissipation inequalities that n stabilizes in a sense yet weaker than claimed in Theorem 1.3, but strong enough to assert a decay property of the forcing term in the Navier-Stokes system in (1.2) which is sufficient to imply decay of u with respect to the norm in $L^p(\Omega)$ for any finite $p \ge 1$ (Lemma 7.5). The eventual integrability property of u thereby implied will enable us to perform a series of arguments based on maximal Sobolev regularity in the Stokes evolution system and inhomogeneous linear heat equations to successively obtain further ultimate regularity properties of u, c and n which by standard Schauder theory imply eventual smoothness (Lemma 7.6–Lemma 7.13). This improved knowledge on regularity thereupon allows for turning the weak decay information previously gathered into the desired uniform convergence statements and thereby complete the proof of Theorem 1.3 in Section 8.

Finally, in proving Theorem 1.2 in Section 9 we shall make use of the fact that our arguments in Section 3 through Section 6 are formulated in a manner slightly more general than used in the mere analysis of eventual energy solutions, namely simultaneously covering also all of the solutions to the approximate systems (9.4), uniformly with respect to the regularization parameter $\varepsilon \in (0,1)$. Since an appropriate sequence of such solutions is known to approach a weak solution of (1.2) satisfying (1.11) and (1.12) for $T_0 := 0$, the additional properties thereby obtained assert that this limit in fact is an eventual energy solution.

Throughout the paper, we let $A := -\mathcal{P}\Delta$ denote the Stokes operator which for any $p \in (1,\infty)$ is sectorial in $L^p_\sigma(\Omega)$ when considered with domain $D(A) \equiv D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap L^p_\sigma(\Omega)$, and hence generates the analytic Stokes semigroup $(e^{-tA})_{t\ge 0}$. Here, by $\mathcal{P}$ we mean the associated Helmholtz projection mapping $L^p(\Omega)$ onto $L^p_\sigma(\Omega)$ (cf. [12], [13], [11]).

2 Weak solutions

The following notion of weak solutions to (1.2) is taken from [39]. Here and in the sequel, for vectors $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$ we let $v \otimes w$ denote the matrix $(a_{ij})_{i,j\in\{1,2,3\}} \in \mathbb{R}^{3\times 3}$ defined on setting $a_{ij} := v_i w_j$ for $i,j \in \{1,2,3\}$.

Definition 2.1 *By a* global weak solution *of (1.2), (1.4), (1.5) we mean a triple* (n,c,u) *of functions*

$$n \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \quad c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \quad u \in L^1_{loc}([0,\infty); W_0^{1,1}(\Omega;\mathbb{R}^3)), \tag{2.1}$$

such that $n \ge 0$ *and* $c \ge 0$ *a.e. in* $\Omega \times (0,\infty)$,

$$\begin{aligned} &nf(c) \in L^1_{loc}(\bar\Omega \times [0,\infty)), \qquad u\otimes u \in L^1_{loc}(\bar\Omega \times [0,\infty);\mathbb{R}^{3\times 3}), \qquad \text{and} \\ &n\chi(c)\nabla c, nu \text{ and } cu \text{ belong to } L^1_{loc}(\bar\Omega \times [0,\infty);\mathbb{R}^3), \end{aligned} \tag{2.2}$$

that $\nabla\cdot u = 0$ *a.e. in* $\Omega\times(0,\infty)$, *and that*

$$-\int_0^\infty \int_\Omega n\phi_t - \int_\Omega n_0\phi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla n\cdot\nabla\phi + \int_0^\infty \int_\Omega n\chi(c)\nabla c\cdot\nabla\phi + \int_0^\infty \int_\Omega nu\cdot\nabla\phi \tag{2.3}$$

for all $\phi \in C_0^\infty(\bar\Omega \times [0,\infty))$,

$$-\int_0^\infty \int_\Omega c\phi_t - \int_\Omega c_0\phi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla c \cdot \nabla\phi - \int_0^\infty \int_\Omega nf(c)\phi + \int_0^\infty \int_\Omega cu \cdot \nabla\phi \tag{2.4}$$

for all $\phi \in C_0^\infty(\bar\Omega \times [0,\infty))$ *as well as*

$$-\int_0^\infty \int_\Omega u \cdot \phi_t - \int_\Omega u_0 \cdot \phi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla\phi + \int_0^\infty u \otimes u \cdot \nabla\phi + \int_0^\infty \int_\Omega n\nabla\Phi \cdot \phi \tag{2.5}$$

for all $\phi \in C_0^\infty(\Omega \times [0,\infty);\mathbb{R}^3)$ *satisfying* $\nabla \cdot \phi \equiv 0$.

The above solution concept meets the basic natural requirement that solutions preserve mass during evolution.

Lemma 2.2 *Suppose that* (n,c,u) *is a global weak solution of (1.2). Then* $n \in L^\infty((0,\infty);L^1(\Omega))$ *with*

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \qquad \textit{for a.e. } t > 0. \tag{2.6}$$

Proof. Let $t_0 > 0$ be a Lebesgue point of $(0,\infty) \ni t \mapsto \int_\Omega n(x,t)dx$. For $\delta \in (0,1)$ we then approximate

$$\zeta_\delta(t) := \begin{cases} 1 & \text{if } t \le t_0, \\ \frac{t_0+\delta-t}{\delta}, & t \in (t_0, t_0+\delta), \\ 0 & \text{if } t \ge t_0+\delta, \end{cases} \tag{2.7}$$

by taking any sequence $(\zeta_{\delta j})_{j\in\mathbb{N}} \subset C^\infty([0,\infty))$ fulfilling $\zeta_{\delta j} \equiv 1$ in $[0,t_0)$, $\zeta_{\delta j} \equiv 0$ in (t_0+1,∞) and $\zeta_{\delta j} \overset{\star}{\rightharpoonup} \zeta_\delta$ in $W^{1,\infty}((0,t_0+1))$ as $j \to \infty$. For each $\delta \in (0,1)$ and $j \in \mathbb{N}$, we may then use $\phi(x,t) := \zeta_{\delta j}(t)$, $(x,t) \in \bar\Omega \times [0,\infty)$, as a test function in (2.3). In the correspondingly obtained identity

$$-\int_{t_0}^{t_0+1} \int_\Omega \zeta_{\delta j}'(t) n(x,t)dxdt = \int_\Omega n_0(x)dx,$$

we first let $j \to \infty$ to obtain

$$\frac{1}{\delta} \int_{t_0}^{t_0+\delta} \int_\Omega n(x,t)dxdt = \int_\Omega n_0(x)dx,$$

for all $\delta \in (0,1)$, whereupon we take $\delta \searrow 0$ to infer from the assumed Lebesgue point property of t_0 that $\int_\Omega n(\cdot,t_0) = \int_\Omega n_0$. Since the complement in $(0,\infty)$ of the set of all such t_0 has measure zero, this proves (2.6). □

3 A family of chemotaxis problems with prescribed convection

From [39] we already know that (1.2) possesses a global weak solution, and that this solution can be obtained as the limit of smooth solutions to certain regularized problems (cf. (9.4) and Lemma 9.2 below). Verifying Theorem 1.2 thus amounts to showing that these approximate solutions in fact enjoy further regularity features which ensure that the limit in fact will be an eventual energy solution. In view of the circumstance that also Theorem 1.3 requires proving regularity, to avoid repetitions we find it convenient to organize our line of arguments in such a way that in the first part of our analysis we consider a generalized variant of the first two equations in (1.2) which includes both the original version appearing in (1.2) and also the regularized subsystem thereof considered later in Section 9.

Moreover, in order to underline that several asymptotic solution properties, including decay of the component c, are widely independent of the particular structure of the fluid flow, let us in this and the following sections consider the boundary value problem

$$\begin{cases} n_t + \widetilde{u} \cdot \nabla n = \Delta n - \nabla \cdot (nF'(n)\chi(c)\nabla c), & x \in \Omega,\ t > T_0, \\ c_t + \widetilde{u} \cdot \nabla c = \Delta c - F(n)f(c), & x \in \Omega,\ t > T_0, \\ \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, & x \in \partial\Omega,\ t > T_0, \end{cases} \tag{3.1}$$

where $\widetilde{u} \in L^\infty_{loc}((T_0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}((T_0,\infty); W^{1,2}_0(\Omega))$ is a given function, and where

$$F \in C^1([0,\infty)) \text{ is such that } F(0) = 0 \qquad \text{and} \qquad 0 \le F'(s) \le 1 \quad \text{for all } s \ge 0 \tag{3.2}$$

as well as

$$F(s) \ge \frac{s}{2} \qquad \text{for all } s \in [0,1]. \tag{3.3}$$

For proving Theorem 1.3 it would be sufficient to concentrate throughout on the case $F(s) := s,\ s \ge 0$; in the proof of Theorem 1.2, however, we will apply some of the results obtained for (3.1) upon choosing $F(s) := F_\varepsilon(s) := \frac{1}{\varepsilon} \ln(1 + \varepsilon s)$ for $s \ge 0$ and $\varepsilon \in (0,1)$, which is as well consistent with (3.2) and (3.3) (see Section 9).

We shall study (3.1) in the framework of solutions fulfilling the regularity requirements in Definition 1.1:

Definition 3.1 *Let $T_0 \ge 0$ and $\widetilde{u} \in L^\infty_{loc}((T_0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}((T_0,\infty); W^{1,2}_0(\Omega))$, and suppose that F satisfies (3.2). Then a couple (n,c) of nonnegative functions defined a.e. in $\Omega \times (T_0,\infty)$ will be called a* strong solution *of the boundary value problem (3.1) in $\Omega \times (T_0,\infty)$ if*

$$\begin{aligned} & n \in L^\infty_{loc}((T_0,\infty); L^1(\Omega)) \cap L^4_{loc}(\bar{\Omega} \times (T_0,\infty)) \cap L^2_{loc}((T_0,\infty); W^{1,2}(\Omega)) \quad \textit{and} \\ & c \in L^\infty_{loc}(\bar{\Omega} \times (T_0,\infty)) \cap L^4_{loc}((T_0,\infty); W^{1,4}(\Omega)), \end{aligned} \tag{3.4}$$

and if

$$-\int_{T_0}^\infty \int_\Omega n\phi_t = -\int_{T_0}^\infty \int_\Omega \nabla n \cdot \nabla \phi + \int_{T_0}^\infty \int_\Omega nF'(n)\chi(c)\nabla c \cdot \nabla \phi - \int_{T_0}^\infty \int_\Omega \widetilde{u} \cdot \nabla n\, \phi \tag{3.5}$$

as well as

$$-\int_{T_0}^\infty \int_\Omega c\phi_t = -\int_{T_0}^\infty \int_\Omega \nabla c \cdot \nabla \phi - \int_{T_0}^\infty \int_\Omega F(n)f(c)\phi - \int_{T_0}^\infty \int_\Omega \widetilde{u} \cdot \nabla c\, \phi \tag{3.6}$$

hold for all $\phi \in C^\infty_0(\bar{\Omega} \times (T_0,\infty))$.

The following lemma inter alia asserts that under the assumptions in (3.4) the integral identities (3.5) and (3.6) are indeed meaningful, and beyond this it provides some further regularity properties of the sources, fluxes and transport terms therein. These properties will become essential in the proof of Lemma 5.1.

Lemma 3.2 *Let $T_0 \geq 0$, $\widetilde{u} \in L^\infty_{loc}((T_0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}((T_0,\infty); W^{1,2}_0(\Omega))$ and F be such that (3.2) holds.*

i) If n and c are nonnegative and satisfy (3.4), then

$$\begin{aligned} &nF'(n)\chi(c)\nabla c \in L^2_{loc}(\bar\Omega \times (T_0,\infty)), \qquad \widetilde{u}\cdot\nabla n \in L^{\frac{5}{4}}_{loc}(\bar\Omega\times(T_0,\infty)), \\ &F(n)f(c) \in L^4_{loc}(\bar\Omega\times(T_0,\infty)) \qquad \text{and} \qquad \widetilde{u}\cdot\nabla c \in L^{\frac{20}{11}}_{loc}(\bar\Omega\times(T_0,\infty)). \end{aligned} \tag{3.7}$$

In particular, all integrals in (3.5) and (3.6) are well-defined.

ii) If (n,c) is a strong solution of (3.1) in $\Omega\times(T_0,\infty)$, then the identity (3.5) is actually valid for any $\phi \in L^5(\Omega\times(T_0,\infty))$ which has compact support in $\bar\Omega\times(T_0,\infty)$, and for which $\nabla\phi \in L^2(\Omega\times(T_0,\infty))$ and $\phi_t \in L^{\frac{4}{3}}(\Omega\times(T_0,\infty))$.

iii) Whenever (n,c) is a strong solution of (3.1) in $\Omega\times(T_0,\infty)$, the equation (3.6) continues to hold for any $\phi\in L^{\frac{20}{9}}(\Omega\times(T_0,\infty))$ with $\nabla\phi\in L^{\frac{4}{3}}(\Omega\times(T_0,\infty))$ and $\phi_t\in L^1(\Omega\times(T_0,\infty))$, for which $\operatorname{supp}\phi$ *is a compact subset of $\bar\Omega\times(T_0,\infty)$.*

Proof. i) In view of Lemma 10.2, the regularity hypotheses on $\widetilde{u}$ entail that $\widetilde{u}\in L^{\frac{10}{3}}_{loc}(\bar\Omega\times(T_0,\infty))$. Since $\frac14+\frac14=\frac12$ and $\frac{3}{10}+\frac12=\frac45$ as well as $\frac{3}{10}+\frac14=\frac{11}{20}$, and since $n\in L^4_{loc}(\bar\Omega\times(T_0,\infty))$, $c\in L^\infty_{loc}(\bar\Omega\times(T_0,\infty))$ and $\nabla c\in L^4_{loc}(\bar\Omega\times(T_0,\infty))$ again by Definition 3.1, several applications of the Hölder inequality and (3.2) readily yield the claimed integrability properties.

ii) and iii) In view of i), both statements can immediately be obtained upon performing standard approximation procedures. □

The proof of the next lemma on a basic dissipative property of the second equation in (3.1) follows a testing procedure which is well-established in the context of related parabolic problems in their weak formulation ([1]), relying on the convexity of $[0,\infty)\ni s\mapsto s^p$ for $p\geq 1$. As we are not aware of a reference precisely covering the present situation, let us include the main arguments for completeness. Upon slight modification, the argument can be adapted so as to extend the result to any $p\geq1$; for simplicity in presentation, however, we restrict ourselves to the cases $p=1$ and $p\geq 2$ relevant below, using that then $0\leq s\mapsto s^p$ belongs to $C^2([0,\infty))$.
For the following proof, as well as for the reasoning in Lemma 5.1, let us separately introduce a variant of the cut-off function in (2.7) defined by

$$\zeta_\delta(t) := \begin{cases} 0 & \text{if } t\leq t_0-\delta \text{ or } t\geq t_1+\delta, \\ \frac{t-t_0+\delta}{\delta} & \text{if } t\in(t_0-\delta,t_0), \\ 1 & \text{if } t\in[t_0,t_1], \\ \frac{t_1+\delta-t}{\delta} & \text{if } t\in(t_1,t_1+\delta), \end{cases} \tag{3.8}$$

for given $t_0\in\mathbb{R}, t_1>t_0$ and $\delta>0$.

Lemma 3.3 *Let $T_0 \ge 0$ and $\widetilde{u} \in L^\infty_{loc}((T_0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}((T_0,\infty); W_0^{1,2}(\Omega))$ and F be such that (3.2) holds, and suppose that (n,c) is a strong solution of (3.1) in $\Omega \times (T_0,\infty)$. Then for each $p \in \{1\} \cup [2,\infty)$ there exists a null set $N(p) \subset (T_0,\infty)$ such that*

$$\int_\Omega c^p(\cdot,t) + p(p-1)\int_{T_0}^t \int_\Omega c^{p-2}|\nabla c|^2 + p\int_{T_0}^t \int_\Omega F(n)f(c)c^{p-1} \quad \le \quad \int_\Omega c^p(\cdot,t_0)$$

$$\text{for all } t_0 \in (T_0,\infty)\setminus N(p) \text{ and any } t \in (t_0,\infty)\setminus N(p). \qquad (3.9)$$

Proof. For $p \ge 1$, we let $N(p) \subset (T_0,\infty)$ denote the complement of the set of all Lebesgue points of $(T_0,\infty) \ni t \mapsto \int_\Omega c^p(\cdot,t)$, and in order to prove the inequality in (3.9) for all $t_0 \in (T_0,\infty)\setminus N(p)$ and $t = t_1 \in (t_0,\infty)\setminus N(p)$, given any such t_0 and t_1 we let ζ_δ denote the cut-off function defined in (3.8) for $\delta \in (0,\delta_0)$ with $\delta_0 := \min\{1, t_0 - T_0\}$. Then according to Lemma 3.2 iii), for all $h \in (0, \delta_0 - \delta)$ we may apply (3.6) to $\phi(x,t) := \zeta_\delta(t) \cdot S_h[c^{p-1}](x,t)$, $(x,t) \in \Omega \times (T_0,\infty)$, with $p \in \{1\} \cup [2,\infty)$ and the Steklov average operator S_h being defined in (10.1). This leads to the identity

$$\begin{aligned}
I_1(\delta,h) + I_2(\delta,h) + I_3(\delta,h) \quad &:= \quad -\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega c \cdot S_h[c^{p-1}] + \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega c \cdot S_h[c^{p-1}] \\
&\qquad - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t) c(x,t) \cdot \frac{c^{p-1}(x,t+h) - c^{p-1}(x,t)}{h} dxdt \\
&= \quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t) \nabla c(x,t) \cdot \nabla S_h[c^{p-1}](x,t) dxdt \\
&\qquad - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t) F(n(x,t)) f(c(x,t)) S_h[c^{p-1}](x,t) dxdt \\
&\qquad - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t) (\widetilde{u}(x,t) \cdot \nabla c) S_h[c^{p-1}](x,t) dxdt \\
&=: \quad I_4(\delta,h) + I_5(\delta,h) + I_6(\delta,h) \qquad (3.10)
\end{aligned}$$

for all $\delta \in (0,\delta_0)$ and $h \in (0,\delta_0 - \delta)$, in which we observe that since $(p-1)c^{p-2}$ is bounded in both cases $p = 1$ and $p \ge 2$, it follows from the definition of S_h, the inclusion $\nabla c \in L^2_{loc}(\bar\Omega \times (T_0,\infty))$ and Lemma 10.1 that

$$\nabla S_h[c^{p-1}] = (p-1) S_h[c^{p-2}\nabla c] \to (p-1)c^{p-2}\nabla c \quad \text{in } L^2_{loc}(\bar\Omega \times (T_0,\infty)) \qquad \text{as } h \searrow 0.$$

Since Lemma 10.1 also warrants that $S_h[c^{p-1}] \to c^{p-1}$ in $L^{\frac{20}{9}}_{loc}(\bar\Omega \times (T_0,\infty))$ as $h \searrow 0$, and since the required regularity properties of n and c along with (3.2) readily ensure that $F(n)f(c) \in L^4_{loc}(\bar\Omega \times (T_0,\infty)) \subset L^{\frac{20}{11}}_{loc}(\bar\Omega \times (T_0,\infty))$ and also $\widetilde{u} \cdot \nabla c \in L^{\frac{20}{11}}_{loc}(\bar\Omega \times (T_0,\infty))$, in (3.10) we obtain

$$\begin{aligned}
I_4(\delta,h) + I_5(\delta,h) + I_6(\delta,h) \quad &\to \quad -(p-1)\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t) c^{p-2}|\nabla c|^2 - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t) F(n) c^{p-1} f(c) \\
&\qquad - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t) c^{p-1} \widetilde{u} \cdot \nabla c \qquad (3.11)
\end{aligned}$$

as $h \searrow 0$, and likewise

$$I_1(\delta,h) + I_2(\delta,h) \to -\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega c^p + \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega c^p \qquad \text{as } h \searrow 0. \tag{3.12}$$

As for the third integral on the left of (3.10), we estimate using Young's inequality to find upon a substitution that

$$\begin{aligned} I_3 &\geq -\frac{1}{h}\cdot\left\{\frac{1}{p}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)c^p(x,t)dxdt + \frac{p-1}{p}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)c^p(x,t+h)dxdt\right\} \\ &\quad + \frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)c^p(x,t)dxdt \\ &= \frac{p-1}{p}\int_{T_0}^{t_1+1}\int_\Omega \frac{\zeta_\delta(t)-\zeta_\delta(t-h)}{h}\cdot c^p(x,t)dxdt \end{aligned}$$

for all $\delta \in (0,\delta_0)$ and $h \in (0,\delta_0-\delta)$, so that by the dominated convergence theorem we conclude that

$$\liminf_{h\searrow 0} I_3(\delta,h) \geq \frac{p-1}{p}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta'(t)c^p = \frac{p-1}{p\delta}\int_{t_0-\delta}^{t_0}\int_\Omega c^p - \frac{p-1}{p\delta}\int_{t_1}^{t_1+\delta} c^p \tag{3.13}$$

as $h \searrow 0$. Since $\widetilde{u}$ is solenoidal and hence

$$-\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)c^{p-1}\widetilde{u}\cdot\nabla c = -\frac{1}{p}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\widetilde{u}\cdot\nabla c^p = 0,$$

combining (3.10)-(3.13) and rearranging shows that

$$\frac{1}{p\delta}\int_{t_1}^{t_1+\delta}\int_\Omega c^p + (p-1)\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)c^{p-2}|\nabla c|^2 + \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)F(n)c^{p-1}f(c) \leq \frac{1}{p\delta}\int_{t_0-\delta}^{t_0} c^p$$

for all $\delta \in (0,\delta_0)$. Since $\zeta_\delta \equiv 1$ in (t_0,t_1), thanks to the assumed Lebesgue point properties of t_0 and t_1 this readily yields the desired inequality for such t_0 and t_1 on taking $\delta \searrow 0$. □

Evaluating (3.9) for $p=1$ and $p=2$ as well as in the limit case $p\to\infty$ we obtain the following consequence which provides some first, still quite weak, information on decay of c, at least under the assumption that n, and hence $F(n)$ by (3.3), remains positive in an appropriate sense.

Corollary 3.4 *Let $T_0 \geq 0$ and $\widetilde{u} \in L^\infty_{loc}([T_0,\infty);L^2_\sigma(\Omega)) \cap L^2_{loc}([T_0,\infty);W_0^{1,2}(\Omega))$ and F be such that (3.2) holds, and suppose that (n,c) is a strong solution of (3.1) in $\Omega\times(T_0,\infty)$. Then*

$$\int_{T_0}^\infty\int_\Omega F(n)f(c) \leq \operatorname{ess}\liminf_{t\searrow T_0}\int_\Omega c(\cdot,t) \tag{3.14}$$

and

$$\int_{T_0}^\infty\int_\Omega |\nabla c|^2 \leq \operatorname{ess}\liminf_{t\searrow T_0}\frac{1}{2}\int_\Omega c^2(\cdot,t), \tag{3.15}$$

and there exists a null set $N \subset (T_0,\infty)$ such that

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \leq \|c(\cdot,t_0)\|_{L^\infty(\Omega)} \qquad \text{for all } t_0 \in (T_0,\infty)\setminus N \text{ and any } t \in (t_0,\infty)\setminus N. \tag{3.16}$$

PROOF. The inequalities in (3.14) and (3.15) immediately result from applying Lemma 3.3 to $p := 1$ and $p := 2$, respectively. We next invoke Lemma 3.3 for $p_j := j$ for $j \in \mathbb{N}$ to obtain null sets $N(p_j)$ with the properties listed there. For the null set $N := \bigcup_{j\in\mathbb{N}} N(p_j) \subset (T_0, \infty)$ we thus obtain that $\|c(\cdot,t)\|_{L^{p_j}(\Omega)} \le \|c(\cdot,t_0)\|_{L^{p_j}(\Omega)}$ whenever $j \in \mathbb{N}$ as well as $t_0 \in [T_0,\infty)\setminus N$ and $t \in (t_0,\infty)\setminus N$, because F and f are both nonnegative. Taking $j \to \infty$ here shows that $\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \|c(\cdot,t_0)\|_{L^\infty(\Omega)}$ for any such t_0 and t, which yields (3.16). □

4 A doubly uniform decay property of solutions to (3.1)

In this section we shall make sure that any solution (n,c) of (3.1) satisfies $c(\cdot,t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$, which will be a fundamental ingredient for our later regularity arguments. Since apart from considering any fixed eventual energy solution of (1.2) we wish to address the entire family of solutions to the approximate problems (9.4) for $\varepsilon \in (0,1)$, our plan will be to make sure that this convergence is actually uniform with respect to the choice of the considered solution, as well as of $\widetilde{u}$ and F, in an appropriate sense. In order to make this more precise, let us introduce the following notation.

Definition 4.1 *Given $m > 0, M > 0, L > 0$ and $T_0 \ge 0$, we let*

$$\mathcal{S}_{m,M,L,T_0} \tag{4.1}$$

denote the set of all triples (n,c,F) of functions $n : \Omega \times (T_0,\infty) \to \mathbb{R}, c : \Omega \times (0,\infty) \to \mathbb{R}$ and $F : [0,\infty) \to \mathbb{R}$ such that F satisfies (3.2) and (3.3), and that for some $\widetilde{u} \in L^\infty_{loc}((T_0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}((T_0,\infty); W^{1,2}_0(\Omega))$, the pair (n,c) is a strong solution of (3.1) in $\Omega \times (T_0,\infty)$ satisfying

$$\int_\Omega n(\cdot,t) = m \quad \textit{and} \quad \|c(\cdot,t)\|_{L^\infty(\Omega)} \le M \qquad \textit{for a.e. } t > T_0 \tag{4.2}$$

as well as

$$\int_t^{t+1} \int_\Omega \Big\{ \frac{|\nabla n|^2}{n} + |\nabla c|^4 \Big\} \le L \qquad \textit{for all } t > T_0. \tag{4.3}$$

In this framework, an adequate interpretation of (3.14) and (3.15) in Corollary 3.4 yields the following.

Lemma 4.2 *Let $m > 0, M > 0, L > 0$ and $T_0 \ge 0$. Then the set $\mathcal{S}_{m,M,L,T_0}$ in (4.1) has the property that*

$$\sup_{(n,c,F)\in\mathcal{S}_{m,M,L,T_0}} \inf_{t\in[T_0,T_0+\tau]} \int_t^{t+1} \int_\Omega \Big\{ F(n)f(c) + |\nabla c|^2 \Big\} \to 0 \qquad \textit{as } \tau \to \infty. \tag{4.4}$$

PROOF. In order to verify (4.4) we let $\delta > 0$ be given and pick any integer k fulfilling $k > \frac{C_1}{\delta}$, where $C_1 := M|\Omega| + \frac{M^2|\Omega|}{2}$. We claim that then for each $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$ we have

$$\inf_{t\in[T_0,T_0+k]} \int_t^{t+1} \int_\Omega \Big\{ F(n)f(c) + |\nabla c|^2 \Big\} < \delta. \tag{4.5}$$

To see this, given any such (n,c) we first apply Corollary 3.4 to obtain

$$\int_{T_0}^\infty \int_\Omega F(n)f(c) \le \operatorname{ess}\liminf_{t\searrow T_0} \int_\Omega c(\cdot,t) \le M|\Omega|$$

and

$$\int_{T_0}^{\infty}\int_\Omega |\nabla c|^2 \le \operatorname{ess}\liminf_{t\searrow T_0} \frac{1}{2}\int_\Omega c^2(\cdot,t) \le \frac{M^2|\Omega|}{2},$$

whence for $h(t) := \int_\Omega F(n(\cdot,t))f(c(\cdot,t)) + \int_\Omega |\nabla c(\cdot,t)|^2$, $t > T_0$, we obtain

$$\int_{T_0}^{\infty} h(t)dt \le C_1 \tag{4.6}$$

by definition of C_1.
Now if (4.5) was false, then (4.6) would imply that

$$C_1 \ge \int_{T_0}^{T_0+k} h(t)dt = \sum_{j=1}^{k}\int_{T_0+j-1}^{T_0+j} h(t)dt \ge \sum_{j=1}^{k}\delta = k\delta,$$

which is absurd in view of our choice of k. As $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$ and $\delta > 0$ were arbitrary, this establishes (4.4). $\square$

Now a crucial point appears to consist in deriving more substantial decay properties from this without any further knowledge on possible lower bounds for n beyond the weak information that its mass $\int_\Omega n$ remains constantly positive by definition of $\mathcal{S}_{m,M,L,T_0}$. In order to prepare a first step in this direction, we state an elementary observation which will below be related to a lower estimate for the first integral appearing in (4.4).

Lemma 4.3 *Assume that F satisfies (3.2) and (3.3). Then for all $m > 0$ and $B \ge \frac{m}{8}$, each nonnegative $\varphi \in L^3(\Omega)$ fulfilling*

$$\int_\Omega \varphi = m \qquad \text{and} \qquad \int_\Omega \varphi^3 \le B \tag{4.7}$$

has the property that

$$\int_\Omega F(\varphi) \ge \sqrt{\frac{m^3}{128B}}. \tag{4.8}$$

Proof. As $B \ge \frac{m}{8}$, the number $C_1 := \sqrt{\frac{8B}{m}}$ satisfies $C_1 \ge 1$, so that combining (3.3) with (3.2) shows that

$$F(s) \ge \min\left\{\frac{s}{2}, \frac{1}{2}\right\} \ge \frac{s}{2C_1} \qquad \text{for all } s \in [0, C_1].$$

Hence, given any nonnegatve $\varphi \in L^3(\Omega)$ fulfilling (4.7), we have

$$\int_\Omega F(\varphi) \ge \int_{\{\varphi \le C_1\}} F(\varphi) \ge \frac{1}{2C_1}\int_{\{\varphi\le C_1\}} \varphi. \tag{4.9}$$

In order to further estimate the latter integral, we use the Hölder inequality and (4.7) to obtain

$$\int_{\{\varphi > C_1\}} \varphi \le \Big(\int_\Omega \varphi^3\Big)^{\frac{1}{3}} \cdot \Big|\{\varphi > C_1\}\Big|^{\frac{2}{3}} \le B^{\frac{1}{3}} \cdot \Big|\{\varphi > C_1\}\Big|^{\frac{2}{3}},$$

so that since $|\{\varphi > C_1\}| \le \frac{m}{C_1}$ by the Chebyshev inequality, we conclude that

$$\int_{\{\varphi\le C_1\}} \varphi = m - \int_{\{\varphi>C_1\}} \varphi \ge m - B^{\frac{1}{3}} \cdot \Big|\{\varphi > C_1\}\Big|^{\frac{2}{3}} \ge m - B^{\frac{1}{3}} \cdot \Big(\frac{m}{C_1}\Big)^{\frac{2}{3}} = m - B^{\frac{1}{3}} \cdot \Bigg(\frac{m}{\sqrt{\frac{8B}{m}}}\Bigg)^{\frac{2}{3}} = \frac{m}{2}.$$

Therefore, (4.8) results from (4.9). □

By means of appropriate interpolation making use of the regularity property (4.3) jointly shared by all elements of $\mathcal{S}_{m,M,L,T_0}$, we can thereby turn Lemma 4.2 into a statement on decay of $f(c)$ which is uniform with respect to functions from this set.

Lemma 4.4 *Let $m > 0, M > 0, L > 0$ and $T_0 \ge 0$. Then*

$$\sup_{(n,c,F)\in\mathcal{S}_{m,M,L,T_0}} \quad \inf_{\substack{S\subset(T_0,T_0+\tau)\\ S \text{ is measurable with } |S|\ge\frac{1}{2}\\ \text{and } \operatorname{diam} S\le 1}} \int_S\int_\Omega f(c(x,t))dxdt \to 0 \qquad \text{as } \tau\to\infty,$$

where $\mathcal{S}_{m,M,L,T_0}$ is as defined in (4.1).

Proof. We need to show that for each fixed m, M, L and T_0, given $\delta > 0$ we can find $\tau > 0$ with the property that for any $(n, c, F) \in \mathcal{S}_{m,M,L,T_0}$ there exists a measurable set $S \subset (T_0, T_0+\tau)$ such that $|S| \ge \frac{1}{2}$ and $\operatorname{diam} S \le 1$ as well as

$$\int_S\int_\Omega f(c(x,t))dxdt < \delta. \tag{4.10}$$

In order to prepare our definition of τ, let us first make use of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, which in conjunction with the Poincaré inequality yields $C_1 > 0$ and $C_2 > 0$ such that

$$\|\varphi\|^2_{L^6(\Omega)} \le C_1\|\nabla\varphi\|^2_{L^2(\Omega)} + C_1\|\varphi\|^2_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \tag{4.11}$$

and

$$\|\varphi - \overline{\varphi}\|_{L^6(\Omega)} \le C_2\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{4.12}$$

where again we have set $\overline{\varphi} := \fint_\Omega \varphi$ for $\varphi \in L^1(\Omega)$. Next, the Gagliardo-Nirenberg inequality provides $C_3 > 0$ fulfilling

$$\|\varphi\|^4_{L^{\frac{12}{5}}(\Omega)} \le C_3\|\nabla\varphi\|_{L^2(\Omega)}\|\varphi\|^3_{L^2(\Omega)} + C_3\|\varphi\|^4_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{4.13}$$

Finally abbreviating $C_4 := \|f'\|_{L^\infty((0,M))}$, $B := 8 \cdot \Big\{\frac{C_1L}{4} + C_1m\Big\}^3$ and $C_5 := \sqrt{\frac{m^3}{128B}}$, given $\delta > 0$ we can find some small $\delta_0 > 0$ such that

$$\frac{|\Omega|}{C_5} \cdot \delta_0 < \frac{\delta}{2} \tag{4.14}$$

and

$$\frac{C_2C_4|\Omega|}{C_5} \cdot \Bigg\{\frac{C_3m^{\frac{3}{2}}L^{\frac{1}{2}}}{2} + C_3m^2\Bigg\}^{\frac{1}{2}} \cdot \delta_0^{\frac{1}{2}} < \frac{\delta}{2}. \tag{4.15}$$

Then Lemma 4.2 says that there exists $\tau > 0$ such that

$$\sup_{(n,c,F)\in\mathcal{S}_{m,M,L,T_0}} \inf_{t\in[T_0,T_0+\tau]} \int_t^{t+1} \int_\Omega \Big\{ F(n)f(c) + |\nabla c|^2 \Big\} < \delta_0.$$

Thus, if we now pick any $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$, then we can pick $t_0 \in [T_0, T_0+\tau]$ such that

$$\int_{t_0}^{t_0+1} \int_\Omega F(n)f(c) < \delta_0 \qquad \text{and} \qquad \int_{t_0}^{t_0+1} \int_\Omega |\nabla c|^2 < \delta_0. \tag{4.16}$$

With this number t_0 fixed henceforth, we observe that

$$\int_{t_0}^{t_0+1} \|\nabla n^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 dt = \frac{1}{4} \int_{t_0}^{t_0+1} \int_\Omega \frac{|\nabla n|^2}{n} \le \frac{L}{4} \tag{4.17}$$

by definition of $\mathcal{S}_{m,M,L,T_0}$, so that using (4.11) we obtain

$$\begin{aligned}
\int_{t_0}^{t_0+1} \|n(\cdot,t)\|_{L^3(\Omega)} dt &= \int_{t_0}^{t_0+1} \|n^{\frac{1}{2}}(\cdot,t)\|_{L^6(\Omega)}^2 dt \\
&\le C_1 \int_{t_0}^{t_0+1} \|\nabla n^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 dt + C_1 \int_{t_0}^{t_0+1} \|n^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 dt \\
&\le \frac{C_1 L}{4} + C_1 m \\
&= \frac{B^{\frac{1}{3}}}{2},
\end{aligned}$$

because

$$\|n^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 = \int_\Omega n(\cdot,t) = m \qquad \text{for a.e. } t > T_0 \tag{4.18}$$

again due to the definition of $\mathcal{S}_{m,M,L,T_0}$. The measurable set

$$S := \Big\{ t \in (t_0, t_0+1) \ \Big| \ \|n(\cdot,t)\|_{L^3(\Omega)} \le B^{\frac{1}{3}} \Big\}$$

therefore satisfies $|S| \ge \frac{1}{2}$ by the Chebyshev inequality, and in view of our definition of C_5 we infer from Lemma 4.3 that

$$\int_\Omega F(n(x,t))dx \ge C_5 \qquad \text{for all } t \in S. \tag{4.19}$$

For the proof of (4.10), we now decompose the first integral in (4.16) according to

$$\begin{aligned}
\int_{t_0}^{t_0+1} \int_\Omega F(n)f(c) &= \int_{t_0}^{t_0+1} \int_\Omega F(n(x,t)) \cdot \Big\{ f(c(x,t)) - \overline{f(c(\cdot,t))} \Big\} dxdt \\
&\quad + \int_{t_0}^{t_0+1} \int_\Omega F(n(x,t)) \cdot \overline{f(c(\cdot,t))} dxdt,
\end{aligned}$$

where by (4.19),

$$\begin{aligned}
\int_{t_0}^{t_0+1} \int_\Omega F(n(x,t)) \cdot \overline{f(c(\cdot,t))} dxdt &= \int_{t_0}^{t_0+1} \overline{f(c(\cdot,t))} \cdot \Big\{ \int_\Omega F(n(x,t))dx \Big\} dt \\
&\geq \int_S \overline{f(c(\cdot,t))} \cdot \Big\{ \int_\Omega F(n(x,t))dx \Big\} dt \\
&\geq C_5 \int_S \overline{f(c(\cdot,t))} dt \\
&= \frac{C_5}{|\Omega|} \int_S \int_\Omega f(c(x,t))dxdt.
\end{aligned}$$

Hence, for the integral in question we obtain the inequality

$$\begin{aligned}
\int_S \int_\Omega f(c(x,t))dxdt \;\; &\leq \;\; \frac{|\Omega|}{C_5} \int_{t_0}^{t_0+1} \int_\Omega F(n)f(c) \\
&\quad -\frac{|\Omega|}{C_5} \int_{t_0}^{t_0+1} \int_\Omega F(n(x,t)) \cdot \Big\{ f(c(x,t)) - \overline{f(c(\cdot,t))} \Big\} dxdt, \qquad (4.20)
\end{aligned}$$

in which thanks to (4.16) and (4.14),

$$\frac{|\Omega|}{C_5} \int_{t_0}^{t_0+1} \int_\Omega F(n)f(c) < \frac{|\Omega|}{C_5} \cdot \delta_0 < \frac{\delta}{2}. \qquad (4.21)$$

Moreover, invoking the Hölder inequality we can estimate

$$\begin{aligned}
&-\frac{|\Omega|}{C_5} \int_{t_0}^{t_0+1} \int_\Omega F(n(x,t)) \cdot \Big\{ f(c(x,t)) - \overline{f(c(\cdot,t))} \Big\} dxdt \\
&\quad \leq \frac{|\Omega|}{C_5} \int_{t_0}^{t_0+1} \Big\| F(n(\cdot,t)) \Big\|_{L^{\frac{6}{5}}(\Omega)} \cdot \Big\| f(c(\cdot,t)) - \overline{f(c(\cdot,t))} \Big\|_{L^6(\Omega)} dt \\
&\quad \leq \frac{|\Omega|}{C_5} \cdot \bigg(\int_{t_0}^{t_0+1} \Big\| F(n(\cdot,t)) \Big\|^2_{L^{\frac{6}{5}}(\Omega)} dt \bigg)^{\frac{1}{2}} \cdot \bigg(\int_{t_0}^{t_0+1} \Big\| f(c(\cdot,t)) - \overline{f(c(\cdot,t))} \Big\|^2_{L^6(\Omega)} dt \bigg)^{\frac{1}{2}}, \qquad (4.22)
\end{aligned}$$

where as a consequence of (4.12) and our choice of C_4 we have

$$\begin{aligned}
\Big\| f(c(\cdot,t)) - \overline{f(c(\cdot,t))} \Big\|^2_{L^6(\Omega)} &\leq C_2^2 \Big\| \nabla f(c(\cdot,t)) \Big\|^2_{L^2(\Omega)} \\
&= C_2^2 \int_\Omega f'^2(c(\cdot,t)) |\nabla c(\cdot,t)|^2 \\
&\leq C_2^2 C_4^2 \|\nabla c(\cdot,t)\|^2_{L^2(\Omega)} \qquad \text{for a.e. } t \in (t_0, t_0+1), \qquad (4.23)
\end{aligned}$$

because $c \leq M$ a.e. in $\Omega \times (T_0, \infty)$ by definition of $\mathcal{S}_{m,M,L,T_0}$.
As for the factor in (4.22) containing n, we use (4.13) and the fact that $F(n) \leq n$ by (3.2) to see that

$$\Big\| F(n(\cdot,t)) \Big\|^2_{L^{\frac{6}{5}}(\Omega)} \leq \|n(\cdot,t)\|^2_{L^{\frac{6}{5}}(\Omega)}$$

$$
\begin{aligned}
&= \|n^{\frac{1}{2}}(\cdot,t)\|^4_{L^{\frac{12}{5}}(\Omega)} \\
&\le C_3\|\nabla n^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}\|n^{\frac{1}{2}}(\cdot,t)\|^3_{L^2(\Omega)} + C_3\|n^{\frac{1}{2}}(\cdot,t)\|^4_{L^2(\Omega)} \\
&= C_3 m^{\frac{3}{2}}\|\nabla n^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)} + C_3 m^2 \qquad \text{for a.e. } t\in(t_0,t_0+1),
\end{aligned}
\tag{4.24}
$$

again due to (4.18).
In summary, from (4.22), (4.23) and (4.24) we obtain upon employing the Cauchy-Schwarz inequality that

$$
\begin{aligned}
&-\frac{|\Omega|}{C_5}\int_{t_0}^{t_0+1}\int_\Omega F(n(x,t))\cdot\Big\{f(c(x,t))-\overline{f(c(\cdot,t))}\Big\}dxdt \\
&\le \frac{|\Omega|}{C_5}\cdot\Big\{C_3 m^{\frac{3}{2}}\int_{t_0}^{t_0+1}\|\nabla n^{\frac{1}{2}}(\cdot,t)\|_{L^2(\Omega)}dt + C_3 m^2\Big\}^{\frac{1}{2}}\cdot\Big\{C_2^2C_4^2\int_{t_0}^{t_0+1}\|\nabla c(\cdot,t)\|^2_{L^2(\Omega)}dt\Big\}^{\frac{1}{2}} \\
&\le \frac{|\Omega|}{C_5}\cdot\Big\{C_3 m^{\frac{3}{2}}\cdot\Big(\int_{t_0}^{t_0+1}\|\nabla n^{\frac{1}{2}}(\cdot,t)\|^2_{L^2(\Omega)}dt\Big)^{\frac{1}{2}} + C_3 m^2\Big\}^{\frac{1}{2}}\cdot\Big\{C_2^2C_4^2\int_{t_0}^{t_0+1}\|\nabla c(\cdot,t)\|^2_{L^2(\Omega)}dt\Big\}^{\frac{1}{2}},
\end{aligned}
$$

so that (4.17), (4.16) and then (4.15) become applicable so as to warrant that

$$
\begin{aligned}
&-\frac{|\Omega|}{C_5}\int_{t_0}^{t_0+1}\int_\Omega F(n(x,t))\cdot\Big\{f(c(x,t))-\overline{f(c(\cdot,t))}\Big\}dxdt \\
&\qquad\le \frac{|\Omega|}{C_5}\cdot\Big\{C_3 m^{\frac{3}{2}}\cdot\frac{L^{\frac{1}{2}}}{2}+C_3m^2\Big\}^{\frac{1}{2}}\cdot\Big\{C_2^2C_4^2\delta_0\Big\}^{\frac{1}{2}} \\
&\qquad< \frac{\delta}{2}.
\end{aligned}
$$

Combined with (4.21) and (4.20), this shows (4.10) and thereby completes the proof. □

Once more relying on the regularity features in (4.3), we can show that the above convergence of $f(c)$ does not only take place in $L^1(\Omega)$ but actually even in $L^\infty(\Omega)$.

Lemma 4.5 *Let $m>0, M>0, L>0$ and $T_0\ge 0$. Then*

$$
\sup_{(n,c,F)\in\mathcal{S}_{m,M,L,T_0}} \ \inf_{\substack{S\subset(T_0,T_0+\tau) \\ S \text{ is measurable with } |S|\ge\frac{1}{2} \\ \text{and } \operatorname{diam} S\le 1}} \int_S \|f(c(\cdot,t))\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } \tau\to\infty.
$$

Proof. Fixing $m>0, M>0, L>0$ and $T_0\ge 0$, we need to make sure that for each $\delta>0$ there exists $\tau>0$ such that given any $(n,c,F)\in\mathcal{S}_{m,M,L,T_0}$ we can find a measurable $S\subset(T_0,T_0+\tau)$ such that $|S|\ge\frac{1}{2}$ and $\operatorname{diam} S\le 1$ as well as

$$
\int_S \|f(c(\cdot,t))\|_{L^\infty(\Omega)}dt < \delta.
\tag{4.25}
$$

For this purpose, we first use that $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ to interpolate by means of the Gagliardo-Nirenberg inequality to find $C_1 > 0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)} \le C_1 \|\nabla\varphi\|_{L^4(\Omega)}^{\frac{12}{13}} \|\varphi\|_{L^1(\Omega)}^{\frac{1}{13}} + C_1\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,4}(\Omega), \tag{4.26}$$

and abbreviate $C_2 := \|f'\|_{L^\infty((0,M))}$. Then for arbitrary $\delta > 0$ we can pick $\delta_0 > 0$ small enough satisfying

$$C_1\delta_0 < \frac{\delta}{2} \tag{4.27}$$

and

$$C_1 C_2^{\frac{12}{13}} L^{\frac{3}{13}} \delta_0^{\frac{1}{13}} < \frac{\delta}{2}, \tag{4.28}$$

and apply Lemma 4.4 to obtain $\tau > 0$ such that

$$\sup_{(n,c,F)\in\mathcal{S}_{m,M,L,T_0}} \quad \inf_{\substack{S\subset(T_0,T_0+\tau) \\ S \text{ is measurable with } |S|\ge\frac{1}{2} \\ \text{and } \operatorname{diam} \mathrm{S}\le 1}} \int_S\int_\Omega f(c(x,t))dxdt < \delta_0.$$

This means that if we fix $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$, then we can find a measurable $S \subset (T_0, T_0+\tau)$ such that $|S| \ge \frac{1}{2}, \operatorname{diam} \mathrm{S} \le 1$ and

$$\int_S\int_\Omega f(c(x,t))dxdt < \delta_0. \tag{4.29}$$

This entails that if we estimate the integral under consideration by using (4.26) according to

$$\int_S \|f(c(\cdot,t))\|_{L^\infty(\Omega)}dt \le C_1\int_{t_0}^{t_0+1} \|\nabla f(c(\cdot,t))\|_{L^4(\Omega)}^{\frac{12}{13}} \|f(c(\cdot,t))\|_{L^1(\Omega)}^{\frac{1}{13}}dt + C_1\int_{t_0}^{t_0+1} \|f(c(\cdot,t))\|_{L^1(\Omega)}dt, \tag{4.30}$$

then the rightmost term herein satisfies

$$C_1\int_{t_0}^{t_0+1} \|f(c(\cdot,t))\|_{L^1(\Omega)}dt < C_1\delta_0 < \frac{\delta}{2} \tag{4.31}$$

in view of (4.27). As the inclusion $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$ ensures that $c \le M$ a.e. in $\Omega \times (T_0,\infty)$ and hence

$$\|\nabla f(c(\cdot,t))\|_{L^4(\Omega)}^4 = \int_\Omega f'^4(c(\cdot,t))|\nabla c(\cdot,t)|^4 \le C_2^4\int_\Omega |\nabla c(\cdot,t)|^4 \qquad \text{for a.e. } t > T_0$$

by definition of C_2, two applications of the Hölder inequality to the first integral on the right of (4.30) yield

$$\begin{aligned} & C_1\int_{t_0}^{t_0+1} \|\nabla f(c(\cdot,t))\|_{L^4(\Omega)}^{\frac{12}{13}} \|f(c(\cdot,t))\|_{L^1(\Omega)}^{\frac{1}{13}}dt \\ & \quad \le \quad C_1\Big\{\int_{t_0}^{t_0+1} \|\nabla f(c(\cdot,t))\|_{L^4(\Omega)}^4 dt\Big\}^{\frac{3}{13}} \cdot \Big\{\int_{t_0}^{t_0+1} \|f(c(\cdot,t))\|_{L^1(\Omega)}^{\frac{1}{10}}dt\Big\}^{\frac{10}{13}} \end{aligned}$$

$$\leq C_1 \Big\{ \int_{t_0}^{t_0+1} \|\nabla f(c(\cdot,t))\|_{L^4(\Omega)}^4 dt \Big\}^{\frac{3}{13}} \cdot \Big\{ \int_{t_0}^{t_0+1} \|f(c(\cdot,t))\|_{L^1(\Omega)} dt \Big\}^{\frac{1}{13}}$$

$$\leq C_1 \cdot \Big\{ C_2^4 \int_{t_0}^{t_0+1} \int_\Omega |\nabla c|^4 \Big\}^{\frac{3}{13}} \cdot \Big\{ \int_{t_0}^{t_0+1} \|f(c(\cdot,t))\|_{L^1(\Omega)} dt \Big\}^{\frac{1}{13}}.$$

Since $\int_{t_0}^{t_0+1} \int_\Omega |\nabla c|^4 \leq L$ by definition of $\mathcal{S}_{m,M,L,T_0}$, (4.29) and (4.28) guarantee that

$$C_1 \int_{t_0}^{t_0+1} \|\nabla f(c(\cdot,t))\|_{L^4(\Omega)}^{\frac{12}{13}} \|f(c(\cdot,t))\|_{L^1(\Omega)}^{\frac{1}{13}} dt \leq C_1 \cdot \Big\{ C_2^4 L \Big\}^{\frac{3}{13}} \cdot \delta_0^{\frac{1}{13}} < \frac{\delta}{2},$$

which in conjunction with (4.31) and (4.30) establishes (4.25). □

Now thanks to the assumed positivity of f on $(0,\infty)$, and in view of the downward monotonicity of $t \mapsto \|c(\cdot,t)\|_{L^\infty(\Omega)}$ asserted by Corollary 3.4, the latter implies the following doubly uniform decay property of (3.1) which constitutes the main result of this section.

Lemma 4.6 *Let $m > 0, M > 0, L > 0$ and $T_0 \geq 0$. Then*

$$\sup_{(n,c,F) \in \mathcal{S}_{m,M,L,T_0}} \|c\|_{L^\infty(\Omega \times (t,\infty))} \to 0 \qquad \text{as } t \to \infty. \tag{4.32}$$

PROOF. We fix $m > 0, M > 0, L > 0$ and $T_0 \geq 0$ and note that proving (4.32) amounts to showing that for each $\delta > 0$ we can find $t_0 > T_0$ such that whenever $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$, we have

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \leq \delta \qquad \text{for a.e. } t > t_0. \tag{4.33}$$

To verify this, we may assume that $\delta < M$ and then observe that since f is continuous and positive on $(0,\infty)$, the number

$$\delta_0 := \min \Big\{ f(s) \ \Big| \ s \in [\delta, M] \Big\}$$

is well-defined and positive. An application of Lemma 4.5 thus yields $\tau > 0$ with the property that

$$\sup_{(n,c,F) \in \mathcal{S}_{m,M,L,T_0}} \ \inf_{\substack{S \subset (T_0, T_0+\tau) \\ S \text{ is measurable with } |S| \geq \frac{1}{2} \\ \text{and } \operatorname{diam} S \leq 1}} \int_S \|f(c(\cdot,t))\|_{L^\infty(\Omega)} < \frac{\delta_0}{4}. \tag{4.34}$$

In order to show that the desired conclusion holds for $t_0 := T_0 + \tau$, we now fix $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$ and then obtain from (4.34) that there exists a measurable set $S \subset (T_0, T_0+\tau)$ such that $|S| \geq \frac{1}{2}$, $\operatorname{diam} S \leq 1$ and

$$\int_S \|f(c(\cdot,t))\|_{L^\infty(\Omega)} < \frac{\delta_0}{4}.$$

Since

$$\delta_0 \cdot \Big| \Big\{ t \in S \ \Big| \ \|f(c(\cdot,t))\|_{L^\infty(\Omega)} \geq \delta_0 \Big\} \Big| \leq \int_S \|f(c(\cdot,t))\|_{L^\infty(\Omega)} dt$$

by the Chebyshev inequality, this guarantees that

$$S_0 := \Big\{ t \in S \ \Big| \ \|f(c(\cdot,t))\|_{L^\infty(\Omega)} < \delta_0 \Big\}$$

satisfies $|S_0| \ge |S| - \frac{1}{4} \ge \frac{1}{4}$. Moreover, the definition of δ_0 ensures that for each $t \in S_0$ we necessarily have $c(\cdot,t) \le \delta$ a.e. in Ω, that is,

$$\|c(\cdot,t_1)\|_{L^\infty(\Omega)} \le \delta \qquad \text{for all } t_1 \in S_0. \tag{4.35}$$

We now invoke Corollary 3.4 to find a null set $N \subset (T_0,\infty)$ such that $\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \|c(\cdot,t_1)\|_{L^\infty(\Omega)}$ for all $t_1 \in (T_0,\infty) \setminus N$ and each $t \in (t_1,\infty) \setminus N$. Since $|S_0 \setminus N| \ge \frac{1}{4}$, we may thus pick an arbitrary $t_1 \in S_0 \setminus N$ and apply (4.35) to infer that

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \|c(\cdot,t_1)\|_{L^\infty(\Omega)} \le \delta \qquad \text{for all } t \in (t_1,\infty) \setminus N,$$

which implies (4.33) due to the fact that the inclusion $t_1 \in S_0 \subset S \subset (T_0, T_0+\tau)$ along with our choice of t_0 warrants that $t_1 < t_0$. □

5 Dissipation in (3.1) implied by uniform smallness of c

Our next goal is to make sure that as soon as c becomes suitably small, solutions to (3.1) enjoy further regularity properties. This will arise as a consequence of the following lemma which provides a weak formulation of a corresponding differential inequality which can formally be obtained on computing

$$\frac{d}{dt} \int_\Omega \psi(n)\rho(c)$$

for certain convex functions ψ and ρ, and which at this heuristic level can be seen to yield an entropy-type inequality under the structural assumption (5.5) below which is satisfied e.g. for the couple

$$\psi(s) := s^p, \quad s \ge 0, \qquad \text{and} \qquad \rho(\sigma) := \frac{1}{(2\eta-\sigma)^\theta}, \quad \sigma \in [0,2\eta), \tag{5.1}$$

for any fixed $p > 1$ and $\theta \in (0, \frac{p-1}{p+1})$, provided that $\eta = \eta(p,\theta) > 0$ is appropriately small (cf. also [37, Lemma 5.1] for a similar reasoning for classical solutions in a related problem). In view of our weak assumptions on regularity of solutions, our rigorous justification of the respective integrated version, in its overall strategy inspired by the testing procedure presented in [1], will require certain additional restrictions on the growth of ψ with respect to n, which at this stage substantially reduce the range of admissible p in (5.1), but which in a later step can be removed upon an approximation argument so as to finally allow for any choice of p in Lemma 6.2.

Lemma 5.1 *Assume (3.2) and (3.3). Let $\eta > 0$, and suppose that $\psi \in C^2([0,\infty))$ and $\rho \in C^2([0,2\eta))$ are such that*

$$\psi(s) > 0, \quad \psi'(s) \ge 0 \quad \textit{and} \quad \psi''(s) > 0 \qquad \textit{for all } s \ge 0 \tag{5.2}$$

with

$$\limsup_{s\to\infty} s^{\frac{1}{5}} \psi''(s) < \infty, \tag{5.3}$$

that

$$\rho(\sigma) > 0, \quad \rho'(\sigma) \ge 0 \quad \text{and} \quad \rho''(\sigma) > 0 \qquad \text{for all } \sigma \in [0,2\eta), \tag{5.4}$$

and that

$$4\psi'^2(s)\rho'^2(\sigma) + \chi_0^2 s^2 \psi''^2(s)\rho^2(\sigma) \le 2\psi(s)\psi''(s)\rho(\sigma)\rho''(\sigma) \qquad \text{for all } s \ge 0 \text{ and } \sigma \in [0,\eta], \tag{5.5}$$

where $\chi_0 := \|\chi\|_{L^\infty((0,\eta))}$.
Then if $T_0 \ge 0$ *and* $\widetilde{u} \in L^\infty_{loc}((T_0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}((T_0,\infty); W_0^{1,2}(\Omega))$, *given any strong solution* (n,c) *of (3.1) in* $\Omega \times (T_0,\infty)$ *with the additional property that*

$$\|c\|_{L^\infty(\Omega\times(T_0,\infty))} \le \eta, \tag{5.6}$$

one can find a null set $N \subset (T_0,\infty)$ *such that*

$$\int_\Omega \psi(n(\cdot,t))\rho(c(\cdot,t)) + \frac{1}{2}\int_{t_0}^t \int_\Omega \psi''(n)\rho(c)|\nabla n|^2 \le \int_\Omega \psi(\cdot,t_0))\rho(\cdot,t_0)) \quad \text{for each } t_0 \in (T_0,\infty)\setminus N$$
$$\text{and all } t \in (t_0,\infty)\setminus N. \tag{5.7}$$

Proof. In order to collect some regularity properties needed in the course of our testing procedure, we first observe that as a consequence of (5.3), we can find $C_1 > 0$ such that

$$\psi''(s) \le C_1(s+1)^{-\frac{1}{5}} \qquad \text{for all } s \ge 0, \tag{5.8}$$

whence there exist $C_2 > 0$ and $C_3 > 0$ such that

$$\psi'(s) \le C_2(s+1)^{\frac{4}{5}} \quad \text{and} \quad \psi(s) \le C_3(s+1)^{\frac{9}{5}} \qquad \text{for all } s \ge 0. \tag{5.9}$$

Since our hypotheses imply that $\rho \in C^2([0,\eta])$, we can moreover fix positive constants C_4, C_5 and C_6 such that

$$\rho(\sigma) \le C_4, \quad \rho'(\sigma) \le C_5 \quad \text{and} \quad \rho''(\sigma) \le C_6 \qquad \text{for all } \sigma \in [0,\eta]. \tag{5.10}$$

We claim that these inequalities ensure that if (n,c) has the assumed strong solution property and additionally satisfies (5.6), then

$$\psi'(n)\rho(c) \in L^5_{loc}(\bar\Omega \times (T_0,\infty)) \cap L^2_{loc}((T_0,\infty); W^{1,2}(\Omega)) \tag{5.11}$$

and

$$\Big(\psi(n_{-h})\rho'(c)\Big)_{h\in(0,1)} \subset L^{\frac{20}{9}}_{loc}(\bar\Omega \times (T_0,\infty)) \quad \text{with}$$
$$\psi(n_{-h})\rho'(c) \to \psi(n)\rho'(c) \quad \text{in } L^{\frac{20}{9}}_{loc}(\bar\Omega \times (T_0,\infty)) \quad \text{as } h \searrow 0 \tag{5.12}$$

as well as

$$\bigg(\nabla\Big(\psi(n_{-h})\rho'(c)\Big)\bigg)_{h\in(0,1)} \subset L^{\frac{4}{3}}_{loc}(\bar\Omega \times (T_0,\infty)) \quad \text{with}$$
$$\nabla\Big(\psi(n_{-h})\rho'(c)\Big) \to \nabla\Big(\psi(n)\rho'(c)\Big) \quad \text{in } L^{\frac{4}{3}}_{loc}(\bar\Omega \times (T_0,\infty)) \quad \text{as } h \searrow 0, \tag{5.13}$$

where for $h \in (0,1)$ we have set

$$n_{-h}(x,t) := \begin{cases} n(x,t-h), & (x,t) \in \Omega \times (T_0+h,\infty), \\ 0, & (x,t) \in \Omega \times (T_0, T_0+h]. \end{cases}$$

To this end, we first note that since by Young's inequality we obtain as a particular consequence of (5.8)-(5.10) and (5.6) that

$$|\psi'(n)\rho(c)|^5 \le C_2^5 C_4^5 (n+1)^4 \qquad \text{in } \Omega \times (T_0,\infty)$$

and

$$\begin{aligned} \Big|\nabla\Big(\psi'(n)\rho(c)\Big)\Big|^2 &\le \Big\{\psi''(n)\rho(c)|\nabla n| + \psi'(n)\rho'(c)|\nabla c|\Big\}^2 \\ &\le \Big\{C_1C_4|\nabla n| + C_2C_5(n+1)^{\frac{4}{5}}|\nabla c|\Big\}^2 \\ &\le 2C_1^2C_4^2|\nabla n|^2 + 2C_2^2C_5^2(n+1)^{\frac{16}{5}} + 2C_2^2C_5^2|\nabla c|^4 \qquad \text{in } \Omega \times (T_0,\infty), \end{aligned}$$

(5.11) results upon recalling that $(n+1)^4, |\nabla n|^2$ and $|\nabla c|^4$ and hence clearly also $(n+1)^{\frac{16}{5}}$ belong to $L^1_{loc}(\bar\Omega \times (T_0,\infty))$ according to Definition 3.1. Moreover, since (5.9) warrants that

$$\psi^{\frac{20}{9}}(n) \le C_3^{\frac{20}{9}}(n+1)^4 \qquad \text{in } \Omega \times (T_0,\infty),$$

it follows from the fact that $n \in L^4_{loc}(\bar\Omega \times (T_0,\infty))$ that

$$\psi(n_{-h}) \to \psi(n) \quad \text{in } L^{\frac{20}{9}}_{loc}(\bar\Omega \times (T_0,\infty)) \quad \text{as } h \searrow 0, \tag{5.14}$$

and that hence in particular (5.12) holds due to the fact that $\rho'(c)$ is bounded by (5.10). Apart from this, (5.14) also implies that in

$$\nabla\Big(\psi(n_{-h})\rho'(c)\Big) = \psi'(n_{-h})\rho'(c)\nabla n_{-h} + \psi(n_{-h})\rho''(c)\nabla c, \tag{5.15}$$

we have

$$\psi(n_{-h})\rho''(c)\nabla c \to \psi(n)\rho''(c)\nabla c \quad \text{in } L^{\frac{4}{3}}_{loc}(\bar\Omega \times (T_0,\infty)) \quad \text{as } h \searrow 0, \tag{5.16}$$

because $\rho''(c)\nabla c \in L^4_{loc}(\bar\Omega \times (0,\infty))$ according to (5.10) and the requirement $\nabla c \in L^4_{loc}(\bar\Omega \times (T_0,\infty))$ in Definition 3.1, and because $\frac{9}{20} + \frac{1}{4} = \frac{7}{10} < \frac{3}{4}$. Since (5.9) combined with Young's inequality shows that

$$|\psi'(n)\nabla n|^{\frac{4}{3}} \le C_2^{\frac{4}{3}}(n+1)^{\frac{16}{15}}|\nabla n|^{\frac{4}{3}} \le C_2^{\frac{4}{3}}(n+1)^{\frac{16}{5}} + C_2^{\frac{4}{3}}|\nabla n|^2 \qquad \text{in } \Omega \times (T_0,\infty),$$

it follows again from the inclusions $n \in L^4_{loc}(\bar\Omega\times(T_0,\infty))$ and $\nabla n \in L^2_{loc}(\bar\Omega\times(T_0,\infty))$ that $\psi'(n_{-h})\nabla n_{-h} \to \psi'(n)\nabla n$ in $L^{\frac{4}{3}}_{loc}(\bar\Omega \times (T_0,\infty))$ as $h \searrow 0$. Once more using the boundedness of $\rho'(c)$, we thus obtain that

$$\psi'(n_{-h})\rho'(c)\nabla n_{-h} \to \psi'(n)\rho'(c)\nabla n \quad \text{in } L^{\frac{4}{3}}_{loc}(\bar\Omega \times (T_0,\infty)) \quad \text{as } h \searrow 0,$$

which together with (5.15) and (5.16) proves (5.13).

Since (5.9) and (5.10) furthermore ensure that $\psi(n)\rho(c) \le C_3C_4(n+1)^{\frac{9}{5}}$ in $\Omega \times (T_0,\infty)$, it is evident that $\psi(n)\rho(c)$ belongs to $L^1_{loc}(\bar\Omega \times (T_0,\infty))$, so that we can pick a null set $N \subset (T_0,\infty)$ such that $(T_0,\infty) \setminus N$ exclusively contains Lebesgue points of $(T_0,\infty) \ni t \mapsto \int_\Omega \psi(n(\cdot,t))\rho(c(\cdot,t))$. We now fix $t_0 \in (T_0,\infty) \setminus N$ and $t_1 \in (t_0,\infty) \setminus N$, let ζ_δ be as given by (3.8), and define

$$\phi(x,t) := \zeta_\delta(t) \cdot S_h[\psi'(n)\rho(c)](x,t), \qquad x \in \Omega,\ t > T_0,$$

for $\delta \in (0,\delta_0)$ and $h \in (0,\delta_0-\delta)$ with $\delta_0 := \min\{1, \frac{t_0-T_0}{2}\}$, where the averaging operator S_h is as introduced in (10.1). Then ϕ has compact support in $\bar\Omega \times (T_0, t_1+\delta]$, and since evidently $\nabla S_h[\psi'(n)\rho(c)] = S_h[\nabla(\psi'(n)\rho(c))]$, it follows from (5.11) that $\phi \in L^5(\bar\Omega \times (T_0,\infty)) \cap L^2((T_0,\infty); W^{1,2}(\Omega))$. Computing

$$\phi_t(x,t) = \zeta_\delta'(t) \cdot S_h[\psi'(n)\rho(c)](x,t) + \zeta_\delta(t) \cdot \frac{\psi'(n(x,t+h))\rho(c(x,t+h)) - \psi'(n(x,t))\rho(c(x,t))}{h}$$

for a.e. $x \in \Omega$ and $t > T_0$,
from (5.11) we furthermore see that $\phi_t \in L^5(\Omega \times (T_0,\infty)) \subset L^{\frac{4}{3}}(\Omega \times (T_0,\infty))$, so that Lemma 3.2 ii) guarantees that we may use ϕ in (3.5) to gain the identity

$$\begin{aligned}
& I_1(\delta,h) + I_2(\delta,h) + I_3(\delta,h) \\
&:= \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega n(x,t)S_h[\psi'(n)\rho(c)](x,t)dxdt - \frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega n(x,t)S_h[\psi'(n)\rho(c)](x,t)dxdt \\
&\quad -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)n(x,t)\Big\{\psi'(n(x,t+h))\rho(c(x,t+h)) - \psi'(n(x,t))\rho(c(x,t))\Big\}dxdt \\
&= -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla n(x,t) \cdot S_h\Big[\nabla\Big(\psi'(n)\rho(c)\Big)\Big](x,t)dxdt \\
&\quad + \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)n(x,t)F'(n(x,t))\chi(c(x,t))\nabla c(x,t) \cdot S_h\Big[\nabla\Big(\psi'(n)\rho(c)\Big)\Big](x,t)dxdt \\
&\quad - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\Big(\widetilde{u}(x,t) \cdot \nabla n(x,t)\Big)S_h[\psi'(n)\rho(c)](x,t)dxdt \\
&=: I_4(\delta,h) + I_5(\delta,h) + I_6(\delta,h)
\end{aligned} \qquad (5.17)$$

for all $\delta \in (0,\delta_0)$ and $h \in (0,\delta_0-\delta)$. Here thanks to (5.11) and Lemma 10.1 we have

$$S_h\Big[\nabla\Big(\psi'(n)\rho(c)\Big)\Big] \to \nabla\Big(\psi'(n)\rho(c)\Big) \quad \text{in } L^2_{loc}(\bar\Omega \times (T_0,\infty)) \qquad \text{as } h \searrow 0,$$

so that since both ∇n and $nF'(n)\chi(c)\nabla c$ belong to $L^2_{loc}(\bar\Omega \times (T_0,\infty))$ by Definition 3.1 and Lemma 3.2, we obtain

$$I_4(\delta,h) \to -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla n \cdot \nabla\Big(\psi'(n)\rho(c)\Big) \qquad \text{as } h \searrow 0 \qquad (5.18)$$

and

$$I_5(\delta,h) \to \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)nF'(n)\chi(c)\nabla c \cdot \nabla\Big(\psi'(n)\rho(c)\Big) \qquad \text{as } h \searrow 0. \qquad (5.19)$$

We next note that (5.11) in light of Lemma 10.1 also warrants that

$$S_h[\psi'(n)\rho(c)] \to \psi'(n)\rho(c) \qquad \text{in } L^5_{loc}(\bar\Omega \times (T_0,\infty)) \qquad \text{as } h \searrow 0,$$

which entails that

$$I_6(\delta,h) \to -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi'(n)\rho(c)(\widetilde{u}\cdot\nabla n) \qquad \text{as } h \searrow 0, \tag{5.20}$$

because $\widetilde{u}\cdot\nabla n \in L^{\frac{5}{4}}_{loc}(\bar\Omega \times (T_0,\infty))$ according to Lemma 3.2, and that

$$\begin{aligned} I_1(\delta,h) + I_2(\delta,h) &\to \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega n\psi'(n)\rho(c) - \frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega n\psi'(n)\rho(c) \\ &=: I_{12}^\infty(\delta) \qquad \text{as } h \searrow 0, \end{aligned} \tag{5.21}$$

for clearly also n lies in $L^{\frac{5}{4}}_{loc}(\bar\Omega \times (T_0,\infty))$.
As for the remaining term $I_3(\delta,h)$ in (5.17), we follow a well-known argument ([1]) in using the convexity of ψ to firstly obtain the pointwise estimate

$$\psi(n(x,t)) - \psi(n(x,t-h)) \le \psi'(n(x,t))\cdot[n(x,t)-n(x,t-h)] \qquad \text{for a.e. } x\in\Omega \text{ and } t\in(T_0,t_1+1),$$

which on integration implies that

$$\begin{aligned} J(\delta,h) &:= \frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\cdot\Big\{\psi(n(x,t))-\psi(n(x,t-h))\Big\}\cdot\rho(c(x,t))dxdt \\ &\le \frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)n(x,t)\psi'(n(x,t))\rho(c(x,t))dxdt \\ &\quad -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)n(x,t-h)\psi'(n(x,t))\rho(c(x,t))dxdt \\ &= \left\{\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)n(x,t)\psi'(n(x,t))\rho(c(x,t))dxdt\right. \\ &\quad \left. -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)n(x,t-h)\psi'(n(x,t))\rho(c(x,t))dxdt\right\} \\ &\quad +\left\{\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)n(x,t-h)\psi'(n(x,t))\rho(c(x,t))dxdt\right. \\ &\quad \left. -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)n(x,t-h)\psi'(n(x,t))\rho(c(x,t))dxdt\right\} \\ &=: J_1(\delta,h) + J_2(\delta,h) \end{aligned} \tag{5.22}$$

for all $\delta\in(0,\delta_0)$ and $h\in(0,\delta_0-\delta)$. Here the substitution $s=t-h$ reveals that

$$J_1(\delta,h) = I_3(\delta,h), \tag{5.23}$$

whereas with $I_{12}^\infty(\delta)$ as in (5.21) we have

$$
\begin{aligned}
J_2(\delta,h) &= -\int_{T_0}^{t_1+1}\int_\Omega \frac{\zeta_\delta(t-h)-\zeta_\delta(t)}{-h}\cdot n(x,t-h)\psi'(n(x,t))\rho(c(x,t))dxdt \\
&\to -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta'(t)n(x,t)\psi'(n(x,t))\rho(c(x,t))dxdt \\
&= I_{12}^\infty(\delta) \qquad \text{as } h\searrow 0 \qquad (5.24)
\end{aligned}
$$

because clearly $\frac{\zeta_\delta(\cdot-h)-\zeta_\delta}{-h}\overset{\star}{\rightharpoonup}\zeta_\delta'$ in $L^\infty((T_0,t_1+1))$ and $n_{-h}\to n$ in $L^4_{loc}(\bar\Omega\times(T_0,\infty))$ as $h\searrow 0$, and once more because of (5.11).
On the other hand, again by substitution we can rewrite the expression on the left-hand side of (5.22) according to

$$
\begin{aligned}
J(\delta,h) &= \frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n(x,t))\rho(c(x,t))dxdt \\
&\quad -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t+h)\psi(n(x,t))\rho(c(x,t+h))dxdt \\
&= -\int_{T_0}^{t_1+1}\int_\Omega \frac{\zeta_\delta(t+h)-\zeta_\delta(t)}{h}\cdot\psi(n(x,t))\rho(c(x,t+h))dxdt \\
&\quad -\Bigg\{\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n(x,t))\rho(c(x,t+h))dxdt \\
&\quad -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n(x,t))\rho(c(x,t))dxdt\Bigg\} \\
&=: J_3(\delta,h)+J_4(\delta,h) \qquad (5.25)
\end{aligned}
$$

for $\delta\in(0,\delta_0)$ and $h\in(0,\delta_0-\delta)$, where arguing as above we see that

$$
\begin{aligned}
J_3(\delta,h) &\to -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta'(t)\psi(n(x,t))\rho(c(x,t))dxdt \\
&= -\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega\psi(n)\rho(c)+\frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega\psi(n)\rho(c) \qquad \text{as } h\searrow 0.
\end{aligned}
$$

In summary, from (5.17) and (5.18)-(5.25) we thus infer that for all $\delta\in(0,\delta_0)$,

$$
\begin{aligned}
-\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla n\cdot\nabla\Big(\psi'(n)\rho(c)\Big) &+ \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)nF'(n)\chi(c)\nabla c\cdot\nabla\Big(\psi'(n)\rho(c)\Big) \\
&-\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi'(n)\rho(c)(\widetilde{u}\cdot\nabla n) \\
&= \liminf_{h\searrow 0}\Big\{I_1(\delta,h)+I_2(\delta,h)+I_3(\delta,h)\Big\} \\
&= I_{12}^\infty(\delta)+\liminf_{h\searrow 0}J_1(\delta,h)
\end{aligned}
$$

$$
\begin{aligned}
&\geq I_{12}^{\infty}(\delta) + \liminf_{h\searrow 0} \Big\{ J(\delta,h) - J_2(\delta,h) \Big\} \\
&= I_{12}^{\infty}(\delta) + \liminf_{h\searrow 0} J(\delta,h) - I_{12}^{\infty}(\delta) \\
&= \liminf_{h\searrow 0} \Big\{ J_3(\delta,h) + J_4(\delta,h) \Big\} \\
&= -\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega \psi(n)\rho(c) + \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega \psi(n)\rho(c) + \liminf_{h\searrow 0} J_4(\delta,h) \qquad (5.26)
\end{aligned}
$$

Now an estimate for $J_4(\delta,h)$ can be obtained by pursuing a variant of the above strategy: First, by convexity of ρ we see that

$$
\rho(c(x,t+h)) - \rho(c(x,t)) \leq \rho'(c(x,t+h))\cdot[c(x,t+h)-c(x,t)] \qquad \text{for a.e. } x\in\Omega \text{ and } t\in(T_0,t_1+1),
$$

and that hence

$$
\begin{aligned}
J_4(\delta,h) &= -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n(x,t))\cdot\Big\{\rho(c(x,t+h)) - \rho(c(x,t))\Big\}dxdt \\
&\geq -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n(x,t)\rho'(c(x,t+h))\cdot\Big\{c(x,t+h)-c(x,t)\Big\}dxdt \\
&= -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)c(x,t+h)\psi(n(x,t)\rho'(c(x,t+h))dxdt \\
&\quad +\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)c(x,t)\psi(n(x,t))\rho'(c(x,t+h))dxdt \\
&\quad +\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)c(x,t)\psi(n(x,t))\rho'(c(x,t+h))dxdt \\
&\quad -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)c(x,t)\psi(n(x,t))\rho'(c(x,t+h))dxdt \\
&= \frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)c(x,t)\cdot\Big\{\psi(n(x,t))\rho'(c(x,t+h)) - \psi(n(x,t-h))\rho'(c(x,t))\Big\}dxdt \\
&\quad +\int_{T_0}^{t_1+1}\int_\Omega \frac{\zeta_\delta(t-h)-\zeta_\delta(t)}{-h}\cdot c(x,t)\psi(n(x,t))\rho'(c(x,t+h))dxdt \\
&=: J_{41}(\delta,h) + J_{42}(\delta,h) \qquad \text{for all } \delta\in(0,\delta_0) \text{ and } h\in(0,\delta_0-\delta), \qquad (5.27)
\end{aligned}
$$

where we have substituted t by $t-h$ in one of the integrals making up $J_{41}(\delta,h)$. Here, arguing as above we infer that

$$
\begin{aligned}
J_{42}(\delta,h) &\to \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta'(t)c(x,t)\psi(n(x,t))\rho'(c(x,t))dxdt \\
&= -\frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega c\psi(n)\rho'(c) + \frac{1}{\delta}\int_{t_0-\delta}^{t_0} c\psi(n)\rho'(c) \\
&=: \tilde{L}_{21}(\delta) \qquad \text{as } h\searrow 0, \qquad (5.28)
\end{aligned}
$$

and in order to gain appropriate information on $J_{41}(\delta,h)$ through the PDE satisfied by c, we note that since $\nabla c \in L^4_{loc}(\bar\Omega\times(T_0,\infty))$ by assumption and $(\psi(n_{-h})\rho'(c))_{h\in(0,1)} \subset L^{\frac{20}{9}}_{loc}(\bar\Omega\times(T_0,\infty))$ as well as $(\nabla(\psi(n_{-h})\rho'(c)))_{h\in(0,1)} \subset L^{\frac{4}{3}}_{loc}(\bar\Omega\times(T_0,\infty))$ by (5.12) and (5.13), for all $\delta\in(0,\delta_0)$ and $h\in(0,\delta_0-\delta)$ we may invoke Lemma 3.2 iii) to use

$$\tilde\phi(x,t) := \zeta_\delta(t-h)\cdot S_h[\psi(n_{-h})\rho'(c)](x,t), \qquad x\in\Omega,\ t>T_0,$$

as a test function in (3.6). We thereby obtain the identity

$$\begin{aligned}
&\tilde I_1(\delta,h)+\tilde I_2(\delta,h)+\tilde I_3(\delta,h)\\
&:= \frac{1}{\delta}\int_{t_1+h}^{t_1+\delta+h}\int_\Omega c(x,t)S_h[\psi(n_{-h})\rho'(c)](x,t)dxdt\\
&\quad -\frac{1}{\delta}\int_{t_0-\delta+h}^{t_0+h}\int_\Omega c(x,t)S_h[\psi(n_{-h})\rho'(c)](x,t)dxdt\\
&\quad -\frac{1}{h}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)c(x,t)\cdot\Big\{\psi(n_{-h}(x,t+h))\rho'(c(x,t+h))-\psi(n_{-h}(x,t))\rho'(c(x,t))\Big\}dxdt\\
&= \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)\nabla c(x,t)\cdot S_h\Big[\nabla\Big(\psi(n_{-h})\rho'(c)\Big)\Big](x,t)dxdt\\
&\quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)F(n(x,t))f(c(x,t))S_h[\psi(n_{-h})\rho'(c)](x,t)dxdt\\
&\quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t-h)(\widetilde u(x,t)\cdot\nabla c(x,t))S_h[\psi(n_{-h})\rho'(c)](x,t)dxdt\\
&=: \tilde I_4(\delta,h)+\tilde I_5(\delta,h)+\tilde I_6(\delta,h) \qquad \text{for all } \delta\in(0,\delta_0) \text{ and } h\in(0,\delta_0-\delta),
\end{aligned} \tag{5.29}$$

where evidently

$$\tilde I_3(\delta,h) = -J_{41}(\delta,h) \qquad \text{for all } \delta\in(0,\delta_0) \text{ and } h\in(0,\delta_0-\delta). \tag{5.30}$$

We now use that (5.12) and (5.13) along with Lemma 10.1 guarantee that

$$S_h[\psi(n_{-h})\rho'(c)] \to \psi(n)\rho'(c) \qquad \text{in } L^{\frac{20}{9}}_{loc}(\bar\Omega\times(T_0,\infty)) \tag{5.31}$$

and

$$S_h\Big[\nabla\Big(\psi(n_{-h})\rho'(c)\Big)\Big] \to \nabla\Big(\psi(n)\rho'(c)\Big) \qquad \text{in } L^{\frac{4}{3}}_{loc}(\bar\Omega\times(T_0,\infty)) \tag{5.32}$$

as $h\searrow 0$, and that hence

$$\tilde I_4(\delta,h) \to -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla c\cdot\nabla\Big(\psi(n)\rho'(c)\Big) \tag{5.33}$$

and

$$\tilde I_5(\delta,h) \to -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)F(n)f(c)\psi(n)\rho'(c) \tag{5.34}$$

as well as

$$\tilde{I}_6(\delta,h) \to -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n)\rho'(c)(\widetilde{u}\cdot\nabla c) \tag{5.35}$$

as $h \searrow 0$, because $\nabla c \in L^4_{loc}(\bar\Omega\times(T_0,\infty))$, $F(n)f(c) \in L^4_{loc}(\bar\Omega\times(T_0,\infty)) \subset L^{\frac{20}{11}}_{loc}(\bar\Omega\times(T_0,\infty))$ and also $\widetilde{u}\cdot\nabla c \in L^{\frac{20}{11}}_{loc}(\bar\Omega\times(T_0,\infty))$ according to Definition 3.1 and Lemma 3.2.
Next, on the left of (5.29) we may combine (5.31) with the fact that the family $(\mathbf{1}_{(t_1+h,t_1+\delta+h)})_{h\in(0,1)}$ of indicator functions satisfies $\mathbf{1}_{(t_1+h,t_1+\delta+h)} \overset{\star}{\rightharpoonup} \mathbf{1}_{(t_1,t_1+\delta)}$ in $L^\infty(\mathbb{R})$ as $h\searrow 0$ to conclude, once more relying on the boundedness of c, that

$$\begin{aligned}\tilde{I}_1(\delta,h)+\tilde{I}_2(\delta,h) \quad &\to \quad \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega c\psi(n)\rho'(c) - \frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega c\psi(n)\rho'(c) \\ &=: \quad -\tilde{I}_{12}^\infty(\delta) \qquad \text{as } h\searrow 0.\end{aligned} \tag{5.36}$$

Collecting (5.30) and (5.33)-(5.36), from (5.27), (5.28) and (5.29) we all in all infer that

$$\begin{aligned}\liminf_{h\searrow 0} J_4(\delta,h) \quad &\geq \quad \liminf_{h\searrow 0}\Big\{J_{41}(\delta,h)+J_{42}(\delta,h)\Big\} \\ &= \quad \liminf_{h\searrow 0} J_{41}(\delta,h) + \tilde{I}_{12}^\infty(\delta) \\ &= \quad \liminf_{h\searrow 0}\Big\{-\tilde{I}_3(\delta,h)\Big\} + \tilde{I}_{12}^\infty(\delta) \\ &= \quad \lim_{h\searrow 0}\Big\{\tilde{I}_1(\delta,h)+\tilde{I}_2(\delta,h)\Big\} + \lim_{h\searrow 0}\Big\{-\tilde{I}_4(\delta,h)-\tilde{I}_5(\delta,h)-\tilde{I}_6(\delta,h)\Big\} + \tilde{I}_{12}^\infty(\delta) \\ &= \quad \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla c\cdot\nabla\Big(\psi(n)\rho'(c)\Big) + \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)F(n)f(c)\psi(n)\rho'(c) \\ &\qquad + \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n)\rho'(c)(\widetilde{u}\cdot\nabla c) \qquad \text{for all } \delta\in(0,\delta_0).\end{aligned}$$

Consequently, (5.26) implies that

$$\begin{aligned}&\frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega \psi(n)\rho(c) - \frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega \psi(n)\rho(c) \\ &\qquad \leq \quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla n\cdot\nabla\Big(\psi'(n)\rho(c)\Big) + \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)nF'(n)\chi(c)\nabla c\cdot\nabla\Big(\psi'(n)\rho(c)\Big) \\ &\qquad\qquad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi'(n)\rho(c)(\widetilde{u}\cdot\nabla n) - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla c\cdot\nabla\Big(\psi(n)\rho'(c)\Big) \\ &\qquad\qquad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)F(n)f(c)\psi(n)\rho'(c) - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n)\rho'(c)(\widetilde{u}\cdot\nabla c)\end{aligned} \tag{5.37}$$

for all $\delta\in(0,\delta_0)$. Here since $\widetilde{u}$ is solenoidal, two integrations by parts show that

$$-\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi'(n)\rho(c)(\widetilde{u}\cdot\nabla n) - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n)\rho'(c)(\widetilde{u}\cdot\nabla c)$$

$$
\begin{aligned}
&= -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\rho(c)\widetilde{u}\cdot\nabla\psi(n) - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n)\widetilde{u}\cdot\nabla\rho(c) \\
&= -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\widetilde{u}\cdot\nabla\Big(\psi(n)\rho(c)\Big) \\
&= 0 \qquad \text{for all } \delta\in(0,\delta_0),
\end{aligned}
$$

whereas clearly

$$
-\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)F(n)f(c)\psi(n)\rho'(c) \le 0 \qquad \text{for all } \delta\in(0,\delta_0).
$$

Therefore, from (5.37) we infer that

$$
\begin{aligned}
&\frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega \psi(n)\rho(c) - \frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega \psi(n)\rho(c) \\
&\quad\le -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla n\cdot\nabla\Big(\psi'(n)\rho(c)\Big) \\
&\quad\quad +\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)nF'(n)\chi(c)\nabla c\cdot\nabla\Big(\psi'(n)\rho(c)\Big) \\
&\quad\quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\nabla c\cdot\nabla\Big(\psi(n)\rho'(c)\Big) \\
&\quad= -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi''(n)\rho(c)|\nabla n|^2 - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi'(n)\rho'(c)\nabla n\cdot\nabla c \\
&\quad\quad +\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)nF'(n)\psi''(n)\chi(c)\rho(c)\nabla n\cdot\nabla c + \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)nF'(n)\psi'(n)\chi(c)\rho'(c)|\nabla c|^2 \\
&\quad\quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi'(n)\rho'(c)\nabla n\cdot\nabla c - \int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi(n)\rho''(c)|\nabla c|^2 \\
&\quad= -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi''(n)\rho(c)|\nabla n|^2 \\
&\quad\quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\cdot\Big\{2\psi'(n)\rho'(c) - nF'(n)\psi''(n)\chi(c)\rho(c)\Big\}\nabla n\cdot\nabla c \\
&\quad\quad -\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\cdot\Big\{\psi(n)\rho''(c) - nF'(n)\psi'(n)\chi(c)\rho'(c)\Big\}|\nabla c|^2 \qquad \text{for all } \delta\in(0,\delta_0). \qquad (5.38)
\end{aligned}
$$

Here we can estimate the second integral on the right by means of Young's inequality according to

$$
\begin{aligned}
&-\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\cdot\Big\{2\psi'(n)\rho'(c) - nF'(n)\psi''(n)\chi(c)\rho(c)\Big\}\nabla n\cdot\nabla c \\
&\quad\le \frac{1}{2}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi''(n)|\nabla n|^2\rho(c) \\
&\quad\quad +\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\cdot\frac{\Big\{2\psi'(n)\rho'(c) - nF'(n)\psi''(n)\chi(c)\rho(c)\Big\}^2}{2\psi''(n)\rho(c)}\cdot|\nabla c|^2 \qquad (5.39)
\end{aligned}
$$

for all $\delta \in (0,\delta_0)$. Now since $F', \psi', \psi'', \chi, \rho$ and ρ' are all nonnegative, and since $F'^2 \le 1$ on $[0,\infty)$ by (3.2) and $\chi^2(c) \le \chi_0^2$ a.e. in $\Omega \times (T_0,\infty)$ by (5.6), the hypothesis (5.5) warrants that

$$
\begin{aligned}
&\frac{\Big\{2\psi'(n)\rho'(c) - nF'(n)\psi''(n)\chi(c)\rho(c)\Big\}^2}{2\psi''(n)\rho(c)} - \Big\{\psi(n)\rho''(c) - nF'(n)\psi'(n)\chi(c)\rho'(c)\Big\} \\
&= \frac{1}{2\psi''(n)\rho(c)} \cdot \Bigg\{ \Big\{4\psi'^2(n)\rho'^2(c) - 4nF'(n)\psi'(n)\psi''(n)\chi(c)\rho(c)\rho'(c) + n^2F'^2(n)\psi''^2(n)\chi^2(c)\rho^2(c)\Big\} \\
&\qquad - \Big\{2\psi(n)\psi''(n)\rho(c)\rho''(c) + 2nF'(n)\psi'(n)\psi''(n)\chi(c)\rho(c)\rho'(c)\Big\}\Bigg\} \\
&= \frac{1}{2\psi''(n)\rho(c)} \cdot \Big\{4\psi'^2(n)\rho'^2(c) + n^2F'^2(n)\psi''^2(n)\chi^2(c)\rho^2(c) \\
&\qquad -2\psi(n)\psi''(n)\rho(c)\rho''(c) - 2nF'(n)\psi'(n)\psi''(n)\chi(c)\rho(c)\rho'(c)\Big\} \\
&\le \frac{1}{2\psi''(n)\rho(c)} \cdot \Big\{4\psi'^2(n)\rho'^2(c) + n^2\psi''^2(n)\chi_0^2\rho^2(c) - 2\psi(n)\psi''(n)\rho(c)\rho''(c)\Big\} \\
&\le 0 \qquad \text{a.e. in } \Omega \times (T_0,\infty).
\end{aligned}
$$

Inserting (5.39) into (5.38), we thus infer that

$$
\begin{aligned}
&\frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega \psi(n)\rho(c) + \frac{1}{2}\int_{T_0}^{t_1+1}\int_\Omega \zeta_\delta(t)\psi''(n)\rho(c)|\nabla n|^2 \\
&\quad \le \frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega \psi(n)\rho(c) \qquad \text{for all } \delta \in (0,\delta_0),
\end{aligned}
\tag{5.40}
$$

where the Lebesgue point properties of t_0 and t_1 ensure that

$$
\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega \psi(n)\rho(c) \to \int_\Omega \psi(n(\cdot,t_0))\rho(c(\cdot,t_0)) \quad \text{and} \quad \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega \psi(n)\rho(c) \to \int_\Omega \psi(n(\cdot,t_1))\rho(c(\cdot,t_1))
$$

as $h \searrow 0$. Again since ψ'' is nonnegative, using that also $\zeta_\delta \ge 0$ and that $\zeta_\delta \equiv 1$ in (t_0,t_1) we thereby obtain from (5.40) that indeed (5.7) is valid. □

6 Estimates implied by Lemma 5.1

Our application of Lemma 5.1 will be prepared by the following statement which partly explains the particular approximation of $[0,\infty) \mapsto s^p$ to be pursued in Lemma 6.2. It may be worthwhile mentioning here that for given $p \ge 2$ it seems impossible to adjust $q < p$ in such a way that for small $\delta > 0$, the alternative and apparently more straightforward choice $\psi_\delta(s) := \frac{s^p}{1+\delta s^q}$ is admissible in Lemma 5.1. We therefore employ a certain integrated variant thereof, with its precise form and some of its properties described as follows.

Lemma 6.1 *For $p \geq 2, q \in (0, p-1)$ and $\delta \in (0,1)$, we let*

$$\psi_\delta(s) := p \int_0^s \frac{\sigma^{p-1}}{1+\delta\sigma^q} d\sigma, \qquad s \geq 0. \tag{6.1}$$

Then $\psi_\delta > 0, \psi_\delta' > 0$ and $\psi_\delta'' > 0$ on $(0,\infty)$, and for all $s \geq 0$ we have

$$\psi_\delta(s) \nearrow s^p \quad \textit{and} \quad \psi_\delta''(s) \to p(p-1)s^{p-2} \qquad \textit{as } \delta \searrow 0. \tag{6.2}$$

Moreover,

$$\frac{\psi_\delta'^2(s)}{\psi_\delta(s)\psi_\delta''(s)} \leq \frac{p}{p-q-1} \qquad \textit{for all } s > 0 \tag{6.3}$$

and

$$\frac{s^2\psi_\delta''(s)}{\psi_\delta(s)} \leq p(p-1) \qquad \textit{for all } s > 0, \tag{6.4}$$

and for each fixed $\delta \in (0,1)$ we have

$$s^{-p+q+2}\psi_\delta''(s) \to \frac{p(p-q-1)}{\delta} \qquad \textit{as } s \to \infty. \tag{6.5}$$

Proof. We first note that

$$(1+\delta s^q)\psi_\delta(s) = p\int_0^s \frac{1+\delta s^q}{1+\delta\sigma^q} \cdot \sigma^{p-1} d\sigma \geq p\int_0^s \sigma^{p-1} d\sigma = s^p \qquad \text{for all } s \geq 0,$$

so that

$$\psi_\delta(s) \geq \frac{s^p}{1+\delta s^q} \qquad \text{for all } s \geq 0. \tag{6.6}$$

Next, computing

$$\psi_\delta'(s) = \frac{ps^{p-1}}{1+\delta s^q} \quad \text{and} \quad \psi_\delta''(s) = p \cdot \frac{(p-1)s^{p-2} + (p-q-1)\delta s^{p+q-2}}{(1+\delta s^q)^2} \tag{6.7}$$

for $s \geq 0$, using that $q < p-1$ we see that $\psi_\delta, \psi_\delta'$ and ψ_δ'' indeed are all positive on $(0,\infty)$. Furthermore, combining (6.7) with (6.6) shows that

$$\begin{aligned}
\frac{\psi_\delta'^2(s)}{\psi_\delta(s)\psi_\delta''(s)} &\leq \frac{\psi_\delta'^2(s)}{\psi_\delta''(s)} \cdot \frac{1+\delta s^q}{s^p} \\
&= \frac{p^2 s^{2p-2}}{(1+\delta s^q)^2} \cdot \frac{1}{p} \frac{(1+\delta s^q)^2}{(p-1)s^{p-2} + (p-q-1)\delta s^{p+q-2}} \cdot \frac{1+\delta s^q}{s^p} \\
&= p \cdot \frac{1+\delta s^q}{p-1+(p-q-1)\delta s^q} \\
&\leq p \cdot \frac{1+\delta s^q}{p-q-1+(p-q-1)\delta s^q} = \frac{p}{p-q-1} \qquad \text{for all } s > 0,
\end{aligned}$$

because $q > 0$, and that similarly

$$
\begin{aligned}
\frac{s^2\psi_\delta''(s)}{\psi_\delta(s)} &\le s^2\psi_\delta''(s)\cdot\frac{1+\delta s^q}{s^p} \\
&= s^2\cdot p\frac{(p-1)s^{p-2}+(p-q-1)\delta s^{p+q-2}}{(1+\delta s^q)^2}\cdot\frac{1+\delta s^q}{s^p} \\
&= p\cdot\frac{p-1+(p-q-1)\delta s^q}{1+\delta s^q} \\
&\le p\cdot\frac{p-1+(p-1)\delta s^q}{1+\delta s^q} = p(p-1) \qquad \text{for all } s>0.
\end{aligned}
$$

Finally, (6.5) and the second statement in (6.2) are evident from (6.7), whereas the first claim in (6.2) results from Beppo Levi's theorem. □

Along with the originally intended choice (5.1) of ρ, making use of these functions in Lemma 5.1 allows us to deduce an entropy-type inequality under a smallness assumption on c. For simplicity in presentation, we confine ourselves to proving the inequality (6.9) which actually slightly differs from an inequality indicating genuine decrease of the functional $\int_\Omega n^p$ in that it involves a factor 2 on its right, but the boundedness and dissipation properties thereby implied will be sufficient for our purpose.

Lemma 6.2 *For all $p \ge 2$ there exists $\eta > 0$ with the following property: Whenever $T_0 \ge 0, F$ satisfies (3.2) and $\widetilde{u} \in L^2_{loc}((T_0,\infty); W_0^{1,2}(\Omega)) \cap L^\infty_{loc}((T_0,\infty); L^2_\sigma(\Omega))$, for any strong solution (n,c) of (3.1) in $\Omega \times (T_0,\infty)$ fulfilling*

$$\|c\|_{L^\infty(\Omega\times(T_0,\infty))} \le \eta \tag{6.8}$$

there exists a null set $N(p) \subset (T_0,\infty)$ such that

$$
\int_\Omega n^p(\cdot,t) + \frac{p(p-1)}{2}\int_{t_0}^t\int_\Omega n^{p-2}|\nabla n|^2 \le 2\int_\Omega n^p(\cdot,t_0) \qquad \text{for all } t_0 \in (T_0,\infty)\setminus N(p) \text{ and each } t \in (t_0,\infty)\setminus N(p). \tag{6.9}
$$

Proof. Given $p \ge 2$, we first choose $\theta \in (0,1)$ small enough such that

$$\frac{5p\theta}{\theta+1} \le 1 \tag{6.10}$$

and then fix a small number $\eta \in (0,1)$ satisfying

$$\frac{4p(p-1)\chi_1^2\eta^2}{\theta(\theta+1)} \le 1, \tag{6.11}$$

where $\chi_1 := \|\chi\|_{L^\infty((0,1))}$. With any fixed sequence $(\delta_j)_{j\in\mathbb{N}} \subset (0,1)$ satisfying $\delta_j \searrow 0$ as $j \to \infty$, we now let ψ_δ be as defined in Lemma 6.1 with $q := p - \frac{9}{5}$, that is, we let

$$\psi_\delta(s) := p\int_0^s \frac{\sigma^{p-1}}{1+\delta\sigma^{p-\frac{9}{5}}}d\sigma \qquad \text{for } s \ge 0,$$

and moreover we set

$$\rho(\sigma) := \frac{1}{(2\eta-\sigma)^\theta} \qquad \text{for } \sigma \in [0,2\eta). \tag{6.12}$$

Then $\psi_\delta > 0, \psi_\delta' > 0$ and $\psi_\delta'' > 0$ on $(0,\infty)$ by Lemma 6.1, whereas computing

$$\rho'(\sigma) = \frac{\theta}{(2\eta-\sigma)^{\theta+1}} \quad \text{and} \quad \rho''(\sigma) = \frac{\theta(\theta+1)}{(2\eta-\sigma)^{\theta+2}} \qquad \text{for } \sigma \in [0,2\eta) \tag{6.13}$$

we see that also $\rho > 0, \rho' > 0$ and $\rho'' > 0$ throughout $[0,2\eta)$.
By means of (6.3) and (6.13), we now estimate

$$\begin{aligned} \frac{4\psi_\delta'^2(s)\rho'^2(\sigma)}{\psi_\delta(s)\psi_\delta''(s)\rho(\sigma)\rho''(\sigma)} &\le 4 \cdot \frac{5p}{4} \cdot \frac{\rho'^2(\sigma)}{\rho(\sigma)\rho''(\sigma)} \\ &= 5p \cdot \frac{\theta^2(2\eta-\sigma)^{-2\theta-2}}{(2\eta-\sigma)^{-\theta}\cdot\theta(\theta+1)(2\eta-\sigma)^{-\theta-2}} \\ &= \frac{5p\theta}{\theta+1} \\ &\le 1 \qquad \text{for all } s>0 \text{ and } \sigma \in [0,2\eta) \end{aligned} \tag{6.14}$$

thanks to (6.10), while (6.4) in conjunction with (6.12) and (6.13) shows that

$$\begin{aligned} \frac{\chi_1^2 s^2 \psi_\delta''^2(s)\rho^2(\sigma)}{\psi_\delta(s)\psi_\delta''(s)\rho(\sigma)\rho''(\sigma)} &= \chi_1^2 \cdot \frac{s^2\psi_\delta''(s)}{\psi_\delta(s)} \cdot \frac{\rho(\sigma)}{\rho''(\sigma)} \\ &\le \chi_1^2 \cdot p(p-1) \cdot \frac{(2\eta-\sigma)^2}{\theta(\theta+1)} \\ &\le \chi_1^2 \cdot p(p-1) \cdot \frac{4\eta^2}{\theta(\theta+1)} \\ &\le 1 \qquad \text{for all } s>0 \text{ and } \sigma \in [0,2\eta) \end{aligned} \tag{6.15}$$

according to (6.11). Since $\eta < 1$ entails that $\chi_1 \ge \chi_0 := \|\chi\|_{L^\infty((0,\eta))}$, combining (6.14) with (6.15) ensures that

$$4\psi_\delta'^2(s)\rho'^2(\sigma) + \chi_0^2 s^2 \psi_\delta''^2(s)\rho^2(\sigma) \le 2\psi_\delta(s)\psi_\delta''(s)\rho(\sigma)\rho''(\sigma) \qquad \text{for all } s>0 \text{ and } \sigma\in[0,2\eta).$$

As furthermore our choice of q guarantees that $\limsup_{s\to\infty} s^{\frac{1}{5}}\psi_\delta''(s) = \frac{4p}{5\delta}$ is finite for each $\delta \in (\delta_j)_{j\in\mathbb{N}}$, Lemma 5.1 becomes applicable so as to assert that whenever $T_0 \ge 0$ and $\widetilde{u}, n$ and c have the assumed properties, for any $\delta \in (\delta_j)_{j\in\mathbb{N}}$ we can find a null set $N_\delta \subset (T_0,\infty)$ such that

$$\int_\Omega \psi_\delta(n(\cdot,t))\rho(c(\cdot,t)) + \frac{1}{2}\int_{t_0}^t\int_\Omega \psi_\delta''(n)\rho(c)|\nabla n|^2 \le \int_\Omega \psi_\delta(n(\cdot,t_0))\rho(c(\cdot,t_0))$$
$$\text{for all } t_0 \in (T_0,\infty)\setminus N_\delta \text{ and } t \in (t_0,\infty)\setminus N_\delta. \tag{6.16}$$

Here as $0 \le c \le \eta$ a.e. in $\Omega\times(T_0,\infty)$, recalling (6.12) we can estimate

$$\frac{1}{(2\eta)^\theta} \le \rho(c) \le \frac{1}{\eta^\theta} \qquad \text{a.e. in } \Omega\times(T_0,\infty),$$

so that since the countable union $\bigcup_{\delta\in(\delta_j)_{j\in\mathbb{N}}} N_\delta$ has measure zero, it follows from (6.16) that with some null set $N=N(p)\subset(T_0,\infty)$ we have

$$\frac{1}{(2\eta)^\theta}\int_\Omega \psi_\delta(n(\cdot,t)) + \frac{1}{2}\cdot\frac{1}{(2\eta)^\theta}\int_{t_0}^t\int_\Omega \psi''_\delta(n)|\nabla n|^2 \le \frac{1}{\eta^\theta}\int_\Omega \psi_\delta(n(\cdot,t_0)) \qquad \text{for all } t_0\in(T_0,\infty)\setminus N \text{ and } t\in(t_0,\infty)\setminus N$$

and any $\delta\in(\delta_j)_{j\in\mathbb{N}}$. Now since from Lemma 6.1 we know that $\psi_\delta(s)\nearrow s^p$ and $\psi''_\delta(s)\to p(p-1)s^{p-2}$ as $\delta=\delta_j\searrow 0$, we may invoke the monotone convergence theorem and Fatou's lemma to conclude that indeed

$$\begin{aligned}\int_\Omega n^p(\cdot,t)+\frac{1}{2}\cdot p(p-1)\int_{t_0}^t\int_\Omega n^{p-2}|\nabla n|^2 &\le \frac{(2\eta)^\theta}{\eta^\theta}\int_\Omega n^p(\cdot,t_0)\\ &\le 2\int_\Omega n^p(\cdot,t_0)\end{aligned}$$

for all $t\in(T_0,\infty)\setminus N$ and $t\in(t_0,\infty)\setminus N$, because $\theta<1$ entails that $2^\theta<2$. □

Indeed, the above lemma entails the following boundedness properties, uniform with respect to functions in $\mathcal{S}_{m,\eta,L,T_0}$, provided that $\eta>0$ is small. We note that since this result will be applied to finitely many p only, the dependence of η and the number τ therein on p will actually be irrelevant in the sequel.

Lemma 6.3 *Let $p\ge 2$. Then there exist $\eta>0$ and $\tau>0$ with the property that for all $m>0$ and $L>0$ one can find $C(m,L)>0$ such that if $T_0\ge 0$, if F complies with (3.2), if $\widetilde{u}\in L^2_{loc}((T_0,\infty);W^{1,2}_0(\Omega))\cap L^\infty_{loc}((T_0,\infty);L^2_\sigma(\Omega))$, and if (n,c) is a strong solution of (3.1) in $\Omega\times(T_0,\infty)$ with*

$$\|c\|_{L^\infty(\Omega\times(T_0,\infty))}\le\eta \tag{6.17}$$

as well as

$$\|n(\cdot,t)\|_{L^1(\Omega)}\le m \quad \textit{for all } t>T_0 \qquad \textit{and} \qquad \int_{T_0}^{T_0+1}\int_\Omega\frac{|\nabla n|^2}{n}\le L, \tag{6.18}$$

then

$$\int_\Omega n^p(\cdot,t)\le C(m,L) \qquad \textit{for a.e. } t>T_0+\tau \tag{6.19}$$

and

$$\int_{T_0+\tau}^\infty\int_\Omega n^{p-2}|\nabla n|^2\le C(m,L). \tag{6.20}$$

Proof. We first observe that since $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$, (6.18) implies that for some $C_1>0$ we have

$$\begin{aligned}\int_{T_0}^{T_0+1}\|n(\cdot,t)\|_{L^3(\Omega)}dt &= \int_{T_0}^{T_0+1}\|n^{\frac12}(\cdot,t)\|^2_{L^6(\Omega)}dt\\ &\le C_1\int_{T_0}^{T_0+1}\Big\{\|\nabla n^{\frac12}(\cdot,t)\|^2_{L^2(\Omega)}+\|n^{\frac12}(\cdot,t)\|^2_{L^2(\Omega)}\Big\}dt\\ &= \frac{C_1}{4}\int_{T_0}^{T_0+1}\int_\Omega\frac{|\nabla n|^2}{n}+C_1\int_{T_0}^{T_0+1}\int_\Omega n\\ &\le \frac{C_1}{4}L+C_1 m.\end{aligned}$$

In view of a recursive argument, to prove the lemma it is thus sufficient to show that for each $p \ge 2$ there exists $\eta > 0$ such that for any choice of $m > 0$ and $B > 0$ we can fix $C_2(m,B) > 0$ such that if for some $T_1 \ge 0$ and some $\widetilde{u} \in L^2_{loc}((T_1,\infty); W^{1,2}_0(\Omega)) \cap L^\infty_{loc}((T_1,\infty); L^2_\sigma(\Omega))$ we are given a strong solution (n,c) of (3.1) in $\Omega \times (T_1,\infty)$ fulfilling $\|c\|_{L^\infty(\Omega\times(T_1,\infty))} \le \eta$ and

$$\|n(\cdot,t)\|_{L^1(\Omega)} \le m \qquad \text{for a.e. } t > T_1 \tag{6.21}$$

as well as

$$\int_{T_1}^{T_1+1} \|n(\cdot,t)\|_{L^p(\Omega)} dt \le B, \tag{6.22}$$

then

$$\int_\Omega n^p(\cdot,t) \le C_2(m,B) \qquad \text{for a.e. } t > T_1+1 \tag{6.23}$$

and

$$\int_{T_1+1}^\infty \int_\Omega n^{p-2} |\nabla n|^2 \le C_2(m,B) \tag{6.24}$$

as well as

$$\int_t^{t+1} \|n(\cdot,t)\|_{L^{3p}(\Omega)} dt \le C_2(m,B) \qquad \text{for all } t > T_1+1. \tag{6.25}$$

To this end, given any such p we invoke Lemma 6.2 to obtain $\eta > 0$ with the properties listed there. In particular, since (6.22) ensures that $\operatorname{essinf}_{t_0\in(T_1,T_1+1)} \int_\Omega n^p(\cdot,t_0) \le B^p$, applying (6.9) to some appropriately chosen $t_0 \in (T_1, T_1+1)$ shows that

$$\int_\Omega n^p(\cdot,t) \le 2B^p \qquad \text{for a.e. } t > t_0 \qquad \text{and} \qquad \int_{t_0}^\infty \int_\Omega n^{p-2}|\nabla n|^2 \le \frac{4}{p(p-1)} B^p. \tag{6.26}$$

As moreover using the Hölder inequality and again the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ along with (6.21) provides $C_3 > 0$ such that

$$\begin{aligned}
\Big(\int_t^{t+1} \|n(\cdot,t)\|_{L^{3p}(\Omega)} dt \Big)^p &\le \int_t^{t+1} \|n(\cdot,t)\|^p_{L^{3p}(\Omega)} dt \\
&= \int_t^{t+1} \|n^{\frac{p}{2}}(\cdot,t)\|^2_{L^6(\Omega)} dt \\
&\le C_3 \int_t^{t+1} \Big\{ \|\nabla n^{\frac{p}{2}}(\cdot,t)\|^2_{L^2(\Omega)} + \|n^{\frac{p}{2}}(\cdot,t)\|^2_{L^{\frac{2}{p}}(\Omega)} \Big\} dt \\
&\le \frac{p^2 C_3}{4} \int_t^{t+1} \int_\Omega n^{p-2}|\nabla n|^2 + C_3 m^p \qquad \text{for all } t > t_0,
\end{aligned}$$

the inequalities in (6.26) entail (6.23)-(6.25). □

7 Ultimate regularity of eventual energy solutions

We now focus on the asymptotic analysis of a given particular eventual energy solution (n,c,u), thus aiming at proving Theorem 1.3.

7.1 The inclusion $(n,c,id) \in \mathcal{S}_{m,M,L,T_0}$

In order to prepare an appropriate exploitation of the energy inequality (1.12), we first assert that the dissipation rate appearing therein dominates the energy functional $\mathcal{F}_\kappa$ in the following sense.

Lemma 7.1 *For all $m>0$, $M>0$ and $\kappa>0$ there exists $C=C(m,M,\kappa)>0$ such that if $n\in L^1(\Omega)$ and $c\in L^\infty(\Omega)$ are nonnegative with $\int_\Omega n\le m$ and $\|c\|_{L^\infty(\Omega)}\le M$ as well as $n^{\frac{1}{2}}\in W^{1,2}(\Omega)$ and $c^{\frac{1}{4}}\in W^{1,4}(\Omega)$, and if moreover $u\in W_0^{1,2}(\Omega;\mathbb{R}^3)$, then*

$$-\frac{|\Omega|}{e}\le \mathcal{F}_\kappa[n,c,u]\le C\cdot\int_\Omega\left\{\frac{|\nabla n|^2}{n}+\frac{|\nabla c|^4}{c^3}+|\nabla u|^2\right\}+C. \tag{7.1}$$

Proof. We first note that our assumptions $f(0)=0$, $\chi(0)>0$ and $(\frac{f}{\chi})'>0$ on $[0,\infty)$ imply that there exists $C_1=C_1(M)>0$ such that $\frac{f(s)}{\chi(s)}\ge C_1 s$ for all $s\in[0,M]$, which in view of Young's inequality entails that

$$\frac{1}{2}\int_\Omega\frac{\chi(c)}{f(c)}|\nabla c|^2\le\frac{1}{2C_1}\int_\Omega\frac{|\nabla c|^2}{c}\le\int_\Omega\frac{|\nabla c|^4}{c^3}+\frac{1}{16C_1^2}\int_\Omega c\le\int_\Omega\frac{|\nabla c|^4}{c^3}+\frac{M|\Omega|}{16C_1^2}.$$

Next, since $z\ln z\le\frac{3}{2}z^{\frac{5}{3}}$ for all $z\ge0$, using the Gagliardo-Nirenberg inequality we find $C_2>0$ and $C_3=C_3(m)>0$ fulfilling

$$\int_\Omega n\ln n\le\int_\Omega n^{\frac{5}{3}}=\frac{3}{2}\|n^{\frac{1}{2}}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)}\le C_2\|\nabla n^{\frac{1}{2}}\|^2_{L^2(\Omega)}\|n^{\frac{1}{2}}\|^{\frac{4}{3}}_{L^2(\Omega)}+C_2\|n^{\frac{1}{2}}\|^{\frac{10}{3}}_{L^2(\Omega)}\le C_3\int_\Omega\frac{|\nabla n|^2}{n}+C_3,$$

because $\|n^{\frac{1}{2}}\|^2_{L^2(\Omega)}=\int_\Omega n\le m$.

As, finally, the Poincaré inequality provides $C_4>0$ satisfying $\int_\Omega|u|^2\le C_4\int_\Omega|\nabla u|^2$, we all in all obtain

$$\mathcal{F}_\kappa[n,c,u]\le\max\left\{1,C_3,C_4\kappa\right\}\cdot\int_\Omega\left\{\frac{|\nabla n|^2}{n}+\frac{|\nabla c|^4}{c^3}+|\nabla u|^2\right\}+\frac{M|\Omega|}{16C_1^2}+C_3$$

for any such n,c and u. Along with the fact that $\mathcal{F}_\kappa[n,c,u]\ge-\frac{|\Omega|}{e}$, valid due to the inequality $z\ln z\ge-\frac{1}{e}$ for $z>0$, this shows (7.1). □

We can thereupon make sure that all the results of the previous sections can actually be applied to such solutions, because (n,c,id) then belongs to $\mathcal{S}_{m,M,L,T_0}$ for adequately chosen m,M,L and T_0.

Lemma 7.2 *Let (n,c,u) be an eventual energy solution of (1.2). Then there exist $m>0$, $M>0$, $L>0$ and $T_0\ge0$ such that with $F(s):=s, s\ge0$, the triple (n,c,F) belongs to $\mathcal{S}_{m,M,L,T_0}$. In particular,*

$$\|c\|_{L^\infty(\Omega\times(t,\infty))}\to0\qquad\text{as } t\to\infty. \tag{7.2}$$

Proof. From Definition 1.1 and Lemma 2.2 it follows that $\int_\Omega n(\cdot,t)=m:=\int_\Omega n_0$ for a.e. $t>0$, and that there exist $T\ge0$, $\kappa>0$, $K>0$ and $C_1>0$ such that the regularity properties in (1.10) as well as (1.12) hold. In particular, we may therefore integrate by parts in the rightmost integrals in (2.3) and (2.4) for suitably chosen ϕ to see that (n,c) is a strong solution of (3.1) in $\Omega\times(T_0,\infty)$ with

$T_0 := T+1$, $F(s) := s$ for $s \ge 0$ and $\widetilde{u} := u$, which in turn allows us to apply Corollary 3.4 to infer that $\|c(\cdot,t)\|_{L^\infty(\Omega)} \le M := \|c\|_{L^\infty(\Omega\times(T,T+1))}$ for a.e. $t > T_0$.

We next observe that writing $y(t) := \mathcal{F}_\kappa[n,c,u](t)$ and $h(t) := \int_\Omega \{\frac{|\nabla n|^2}{n} + \frac{|\nabla c|^4}{c^3} + |\nabla u|^2\}(\cdot,t)$ for $t > T$, by a completion argument we infer from (1.12) that for any nonnegative $\phi \in W^{1,\infty}((T,\infty))$ with compact support in (T,∞) we have

$$-\int_T^\infty y(t)\phi'(t)dt + \frac{1}{K}\int_T^\infty h(t)\phi(t) \le K\int_T^\infty \phi(t)dt, \tag{7.3}$$

because both y and h belong to $L^1_{loc}((T,\infty))$ due to Definition 1.1 and Lemma 7.1.

In order to make sure that this implies boundedness of y on $(T+1,\infty)$, we make use of the estimate provided by Lemma 7.1 when applied to $m := \int_\Omega n_0$ and M as defined above to infer from (7.3) the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$-\int_T^\infty y(t)\phi'(t)dt + C_1\int_T^\infty y(t)\phi(t) \le C_2\int_T^\infty \phi(t)dt$$

for any such ϕ. Here we take $\phi(t) := e^{C_1 t}\zeta_\delta(t)$, $t > T$, where ζ_δ is as introduced in (3.8) for $\delta \in (0, t_0 - T)$, with arbitrary Lebesgue points $t_0 \in (T, T+1)$ and $t_1 > T+1$ of $(T,\infty) \ni t \mapsto e^{C_1 t}y(t)$. Since $-\phi'(t) + C_1\phi(t) = -e^{C_1 t}\zeta'_\delta(t)$ for a.e. $t > T$, we thereby gain the inequality

$$\frac{1}{\delta}\int_{t_1}^{t_1+\delta} e^{C_1 t}y(t)dt - \frac{1}{\delta}\int_{t_0-\delta}^{t_0} e^{C_1 t}y(t)dt \le C_2\int_T^\infty e^{C_1 t}\zeta_\delta(t)dt \qquad \text{for all } \delta \in (0, t_0 - T),$$

which on taking $\delta \searrow 0$ shows that

$$y(t_1) \le e^{-C_1(t_1-t_0)}y(t_0) + C_2\int_{t_0}^{t_1} e^{-C_1(t_1-t)}dt \le e^{-C_1(t_1-t_0)}y(t_0) + \frac{C_2}{C_1}.$$

As the set of such Lebesgue points complements a null set in (T,∞), this implies that indeed

$$y(t) \le C_3 := \operatorname*{ess\,inf}_{t_0\in(T,T+1)} y(t_0) + \frac{C_2}{C_1} \qquad \text{for a.e. } t > T+1. \tag{7.4}$$

We next pick any $t > T+2$ and let $\phi(t) := \zeta_\delta(t)$, $t > T$, where again ζ_δ is taken from (3.8), now with $t_0 := t, t_1 := t+1$ and $\delta := 1$. From (7.3) we thus obtain that

$$-\int_{t-1}^t y(s)ds + \int_{t+1}^{t+2} y(s)ds + \frac{1}{K}\int_{t-1}^{t+2} h(s)\phi(s)ds \le K\int_{t-1}^{t+2}\phi(s)ds,$$

so that since $\phi \equiv 1$ in $(t,t+1)$ and $0 \le \phi \le 1$ on (T,∞), using (7.4) and the left inequality in (7.1) we conclude that with $C_4 := \max\{1, M^3\}$ we have

$$\begin{aligned}
\frac{1}{C_4K}\int_t^{t+1}\int_\Omega \left\{\frac{|\nabla n|^2}{n} + |\nabla c|^4\right\} &\le \frac{1}{K}\int_t^{t+1}\int_\Omega \left\{\frac{|\nabla n|^2}{n} + \frac{|\nabla c|^4}{c^3}\right\} \\
&\le \frac{1}{K}\int_t^{t+1} h(s)ds \\
&\le C_3 + \frac{|\Omega|}{e} + 3K \qquad \text{for all } t > T+2.
\end{aligned}$$

Therefore, (n,c,id) belongs to $\mathcal{S}_{m,M,L,T_0}$ with $L := C_4K\cdot(C_3 + \frac{|\Omega|}{e} + 3K)$ and $T_0 := T+2$, whereupon (7.2) becomes a consquence of Lemma 4.6. □

7.2 Preliminary statements on decay of ∇n and u

Thus knowing that c decays uniformly, invoking Lemma 6.3 we obtain the following.

Lemma 7.3 *For any eventual energy solution (n,c,u) of (1.2), one can find $T>0$ such that*

$$\int_T^\infty \int_\Omega |\nabla n|^2 < \infty. \tag{7.5}$$

Moreover, for all $p \ge 2$ there exist $T(p)>0$ and $C(p)>0$ such that

$$\|n(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for a.e. } t > T(p), \tag{7.6}$$

and we have

$$\int_t^{t+1} \|n(\cdot,s) - \overline{n_0}\|_{L^p(\Omega)} dt \to 0 \qquad \textit{as } t \to \infty. \tag{7.7}$$

Proof. In view of Lemma 7.2, we may apply Lemma 6.3 firstly to $p := 2$ and $F(s) := s,\ s \ge 0$, to obtain (7.5), whereas (7.6) similarly results on invoking Lemma 6.3 to general $p \ge 2$.
Next, given $p \ge 2$ we take $T(p)$ and $C(p)$ as in (7.6) and invoke the Poincaré inequality to find $C_1 > 0$ such that

$$\Big\|\varphi - \fint_\Omega \varphi\Big\|_{L^2(\Omega)}^2 \le C_1 \int_\Omega |\nabla\varphi|^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Then two applications of the Hölder inequality show that

$$\begin{aligned}
\int_t^{t+1} \|n(\cdot,t) - \overline{n_0}\|_{L^p(\Omega)} dt &\le \int_t^{t+1} \|n(\cdot,t) - \overline{n_0}\|_{L^{2p}(\Omega)}^{\frac{p-2}{p-1}} \|n(\cdot,t) - \overline{n_0}\|_{L^2(\Omega)}^{\frac{1}{p-1}} dt \\
&\le \Big(C(2p)\Big)^{\frac{p-2}{p-1}} \int_t^{t+1} \|n(\cdot,t) - \overline{n_0}\|_{L^2(\Omega)}^{\frac{1}{p-1}} dt \\
&\le \Big(C(2p)\Big)^{\frac{p-2}{p-1}} \int_t^{t+1} \|n(\cdot,t) - \overline{n_0}\|_{L^2(\Omega)}^2 dt \\
&\le \Big(C(2p)\Big)^{\frac{p-2}{p-1}} C_1 \int_t^{t+1} \int_\Omega |\nabla n|^2 \qquad \text{for all } t > T(p),
\end{aligned}$$

whence (7.7) is implied by (7.5). □

The stabilization property implied by (7.5) can now be turned into a preliminary statement on decay of u by making use of the energy inequality (1.11).

Lemma 7.4 *Let (n,c,u) be an eventual energy solution of (1.2). Then there exists $T>0$ such that*

$$\int_T^\infty \int_\Omega |\nabla u|^2 < \infty. \tag{7.8}$$

In particular, for all $p \in [1,6]$ we have

$$\int_t^{t+1} \|u(\cdot,t)\|_{L^p(\Omega)} dt \to 0 \qquad \textit{as } t \to \infty. \tag{7.9}$$

PROOF. We combine Lemma 7.3 with Definition 1.1 to find $T_1 > 0$ such that $C_1 := \int_{T_1}^\infty \int_\Omega |\nabla n|^2$ is finite, and such that

$$\frac{1}{2}\int_\Omega |u(\cdot,t)|^2 + \int_{t_0}^t \int_\Omega |\nabla u|^2 \le \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 + \int_{t_0}^t \int_\Omega nu\cdot\nabla\Phi \qquad \text{for a.e. } t_0 > T_1 \text{ and all } t > t_0.$$

Since $\nabla\cdot u = 0$, and since with some $C_2 > 0$ we have $\int_\Omega |u|^2 \le C_2 \int_\Omega |\nabla u|^2$ for a.e. $t > T_1$ by the Poincaré inequality, integrating by parts in the rightmost integral and using Young's inequality we see that for any such t_0 and t we have

$$\begin{aligned}\int_{t_0}^t \int_\Omega |\nabla u|^2 &\le \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 - \int_{t_0}^t \int_\Omega \Phi u\cdot\nabla n \\ &\le \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 + \frac{1}{2C_2}\int_{t_0}^t \int_\Omega |u|^2 + \frac{C_2\|\Phi\|_{L^\infty(\Omega)}^2}{2}\int_{t_0}^t \int_\Omega |\nabla n|^2 \\ &\le \frac{1}{2}\int_\Omega |u(\cdot,t_0)|^2 + \frac{1}{2}\int_{t_0}^t \int_\Omega |\nabla u|^2 + \frac{C_1C_2\|\Phi\|_{L^\infty(\Omega)}^2}{2}.\end{aligned}$$

This implies that

$$\int_{T_1+1}^t \int_\Omega |\nabla u|^2 \le \operatorname*{ess\,inf}_{t_0\in(T_1,T_1+1)} \int_\Omega |u(\cdot,t_0)|^2 + C_1C_2\|\Phi\|_{L^\infty(\Omega)}^2 \qquad \text{for all } t > T_1+1$$

and hence establishes (7.8), from which in turn one can readily derive (7.9), once again because $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $p \le 6$. □

The latter implies smallness of $u(\cdot,t_\star)$ at some conveniently large $t_\star > 0$ in some of the spaces $L^{3+\varepsilon}(\Omega)$, $\varepsilon > 0$, which are supercritical with respect to the current knowledge on the global existence of smooth small-data solutions to the unforced three-dimensional Navier-Stokes system ([34]). Thanks to the decay property of $n - \overline{n_0}$ formulated in (7.7), this actually entails a certain eventual regularity and decay of u also in the present situation. More precisely, by means of a contraction mapping argument we can achieve the following.

Lemma 7.5 *Let (n,c,u) be an eventual energy solution of (1.2). Then for all $p > 3$ we have*

$$\|u\|_{L^\infty((t,\infty);L^p(\Omega))} \to 0 \qquad \text{as } t \to \infty. \tag{7.10}$$

PROOF. First, since $p > 3$ we can pick any $p_0 \in (3,p)$ such that $p_0 \le 6$ to achieve that then $\gamma := \frac{3}{2}(\frac{1}{p_0} - \frac{1}{p})$ satisfies $\gamma < \frac{1}{2} - \frac{3}{2p}$, whence in particular

$$C_1 := \int_0^1 (1-\sigma)^{-\frac{1}{2}-\frac{3}{2p}}\sigma^{-2\gamma}d\sigma$$

is finite. We next recall known facts on the regularizing action of the Stokes semigroup ([12]) to fix positive constants C_2, C_3 and C_4 satisfying

$$\|e^{-tA}\varphi\|_{L^p(\Omega)} \le C_2 t^{-\gamma}\|\varphi\|_{L^{p_0}(\Omega)} \qquad \text{for all } \varphi \in C_0^\infty(\Omega)\cap L^2_\sigma(\Omega) \quad \text{and } t \in (0,2) \tag{7.11}$$

and

$$\|e^{-tA}\mathcal{P}[\nabla\cdot\varphi]\|_{L^p(\Omega)} \le C_3 t^{-\frac{1}{2}-\frac{3}{2p}}\|\varphi\|_{L^{\frac{p}{2}}(\Omega)} \qquad \text{for all } \varphi\in C_0^\infty(\Omega)\cap L^2_\sigma(\Omega) \quad \text{and } t\in(0,2) \tag{7.12}$$

as well as

$$\Big\|e^{-tA}\mathcal{P}\Big[\varphi - \fint_\Omega \varphi\Big]\Big\|_{L^p(\Omega)} \le C_4\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi\in C_0^\infty(\Omega)\cap L^2_\sigma(\Omega) \quad \text{and } t\in(0,2), \tag{7.13}$$

where (7.12) in particular implies that for each $t\in(0,2)$ the operator $e^{-tA}\mathcal{P}[\nabla\cdot()]$ admits a continuous extension to all of $L^{\frac{p}{2}}_\sigma(\Omega)$ with norm controlled according to (7.12).
We finally invoke the Cauchy-Schwarz inequality to fix $C_5>0$ such that

$$\|\varphi\otimes\psi\|_{L^{\frac{p}{2}}(\Omega)} \le C_5\|\varphi\|_{L^p(\Omega)}\|\psi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \text{ and } \psi \text{ belonging to } \in L^p(\Omega). \tag{7.14}$$

We now take any positive $\delta\le\delta_0 := \{3\cdot 2^{\frac{1}{2}-\frac{3}{2p}-\gamma}\cdot C_1C_3C_5\}^{-1}$ and let $\delta_1 := \frac{\delta}{3C_2}$ and $\delta_2 := \frac{\delta}{3\cdot 2^\gamma C_4}$, and note that as a consequence of Lemma 7.4 and Lemma 7.3, for any such δ we can pick $T(\delta)>2$ such that

$$\int_t^{t+1}\|u(\cdot,s)\|^2_{L^{p_0}(\Omega)}ds \le \delta_1^2 \qquad \text{for all } t>T(\delta)-2 \tag{7.15}$$

and

$$\|\nabla\Phi\|_{L^\infty(\Omega)}\cdot\int_t^{t+2}\|n(\cdot,s)-\overline{n_0}\|_{L^p(\Omega)}ds \le \delta_2 \qquad \text{for all } t>T(\delta)-2. \tag{7.16}$$

To see that these choices ensure that

$$\|u(\cdot,t_0)\|_{L^p(\Omega)} \le \delta \qquad \text{for all } t_0>T(\delta), \tag{7.17}$$

we fix any such t_0 and then infer from (7.15) that there exists $t_\star\in(t_0-2,t_0-1)$ such that $\|u(\cdot,t_\star)\|_{L^{p_0}(\Omega)} \le \delta_1$. We now follow a standard reasoning to construct, independently of u, another weak solution $\widehat{u}$ of the initial value problem associated with the Navier-Stokes system $\widehat{u}_t + A\widehat{u} = -\mathcal{P}[\nabla\cdot(\widehat{u}\otimes\widehat{u})] + \mathcal{P}[n\nabla\Phi]$ in $\Omega\times(t_\star,t_\star+2)$ with $\widehat{u}(\cdot,t_\star) = u(\cdot,t_\star)$ and some favorable additional properties, finally implying by a uniqueness argument that actually $u=\widehat{u}$ and that hence u itself has these properties. To this end, in the Banach space

$$X := \Big\{\varphi\in C^0((t_\star,t_\star+2];L^p_\sigma(\Omega)) \ \Big|\ \|\varphi\|_X := \sup_{t\in(t_\star,t_\star+2)}(t-t_\star)^\gamma\|\varphi(\cdot,t)\|_{L^p(\Omega)}<\infty\Big\}$$

we consider the mapping Ψ defined by

$$\begin{aligned}(\Psi\varphi)(\cdot,t) \ &:= \ e^{-(t-t_\star)A}u(\cdot,t_\star) - \int_{t_\star}^t e^{-(t-s)A}\mathcal{P}\Big[\nabla\cdot(\varphi(\cdot,s)\otimes\varphi(\cdot,s))\Big]ds \\ &\quad + \int_{t_\star}^t e^{-(t-s)A}\mathcal{P}[n(\cdot,s)\nabla\Phi]ds, \qquad t\in(t_\star,t_\star+2],\end{aligned}$$

for φ belonging to the closed subset

$$S := \Big\{\varphi\in X \ \Big|\ \|\varphi\|_X\le\delta\Big\}$$

of X. Then for $\varphi \in S$ we can use (7.11)-(7.13) and (7.16) to estimate

$$\begin{aligned}
\|(\Psi\varphi)(\cdot,t)\|_{L^p(\Omega)} &\le C_2(t-t_\star)^{-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{p})}\|u(\cdot,t_\star)\|_{L^{p_0}(\Omega)} + C_3\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\Big\|\varphi(\cdot,s)\otimes\varphi(\cdot,s)\Big\|_{L^{\frac{p}{2}}(\Omega)}ds \\
&\quad + C_4\int_{t_\star}^t \Big\|[n(\cdot,s)-\overline{n_0}]\nabla\Phi\Big\|_{L^p(\Omega)}ds \\
&\le C_2\delta_1(t-t_\star)^{-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{p})} + C_3C_5\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\|\varphi(\cdot,s)\|_{L^p(\Omega)}^2 ds \\
&\quad + C_4\|\nabla\Phi\|_{L^\infty(\Omega)}\int_{t_\star}^t \|n(\cdot,s)-\overline{n_0}\|_{L^p(\Omega)}ds \\
&\le C_2\delta_1(t-t_\star)^{-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{p})} + C_3C_5\delta^2\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}s^{-2\gamma}ds \\
&\quad + C_4\|\nabla\Phi\|_{L^\infty(\Omega)}\int_{t_\star}^{t_\star+2} \|n(\cdot,s)-\overline{n_0}\|_{L^p(\Omega)}ds \\
&\le C_2\delta_1(t-t_\star)^{-\frac{3}{2}(\frac{1}{p_0}-\frac{1}{p})} + C_3C_5\delta^2\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}s^{-2\gamma}ds \\
&\quad + C_4\delta_2 \qquad \text{for all } t\in(t_\star,t_\star+2],
\end{aligned}$$

so that according to our choice of γ we obtain

$$\begin{aligned}
(t-t_\star)^\gamma\|(\Psi\varphi)(\cdot,t)\|_{L^p(\Omega)} &\le C_2\delta_1 + C_3C_5\delta^2(t-t_\star)^{\gamma-\frac{1}{2}-\frac{3}{2p}-2\gamma+1}\int_0^1(1-\sigma)^{-\frac{1}{2}-\frac{3}{2p}}\sigma^{-2\gamma}d\sigma + C_4\delta_2(t-t_\star)^\gamma \\
&= C_2\delta_1 + C_1C_3C_5\delta^2(t-t_\star)^{\frac{1}{2}-\frac{3}{2p}-\gamma} + C_4\delta_2(t-t_\star)^\gamma \\
&\le C_2\delta_1 + 2^{\frac{1}{2}-\frac{3}{2p}-\gamma}C_1C_3C_5\delta^2 + 2^\gamma C_4\delta_2 \\
&\le \frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3} = \delta \qquad \text{for all } t\in(t_\star,t_\star+2],
\end{aligned}$$

from which it readily follows that $\Psi S\subset S$. Likewise, for $\varphi\in S$ and $\psi\in S$ we can use (7.12) and (7.14) to find that

$$\begin{aligned}
\Big\|(\Psi\varphi-\Psi\psi)(\cdot,t)\Big\|_{L^p(\Omega)} &= \Bigg\|\int_{t_\star}^t e^{-(t-s)A}\mathcal{P}\Big[-\nabla\cdot\Big\{\varphi(\cdot,s)\otimes[\varphi(\cdot,s)-\psi(\cdot,s)]\Big\} \\
&\qquad -\nabla\cdot\Big\{[\varphi(\cdot,s)-\psi(\cdot,s)]\otimes\psi(\cdot,s)\Big\}\Big]ds\Bigg\|_{L^p(\Omega)} \\
&\le C_3\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\Big\{\Big\|\varphi(\cdot,s)\otimes[\varphi(\cdot,s)-\psi(\cdot,s)]\Big\|_{L^{\frac{p}{2}}(\Omega)} \\
&\qquad +\Big\|[\varphi(\cdot,s)-\psi(\cdot,s)]\otimes\psi(\cdot,s)\Big\|_{L^{\frac{p}{2}}(\Omega)}\Big\}ds \\
&\le C_3C_5\int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}}\Big\{\|\varphi(\cdot,s)\|_{L^p(\Omega)}+\|\psi(\cdot,s)\|_{L^p(\Omega)}\Big\}\times
\end{aligned}$$

$$\times \|\varphi(\cdot,s) - \psi(\cdot,s)\|_{L^p(\Omega)} ds$$

$$\leq C_3 C_5 \int_{t_\star}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \cdot 2\delta s^{-\gamma} \cdot s^{-\gamma} \|\varphi - \psi\|_X ds \qquad \text{for all } t \in (t_\star, t_\star + 2],$$

implying that

$$(t-t_\star)^\gamma \Big\| (\Psi\varphi - \Psi\psi)(\cdot,t) \Big\|_{L^p(\Omega)} \leq 2 \cdot 2^{\frac{1}{2}-\frac{3}{2p}-\gamma} C_1 C_3 C_5 \delta \|\varphi - \psi\|_X \qquad \text{for all } t \in (t_\star, t_\star + 2].$$

As $2 \cdot 2^{\frac{1}{2}-\frac{3}{2p}-\gamma} C_1 C_3 C_5 \delta \leq \frac{2}{3} < 1$, this proves that Φ acts as a contraction on S and hence possesses a unique fixed point $\widehat{u}$. By standard arguments ([26]), it follows that $\widehat{u}$ in fact is a weak solution of the Navier-Stokes subsystem of (1.2) in $\Omega \times (t_\star, t_\star + 2)$ subject to the initial condition $\widehat{u}(\cdot, t_\star) = u(\cdot, t_\star)$. Since furthermore our choice of γ ensures that with $q := \frac{2p}{p-3}$ we have $q\gamma < q \cdot (\frac{1}{2} - \frac{3}{2p}) = 1$, it follows from the inclusion $\widehat{u} \in S$ that

$$\int_{t_\star}^{t_\star+2} \|\widehat{u}(\cdot,t)\|_{L^p(\Omega)}^q dt \leq \delta^q \int_{t_\star}^{t_\star+2} (t-t_\star)^{-q\gamma} dt < \infty.$$

Using that p and q satisfy the Serrin condition $\frac{2}{q} + \frac{3}{p} = 1$, a well-known uniqueness property of the Navier-Stokes equations ([26]) entails that $\widehat{u}$ must coincide with u in $\Omega \times (t_\star, t_\star + 2)$. In particular, since $t_0 \in (t_\star + 1, t_\star + 2)$, this implies that

$$\|u(\cdot,t_0)\|_{L^p(\Omega)} = \|\widehat{u}(\cdot,t_0)\|_{L^p(\Omega)} \leq \delta (t_0 - t_\star)^{-\gamma} \leq \delta$$

and thereby establishes (7.17), which in turn proves (7.10), because $\delta \in (0, \delta_0]$ was arbitrary. □

7.3 Eventual Hölder regularity of u and ∇u

We next plan to derive some higher order regularity properties of a given eventual energy solution. Here we first combine the boundedness feature of u implied by Lemma 7.5 with the integrability properties of the forcing term $n\nabla\Phi$ in the Navier-Stokes equations in (1.2), as obtained from Lemma 7.3, to achieve spatio-temporal L^p bounds for $u, \nabla u, D^2 u$ and u_t for any $p \geq 1$ by means of a bootstrap argument based on maximal Sobolev regularity in the Stokes evolution system.

For use in this and also the following sections, we fix a function $\xi_0 \in C^\infty(\mathbb{R})$ such that

$$0 \leq \xi_0 \leq 1 \text{ in } \mathbb{R}, \quad \xi_0 \equiv 0 \text{ in } (-\infty, \tfrac{1}{2}] \quad \text{and} \quad \xi_0 \equiv 1 \text{ in } [1, \infty), \tag{7.18}$$

and for $t_0 > 1$, we introduce

$$\xi_{t_0}(t) := \xi_0(t - t_0), \qquad t \in \mathbb{R}. \tag{7.19}$$

Lemma 7.6 *Let (n, c, u) be an eventual energy solution of (1.2). Then for all $p \geq 1$ there exist $T > 0$ and $C > 0$ such that*

$$\|u\|_{L^p((t,t+1);W^{2,p}(\Omega))} + \|u_t\|_{L^p(\Omega\times(t,t+1))} \leq C \qquad \textit{for all } t > T. \tag{7.20}$$

PROOF. We first claim that there exist $T_1 > 0$ and $C_1 > 0$ such that

$$\|u\|_{L^2((t_0,t_0+1);W^{2,2}(\Omega))} \le C_1 \qquad \text{for all } t > T_1. \tag{7.21}$$

To see this, let us apply Lemma 7.3, Lemma 7.5 and Lemma 7.4 to fix $T_1 > 1$ and positive constants C_2, C_3 and C_4 such that

$$\|n\|_{L^2((t,t+2);L^2(\Omega))} \le C_2 \qquad \text{for all } t > T_1 - 1 \tag{7.22}$$

and

$$\|u\|_{L^\infty((t,t+2);L^4(\Omega))} \le C_3 \qquad \text{for all } t > T_1 - 1 \tag{7.23}$$

and

$$\|\nabla u\|_{L^2((t,t+2);L^2(\Omega))} \le C_4 \qquad \text{for all } t > T_1 - 1. \tag{7.24}$$

Then (7.24) in particular implies that for any choice of $t_0 > T_1$ we can find $t_\star \in (t_0 - 1, t_0)$ fulfilling $\|\nabla u(\cdot, t_\star)\|_{L^2(\Omega)} \le C_3$, and upon an interpolation using the Hölder inequality and the Gagliardo-Nirenberg inequality, (7.24) combined with (7.23) shows that with some $C_5 > 0$ and $C_6 > 0$ we have

$$\begin{aligned}
\int_{t_\star}^{t_\star+2} \Big\|\mathcal{P}[(u\cdot\nabla)u](\cdot,t)\Big\|^2_{L^2(\Omega)} dt &\le C_5 \int_{t_\star}^{t_\star+2} \|\nabla u(\cdot,t)\|^2_{L^4(\Omega)} \|u(\cdot,t)\|^2_{L^4(\Omega)} dt \\
&\le C_3^2 C_5 \int_{t_\star}^{t_\star+2} \|\nabla u(\cdot,t)\|^2_{L^4(\Omega)} dt \\
&\le C_6 \int_{t_\star}^{t_\star+2} \|u(\cdot,t)\|^{\frac{8}{5}}_{W^{2,2}(\Omega)} \|u(\cdot,t)\|^{\frac{2}{5}}_{L^4(\Omega)} dt \\
&\le C_3^{\frac{2}{5}} C_6 \int_{t_\star}^{t_\star+2} \|u(\cdot,t)\|^{\frac{8}{5}}_{W^{2,2}(\Omega)} dt.
\end{aligned}$$

As moreover

$$\begin{aligned}
\int_{t_\star}^{t_\star+2} \Big\|\mathcal{P}[n(\cdot,t)\nabla\Phi]\Big\|^2_{L^2(\Omega)} dt &\le \|\nabla\Phi\|^2_{L^\infty(\Omega)} \int_{t_\star}^{t_\star+2} \|n(\cdot,t)\|^2_{L^2(\Omega)} dt \\
&\le C_2^2 \|\nabla\Phi\|^2_{L^\infty(\Omega)}
\end{aligned}$$

by (7.22), it follows from a well-known maximal Sobolev regularity property of the Stokes evolution equation ([13]) and a corresponding uniqueness argument ([26]) that there exist $C_7 > 0$ and $C_8 > 0$ satisfying

$$\begin{aligned}
\int_{t_\star}^{t_\star+2} \|u(\cdot,t)\|^2_{W^{2,2}(\Omega)} dt &\le C_7 \cdot \Big\{ \int_\Omega |\nabla u(\cdot,t_\star)|^2 + \int_{t_\star}^{t_\star+2} \Big\| -\mathcal{P}[(u\cdot\nabla)u](\cdot,t) + \mathcal{P}[n(\cdot,t)\nabla\Phi]\Big\|^2_{L^2(\Omega)} dt \Big\} \\
&\le C_8 \cdot \Big\{ 1 + \int_{t_\star}^{t_\star+2} \|u(\cdot,t)\|^{\frac{8}{5}}_{W^{2,2}(\Omega)} dt \Big\}.
\end{aligned}$$

Since $\frac{8}{5} < 2$, Young's inequality becomes applicable here to warrant that (7.21) indeed holds if we let $C_1 > 0$ be apporopriately large, because by construction we have $(t_0, t_0 + 1) \subset (t_\star, t_\star + 2)$.

In order to prove the lemma, upon a recursive argument it is hence sufficient to show that whenever $p \geq 2$ is such that

$$\|u\|_{L^p((t,t+2);W^{2,p}(\Omega))} \leq C_9 \qquad \text{for all } t > T_2 \tag{7.25}$$

with certain $T_2 > 1$ and $C_9 > 0$, there exist $T_3 > 1$ and $C_{10} > 0$ fulfilling

$$\|u\|_{L^{\frac{3p}{2}}((t_0,t_0+1);W^{2,\frac{3p}{2}}(\Omega))} + \|u_t\|_{L^{\frac{3p}{2}}(\Omega\times(t_0,t_0+1))} \leq C_{10} \qquad \text{for all } t > T_3. \tag{7.26}$$

To see that this implication actually holds, under the assumption therein we invoke Lemma 7.5 and 7.3 to fix $T_3 > T_2$, $C_{11} > 0$ and $C_{12} > 0$ such that with $q := \max\{6p, \frac{6p}{2p-3}\}$ we have

$$\|n\|_{L^\infty((t-1,t+1);L^{\frac{3p}{2}}(\Omega))} \leq C_{11} \qquad \text{for all } t > T_3 \tag{7.27}$$

and

$$\|u\|_{L^\infty((t-1,t+1);L^q(\Omega))} \leq C_{12} \qquad \text{for all } t > T_3, \tag{7.28}$$

and given $t_0 > T_3$ we let ξ_{t_0} be as defined in (7.19). Then the function $v : \Omega \times (t_0-1,\infty) \to \mathbb{R}^3$ defined by $v(x,t) := \xi_{t_0}(t)u(x,t)$, $(x,t) \in \Omega \times (t_0-1,\infty)$, is a weak solution in $L^\infty_{loc}([t_0-1,\infty);L^2_\sigma(\Omega)) \cap L^2_{loc}([t_0-1,\infty);W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega))$ of

$$v_t + Av = h(x,t) := \xi_{t_0}(t)\mathcal{P}[-(u\cdot\nabla)u + n\nabla\Phi] + \xi'_{t_0}(t)u \qquad \text{in } \Omega \times (t_0-1,\infty) \tag{7.29}$$

with $v(\cdot,t_0-1) \equiv 0$. To estimate the inhomogeneity h herein, we first note that the boundedness of the Helmholtz projection in $L^{\frac{3p}{2}}(\Omega)$, (7.27) and (7.28) imply that there exist positive constants C_{13}, C_{14} and C_{15} such that

$$\begin{aligned}
\int_{t_0-1}^{t_0+1} \Big\|\xi_{t_0}(t)\mathcal{P}[n(\cdot,t)\nabla\Phi]\Big\|^{\frac{3p}{2}}_{L^{\frac{3p}{2}}(\Omega)} dt &\leq C_{13}\int_{t_0-1}^{t_0+1} \|n(\cdot,t)\nabla\Phi\|^{\frac{3p}{2}}_{L^{\frac{3p}{2}}(\Omega)} dt \\
&\leq C_{13}\|\nabla\Phi\|^{\frac{3p}{2}}_{L^\infty(\Omega)}\int_{t_0-1}^{t_0+1} \|n(\cdot,t)\|^{\frac{3p}{2}}_{L^{\frac{3p}{2}}(\Omega)} dt \\
&\leq C_{14}
\end{aligned} \tag{7.30}$$

and

$$\int_{t_0-1}^{t_0+1} \|\xi'_{t_0}(t)u(\cdot,t)\|^{\frac{3p}{2}}_{L^{\frac{3p}{2}}(\Omega)} dt \leq C_{15}. \tag{7.31}$$

Moreover, by means of another interpolation on the basis of the Hölder inequality and the Gagliardo-Nirenberg inequality we may use (7.27) and then (7.25) to find positive constants C_{16}, C_{17}, C_{18} and C_{19} such that

$$\begin{aligned}
\int_{t_0-1}^{t_0+1} \Big\|\xi_{t_0}(t)\mathcal{P}[(u\cdot\nabla)u](\cdot,t)\Big\|^{\frac{3p}{2}}_{L^{\frac{3p}{2}}(\Omega)} dt &\leq C_{16}\int_{t_0-1}^{t_0+1} \|\nabla u(\cdot,t)\|^{\frac{3p}{2}}_{L^{2p}(\Omega)}\|u(\cdot,t)\|^{\frac{3p}{2}}_{L^{6p}(\Omega)} dt \\
&\leq C_{17}\int_{t_0-1}^{t_0+1} \|\nabla u(\cdot,t)\|^{\frac{3p}{2}}_{L^{2p}(\Omega)} dt \\
&\leq C_{18}\int_{t_0-1}^{t_0+1} \|u(\cdot,t)\|^p_{W^{2,p}(\Omega)}\|u(\cdot,t)\|^{\frac{p}{2}}_{L^{\frac{6p}{2p-3}}(\Omega)} dt \\
&\leq C_{19}\int_{t_0-1}^{t_0+1} \|u(\cdot,t)\|^p_{W^{2,p}(\Omega)} dt \\
&\leq C_{19}C_9^p.
\end{aligned} \tag{7.32}$$

As a consequence of (7.30), (7.31) and (7.32), once more by maximal Sobolev regularity estimates, now applied to (7.29), we obtain $C_{20} > 0$ and $C_{21} > 0$ satisfying

$$
\begin{aligned}
\int_{t_0}^{t_0+1} \|u(\cdot,t)\|_{W^{2,\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt + \int_{t_0}^{t_0+1} \|u_t(\cdot,t)\|_{L^{\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt \\
&\le \int_{t_0-1}^{t_0+1} \|v(\cdot,t)\|_{W^{2,\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt + \int_{t_0-1}^{t_0+1} \|v_t(\cdot,t)\|_{L^{\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt \\
&\le C_{20} \int_{t_0-1}^{t_0+1} \|h(\cdot,t)\|_{L^{\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt \\
&\le C_{21}.
\end{aligned}
$$

This establishes (7.26) and thereby completes the proof. □

Since the exponent p in Lemma 7.6 can be chosen arbitrarily large, an immediate consequence is the following.

Corollary 7.7 *Suppose that (n,c,u) is an eventual energy solution of (1.2). Then one can find $\alpha \in (0,1)$, $T > 0$ and $C > 0$ such that*

$$\|u\|_{C^{1+\alpha,\alpha}(\bar{\Omega}\times[t,t+1])} \le C \qquad \textit{for all } t > T.$$

PROOF. According to a well-known embedding result ([2]), for any $\alpha > 0$ and $\beta > 0$ fulfilling $\alpha + 2\beta < 2$ there exists $p > 1$ such that for each bounded interval $J \subset \mathbb{R}$, the space of functions φ on $\Omega \times (0,T)$ having finite norm $\|\varphi\|_{L^p(J;W^{2,p}(\Omega))} + \|\varphi_t\|_{L^p(J;L^p(\Omega))}$ is continuously embedded into $C^{\alpha,\beta}(\bar{\Omega} \times \bar{J})$. Therefore, the claim is an immediate consequence of Lemma 7.6 when applied to conveniently large $p \ge 2$. □

7.4 Eventual L^p regularity of $c, \nabla c$ and D^2c. Hölder regularity of c and ∇c

By pursuing a similar overall strategy, we can derive the counterpart of the statement in Lemma 7.6 for the second component c. As compared to the situation in the previous section, however, the different structure of the inhomogeneity $h(x,t) = -nf(c) - u \cdot \nabla c$ in $c_t - \Delta c = h(x,t)$, and especially its dependence on c, require modifications in the argument.

Lemma 7.8 *Let (n,c,u) be an eventual energy solution of (1.2). Then for all $p \ge 1$ there exist $T > 0$ and $C > 0$ such that*

$$\|c\|_{L^p((t,t+1);W^{2,p}(\Omega))} + \|c_t\|_{L^p(\Omega\times(t,t+1))} \le C \qquad \textit{for all } t > T. \tag{7.33}$$

PROOF. Let us first make sure that with some $T_1 > 0$ and $C_1 > 0$ we have

$$\|c\|_{L^4((t_0,t_0+1);W^{2,4}(\Omega))} \le C \qquad \text{for all } t > T. \tag{7.34}$$

To this end, we observe that in view of Lemma 7.3, Definition 1.1, (3.16) and Corollary 7.7 there exist $T_1 > 1$ and positive constants C_2, C_3 and C_4 such that

$$\|n\|_{L^4(\Omega\times(t-1,t+1))} \le C_2 \qquad \text{for all } t > T_1 \tag{7.35}$$

and

$$\|c\|_{L^4((t-1,t+1);W^{1,4}(\Omega))} \le C_3 \qquad \text{for all } t > T_1 \tag{7.36}$$

as well as

$$\|u\|_{L^\infty(\Omega\times(t-1,t+1))} \le C_4 \qquad \text{for all } t > T_1. \tag{7.37}$$

Now for fixed $t_0 > T_1$ we let ξ_{t_0} be as given by (7.19) and consider the problem

$$\begin{cases} z_t - \Delta z = h(x,t) := -\xi_{t_0}(t)\cdot\{nf(c) + u\cdot\nabla c\} + \xi'_{t_0}(t)c, \qquad x\in\Omega,\ t > t_0-1, \\ \frac{\partial z}{\partial\nu} = 0, \qquad x\in\partial\Omega,\ t > t_0+1, \\ z(x,t_0-1) = 0, \qquad x\in\Omega. \end{cases} \tag{7.38}$$

Here using (7.35)-(7.37) and (3.16) we see that for some $C_5 > 0$ we have

$$\begin{aligned} \int_{t_0-1}^{t_0+1} \Big\|\xi_{t_0}(t)\Big\{n(\cdot,t)f(c(\cdot,t)) + u(\cdot,t)\cdot\nabla c(\cdot,t)\Big\}\Big\|_{L^4(\Omega)}^4 dt \\ \le\ & C_5\int_{t_0-1}^{t_0+1}\Big\{\|n(\cdot,t)\|_{L^4(\Omega)}^4 + \|u(\cdot,t)\|_{L^\infty(\Omega)}^4\|\nabla c(\cdot,t)\|_{L^4(\Omega)}^4\Big\}dt \\ \le\ & C_5\cdot(C_2^4 + C_4^4C_3^4) \end{aligned} \tag{7.39}$$

and

$$\int_{t_0-1}^{t_0+1}\|\xi'_{t_0}(t)c(\cdot,t)\|_{L^4(\Omega)}^4 dt \le \|\xi'_0\|_{L^\infty(\mathbb{R})}^4\cdot C_3^4. \tag{7.40}$$

According to well-known results on maximal Sobolev regularity properties of the Neumann heat semigroup ([13]), we thus infer from (7.39) and (7.40) that (7.38) possesses a unique strong solution $z\in L^4((t_0-1,t_0+1);W^{2,4}(\Omega))$ with $z_t\in L^4(\Omega\times(t_0-1,t_0+1))$ which satisfies

$$\int_{t_0-1}^{t_0+1}\|z(\cdot,t)\|_{W^{2,4}(\Omega)}^4 dt + \int_{t_0-1}^{t_0+1}\|z_t(\cdot,t)\|_{L^4(\Omega)}^4 dt \le C_6\int_{t_0-1}^{t_0+1}\|h(\cdot,t)\|_{L^4(\Omega)}^4 dt \le C_7 \tag{7.41}$$

with some $C_6 > 0$ and $C_7 > 0$. Since clearly both z and the function $\Omega\times(t_0-1,t_0+1)\ni(x,t)\mapsto \xi_{t_0}(t)c(x,t)$ are weak solutions of (7.38) in the class of functions from $L^2((t_0-1,t_0+1);W^{1,2}(\Omega))$, it follows from a corresponding uniqueness property that $z(x,t) = \xi_{t_0}(t)c(x,t)$ for a.e. $(x,t)\in\Omega\times(t_0-1,t_0+1)$, whereupon (7.41) implies (7.34), because $\xi_{t_0}\equiv 1$ in (t_0,t_0+1).
Let us next verify that if

$$\|c\|_{L^p((t,t+1);W^{2,p}(\Omega))} \le C_8 \qquad \text{for all } t > T_2 \tag{7.42}$$

with some $p > \frac{3}{2}, T_2 > 1$ and $C_8 > 0$, then there exist $T_3 > T_2$ and $C_9 > 0$ such that

$$\|c\|_{L^{2p}((t_0,t_0+1);W^{2,2p}(\Omega))} + \|c_t\|_{L^{2p}(\Omega\times(t_0,t_0+1))} \le C_9 \qquad \text{for all } t > T_3. \tag{7.43}$$

Indeed, assuming (7.42) we once again invoke Lemma 7.3 to obtain $T_3 > T_2$ and $C_{10} > 0$ such that $T_3 > T_1$ and

$$\|n\|_{L^{2p}(\Omega\times(t_0-1,t_0+1))} \le C_{10} \qquad \text{for all } t > T_3. \tag{7.44}$$

Then given $t_0 > T_3$ we define z in the same manner as before and the see that in (7.38) we can use (7.44), (7.37) and (3.16) to find $C_{11} > 0$ fulfilling

$$\begin{aligned}
\int_{t_0-1}^{t_0+1} \Big\| \xi_{t_0}(t) \Big\{ n(\cdot,t) f(c(\cdot,t)) + u(\cdot,t)\cdot\nabla c(\cdot,t) \Big\} \Big\|_{L^{2p}(\Omega)}^{2p} dt \\
\le \ C_{11} \int_{t_0-1}^{t_0+1} \Big\{ \|n(\cdot,t)\|_{L^{2p}(\Omega)}^{2p} + \|u(\cdot,t)\|_{L^\infty(\Omega)}^{2p} \|\nabla c(\cdot,t)\|_{L^{2p}(\Omega)}^{2p} \Big\} dt \\
\le \ C_{10}^{2p} C_{11} + C_4^{2p} C_{11} \int_{t_0-1}^{t_0+1} \|\nabla c(\cdot,t)\|_{L^{2p}(\Omega)}^{2p} dt,
\end{aligned}$$

where combining the Gagliardo-Nirenberg inequality with (7.42) and (3.16) provides $C_{12} > 0$ and $C_{13} > 0$ such that

$$\begin{aligned}
\int_{t_0-1}^{t_0+1} \|\nabla c(\cdot,t)\|_{L^{2p}(\Omega)}^{2p} dt \ &\le \ C_{12} \int_{t_0-1}^{t_0+1} \|c(\cdot,t)\|_{W^{2,p}(\Omega)}^{p} \|c(\cdot,t)\|_{L^\infty(\Omega)}^{p} dt \\
&\le \ C_{13}.
\end{aligned}$$

Since again from (3.16) we obtain $C_{14} > 0$ such that

$$\int_{t_0-1}^{t_0+1} \|\xi_{t_0}'(t) c(\cdot,t)\|_{L^{2p}(\Omega)}^{2p} dt \le 2\|\xi_0'\|_{L^\infty(\mathbb{R})}^{2p} \|c\|_{L^\infty(\Omega\times(t_0-1,t_0+1))}^{2p} \le C_{14},$$

we may argue as above to conclude from maximal Sobolev regularity estimates that there exist $C_{15} > 0$ and $C_{16} > 0$ such that

$$\int_{t_0-1}^{t_0+1} \|z(\cdot,t)\|_{W^{2,2p}(\Omega)}^{2p} dt + \int_{t_0-1}^{t_0+1} \|z_t(\cdot,t)\|_{L^{2p}(\Omega)}^{2p} dt \le C_{15} \int_{t_0-1}^{t_0+1} \|h(\cdot,t)\|_{L^{2p}(\Omega)}^{2p} dt \le C_{16}.$$

Again since $z = c$ a.e. in $\Omega \times (t_0, t_0+1)$ by definition of ξ_{t_0}, this shows (7.43) and thereby completes the proof of (7.33) upon iteration. □

Again, this implies Hölder estimates as follows.

Corollary 7.9 *Let (n, c, u) be an eventual energy solution of (1.2). Then there exist $\alpha \in (0,1), T > 0$ and $C > 0$ such that*

$$\|c\|_{C^{1+\alpha,\alpha}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t > T.$$

PROOF. In precisely the same manner as Corollary 7.7 was derived from Lemma 7.6, this follows from Lemma 7.8 by application of a standard embedding result ([2]). □

7.5 Eventual L^p regularity of $n, \nabla n$ and $D^2 n$. Hölder regularity of n and ∇n

The above estimates, inter alia asserting a uniform pointwise bound for ∇c and a space-time bound for Δc in any L^p norm for $p < \infty$, now provide sufficient information for the derivation of the following analogue of Lemma 7.8 for n.

Lemma 7.10 *Given an eventual energy solution* (n, c, u) *of (1.2), for each* $p \geq 1$ *one can find* $T > 0$ *and* $C > 0$ *such that*

$$\|n\|_{L^p((t,t+1);W^{2,p}(\Omega))} + \|n_t\|_{L^p(\Omega\times(t,t+1))} \leq C \qquad \textit{for all } t > T. \tag{7.45}$$

Proof. The proof is quite similar to that of Lemma 7.8, so that we may confine ourselves with outlining the main steps only. We first show that there exist $T_1 > 0$ and $C_1 > 0$ such that

$$\|n\|_{L^2((t_0,t_0+1);W^{2,2}(\Omega))} \leq C_1 \qquad \text{for all } t_0 > T_1. \tag{7.46}$$

To see this, we recall Lemma 7.3, Corollary 7.9, Lemma 7.8 and Corollary 7.7 to fix $T_1 > 1$ and $C_2 > 0$ such that

$$\|n\|_{L^2((t-1,t+1);W^{1,2}(\Omega))} + \|n\|_{L^4(\Omega\times(t-1,t+1))} + \|\nabla c\|_{L^\infty(\Omega\times(t-1,t+1))} + \|\Delta c\|_{L^4(\Omega\times(t-1,t+1))} + \|u\|_{L^\infty(\Omega\times(t-1,t+1))} \leq C_1 \qquad \text{for all } t > T_1. \tag{7.47}$$

Then given $t_0 > T_1$, with ξ_{t_0} taken from (7.19) we see that the source term h in

$$\begin{cases} w_t - \Delta w = h(x,t) := -\xi_{t_0}(t) \cdot \Big\{\nabla n \cdot \nabla c + n\Delta c + u \cdot \nabla n\Big\} + \xi'_{t_0}(t) n, \qquad x \in \Omega,\ t > t_0 - 1, \\ \frac{\partial w}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > t_0 - 1, \\ w(x, t_0 - 1) = 0, \qquad x \in \partial\Omega, \end{cases} \tag{7.48}$$

satisfies

$$\begin{aligned} \int_{t_0-1}^{t_0+1} \|h(\cdot,t)\|^2_{L^2(\Omega)} dt \ &\leq \ \|\nabla c\|^2_{L^\infty(\Omega\times(t_0-1,t_0+1))} \int_{t_0-1}^{t_0+1} \|\nabla n(\cdot,t)\|^2_{L^2(\Omega)} dt \\ &\quad + \Big(\int_{t_0-1}^{t_0+1} \|n(\cdot,t)\|^2_{L^4(\Omega)}\Big)^{\frac12} \cdot \Big(\int_{t_0-1}^{t_0+1} \|\Delta c(\cdot,t)\|^2_{L^4(\Omega)}\Big)^{\frac12} \\ &\quad + \|u\|_{L^\infty(\Omega\times(t_0-1,t_0+1))} \int_{t_0-1}^{t_0+1} \|\nabla n(\cdot,t)\|^2_{L^2(\Omega)} dt \\ &\quad + \|\xi'_0\|^2_{L^\infty(\mathbb{R})} \int_{t_0-1}^{t_0+1} \|n(\cdot,t)\|^2_{L^2(\Omega)} dt \\ &\leq \ C_2 \end{aligned}$$

for some $C_2 > 0$, where we have used the Cauchy-Schwarz inequality. Therefore, (7.46) is a consequence of a maximal Sobolev regularity inequality along with a uniqueness argument applied to (7.48). Next, assuming that with some $p > 1, T_2 > 1$ and $C_3 > 0$ we have

$$\|n\|_{L^p((t-1,t+1);W^{2,p}(\Omega))} \leq C_3 \qquad \text{for all } t > T_2, \tag{7.49}$$

we can infer the existence of $T_3 > 0$ and $C_4 > 0$ fulfilling

$$\|n\|_{L^{\frac{3p}{2}}((t_0,t_0+1);W^{2,\frac{3p}{2}}(\Omega))} + \|n_t\|_{L^{\frac{3p}{2}}(\Omega\times(t_0,t_0+1))} \leq C_4 \qquad \text{for all } t_0 > T_3. \tag{7.50}$$

Indeed, thanks to Lemma 7.3 and Lemma 7.8 we may pick $T_3 > \max\{T_1, T_2\}$ and $C_5 > 0$ such that

$$\|n\|_{L^{3p}(\Omega\times(t-1,t+1))} + \|n\|_{L^\infty((t-1,t+1);L^3(\Omega))} + \|\Delta c\|_{L^{3p}(\Omega\times(t-1,t+1))} \le C_5 \qquad \text{for all } t > T_3,$$

whence by (7.49) and the Gagliardo-Nirenberg inequality we obtain $C_6 > 0$ satisfying

$$\int_{t-1}^{t+1} \|n(\cdot,s)\|_{W^{1,\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} ds \le C_6 \int_{t-1}^{t+1} \|n(\cdot,s)\|_{W^{2,p}(\Omega)}^p \|n(\cdot,s)\|_{L^3(\Omega)}^{\frac{p}{2}} ds \le C_6 C_3^p C_5^{\frac{p}{2}} \qquad \text{for all } t > T_3.$$

Accordingly, if $t_0 > T_3$ and ξ_{t_0} is as in (7.19), then in (7.48) we can once more use (7.47) and the Cauchy-Schwarz inequality to estimate

$$\begin{aligned}
\int_{t_0-1}^{t_0+1} \|h(\cdot,t)\|_{L^{\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt &\le \|\nabla c\|_{L^\infty(\Omega\times(t_0-1,t_0+1))}^{\frac{3p}{2}} \int_{t_0-1}^{t_0+1} \|\nabla n(\cdot,t)\|_{L^{\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt \\
&\quad + \Big(\int_{t_0-1}^{t_0+1} \|n(\cdot,t)\|_{L^{3p}(\Omega)}^{3p} dt \Big)^{\frac12} \cdot \Big(\int_{t_0-1}^{t_0+1} \|\Delta c(\cdot,t)\|_{L^{3p}(\Omega)}^{3p} \Big)^{\frac12} \\
&\quad + \|u\|_{L^\infty(\Omega\times(t_0-1,t_0+1))}^{\frac{3p}{2}} \int_{t_0-1}^{t_0+1} \|\nabla n(\cdot,t)\|_{L^{\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt \\
&\quad + \|\xi_0'\|_{L^\infty(\mathbb{R})}^{\frac{3p}{2}} \int_{t_0-1}^{t_0+1} \|n(\cdot,t)\|_{L^{\frac{3p}{2}}(\Omega)}^{\frac{3p}{2}} dt \\
&\le C_7
\end{aligned}$$

with some $C_7 > 0$. Another application of maximal Sobolev regularity theory thus yields (7.50) and hence proves (7.45), because $p > 1$ was arbitrary. $\square$

Once more, this entails a certain Hölder regularity.

Corollary 7.11 *Let (n, c, u) be an eventual energy solution of (1.2). Then there exist $\alpha \in (0,1), T > 0$ and $C > 0$ such that*

$$\|n\|_{C^{1+\alpha,\alpha}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t > T.$$

PROOF. In precisely the same manner as Corollary 7.7 was derived from Lemma 7.6, this follows from Lemma 7.10 by application of a standard embedding result ([2]). $\square$

7.6 Estimates in $C^{2+\alpha,1+\frac{\alpha}{2}}$

Straightforward applications of standard Schauder estimates for the Stokes evolution equation and the heat equation, respectively, finally yield eventual smoothness of the solution components u as well as n and c, respectively.

Lemma 7.12 *For any eventual energy solution (n, c, u) (1.2) one can find $\alpha \in (0,1), T > 0$ and $C > 0$ such that*

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t > T. \tag{7.51}$$

Proof. According to Corollary 7.11 and Corollary 7.7, it is possible to fix $\alpha' \in (0,1)$, $T_1 > 0$ and $C_2 > 0$ such that

$$\|n\|_{C^{\alpha',\frac{\alpha'}{2}}(\bar\Omega\times[t,t+1])} \le C_1 \qquad \text{for all } t > T_1 \tag{7.52}$$

and

$$\|u\|_{C^{1+\alpha',\alpha'}(\bar\Omega\times[t,t+1])} \le C_2 \qquad \text{for all } t > T_1. \tag{7.53}$$

We next set $T := T_1 + 1$ and let $t_0 > T$ be given. Then with ξ_{t_0} taken from (7.19), we again use that $v(\cdot,t) := \xi_{t_0}(t)u(\cdot,t)$, $t > t_0 - 1$, is a solution of

$$\begin{cases} v_t + Av = h(x,t) := \xi_{t_0}(t)\mathcal{P}[(u\cdot\nabla)u + n\nabla\Phi] + \xi'_{t_0}(t)u, & x\in\Omega,\ t > t_0 - 1, \\ v(x,t_0) = 0, & x \in \Omega, \end{cases} \tag{7.54}$$

which hence, in particular, satisfies the associated first-order compatibility condition at $t = t_0 - 1$ ([27]).
Now from (7.52), (7.53) and the smoothness of ξ_0 we readily obtain $\alpha \in (0,1)$ and $C_3 > 0$ fulfilling

$$\|h\|_{C^{\alpha,\frac{\alpha}{2}}(\bar\Omega\times[t_0-1,t_0+1])} \le C_3,$$

so that regularity estimates from Schauder theory for the Stokes evolution equation ([27]) ensure that (7.54) possesses a classical solution $\tilde v \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t_0-1,t_0+1])$ satisfying

$$\|\tilde v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t_0-1,t_0+1])} \le C_4$$

with some $C_4 > 0$ which is independent of t_0. As clearly $\tilde v \equiv v$ by an evident uniqueness property of (7.54), this proves (7.51). □

Lemma 7.13 *Assume that (n,c,u) be an eventual energy solution of (1.2). Then there exist $\alpha \in (0,1), T > 0$ and $C > 0$ such that*

$$\|n\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} + \|c\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t > T. \tag{7.55}$$

Proof. We first combine Corollary 7.9 with Corollary 7.11 and Corollary 7.7 to infer the existence of $\alpha' \in (0,1)$, $T_1 > 0$ and $C_1 > 0$ such that

$$\|nf(c)\|_{C^{\alpha',\frac{\alpha'}{2}}(\bar\Omega\times[t,t+1])} + \|u\cdot\nabla c\|_{C^{\alpha',\frac{\alpha'}{2}}(\bar\Omega\times[t,t+1])} \le C_1 \qquad \text{for all } t > T_1.$$

Standard parabolic Schauder estimates applied to the second equation in (1.2) ([22]) thus provide $C_2 > 0$ fulfilling

$$\|c\|_{C^{2+\alpha',1+\frac{\alpha'}{2}}(\bar\Omega\times[t,t+1])} \le C_2 \qquad \text{for all } t > T_1 + 1. \tag{7.56}$$

Again in view of Corollary 7.11 and Corollary 7.7, this in turn warrants that for some $\alpha'' \in (0,1), T_2 > 0$ and $C_3 > 0$ we have

$$\|\nabla\cdot(n\nabla c)\|_{C^{\alpha'',\frac{\alpha''}{2}}(\bar\Omega\times[t,t+1])} + \|u\cdot\nabla n\|_{C^{\alpha'',\frac{\alpha''}{2}}(\bar\Omega\times[t,t+1])} \le C_3 \qquad \text{for all } t > T_2,$$

whereupon Schauder theory says that

$$\|n\|_{C^{2+\alpha'',1+\frac{\alpha''}{2}}(\bar\Omega\times[t,t+1])} \le C_4 \qquad \text{for all } t > T_2 + 1.$$

Along with (7.56), this proves (7.55) with $\alpha := \min\{\alpha',\alpha''\}, T := \max\{T_1,T_2\}$ and some suitably large $C > 0$. □

8 Stabilization of n and u. Proof of Theorem 1.3

On the basis of the eventual uniform continuity properties implied by the estimates in the previous section, we can now turn the weak stabilization properties of n and u from Lemma 7.3 and Lemma 7.5 into convergence with regard to the norm in $L^\infty(\Omega)$. In the proofs of our results in Lemma 8.2 and Lemma 8.3 in this direction we shall make use of the following statement, the elementary proof of which may be omitted here.

Lemma 8.1 *Let $T \in \mathbb{R}$, and assume that $h : (T, \infty) \to [0, \infty)$ is uniformly continuous and such that $\int_t^{t+1} h(s)ds \to 0$ and $t \to \infty$. Then $h(t) \to 0$ as $t \to \infty$.*

A first application thereof shows that Corollary 7.9 and Lemma 7.3 entail the following.

Lemma 8.2 *Assume that (n, c, u) be an eventual energy solution of (1.2). Then with $\overline{n_0} := \fint_\Omega n_0 > 0$, we have*

$$\|n(\cdot, t) - \overline{n_0}\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{8.1}$$

Proof. By means of Corollary 7.9 and the Arzelà-Ascoli theorem, we can fix $T_1 > 0$ such that

$$(n(\cdot, t))_{t>T_1} \text{ is relatively compact in } C^0(\bar{\Omega}), \tag{8.2}$$

and that the function $\varphi : (T_1, \infty) \to \mathbb{R}$ defined by $h(t) := \|n(\cdot, t) - \overline{n_0}\|^2_{L^2(\Omega)}$, $t > T_1$, is uniformly continuous. Moreover, from Lemma 7.3 we know that with some $T_2 > T_1$ we have $\int_{T_2}^\infty \int_\Omega |\nabla n|^2 < \infty$, so that since the Poincaré inequality combined with (2.6) provides $C_1 > 0$ such that

$$\int_t^{t+1} \|n(\cdot, s) - \overline{n_0}\|^2_{L^2(\Omega)} ds \le C_1 \int_t^{t+1} \int_\Omega |\nabla n|^2 \qquad \text{for all } t > T_2,$$

we obtain that $\int_t^{t+1} h(s)ds \to 0$ as $t \to \infty$. In view of Lemma 8.1, this asserts that $h(t) \to 0$ and hence $n(\cdot, t) \to \overline{n_0}$ in $L^2(\Omega)$ as $t \to \infty$. Along with (8.2), this proves (8.1). □

Next, combining Corollary 7.7 with Lemma 7.3 yields decay of u in $L^\infty(\Omega)$.

Lemma 8.3 *If (n, c, u) is an eventual energy solution of (1.2), then*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{8.3}$$

Proof. We first observe that as a consequence of Lemma 7.4,

$$\int_t^{t+1} \int_\Omega |u|^2 \to 0 \qquad \text{as } t \to \infty, \tag{8.4}$$

meaning that if with some conveniently large $T_1 > 0$ we let $h(t) := \|u(\cdot, t)\|_{L^2(\Omega)}$, $t > T_1$, then $\int_t^{t+1} h(s)ds \to 0$ as $t \to \infty$. We now follow the reasoning from Lemma 8.2 and invoke Corollary 7.7 in choosing $T_2 > T_1$ such that

$$(u(\cdot, t))_{t>T_2} \text{ is relatively compact in } C^0(\bar{\Omega}) \text{ and } h \text{ is uniformly continuous for } t > T_2, \tag{8.5}$$

where the latter along with (8.4) entails that $h(t) \to 0$ as $t \to \infty$ according to Lemma 8.1. Thus knowing that $u(\cdot,t) \to 0$ in $L^2(\Omega)$ as $t \to \infty$, using the compactness property stated in (8.5) we readily end up with (8.3). □

Proving our final result on large time behavior of arbitrary eventual energy solutions now reduces to collecting the above convergence and regularity properties.

Proof of Theorem 1.3 The regularity properties in (1.13) are immediate consequences of Lemma 7.13 and Lemma 7.12. In order to verify (1.14), we note that on choosing $\widetilde{u} := u$ and $F(s) := s$ for $s \ge 0$ we obtain from Definition 1.1, (2.6) and (3.16) that $(n,c,F) \in \mathcal{S}_{m,M,L,T_0}$ with $m := \int_\Omega n_0$, $M := \|c_0\|_{L^\infty(\Omega)}$ and some $L > 0$ and $T_0 > 0$. Therefore, Lemma 4.6 in particular implies that $c(\cdot,t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$. Combined with the results of Lemma 8.2 and Lemma 8.3, this proves the claimed stabilization properties. □

9 Existence of an eventual energy solution. Proof of Theorem 1.2

Following [39], we regularize the original problem (1.2) by fixing families of approximate initial data $n_{0\varepsilon}, c_{0\varepsilon}$ and $u_{0\varepsilon}$, $\varepsilon \in (0,1)$, with the properties that

$$\begin{cases} n_{0\varepsilon} \in C_0^\infty(\Omega), \quad n_{0\varepsilon} \ge 0 \text{ in } \Omega \text{ and } \quad \int_\Omega n_{0\varepsilon} = \int_\Omega n_0 \quad \text{for all } \varepsilon \in (0,1) \qquad \text{and} \\ n_{0\varepsilon} \to n_0 \quad \text{in } L\log L(\Omega) \qquad \text{as } \varepsilon \searrow 0, \end{cases} \tag{9.1}$$

that

$$\begin{cases} c_{0\varepsilon} \ge 0 \text{ in } \Omega \text{ is such that } \sqrt{c_{0\varepsilon}} \in C_0^\infty(\Omega) \quad \text{and} \quad \|c_{0\varepsilon}\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } \varepsilon \in (0,1) \qquad \text{and} \\ \sqrt{c_{0\varepsilon}} \to \sqrt{c_0} \quad \text{a.e. in } \Omega \text{ and in } W^{1,2}(\Omega) \qquad \text{as } \varepsilon \searrow 0, \end{cases} \tag{9.2}$$

and that

$$\begin{cases} u_{0\varepsilon} \in C_0^\infty(\Omega) \cap L^2_\sigma(\Omega) \quad \text{with} \quad \|u_{0\varepsilon}\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)} \quad \text{for all } \varepsilon \in (0,1) \qquad \text{and} \\ u_{0\varepsilon} \to u_0 \quad \text{in } L^2(\Omega) \qquad \text{as } \varepsilon \searrow 0. \end{cases} \tag{9.3}$$

Then for $\varepsilon \in (0,1)$, we consider

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon), & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) f(c_\varepsilon), & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} = \Delta u_\varepsilon - \nabla P_\varepsilon + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon + n_\varepsilon \nabla \phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_{0\varepsilon}(x), \quad c_\varepsilon(x,0) = c_{0\varepsilon}(x), \quad u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{9.4}$$

where

$$F_\varepsilon(s) := \frac{1}{\varepsilon} \ln(1 + \varepsilon s) \qquad \text{for } s \ge 0, \tag{9.5}$$

and where

$$Y_\varepsilon v := (1 + \varepsilon A)^{-1} v \qquad \text{for } v \in L^2_\sigma(\Omega). \tag{9.6}$$

The following lemma summarizes some of the results for (9.4) obtained in [39, Lemma 2.2, Lemma 2.3, Lemma 3.9, Lemma 3.6].

Lemma 9.1 *For each $\varepsilon \in (0,1)$, there exist uniquely determined functions*

$$n_\varepsilon \in C^{2,1}(\bar\Omega \times [0,\infty)), \quad c_\varepsilon \in C^{2,1}(\bar\Omega \times [0,\infty)) \quad \text{and} \quad u_\varepsilon \in C^{2,1}(\bar\Omega \times [0,\infty);\mathbb{R}^3) \tag{9.7}$$

which are such that $n_\varepsilon > 0$ and $c_\varepsilon > 0$ in $\bar\Omega \times (0, T_{max,\varepsilon})$, and such that with some $P_\varepsilon \in C^{1,0}(\Omega \times (0, T_{max,\varepsilon}))$, the quadruple $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ solves (9.4) classically in $\Omega \times (0, T_{max,\varepsilon})$.
These solutions satisfy

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \qquad \text{for all } t > 0 \tag{9.8}$$

as well as

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0, \tag{9.9}$$

and there exist $\kappa > 0$ an $K > 0$ such that

$$\frac{d}{dt}\mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t) + \frac{1}{K}\bigg\{ \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2 \bigg\} \le K \qquad \text{for all } t > 0. \tag{9.10}$$

Furthermore, Theorem 1.1 in [39] asserts that these solutions approach a global weak solution of (1.2) in the following sense.

Lemma 9.2 *There exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and a global weak solution (n,c,u) of (1.2) such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and $(n_\varepsilon, c_\varepsilon, u_\varepsilon) \to (n,c,u)$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$. For this solution, we moreover have*

$$\begin{aligned}
&n \in L^\infty((0,\infty); L^1(\Omega)) \quad \text{with} \quad n^{\frac12} \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)),\\
&c \in L^\infty(\Omega \times (0,\infty)) \quad \text{with} \quad c^{\frac14} \in L^4_{loc}([0,\infty); W^{1,4}(\Omega)), \qquad \text{and}\\
&u \in L^\infty_{loc}([0,\infty); L^2_\sigma(\Omega)) \cap L^2_{loc}([0,\infty); W^{1,2}_0(\Omega)),
\end{aligned} \tag{9.11}$$

and there exist $\kappa > 0$ and $K > 0$ such that (1.11) and (1.12) hold with $T := 0$.

It remains to verify that the component n of this limit function has the additional regularity properties $\nabla n \in L^2_{loc}(\bar\Omega \times (T,\infty))$ and $n \in L^4_{loc}(\bar\Omega \times (T,\infty))$ required in Definition 1.1. Thanks to all our previous analysis, without substantial further efforts these will result from the fact that for each of the approximate solutions $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$, the triple $(n_\varepsilon, c_\varepsilon, F_\varepsilon)$ lies in $\mathcal{S}_{m,M,L,0}$ with suitable $m > 0, M > 0$ and $L > 0$:

Lemma 9.3 *There exist $T > 0$ and $C > 0$ such that for all $\varepsilon \in (0,1)$, the solution of (9.4) satisfies*

$$\int_T^\infty \int_\Omega |\nabla n_\varepsilon|^2 \le C \tag{9.12}$$

and

$$\int_T^{T+\tilde T} \int_\Omega n_\varepsilon^4 \le C \cdot (\tilde T + 1) \qquad \text{for all } \tilde T > 0. \tag{9.13}$$

PROOF. In order to prepare an application of Lemma 6.3, we recall that $\int_\Omega n_\varepsilon(\cdot,t) = m := \int_\Omega n_0$ and $\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le M := \|c_0\|_{L^\infty(\Omega)}$ for all $t>0$ and $\varepsilon \in (0,1)$ according to (9.8) and (9.9). Next, Lemma 9.1 combined with Lemma 7.1 says that there exist positive constants C_1 and C_2, independent of $\varepsilon \in (0,1)$, such that with $\kappa > 0$ as given there, the function y defined by $y(t) := \mathcal{F}_\kappa[n_\varepsilon, c_\varepsilon, u_\varepsilon](t)$, $t \ge 0$, satisfies $y'(t) + C_1 y(t) \le C_2$ for all $t \ge 0$, which shows that $|y(t)| \le C_3$ for all $t \ge 0$ with some $C_3 > 0$ independent of ε. Therefore, integrating (9.10) in time yields

$$\int_t^{t+1} \int_\Omega \left\{ \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + |\nabla c_\varepsilon|^4 \right\} \le K \max\{1, M^3\} \cdot (K + 2C_3) \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1), \tag{9.14}$$

from which we infer that $(n_\varepsilon, c_\varepsilon, F_\varepsilon) \in \mathcal{S}_{m,M,L,0}$ for all $\varepsilon \in (0,1)$ if we let $L := K\max\{1,M^3\}\cdot(K+2C_3)$. Therefore, Lemma 4.6 applies so as to assert the doubly uniform decay property

$$\sup_{\varepsilon \in (0,1)} \|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega \times (T_0,\infty))} \to 0 \qquad \text{as } T_0 \to \infty. \tag{9.15}$$

In particular, if we let $\eta > 0$ and $\tau > 0$ denote the numbers obtained from Lemma 6.3 upon the specific choice $p := 2$, we can fix $T_0 > 0$ such that

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega \times (T_0,\infty))} \le \eta \qquad \text{for all } \varepsilon \in (0,1).$$

Combining this with (9.14) and using the outcome of Lemma 6.3, we thus infer that

$$\|n_\varepsilon(\cdot,t)\|_{L^3(\Omega)} \le C_1 \quad \text{for all } t > T \qquad \text{and} \qquad \int_T^\infty \int_\Omega |\nabla n_\varepsilon|^2 \le C_1 \tag{9.16}$$

with $T := T_0 + \tau$ and some $C_1 > 0$ possibly depending on m and L but not on $\varepsilon \in (0,1)$. Since using the Gagliardo-Nirenberg inequality we obtain $C_2 > 0$ such that for each $\tilde{T} > 0$ we have

$$\int_T^{T+\tilde{T}} \|n_\varepsilon(\cdot,t)\|^4_{L^4(\Omega)} dt \le \int_T^{T+\tilde{T}} \Big\{ C_2 \|\nabla n_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} \|n_\varepsilon(\cdot,t)\|^2_{L^3(\Omega)} + C_2 \|n_\varepsilon(\cdot,t)\|^4_{L^3(\Omega)} \Big\} dt,$$

both (9.12) and (9.13) result from (9.16). □

We thereby immediately obtain our final result on global existence of an eventual energy solution.

PROOF of Theorem 1.2. We let (n,c,u) denote the limit function gained in Lemma 9.2. Then as a consequence of Lemma 9.3 we know that there exist $T>0$ and $C_1>0$ such that

$$\int_T^\infty \int_\Omega |\nabla n|^2 < \infty$$

and

$$\int_T^{T+\tilde{T}} \int_\Omega n^4 \le C_1 \cdot (\tilde{T}+1) \qquad \text{for all } \tilde{T} > 0.$$

In conjunction with (9.11), this shows that (n,c,u) enjoys all the regularity properties required in Definition 1.1 and therefore implies that (n,c,u) indeed is an eventual energy solution of (1.2). □

10 Appendix

For convenience, let us include a short proof of a stability property of Steklov averages which is essential to our arguments in both Lemma 5.1 and also Lemma 3.3.

Lemma 10.1 *Let $-\infty < T_0 < T_1 < \infty$, and for $h \in (0,1)$ define*

$$S_h[\varphi](x,t) := \frac{1}{h}\int_t^{t+h} \varphi(x,s)ds, \qquad x \in \Omega,\ t \in (T_0,T_1), \tag{10.1}$$

for $\varphi \in L^1(\Omega \times (T_0,T_1))$.

i) If $\varphi \in L^p(\Omega \times (T_0,T_1+1))$ for some $p \in [1,\infty)$, then

$$\|S_h[\varphi]\|_{L^p(\Omega\times(T_0,T_1))} \le \|\varphi\|_{L^p(\Omega\times(T_0,T_1+h))} \qquad \textit{for all } h \in (0,1). \tag{10.2}$$

ii) Suppose that for some $p \in [1,\infty)$, $(\varphi_h)_{h\in(0,1)} \subset L^p(\Omega \times (T_0,T_1+1))$ and $\varphi \in L^p(\Omega \times (T_0,T_1+1))$ are such that

$$\|\varphi_h - \varphi\|_{L^p(\Omega\times(T_0,T_1+h))} \to 0 \qquad \textit{as } h \searrow 0. \tag{10.3}$$

Then

$$S_h[\varphi_h] \to \varphi \quad \textit{in } L^p(\Omega \times (T_0,T_1)) \qquad \textit{as } h \searrow 0. \tag{10.4}$$

PROOF. We may assume that $T_0 = 0$ and write $T := T_1$.

i) Using the Hölder inequality and the Fubini theorem, we directly see that

$$\begin{aligned}
\int_0^T \int_\Omega \Big|S_h[\varphi](x,t)\Big|^p dxdt &= \frac{1}{h^p}\int_\Omega \int_0^T \bigg|\int_t^{t+h}\varphi(x,s)ds\bigg|^p dtdx \\
&\le \frac{1}{h}\int_\Omega\int_0^T\int_t^{t+h} |\varphi(x,s)|^p dsdtdx \\
&= \frac{1}{h}\int_\Omega \bigg\{ \int_h^T \int_{s-h}^s |\varphi(x,s)|^p dtds + \int_0^h\int_0^s |\varphi(x,s)|^p dtds + \int_T^{T+h}\int_{s-h}^T |\varphi(x,s)|^p dtds \bigg\} dx \\
&= \frac{1}{h}\int_\Omega \bigg\{ h\int_h^T |\varphi(x,s)|^p ds + \int_0^h s|\varphi(x,s)|^p ds + \int_T^{T+h} (T+h-s)|\varphi(x,s)|^p ds \bigg\} dx \\
&\le \frac{1}{h}\int_\Omega \bigg\{ h\int_h^T |\varphi(x,s)|^p ds + h\int_0^h |\varphi(x,s)|^p ds + h\int_T^{T+h} |\varphi(x,s)|^p ds \bigg\} dx \\
&= \int_0^{T+h}\int_\Omega |\varphi(x,s)|^p dxds
\end{aligned}$$

for all $h \in (0,1)$.

ii) By linearity of S_h and i), we have

$$\begin{aligned}
\|S_h[\varphi_h] - \varphi\|_{L^p(\Omega\times(0,T))} &= \|S_h[\varphi_h - \varphi] + (S_h[\varphi] - \varphi)\|_{L^p(\Omega\times(0,T))} \\
&\le \|S_h[\varphi_h - \varphi]\|_{L^p(\Omega\times(0,T))} + \|S_h[\varphi] - \varphi\|_{L^p(\Omega\times(0,T))} \\
&\le \|\varphi_h - \varphi\|_{L^p(\Omega\times(0,T+h))} + \|S_h[\varphi] - \varphi\|_{L^p(\Omega\times(0,T))} \qquad \text{for all } h \in (0,1).
\end{aligned}$$

Since $S_h[\varphi] \to \varphi$ in $L^p(\Omega \times (0,T))$ by a well-known result (see e.g. [5, Lemma I.3.2]), (10.3) therefore entails (10.4). $\square$

We also separately state the following interpolation lemma which is used in several places.

Lemma 10.2 *There exists $C > 0$ such that whenever $J \subset \mathbb{R}$ is an interval, the following holds.*
i) Any function $n \in L^\infty(J; L^1(\Omega)) \cap L^2(J; W_0^{1,2}(\Omega))$ belongs to $L^{\frac{8}{3}}(\Omega \times J)$ and satisfies

$$\int_J \int_\Omega n^{\frac{8}{3}} \le C \|\nabla n\|^2_{L^2(\Omega\times J)} \|n\|^{\frac{2}{3}}_{L^\infty(J;L^1(\Omega))}. \tag{10.5}$$

ii) If $u \in L^\infty(J; L^2(\Omega;\mathbb{R}^3)) \cap L^2(J; W_0^{1,2}(\Omega;\mathbb{R}^3))$, then $u \in L^{\frac{10}{3}}(\Omega \times J;\mathbb{R}^3)$ with

$$\int_J \int_\Omega |u|^{\frac{10}{3}} \le C \|\nabla u\|^2_{L^2(\Omega\times J)} \|u\|^{\frac{4}{3}}_{L^\infty(J;L^2(\Omega))}. \tag{10.6}$$

Proof. Both statements can be obtained upon straightforward interpolation using the Gagliardo-Nirenberg inequality. $\square$

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