Notre Dame Journal of Formal Logic
Volume 50, Number 4, 2009

How Incomputable Is the Separable Hahn-Banach Theorem?

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Abstract We determine the computational complexity of the Hahn-Banach Extension Theorem. To do so, we investigate some basic connections between reverse mathematics and computable analysis. In particular, we use Weak König's Lemma within the framework of computable analysis to classify incomputable functions of low complexity. By defining the multivalued function **Sep** and a natural notion of reducibility for multivalued functions, we obtain a computational counterpart of the subsystem of second-order arithmetic WKL_0. We study analogies and differences between WKL_0 and the class of **Sep**-computable multivalued functions. Extending work of Brattka, we show that a natural multivalued function associated with the Hahn-Banach Extension Theorem is **Sep**-complete.

1 Introduction

In this paper we tackle a problem in computable analysis ([21] is the main reference in the area) borrowing ideas and proof techniques from the research program of reverse mathematics ([19] is the standard reference). The two subjects share the goal of classifying complexity of mathematical practice. Reverse mathematics was started by Friedman [7] in the 1970s. It adopts a proof-theoretic viewpoint (although techniques from computability theory are increasingly important in the subject) and investigates which axioms are needed to prove a given theorem (see Section 3 for details). On the other hand, computable analysis extends to computable separable metric spaces the notions of computability and incomputability by combining concepts of approximation and of computation. To this end the representation approach (Type-2 Theory of Effectivity, TTE), introduced for real functions by Grzegorczyk

Received December 29, 2008; accepted January 15, 2009; printed January 8, 2010
2010 Mathematics Subject Classification: Primary, 03F60; Secondary, 03B30, 46A22, 46S30
Keywords: computable analysis, reverse mathematics, weak König's lemma, Hahn-Banach extension theorem, multivalued functions
 10.1215/00294527-2009-018

and Lacombe [10; 13], is used. This approach provides a realistic and flexible model of computation.

One of the goals of computable analysis is to study and compare degrees of incomputability of (possibly multivalued) functions between separable metric spaces. Multivalued functions are the appropriate way of dealing with situations where problems have nonunique solutions and they have been studied in computable analysis since [21]. We use the notation $f : \subseteq X \rightrightarrows Y$ to mean that f is a multivalued function with $\mathrm{dom}(f) \subseteq X$ and $\mathrm{ran}(f) \subseteq Y$. Following [21, §1.4], we view a partial multivalued function $f : \subseteq X \rightrightarrows Y$ as a subset of $X \times Y$. Then $\mathrm{dom}(f) = \{x \in X \mid \exists y \in Y\, (x, y) \in f\}$ and, when $x \in \mathrm{dom}(f)$, we have $f(x) = \{ y \in Y \mid (x, y) \in f \}$.

In this paper we introduce a notion of computable reducibility for multivalued functions which generalizes at once both notions of reducibility for single-valued functions extensively studied by Brattka in [1]. Let $f : \subseteq X \rightrightarrows Y$ and $g : \subseteq U \rightrightarrows V$ be two (partial) multivalued functions, where X, Y, U, V are separable metric spaces. We say that f is computably reducible to g and write $f \leqslant_c g$ if there are computable multivalued functions $h : \subseteq X \rightrightarrows U$ and $k : \subseteq X \times V \rightrightarrows Y$ such that $\varnothing \neq k(x, g(h(x))) \subseteq f(x)$ (see Definition 4.1 below for the definition of composition of multivalued functions) for all $x \in \mathrm{dom}(f)$. We use $<_c$ and $\cong_c$ to denote the strict order and the equivalence relation defined in the obvious way.

In [2] Brattka started the study of the separable Hahn-Banach Theorem from the viewpoint of computable analysis. Given a computable separable Banach space X, consider the multivalued function H_X mapping a closed linear subspace A of X and a bounded linear functional $f : A \to \mathbb{R}$ with $\|f\| = 1$ to the set of all bounded linear functionals $g : X \to \mathbb{R}$ which extend f and are such that $\|g\| = 1$. For many computable separable Banach spaces X, it turns out that H_X is incomputable. Brattka does not establish precisely the degree of incomputability of these functions, as he shows, in our notation, that $H_X <_c C_1$ for every X. Here C_1 is a standard function considered in computable analysis, the first in a sequence of increasingly incomputable functions (see Definition 2.10 below).

We generalize Brattka's approach and consider the following "global separable Hahn-Banach multivalued function" **HB**: **HB** takes as input a separable Banach space X, a closed linear subspace $A \subseteq X$, and a bounded linear functional $f : A \to \mathbb{R}$ of norm 1 and gives as output the bounded linear functionals $g : X \to \mathbb{R}$ which extend f and are such that $\|g\| = 1$.

Reverse mathematics suggests a plausible representative for the degree of incomputability of **HB**. To see this, recall that reverse mathematics singled out five subsystems of second-order arithmetic: in order of increasing strength these are RCA_0, WKL_0, ACA_0, ATR_0 and $\mathbf{\Pi}^1_1$-CA_0. Most theorems of ordinary mathematics are either provable in the weak base system RCA_0 or are equivalent, over RCA_0, to one of the other systems. Computable functions naturally correspond to RCA_0 and it is easy to see that C_1 (and indeed any C_k with $k > 0$) corresponds to ACA_0 (the correspondence between a system and a function will be made precise in Section 5 below). Brown and Simpson [6] showed that, over the base theory RCA_0, the Hahn-Banach Theorem for separable Banach spaces is equivalent to WKL_0. Thus to define a representative for the incomputability degree of **HB** we could look for a function in computable analysis corresponding to WKL_0.

We consider the multivalued function **Sep** $: \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ defined on $\{ (p, q) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \mid \forall n \, \forall m \; p(n) \neq q(m) \}$ by

$$\mathbf{Sep}(p, q) = \{ r \in 2^{\mathbb{N}} \mid \forall n (r(p(n)) = 0 \wedge r(q(n)) = 1) \}.$$

In other words, the domain of **Sep** is the collection of pairs of functions from the natural numbers into themselves (i.e., of elements of Baire space) with disjoint ranges, and, for any such pair (p, q), $\mathbf{Sep}(p, q)$ is the set of the characteristic functions of sets of natural numbers (i.e., elements of Cantor space) separating the range of p and the range of q. Thus **Sep** corresponds to a statement (strictly connected with $\mathbf{\Sigma}^0_1$-separation) which is well known to be equivalent to WKL_0 (see [19, Lemma IV.4.4]). **Sep** is not computable and, using our definition of computable reducibility between multivalued functions, we obtain, as expected, $\mathbf{Sep} <_c C_1$. We also show that $\mathbf{Sep} \cong_c \mathbf{Path}_2$, where $\mathbf{Path}_2$ is the multivalued function associating to an infinite subtree of $2^{<\mathbb{N}}$ the set of its infinite paths. Moreover, we prove that $\mathbf{Sep} \cong_c \mathbf{HB}$, establishing the degree of incomputability of the separable Hahn-Banach Theorem.

The "reversal" in Brown and Simpson's result (i.e., the proof that the separable Hahn-Banach Theorem implies WKL_0) is based on a construction due to Bishop, Metakides, Nerode, and Shore [14] and appears also in [19, Theorem IV.9.4]. We exploit the ideas of this proof to show $\mathbf{Sep} \leqslant_c \mathbf{HB}$. The original proof by Brown and Simpson of the "forward direction" (showing that WKL_0 proves the separable Hahn-Banach Theorem) has been simplified first by Shioji and Tanaka ([17], this is essentially the proof contained in [19, §IV.9]) and then by Humphreys and Simpson [11]. No details of these or other proofs of the Hahn-Banach Theorem are needed for showing $\mathbf{HB} \leqslant_c \mathbf{Sep}$. Brattka noticed the possibility of avoiding these details in [2] and wrote, "Surprisingly, the proof of this theorem does not require a constructivization of the classical proof but just an 'external analysis'." We explain this fact by observing that the computable analyst is allowed to conduct an unbounded search for an object that is guaranteed to exist by (nonconstructive) mathematical knowledge, whereas the reverse mathematician has the burden of an existence proof with limited means. We give another instance of this phenomenon in Example 5.9 below.

Of course, each of the mathematical objects mentioned above needs some "coding" (in reverse mathematics jargon) or "representation" (using computable analysis terminology). In this respect the computable analysis and the reverse mathematics traditions have developed slightly different approaches to separable Banach spaces.

The plan of the paper is as follows. Sections 2 and 3 are brief introductions to computable analysis and reverse mathematics, respectively. The reader with some basic knowledge in one of these fields can safely skip the corresponding section and refer back to it when needed. Section 4 deals with multivalued functions and computable reductions among them. In Section 5 we compare reverse mathematics and computable analysis. We show the similarities of the two approaches but also note that results cannot be translated automatically in either direction. The multivalued function **Sep** is studied in Section 6. Section 7 sets up the study of Banach spaces in computable analysis, while Section 8 contains the proof of $\mathbf{HB} \cong_c \mathbf{Sep}$.

1.1 Notation for sequences We finish this introduction by establishing our notation for finite and infinite sequences of natural numbers. Let $\mathbb{N}^{<\mathbb{N}}$, respectively, $\mathbb{N}^{\mathbb{N}}$, be the sets of all finite, respectively, infinite, sequences of natural numbers. When $s \in \mathbb{N}^{<\mathbb{N}}$ we use $|s|$ to denote its length and, for $i < |s|$, $s(i)$ to denote the $(i + 1)$th element in the sequence. Similarly, $p(i)$ is defined for every i when $p \in \mathbb{N}^{\mathbb{N}}$. Let $\mathbb{N}^n$ be the set of all $s \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = n$. We use λ to denote the empty sequence, that is, the only element of $\mathbb{N}^0$, and $\bar{0}$ to denote the infinite sequence which always takes value 0. When $s, t \in \mathbb{N}^{<\mathbb{N}}$ we write $s \sqsubseteq t$ to mean that s is an initial segment of t. $s^\frown t$ is the sequence obtained by concatenating t after s, and when $k \in \mathbb{N}$, $s * k$ abbreviates $s^\frown \langle k \rangle$, and $k * s$ abbreviates $\langle k \rangle {}^\frown s$. When $p \in \mathbb{N}^{\mathbb{N}}$ we write also $s^\frown p$ and $k * p$, which are the obvious elements of $\mathbb{N}^{\mathbb{N}}$. If $p \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we write $p[n]$ for the sequence $\langle p(0), p(1), \dots, p(n-1) \rangle \in \mathbb{N}^n$. If $p, q \in \mathbb{N}^{\mathbb{N}}$, we let $p \oplus q \in \mathbb{N}^{\mathbb{N}}$ be such that $(p \oplus q)(2i) = p(i)$ and $(p \oplus q)(2i+1) = q(i)$.

We define $2^{\mathbb{N}}$, $2^{<\mathbb{N}}$, and 2^n as the subsets of $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}$, and $\mathbb{N}^n$ whose elements take values in $\{0, 1\}$. We fix a bijection between $\mathbb{N}^{<\mathbb{N}}$ and $\mathbb{N}$ and, as usual in the literature, we identify an element of $\mathbb{N}^{<\mathbb{N}}$ with the corresponding natural number. We assume that the maps $s \mapsto |s|$, $(s, i) \mapsto s(i)$, $k \mapsto \langle k \rangle$, and $(s, t) \mapsto s^\frown t$ are all computable. Of course, $\mathbb{N}^{\mathbb{N}}$ has a natural topology, namely, the product topology starting from the discrete topology on $\mathbb{N}$. When we view $\mathbb{N}^{\mathbb{N}}$ as a topological space, we call it the *Baire space*. Similarly, $2^{\mathbb{N}}$ with the relative topology is the *Cantor space*.

2 Computable Analysis

2.1 TTE computability In contrast with the case of natural numbers, several nonequivalent approaches to computability theory for the reals have been proposed in the literature. We work in the framework of the so-called Type-2 Theory of Effectivity (TTE), which finds a systematic foundation in [21]. TTE extends the ordinary notion of Turing computability to second countable T_0-topological spaces, and therefore deals with computability over the reals as a particular case within a more general theory.

The basic idea of TTE is that concrete computing machines do not manipulate directly abstract mathematical objects, but they perform computations on sequences of digits which are codings for such objects. In general, mathematical objects require an infinite amount of information to be completely described, and it is therefore natural to extend the ordinary theory of computation to infinite sequences. This extension does not compromise the concreteness of the model, since computations on infinite sequences have a very natural translation in terms of ordinary Turing computations on finite sequences (see [21, Lemma 2.1.11]). The most important feature that differentiates TTE Turing machines from ordinary Turing machines is the fact that they are conceived to work on infinite strings of 0s and 1s, and they do that according to the following specifications. TTE Turing machines have one input tape, one working tape, and one output tape. Each tape is equipped with a head. All ordinary instructions for Turing machines are allowed for the working tape, while the head of the input tape can only read and move rightward, and the head of the output tape can only write and move rightward. These limitations (in particular, those for the output tape) imply the impossibility of correcting the output; once a digit is written, it cannot be canceled or changed. Hence at each stage of the computation the partial output is reliable (this is the most we can ask, since in finitely many steps we never obtain a complete output).

It is straightforward to enumerate all TTE Turing machines and let M_k be the kth such machine. Let $\xi_k : \subseteq\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the partial function computed by M_k as follows. Given $p \in \mathbb{N}^{\mathbb{N}}$, let p' consist of $p(0)$ 1s followed by a single 0, $p(1)$ 1s, and so on; write p' on the input tape and start M_k. If the computation is infinite and the output tape eventually contains an infinite sequence of 0s and 1s with infinitely many 0s, we translate back to an element of $\mathbb{N}^{\mathbb{N}}$ which is $\xi_k(p)$; otherwise, $p \notin \operatorname{dom}(\xi_k)$. Notice that $\operatorname{dom}(\xi_k)$ is a G_δ subset of $\mathbb{N}^{\mathbb{N}}$ for every k.

Definition 2.1 (Computable functions on $\mathbb{N}^{\mathbb{N}}$) We say that a function $F : \subseteq\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is *computable* if there exists k such that $\operatorname{dom}(F) \subseteq \operatorname{dom}(\xi_k)$ and $F(p) = \xi_k(p)$ for every $p \in \operatorname{dom}(F)$.

As noticed by Weihrauch ([21, p. 38]), TTE Turing machines can be viewed as ordinary oracle Turing machines; the oracle supplies the information about the input and the nth bit of the output is computed when we give n as input to the oracle Turing machine. Therefore, the computable partial functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ coincide with the computable (or recursive) functionals (beware that in some literature "functional" means function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$, rather than function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ as here) of classical computability (or recursion) theory, also known as Lachlan functionals.

The restrictions on the instructions allowed in TTE Turing machines imply the following fact ([21, Theorem 2.2.3]).

Lemma 2.2 *Every computable function $F : \subseteq\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous.*

We transfer the notion of computability for the Baire space to spaces with cardinality less than or equal to the continuum using the notion of representation.

Definition 2.3 (Representations and represented spaces) A *representation* σ_X of a set X is a surjective function $\sigma_X : \subseteq\mathbb{N}^{\mathbb{N}} \to X$. The pair (X, σ_X) is a *represented space*. If $x \in X$, a *σ_X-name* for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$. We say that x is *σ_X-computable* when it has a computable σ_X-name p (i.e., $graph(p)$ is a computable set).

2.2 Effective metric spaces The definition of representation is too general for practical purposes as it allows an object in X to be coded by arbitrary sequences. However, there are important cases in which we can find meaningful representations, for example, when X is a separable metric space.

Definition 2.4 (Effective metric space) An *effective metric space* is a triple (X, d, a) where

(i) (X, d) is a separable metric space;
(ii) $a : \mathbb{N} \to X$ is a dense sequence in X.

If there is no danger of confusion, we often write X in place of (X, d, a).

We equip every effective metric space (X, d, a) with the *Cauchy representation* $\delta_X : \subseteq\mathbb{N}^{\mathbb{N}} \to X$ such that $p \in \operatorname{dom}(\delta_X)$ if and only if for all i and all $j \geq i$, $d(a(p(i)), a(p(j))) \leq 2^{-i}$, and $\delta_X(p) = x$ if and only if $\lim a(p(n)) = x$. In other words, $p \in \mathbb{N}^{\mathbb{N}}$ is a name for x when p encodes a Cauchy sequence of elements in the fixed dense subset of X which *converges effectively* to x.

A *rational open ball* in (X, d, a) is an open ball of the form $B^X(c; \alpha) = \{x \in X \mid d(x, c) < \alpha\}$ with $c \in \operatorname{ran}(a)$, and $\alpha \in \mathbb{Q}^+ \cup \{0\}$.

In particular, we have the effective metric space $(\mathbb{R}, d, a_{\mathbb{Q}})$, where $d(x, y) = |x - y|$ and $a_{\mathbb{Q}}$ is a standard computable enumeration of the set of the rational numbers (it is convenient to assume $a_{\mathbb{Q}}(0) = 0$ and $a_{\mathbb{Q}}(1) = 1$). The notion of effective metric space can be generalized.

Definition 2.5 (Effective topological space) An *effective topological space* is a triple (X, τ, u), where τ is a second countable T_0-topology on X and $u : \mathbb{N} \to \mathcal{P}(X)$ is an enumeration of a subbase of τ. Each effective topological space (X, τ, u) has a *standard representation* δ_X such that $\delta_X(p) = x \in X$ if and only if $\operatorname{ran}(p) = \{ n \mid x \in u(n) \}$.

It is immediate that effective metric spaces are particular examples of effective T_0-topological spaces. In fact, if (X, d, a) is an effective metric space we let u enumerate the rational open balls of X. We will always assume that there exist computable functions c and r such that $u(n)$ has center $a(c(n))$ and radius $a_{\mathbb{Q}}(r(n))$. In this context we usually write B_n^X in place of $u(n)$.

The Cauchy representation of an effective metric space X is equivalent to the representation of X considered as an effective topological space. This equivalence means that each representation is reducible to the other, where a representation δ of a set X is *reducible* to a representation σ of the same set when there is a continuous function $F : \subseteq\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\delta(p) = \sigma(F(p))$ for all $p \in \operatorname{dom}(\delta)$. A representation of X which is equivalent to the standard representation is said to be *admissible* for X.

Definition 2.6 (Realizers) Given represented spaces (X, σ_X) and (Y, σ_Y) and a partial function $f : \subseteq X \to Y$, we say that $F : \subseteq\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a (σ_X, σ_Y)*-realizer* of f when $f(\sigma_X(p)) = \sigma_Y(F(p))$, for all $p \in \operatorname{dom}(f \circ \sigma_X)$. The function f is said to be (σ_X, σ_Y)*-computable* if it has a computable (σ_X, σ_Y)-realizer. In practice we often omit explicit mention of the representations and write just computable.

Using the notion of realizer we thus extend the notion of computable from the Baire space to the effective topological spaces. This extension is particularly successful when we use admissible representations, as the following results (due to Kreitz and Weihrauch) show.

Theorem 2.7 *Let X and Y be effective topological spaces with admissible representations σ_X and σ_Y. A function $f : \subseteq X \to Y$ is continuous if and only if it has a continuous (σ_X, σ_Y)-realizer.*

Corollary 2.8 *Let X and Y be effective topological spaces with admissible representations σ_X and σ_Y. Then every function $f : \subseteq X \to Y$ which is (σ_X, σ_Y)-computable is continuous.*

Corollary 2.8 is an extension of Lemma 2.2. We point out that Theorem 2.7 and Corollary 2.8 hold in particular for effective metric spaces and Cauchy representations.

The notions of effective metric and effective topological spaces in their complete generality have no computational content. In fact, notwithstanding the established terminology [21], we are not requiring any "effectivity" property (even the computable enumeration of the rational open balls of an effective metric space is nothing but an enumeration of pairs of natural numbers). In the case of effective metric spaces, the natural "effective" requirement is the computability of the distance between points.

Definition 2.9 (Computable metric space) A *computable metric space* is an effective metric space (X, d, a) such that the function $(n, m) \mapsto d(a(n), a(m))$ is computable.

If X is a computable metric space it is straightforward that the distance function is $((\delta_X, \delta_X), \delta_{\mathbb{R}})$-computable. Typical examples of computable metric spaces are $\mathbb{R}$ and the Baire space (recall that for $p, q \in \mathbb{N}^{\mathbb{N}}$ such that $p \neq q$ we let $d(p, q) = 2^{-i}$ for the least i such that $p(i) \neq q(i)$). In the case of effective T_0-topological spaces, the "effective" requirement is the computability of the operation of intersection of open sets (see [9]).

2.3 Representations of continuous functions Notice that the set of all continuous partial functions on the Baire space is too large to have a representation. However, every partial continuous function $f : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ has a continuous extension to a G_δ set ([21, Theorem 2.3.8]; this is an instance of a classical result due to Kuratowski; see, e.g., [12, Theorem 3.8]). Thus it suffices to represent

$$\mathrm{Cont} = \{ F : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \mid F \text{ is continuous and } \mathrm{dom}(F) \text{ is } G_\delta \}.$$

Lemma 2.2 implies that each computable $F : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ has an extension in Cont. Define $\eta : \mathbb{N}^{\mathbb{N}} \to \mathrm{Cont}$ by $\eta(k * p)(q) = \xi_k(p \oplus q)$, for $k \in \mathbb{N}$ and $p, q \in \mathbb{N}^{\mathbb{N}}$. η is a representation of Cont.

Given effective metric spaces X and Y, we define a representation $[\delta_X \to \delta_Y]$ of the set $\mathcal{C}(X, Y)$ of total continuous functions from X into Y by $[\delta_X \to \delta_Y](p) = f$ if and only if $\eta(p)$ is a (δ_X, δ_Y)-realizer of f. This representation satisfies the following fundamental properties:

Evaluation the map $(f, x) \mapsto f(x)$ is $(([\delta_X \to \delta_Y], \delta_X), \delta_Y)$-computable;

Type conversion let (Z, σ_Z) be a represented space; every function $g : Z \times X \to Y$ is $((\sigma_Z, \delta_X), \delta_Y)$-computable if and only if $\hat{g} : Z \to \mathcal{C}(X, Y)$, defined by $\hat{g}(z)(x) = g(z, x)$, is $(\sigma_Z, [\delta_X \to \delta_Y])$-computable.

The evaluation and type conversion properties witness the reliability of the simulation of continuous functions on separable metric spaces via realizers.

2.4 Borel complexity Computable analysis provides a method to classify incomputable functions between separable metric spaces in complexity hierarchies, analogously to the classification of functions from $\mathbb{N}$ to $\mathbb{N}$ pursued in classical computability theory. In particular, [1] studied the following functions of strictly increasing complexity.

Definition 2.10 (The C_ks) For every $k \in \mathbb{N}$, let $C_k : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be defined by

$$C_k(p)(n) = \begin{cases} 0 & \text{if } \exists n_k\, \forall n_{k-1}\, \exists n_{k-2} \ldots Q n_1\; p(\langle n, n_k, n_{k-1}, \ldots, n_1 \rangle) \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

where Q is $\exists$ when k is odd and $\forall$ when k is even.

Using natural representations for Borel sets of each given finite level, Brattka [1] says that a function $f : \subseteq X \to Y$, for X and Y computable metric spaces, is $\boldsymbol{\Sigma}^0_k$-*computable* (for $k \geq 1$) if there exists a computable function that maps every name of an open set $U \subseteq Y$ to a name of a $\boldsymbol{\Sigma}^0_k$ set $V \subseteq X$ such that $f^{-1}(U) = V \cap \mathrm{dom}(f)$. It follows immediately that every $\boldsymbol{\Sigma}^0_k$-computable function is $\boldsymbol{\Sigma}^0_k$-measurable (equivalently, of Baire class $k - 1$). Brattka shows that f is $\boldsymbol{\Sigma}^0_{k+1}$-computable if and only if f is computably reducible to C_k. (We refer the reader to Section 4 below for the definition of reducibility.)

3 Reverse Mathematics

In the 1970s Friedman started the research program of reverse mathematics, which was pursued in the two next decades by Simpson and his students and increasingly by other researchers. Nowadays reverse mathematics is an important area of mathematical logic, crossing the boundary between computability theory and proof theory but employing ideas and techniques also from model theory and set theory. We refer the reader to [19] for details about the topics we will sketch in this section (the collection [18] documents more recent advances).

Reverse mathematics searches in a systematic way for equivalences between different statements with respect to some base theory (which does not prove any of them) in the context of subsystems of second-order arithmetic. Recall that the language $\mathcal{L}_2$ of second-order arithmetic has variables for natural numbers and variables for sets of natural numbers, constant symbols 0 and 1, binary function symbols for addition and product of natural numbers, symbols for equality and the order relation on the natural numbers and for membership between a natural number and a set. Second-order arithmetic is the $\mathcal{L}_2$-theory with classical logic consisting of the axioms stating that $(\mathbb{N}, 0, 1, +, \cdot, <)$ is a commutative ordered semiring with identity, the induction scheme for arbitrary formulas, and the comprehension scheme for sets of natural numbers defined by arbitrary formulas. Weyl and Hilbert and Bernays already noticed that $\mathcal{L}_2$ was rich enough to express, using appropriate codings, significant parts of mathematical practice, and that many mathematical theorems were provable in (fragments of) second-order arithmetic.

Formulas of $\mathcal{L}_2$ are classified in the usual hierarchies: those with no set quantifiers and only bounded number quantifiers are $\mathbf{\Delta}^0_0$, while counting the number of alternating unbounded number quantifiers we obtain the classification of all arithmetical (= without set quantifiers) formulas as $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ formulas (one uses $\mathbf{\Sigma}$ or $\mathbf{\Pi}$ depending on the type of the first quantifier in the formula, existential in the former, universal in the latter). Formulas with set quantifiers in front of an arithmetical formula are classified by counting their alternations as $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$. A formula is $\mathbf{\Delta}^i_n$ a certain theory if it is equivalent in that theory both to a $\mathbf{\Sigma}^i_n$ formula and to a $\mathbf{\Pi}^i_n$ formula.

Reverse mathematics starts with the fairly weak base theory RCA_0, where the induction scheme and the comprehension scheme are restricted, respectively, to $\mathbf{\Sigma}^0_1$ and $\mathbf{\Delta}^0_1$ formulas. RCA_0 is strong enough to prove some basic results about many mathematical structures but too weak for many others.

If a theorem T is expressible in $\mathcal{L}_2$ but unprovable in RCA_0, reverse mathematics asks the question, What is the weakest axiom we can add to RCA_0 to obtain a theory that proves T? In principle, we could expect that this question has a different answer for each T. The "discovery" of reverse mathematics is that this is not the case. In fact, most theorems of ordinary mathematics expressible in $\mathcal{L}_2$ are either provable in RCA_0 or equivalent over RCA_0 to one of the following four subsystems of second-order arithmetic, listed in order of increasing strength: WKL_0, ACA_0, ATR_0, and $\mathbf{\Pi}^1_1$-CA_0. This leads to a neat picture where theorems belonging to quite different areas of mathematics are classified in five levels, roughly corresponding to the mathematical principles used in their proofs. RCA_0 corresponds to "computable mathematics," WKL_0 embodies a compactness principle, ACA_0 is linked to sequential compactness, ATR_0 allows for transfinite arguments, $\mathbf{\Pi}^1_1$-CA_0 includes impredicative principles.

In this paper we will refer extensively to WKL_0 and, in passing, to ACA_0. Therefore, we describe these two theories in a little more detail. ACA_0 is obtained from RCA_0 by extending the comprehension scheme to all arithmetical formulas. The statements without set variables provable in ACA_0 coincide exactly with the theorems of Peano arithmetic so that, in particular, the consistency strength of the two theories is the same. Within ACA_0 one can develop a fairly extensive theory of continuous functions, using the completeness of the real line as an important tool. ACA_0 proves (and often turns out to be equivalent to) also many basic theorems about countable fields, rings, and vector spaces.

To obtain WKL_0 we add to RCA_0 the statement of Weak König's Lemma; that is, every infinite binary tree has a path, which is essentially the compactness of Cantor space. An equivalent statement, clearly showing that WKL_0 is stronger than RCA_0 and weaker than ACA_0, is $\mathbf{\Sigma}^0_1$-separation: if $\varphi(n)$ and $\psi(n)$ are $\mathbf{\Sigma}^0_1$-formulas such that $\forall n \neg(\varphi(n) \wedge \psi(n))$, there exists a set X such that $\varphi(n) \implies n \in X$ and $\psi(n) \implies n \notin X$ for all n. WKL_0 and RCA_0 have the same consistency strength of Primitive Recursive Arithmetic and are thus proof-theoretically fairly weak. Nevertheless, WKL_0 proves (and often turns out to be equivalent to) a substantial amount of classical mathematical theorems, including many results about real-valued functions, basic Banach space facts, and so on. For example, WKL_0 is equivalent, over RCA_0, to the Peano-Cauchy existence theorem for solutions of ordinary differential equations.

4 Multivalued Functions in Computable Analysis

The main goal of this section is to give the definition of reducibility of multivalued functions. Since we will often compose multivalued functions, we spell out Weihrauch's definition for this operation.

Definition 4.1 (Composition of multivalued functions) Given two (partial) multivalued functions $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Y \rightrightarrows Z$, the *composition* $g \circ f :\subseteq X \rightrightarrows Z$ is the multivalued function defined by

(i) $\mathrm{dom}(g \circ f) = \{ x \in \mathrm{dom}(f) \mid f(x) \subseteq \mathrm{dom}(g) \}$;
(ii) $\forall x \in \mathrm{dom}(g \circ f)\, (g \circ f)(x) = \bigcup_{y \in f(x)} g(y)$.

To define the notion of computable multivalued function we look at realizers.

Definition 4.2 (Realizers of multivalued functions) Let (X, σ_X) and (Y, σ_Y) be represented spaces and $f :\subseteq X \rightrightarrows Y$. A (σ_X, σ_Y)-*realizer* for f is a (single-valued) function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\sigma_Y(F(p)) \in f(\sigma_X(p))$ for every $p \in \mathrm{dom}(f \circ \sigma_X)$.

Notice that in Definition 4.2 we do not require that $\sigma_X(p) = \sigma_X(p')$ implies $\sigma_Y(F(p)) = \sigma_Y(F(p'))$. In other words, a realizer does not, in general, lift to a single-valued selector for the multivalued function.

Definition 4.3 (Computability of multivalued functions) Let (X, σ_X) and (Y, σ_Y) be represented spaces. A multivalued function $f :\subseteq X \rightrightarrows Y$ is (σ_X, σ_Y)-*computable* if it has a computable (σ_X, σ_Y)-realizer. In practice we often omit explicit mention of the representations and write just computable.

Our definition of computable multivalued function agrees with [21, Definition 3.1.3.4] and [1, p. 21]. Notice, however, that Brattka's paper includes also the definition of $\mathbf{\Sigma}^0_1$-computable multivalued function; for single-valued functions the notions of computable and $\mathbf{\Sigma}^0_1$-computable coincide, but for arbitrary multivalued functions the latter is stronger.

4.1 Reducibility of multivalued functions We now define the notion of computable reducibility for multivalued functions. The intuitive idea is that one problem is reducible to another, provided that whenever we have a method to compute a solution for the second problem, we can uniformly find a way to compute a solution for the first one. This generalizes the notion of reducibility between single-valued functions investigated in [1] and extensively used in recent work in computable analysis. Actually, in [1] there are two distinct notions, introduced in Definitions 5.1 and 7.1, of computable reducibility between single-valued functions. Our definition generalizes the former, and Lemma 4.5 below shows that the generalization of the latter (realizer reducibility) leads to an equivalent concept.[1] Thus the notion of computable reducibility appears to be more robust in the multivalued setting.

Definition 4.4 (Reducibility of multivalued functions) Let (X, σ_X), (Y, σ_Y), (Z, σ_Z), (W, σ_W) be represented spaces. Let $f : \subseteq X \rightrightarrows Y$ and $g : \subseteq Z \rightrightarrows W$ be multivalued functions. We say that f is *computably reducible* to g and write $f \leqslant_c g$ if there exist computable multivalued functions $h : \subseteq X \rightrightarrows Z$ and $k : \subseteq X \times W \rightrightarrows Y$ such that $k(x, (g \circ h)(x)) \subseteq f(x)$ for all $x \in \mathrm{dom}(f)$.

Notice that when f and g are single-valued, k is single-valued on $\{ (x, (g \circ h)(x)) \mid x \in \mathrm{dom}(f) \}$, but it may be the case that h is not single-valued. Therefore, the restriction of our notion of computable reducibility to single-valued functions is weaker than Brattka's notion of computable reducibility for single-valued functions. However, when dealing with multivalued functions it is natural to allow h and k to be multivalued as well. As we have pointed out, the following lemma gives further support to our definition by showing that it coincides with the natural generalization of Brattka's notion of realizer reducibility.

Lemma 4.5 *Let (X, σ_X), (Y, σ_Y), (Z, σ_Z), (W, σ_W) be represented spaces. Let $f : \subseteq X \rightrightarrows Y$ and $g : \subseteq Z \rightrightarrows W$ be multivalued functions. The following are equivalent:*

(i) $f \leqslant_c g$;

(ii) there exist computable functions $H : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $K : \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $p \mapsto K(p, (G \circ H)(p))$ is a realizer for f whenever G is a realizer for g.

Proof First assume $f \leqslant_c g$ and let the computable multivalued functions h and k witness this. Let H and K, respectively, be computable realizers for h and k. Suppose G is a realizer for g; we claim that $p \mapsto K(p, (G \circ H)(p))$ is a realizer for f. In fact, if $p \in \mathrm{dom}(f \circ \sigma_X)$, then $(\sigma_Z \circ H)(p) \in (h \circ \sigma_X)(p)$, and hence $(\sigma_W \circ G \circ H)(p) \in (g \circ h \circ \sigma_X)(p)$ so that $(\sigma_Y \circ K)(p, (G \circ H)(p)) \in k(\sigma_X(p), (g \circ h \circ \sigma_X)(p)) \subseteq (f \circ \sigma_X)(p)$.

Now suppose (ii) holds and let H and K witness this. Define h and k by

$$h(x) = \{ (\sigma_Z \circ H)(p) \mid \sigma_X(p) = x \} \quad \text{and}$$
$$k(x, w) = \{ (\sigma_Y \circ K)(p, p') \mid \sigma_X(p) = x \wedge \sigma_W(p') = w \}.$$

Since H and K are computable realizers for h and k, respectively, the latter are computable multivalued functions.

To check that h and k witness $f \leqslant_c g$, let $x \in \mathrm{dom}(f)$ and suppose $y \in k(x, (g \circ h)(x))$. There exist $z \in h(x)$ and $w \in g(z)$ such that $y \in k(x, w)$. By definition of k, let p, p' be such that $\sigma_X(p) = x$, $\sigma_W(p') = w$, and $y = (\sigma_Y \circ K)(p, p')$. Let G be a realizer for g such that $p' = (G \circ H)(p)$. Then

$$y = (\sigma_Y \circ K)(p, p') = (\sigma_Y \circ K)(p, (G \circ H)(p)) \in (f \circ \sigma_X)(p) = f(x),$$

where membership follows from the fact that $p \mapsto K(p, (G \circ H)(p))$ is a realizer for f. We have thus shown $k(x, (g \circ h)(x)) \subseteq f(x)$, as needed. □

Since transitivity of $\leqslant_c$ for multivalued functions is not immediately obvious, we state it explicitly.

Lemma 4.6 *$\leqslant_c$ is transitive.*

Proof Let $f : \subseteq X \rightrightarrows Y$, $g : \subseteq Z \rightrightarrows W$, and $\ell : \subseteq U \rightrightarrows V$ be multivalued functions. Let h and k witness $f \leqslant_c g$, while h' and k' witness $g \leqslant_c \ell$. It is easy to check that $h' \circ h$ and the map $(x, v) \mapsto k(x, k'(h(x), v))$ witness $f \leqslant_c \ell$. □

Thus $\leqslant_c$ is a preorder (reflexivity is obvious) and we can give the usual definitions.

Definition 4.7 As usual we use $<_c$ and $\cong_c$ for the strict relation and the equivalence relation arising from $\leqslant_c$.

We now prove two simple lemmas about $\leqslant_c$.

Lemma 4.8 *Let $f, g : \subseteq X \rightrightarrows Y$ be multivalued functions such that* $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ *and $g(x) \subseteq f(x)$ for every $x \in$* $\mathrm{dom}(f)$. *Then $f \leqslant_c g$.*

Proof It is straightforward to check that the identity on X and projection on the second coordinate from $X \times Y$ witness this. □

Lemma 4.9 *Let $h : \subseteq X \rightrightarrows Y$ and $g : \subseteq Z \rightrightarrows W$ be computable multivalued functions. For any multivalued function $f : \subseteq Y \rightrightarrows Z$, we have $(g \circ f \circ h) \leqslant_c f$.*

Proof It is straightforward to check that h and $(x, z) \mapsto g(z)$ witness this. □

Our definition of $\mathbf{\Sigma}^0_k$-computability for multivalued functions is motivated by the characterization of this notion for singled-valued functions of Theorems 5.5 and 7.6 (one for each notion of reducibility) in [1]. The reader should, however, be aware that Brattka defined a notion of $\mathbf{\Sigma}^0_k$-computability for multivalued functions which is properly stronger than ours [1, Definition 3.5].

Definition 4.10 ($\mathbf{\Sigma}^0_k$-computable and $\mathbf{\Sigma}^0_k$-complete) Let $k \geq 1$ and (X, σ_X), (Y, σ_Y) be represented spaces. A multivalued function $f : \subseteq X \rightrightarrows Y$ is *$\mathbf{\Sigma}^0_k$-computable* if $f \leqslant_c C_{k-1}$ and *$\mathbf{\Sigma}^0_k$-complete* if $f \cong_c C_{k-1}$.

Lemma 4.5 above and Theorem 7.6 in [1] imply that a multivalued function is $\mathbf{\Sigma}^0_k$-computable if and only if it has a $\mathbf{\Sigma}^0_k$-computable realizer.

5 Reverse Mathematics and Computable Analysis

5.1 Statements of second-order arithmetic and functions Many mathematical statements expressed in $\mathcal{L}_2$ have the form

$$\forall X(\psi(X) \implies \exists Y\, \varphi(X, Y))$$

where X and Y range over sets of natural numbers. Here are a few examples (we use the standard coding techniques for expressing in RCA_0 functions, real numbers, sequences, and so on).

(1) The statement of Weak König's Lemma (the main axiom of WKL_0) is

$$\forall T\,(T \text{ is an infinite binary tree } \implies \exists p\; p \text{ is an infinite path in } T).$$

(2) The existence of the range of any function is

$$\forall p(p : \mathbb{N} \to \mathbb{N} \implies \exists Y\, \forall m(m \in Y \iff \exists n\, m = p(n))).$$

(3) The existence of the least upper bound for any sequence in $I = [0, 1]$ is

$$\forall \langle x_n : n \in \mathbb{N}\rangle\, (\forall n\, x_n \in I \implies \exists x(\forall n\, x_n \leq x \wedge \forall k\, \exists n\, x < x_n + 2^{-k})).$$

(4) Separation of disjoint ranges is

$$\forall p, q(p, q : \mathbb{N} \to \mathbb{N} \wedge \forall n, m\; p(n) \neq q(m) \implies \exists Y\, \forall n(p(n) \in Y \wedge q(n) \notin Y)).$$

(5) The statement of the Heine-Borel compactness of the interval I is

$$\forall \langle I_k : k \in \mathbb{N}\rangle\, (\forall k\, I_k \subseteq I \text{ is an interval with rational endpoints } \wedge \\ I = \bigcup_{k \in \mathbb{N}} I_k \implies \exists n\, I = \bigcup_{k<n} I_k).$$

(6) The statement of the separable Hahn-Banach Theorem is

$$\forall X, A, f(X \text{ is a separable Banach space } \wedge A \text{ is a closed linear subspace of } X \wedge \\ f \text{ is a bounded linear functional on } A \implies \\ \exists g(g \text{ is a bounded linear functional on } X \text{ extending } f \wedge \|g\| = \|f\|)).$$

If $\forall X(\psi(X) \implies \exists! Y\, \varphi(X, Y))$ holds (this is the case in (2) and (3) above) it is natural to consider the partial function $f : \subseteq\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ with $\mathrm{dom}(f) = \{X \in \mathcal{P}(\mathbb{N}) \mid \psi(X)\}$ such that $\varphi(X, f(X))$ for every $X \in \mathrm{dom}(f)$. When the uniqueness condition fails we could consider all possible functions with the properties above. However, it seems more useful to study the multivalued function $f : \subseteq\mathcal{P}(\mathbb{N}) \rightrightarrows \mathcal{P}(\mathbb{N})$ defined by $f(X) = \{Y \mid \varphi(X, Y)\}$ for all X such that $\psi(X)$.

Remark 5.1 In many cases, including some of the examples given above, it is best to view the domain and the range of f as represented spaces different from $\mathcal{P}(\mathbb{N})$, thus unraveling the coding used in the reverse mathematics approach. For example, the functions arising from examples (1) and (3) are best viewed, respectively, as a partial multivalued function from $\mathcal{P}(2^{<\mathbb{N}})$ to $2^{\mathbb{N}}$ and a total single-valued function from $I^{\mathbb{N}}$ to I.

We have thus associated to the mathematical statement expressed in $\mathcal{L}_2$ a function between represented spaces which can be studied within the framework of computable analysis. Notice that the lack of restrictions on the complexity of ψ corresponds to the principle of computable analysis stating that "the user is responsible for the correctness of the input" (see [8, §6] for a discussion).

We can also reverse the procedure. If we want to study from the viewpoint of computable analysis a multivalued function $f : \subseteq X \rightrightarrows Y$, we can look at the reverse mathematics of the statement,

$$\forall x(x \in \mathrm{dom}(f) \implies \exists y \in Y\ y \in f(x)),$$

with the hope of gaining some useful insight. For example, if $k \geq 1$, from C_k we obtain the statement,

$$\forall p(p \in \mathbb{N}^{\mathbb{N}} \implies \exists q \in 2^{\mathbb{N}}\, \forall n \\ (q(n) = 0 \iff \exists n_k\, \forall n_{k-1}\, \exists n_{k-2} \ldots Q n_1\, p(\langle n, n_k, n_{k-1}, \ldots, n_1 \rangle) \neq 0)),$$

which is easily seen to be equivalent (over RCA_0) to $\mathbf{\Sigma}^0_k$-comprehension. In any case, we expect some connection between the proof-theoretic strength of the statement and the computability strength of the function.

Notice that statements corresponding to functions belonging to different degrees of incomputability may collapse into a single system of reverse mathematics. Indeed, for any $k \geq 1$, the statement obtained above in correspondence with C_k is equivalent to arithmetic comprehension. This means that each C_k with $k \geq 1$ corresponds to ACA_0, while it is well known that $C_k <_c C_{k+1}$. In other words, at the level of ACA_0 computable analysis is finer than reverse mathematics.

The correspondence between proof-theoretic and computable equivalence is more useful when we are at the level of RCA_0 or WKL_0. First, the computable sets are the intended ω-model of RCA_0, which is therefore a formal version of computable mathematics. Hence we expect that a statement provable in RCA_0 gives rise to a computable function. Second, we expect most statements equivalent to WKL_0 to give rise to computably equivalent uncomputable functions.

Sometimes these expectations are fulfilled, and some reverse mathematics proofs even translate naturally into a computable analysis proof. This is the case with Theorems 6.7 and 8.12 below. However, the existence of this translation cannot be taken for granted, and for each direction of the correspondence we will give examples of failures. In other words, no automatic translation from the reverse mathematics literature into computable analysis, or vice versa, is possible. This phenomenon is a consequence of the different methods and goals of the two approaches. On one hand, the subsystems of second-order arithmetic studied in reverse mathematics use freely classical principles with no algorithmic content, such as excluded middle and proofs by contraposition. On the other hand, the algorithms of computable analysis assume the existence of the objects they have to compute, without the need of proving it. The examples of failure of the correspondence below highlight these differences.

5.2 Success of the correspondence An often-used equivalent of ACA_0 is the statement that the range of every one-to-one function from $\mathbb{N}$ to $\mathbb{N}$ is a set. Using the approach described above, this translates into the following function.

Definition 5.2 (Range) Let **Range** : $\subseteq\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the function that maps any one-to-one function to the characteristic function of its range; that is,

$$\mathbf{Range}(p)(n) = \begin{cases} 1, & \text{if } \exists m\ p(m) = n; \\ 0, & \text{otherwise,} \end{cases}$$

for every injective $p : \mathbb{N} \to \mathbb{N}$ and every n.

As expected, we have the following lemma.

Lemma 5.3 $\mathbf{Range} \cong_c C_1$.

Proof First we show $\mathbf{Range} \leqslant_c C_1$. Given $p \in \mathrm{dom}(\mathbf{Range})$, let $H(p) \in 2^{\mathbb{N}}$ be defined by $H(p)(\langle n, m\rangle) = 1$ if and only if $p(m) = n$. $H : \subseteq\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}$ is computable and it is immediate that $\mathbf{Range}(p)(n) = 1 - C_1(H(p))(n)$ for every injective p and n.

We now show that $C_1 \leqslant_c \mathbf{Range}$. Given $p \in \mathbb{N}^{\mathbb{N}}$, let $H(p) \in \mathbb{N}^{\mathbb{N}}$ be defined by

$$H(p)(\langle n, m\rangle) = \begin{cases} \langle n, 0\rangle & \text{if } p(\langle n, m\rangle) \neq 0 \text{ and } \forall k < m\ p(\langle n, k\rangle) = 0; \\ \langle n, m+1\rangle & \text{otherwise.} \end{cases}$$

The function $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is computable with $\mathrm{ran}(H) \subseteq \mathrm{dom}(\mathbf{Range})$ (i.e., each $H(p)$ is one-to-one). Moreover, $C_1(p)(n) = 1 - \mathbf{Range}(H(p))(\langle n, 0\rangle)$. □

A basic example of reverse mathematics deals with the existence of least upper bounds of bounded sequences of real numbers. Indeed, this mathematical principle turns out to be equivalent to ACA_0 [19, Theorem III.2.2]. We now show how this equivalence translates into computable analysis.

Definition 5.4 (Sup) Let **Sup** : $I^{\mathbb{N}} \to I$ be the function that maps any sequence in $I^{\mathbb{N}}$ to its least upper bound.

Theorem 5.5 $C_1 \cong_c \mathbf{Sup}$.

Proof We start by showing that $\mathbf{Sup} \leqslant_c C_1$. Given $(x_n) \in I^{\mathbb{N}}$ observe that it is easy to use C_1 to compute the (characteristic function of the) set $A = \{\alpha \in \mathbb{Q} \mid \exists n\ \alpha < x_n\}$. Now we can computably define a sequence of rationals (α_k), where $\alpha_k = \frac{i}{2^k}$ is such that $\frac{i}{2^k} \in A$ and $\frac{i+1}{2^k} \notin A$. Clearly (α_k) is a Cauchy representation of the real number $\mathbf{Sup}(x_n)$.

By Lemma 5.3, to prove $C_1 \leqslant_c \mathbf{Sup}$, it suffices to show that $\mathbf{Range} \leqslant_c \mathbf{Sup}$. Given $p \in \mathrm{dom}(\mathbf{Range})$, define $(x_m) \in I^{\mathbb{N}}$ by setting $x_m = \sum_{k \leq m} 2^{-(p(k)+1)}$. Given $x = \mathbf{Sup}(x_m)$ we can define $q : \mathbb{N} \to \mathbb{N}$ by letting $q(n)$ be the least k satisfying $x - x_k < 2^{-(n+1)}$. Then for every n we have

$$\exists m\ p(m) = n \iff \exists m \leq q(n)\ p(m) = n,$$

and hence

$$\mathbf{Range}(p)(n) = \begin{cases} 1 & \text{if } \exists m \leq q(n)\ p(m) = n; \\ 0 & \text{otherwise.} \end{cases}$$

This shows that, after using **Sup** to obtain x, we can establish whether $n \in \mathbf{Range}(p)$ by first computing $q(n)$ by search, and then checking finitely many values of m. □

5.3 Failure of the correspondence We now exhibit some examples where the correspondence outlined above fails. We first show that sometimes functions arising from statements provable in RCA_0 are incomputable.

Example 5.6 The following function, known as the *Allwissenheitsprinzip* (Principle of Omniscience), has been studied in detail from the viewpoint of computable analysis [20; 15].

Let $\Omega : \mathbb{N}^{\mathbb{N}} \to \{0, 1\}$ be defined by

$$\Omega(p) = \begin{cases} 0 & \text{if } p = \bar{0}; \\ 1 & \text{otherwise.} \end{cases}$$

The incomputability of Ω follows immediately from Lemma 2.2. On the other hand, the statement corresponding to Ω is

$$\forall p \in \mathbb{N}^{\mathbb{N}} \, \exists i \in \{0, 1\}(i = 0 \iff \forall n \, p(n) = 0),$$

which is obviously provable in RCA_0 (and indeed just from the excluded middle, except for the coding of functions in the language of second-order arithmetic).

We now give another example, which is more mathematical, but again has its roots in the use of classical logic in reverse mathematics.

Example 5.7 Let $\mathcal{A}_-(2^{\mathbb{N}})$ be the hyperspace of closed subsets of $2^{\mathbb{N}}$ represented by negative information (see Definition 7.3 below) and $\mathbf{Sel} : \subseteq \mathcal{A}_-(2^{\mathbb{N}}) \rightrightarrows 2^{\mathbb{N}}$ be the multivalued function which selects a point from nonempty closed subsets of $2^{\mathbb{N}}$. In other words, $\mathbf{Sel}(A) = A$, but on the left-hand side of this equality A is a closed set (and hence a single element in the hyperspace), while on the right-hand side it is a set of points in the space $2^{\mathbb{N}}$.

The statement corresponding to $\mathbf{Sel}$ is $\forall A \in \mathcal{A}_-(2^{\mathbb{N}})(A \neq \varnothing \implies \exists x \, x \in A)$, which is a tautology, since $A \neq \varnothing$ is an abbreviation for $\exists x \, x \in A$, and hence provable in RCA_0. On the other hand, it follows from Theorem 8.3 below that if we represent closed sets with respect to negative information (coherently with the reverse mathematics definition of closed set), $\mathbf{Sel} \cong_c \mathbf{Sep}$ and hence $\mathbf{Sel}$ is incomputable.

Example 5.8 It is well known that the intermediate value theorem is not constructive, and it can be shown that the corresponding multivalued function is not computable (Brattka and Gherardi have forthcoming results about the incomputability strength of this function). On the other hand, a standard proof of the intermediate value theorem which uses the excluded middle can be carried out in RCA_0 [19, Theorem II.6.6].

We now give an example of the opposite phenomena, that is, a theorem which is not provable in RCA_0 but corresponds to a computable function.

Example 5.9 The Heine-Borel compactness of the interval I is Example (5) at the beginning of this section. In reverse mathematics it is well known that this statement is equivalent to WKL_0 [19, Theorem IV.1.2]. On the other hand, in computable analysis it is well known that the function which maps each countable open covering of I consisting of intervals with rational endpoints to a finite subcovering is computable [21]. We sketch the proof to emphasize the difference between the reverse mathematics and the computable analysis approaches in this case.

There exists a computable enumeration $(\mathfrak{C}_n)$ of all finite open coverings of I consisting of intervals with rational endpoints (in RCA_0 we can even prove by using, for example, the ideas of the last part of the proof of Lemma 8.8 below, that the set of all these finite open coverings does exist). If we are given an (infinite) open covering (U_k) of I, where each U_k is an interval with rational endpoints, it suffices to search for $j, n \in \mathbb{N}$ such that any interval in $\mathfrak{C}_n$ is U_k for some $k \leq j$. Then $\langle U_k : k \leq j \rangle$ is the desired finite subcovering.

In this proof our knowledge of the compactness of I insures that the search will sooner or later succeed. From the reverse mathematics viewpoint, the algorithm can be defined in RCA_0, but the proof of its termination requires WKL_0.

6 The Multivalued Function Sep

For the reader's convenience, we repeat here the definition of **Sep** given in the introduction.

Definition 6.1 (Sep) Let $\mathbf{Sep} : \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ be defined by $\mathrm{dom}(\mathbf{Sep}) = \{ (p, q) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \mid \forall n \, \forall m \; p(n) \neq q(m) \}$,

$$\mathbf{Sep}(p, q) = \{ r \in 2^\mathbb{N} \mid \forall n (r(p(n)) = 0 \land r(q(n)) = 1) \}.$$

Thus $\mathbf{Sep}(p, q)$ is the set of the characteristic functions of the sets separating $\mathrm{ran}(p)$ and $\mathrm{ran}(q)$.

6.1 Sep, Path, and other incomputable functions

The following fact follows from standard facts in computability theory (Ω was defined in Example 5.6).

Lemma 6.2 $\mathbf{Sep} \not\leqslant_c \Omega$.

Proof We show that $\mathbf{Sep} \not\leqslant_c f$ for any $f : \subseteq X \to \mathbb{N}$. Toward a contradiction, suppose $\mathbf{Sep} \leqslant_c f$ and let $h : \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightrightarrows X$ and $k : \subseteq (\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}) \times \mathbb{N} \rightrightarrows 2^\mathbb{N}$ be such that $k((p, q), (f \circ h)(p, q)) \subseteq \mathbf{Sep}(p, q)$ for every $(p, q) \in \mathrm{dom}(\mathbf{Sep})$. In particular, this holds for (p_0, q_0), where $p_0, q_0 \in \mathbb{N}^\mathbb{N}$ are computable functions with disjoint, yet computably inseparable, ranges. Since $(f \circ h)(p_0, q_0) \subseteq \mathbb{N}$, to compute an element of $\mathbf{Sep}(p_0, q_0)$ we can give as input to k the pair $((p_0, q_0), n)$ for some $n \in (f \circ h)(p_0, q_0)$. The resulting characteristic function is computable, a contradiction. □

Corollary 6.3 **Sep** *is not computable.*

Proof It is easy to see that $f \leqslant_c \Omega$ for all computable multivalued functions f. □

On the other hand, **Sep** is computably reducible to C_1 (we will show in Corollary 6.11 that $\mathbf{Sep} <_c C_1$).

Lemma 6.4 $\mathbf{Sep} \leqslant_c C_1$.

Proof We define the computable function $h : \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by

$$h(p, q)(\langle n, m \rangle) = \begin{cases} 1 & \text{if } p(m) = n; \\ 0 & \text{otherwise.} \end{cases}$$

When $(p, q) \in \mathrm{dom}(\mathbf{Sep})$ it is immediate that $C_1(h(p, q)) \in \mathbf{Sep}(p, q)$. In fact,

$$C_1(h(p, q))(n) = \begin{cases} 0 & \text{if } n \in \mathrm{ran}(p); \\ 1 & \text{otherwise.} \end{cases}$$ □

We intend to use computable reducibility to **Sep** as a way of assessing incomputability of other functions.

Definition 6.5 (Sep-computable and Sep-complete) Let (X, σ_X), (Y, σ_Y) be represented spaces. Then a multivalued (possibly single-valued) function $f : \subseteq X \rightrightarrows Y$ is **Sep**-*computable* if $f \leqslant_c$ **Sep**, **Sep**-*complete* if $f \cong_c$ **Sep**.

To study **Sep** we introduce the function $\mathbf{Path}_2$, which corresponds to Weak König's Lemma, that is, the statement asserting the existence of infinite paths in any infinite binary tree.

Definition 6.6 ($\mathbf{Path}_2$) Let $\mathbf{InfTr}_2 \subseteq \mathcal{P}(2^{<\mathbb{N}})$ be the set of all infinite binary trees:

$$\mathbf{InfTr}_2 = \{ T \subseteq 2^{<\mathbb{N}} \mid \forall t \in T\, \forall s \sqsubseteq t\; s \in T \wedge \forall n\; T \cap 2^n \neq \varnothing \}.$$

Let $\mathbf{Path}_2 : \mathcal{P}(2^{<\mathbb{N}}) \rightrightarrows 2^{\mathbb{N}}$ with $\mathrm{dom}(\mathbf{Path}_2) = \mathbf{InfTr}_2$ be the multivalued function mapping each infinite binary tree to the set of its infinite paths:

$$\mathbf{Path}_2(T) = \{ q \in 2^{\mathbb{N}} \mid \forall n\; q[n] \in T \}.$$

The proof of the next theorem follows closely the proof of [19, Lemma IV.4.4].

Theorem 6.7 $\mathbf{Sep} \cong_c \mathbf{Path}_2$.

Proof We start by showing that $\mathbf{Sep} \leqslant_c \mathbf{Path}_2$. Let $h : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(2^{<\mathbb{N}})$ be defined by

$$\begin{aligned} h(p, q) = \{ t \in 2^{<\mathbb{N}} | \forall i < |t| [(\exists j < |t|\; p(j) = i &\implies t(i) = 0) \\ &\wedge (\exists j < |t|\; q(j) = i \implies t(i) = 1)] \}. \end{aligned}$$

The function h is clearly computable.

If $(p, q) \in \mathrm{dom}(\mathbf{Sep})$ it is easy to see that $h(p, q) \in \mathbf{InfTr}_2$ and any infinite path in $h(p, q)$ is the characteristic function of a set separating $\mathrm{ran}(p)$ and $\mathrm{ran}(q)$. Thus $\mathbf{Path}_2(h(p, q)) \subseteq \mathbf{Sep}(p, q)$ for every $(p, q) \in \mathrm{dom}(\mathbf{Sep})$, showing $\mathbf{Sep} \leqslant_c \mathbf{Path}_2$.

We now prove $\mathbf{Path}_2 \leqslant_c \mathbf{Sep}$. Given any $T \in \mathcal{P}(2^{<\mathbb{N}})$ let, for $s \in 2^{<\mathbb{N}}$ and $i < 2$,

$$\begin{aligned} \theta_T(n, s) &\iff \exists t \in 2^n (t \in T \wedge s \sqsubseteq t); \\ \varphi_T(s, i) &\iff \exists n (\theta_T(n, s * i) \wedge \neg\theta_T(n, s * (1 - i))). \end{aligned}$$

Notice that if $T \in \mathbf{InfTr}_2$ we have $\neg(\varphi_T(s, 0) \wedge \varphi_T(s, 1))$ for all $s \in 2^{<\mathbb{N}}$.

It is easy to define a computable function $h : \mathcal{P}(2^{<\mathbb{N}}) \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $h(T) = (p_T, q_T)$ with

$$\begin{aligned} \mathrm{ran}(p_T) &= \{ s + 2 \mid \varphi_T(s, 0) \} \cup \{0\} \text{ and} \\ \mathrm{ran}(q_T) &= \{ s + 2 \mid \varphi_T(s, 1) \} \cup \{1\}. \end{aligned}$$

If $T \in \mathbf{InfTr}_2$ the observation above implies $\mathrm{ran}(p_T) \cap \mathrm{ran}(q_T) = \varnothing$; that is, $h(T) \in \mathrm{dom}(\mathbf{Sep})$.

Given $r \in 2^{\mathbb{N}}$, we can recursively define $k(r) \in 2^{\mathbb{N}}$ by

$$k(r)(m) = r(k(r)[m] + 2).$$

We have thus defined a computable function $k : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

If $T \in \mathbf{InfTr}_2$ and $r \in \mathbf{Sep}(h(T))$ we show by induction on m that $\theta_T(n, k(r)[m])$ holds for all m and $n \geq m$. To simplify the notation, let $s_m = k(r)[m]$.

1. We have $s_0 = \lambda$ (recall that λ is the empty sequence), and $\theta_T(n, \lambda)$ for all $n \geq 0$ follows immediately from the fact that T is an infinite tree.

2. Now suppose that $\theta_T(n, s_m)$ holds for all $n \geq m$. We want to show that $\theta_T(n, s_{m+1})$ holds for all $n > m$.
   (a) If $\varphi_T(s_m, 0)$, then $s_m + 2 \in \operatorname{ran}(p_T)$ and from $r \in \mathbf{Sep}(h(T))$ it follows that $r(s_m + 2) = 0$. Therefore, $s_{m+1} = s_m * 0$. Let $N \in \mathbb{N}$ be such that $\theta_T(N, s_m * 0)$ and $\neg\theta_T(N, s_m * 1)$. For $n \geq N$, it cannot be $\theta_T(n, s_m * 1)$ (because T is a tree), but by induction hypothesis $\theta_T(n, s_m)$ holds. Hence we have $\theta_T(n, s_m * 0)$, that is, $\theta_T(n, s_{m+1})$. When $m \leq n < N$, $\theta_T(n, s_{m+1})$ follows from $\theta_T(N, s_{m+1})$ and T being a tree.
   (b) When $\varphi_T(s_m, 1)$, the argument is similar to the previous case.
   (c) If $\varphi_T(s_m, 0)$ and $\varphi_T(s_m, 1)$ both fail, then for every $n > m$ either

$$\neg\theta_T(n, s_m * 0) \wedge \neg\theta_T(n, s_m * 1) \text{ or}$$
$$\theta_T(n, s_m * 0) \wedge \theta_T(n, s_m * 1).$$

   The first case is impossible, since $\theta_T(n, s_m)$ for all $n \geq m$. Therefore, only the second case is possible, which means that no matter what is s_{m+1} (i.e., whatever is the value of $r(s_m + 2)$) we have $\theta_T(n, s_{m+1})$ for all $n > m$.

In particular, for all n we have $\theta_T(n, k(r)[n])$ and thus $k(r)[n] \in T$. Hence $k(r) \in \mathbf{Path}_2(T)$. We have thus shown that $k(\mathbf{Sep}(h(T))) \subseteq \mathbf{Path}_2(T)$, which shows that $\mathbf{Path}_2 \leqslant_c \mathbf{Sep}$. □

We will need to consider also paths in bounded trees. These are the finitely branching trees for which there is an explicit bound, depending on the level, for the values attained by the sequences occurring in the tree.

Definition 6.8 ($\mathbf{Path}_\mathrm{B}$) Let $\mathbf{InfTr}_\mathrm{B} \subseteq \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \times \mathbb{N}^\mathbb{N}$ be the set of all infinite "bounded trees." $(T, b) \in \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \times \mathbb{N}^\mathbb{N}$ belongs to $\mathbf{InfTr}_\mathrm{B}$ if and only if

$$\forall t \in T\, \forall s \sqsubseteq t\; s \in T \wedge \forall i\, \forall t \in T \cap \mathbb{N}^{i+1}\; t(i) < b(i) \wedge \forall n\; T \cap \mathbb{N}^n \neq \varnothing.$$

Let $\mathbf{Path}_\mathrm{B} : \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \times \mathbb{N}^\mathbb{N} \rightrightarrows 2^\mathbb{N}$ with $\operatorname{dom}(\mathbf{Path}_\mathrm{B}) = \mathbf{InfTr}_\mathrm{B}$ be the multivalued function mapping each infinite bounded tree to the set of its infinite paths:

$$\mathbf{Path}_\mathrm{B}(T, b) = \{\, p \in \mathbb{N}^\mathbb{N} \mid \forall n\; p[n] \in T \,\}.$$

The following result is the computable analysis equivalent of Lemma IV.1.4 in [19]. We omit the proof, which is a straightforward adaptation of the proof in the reverse mathematics setting.

Lemma 6.9 $\mathbf{Path}_\mathrm{B} \cong_c \mathbf{Path}_2$ *and hence* $\mathbf{Path}_\mathrm{B} \cong_c \mathbf{Sep}$.

We now show the incomparability of $\mathbf{Sep}$ and Ω. We already know from Lemma 6.2 that $\mathbf{Sep} \nleqslant_c \Omega$.

Theorem 6.10 $\Omega \nleqslant_c \mathbf{Sep}$.

Proof By Theorem 6.7 it suffices to show that $\Omega \nleqslant_c \mathbf{Path}_2$. Suppose that $\Omega \leqslant_c \mathbf{Path}_2$ and let $h : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbf{InfTr}_2$ and $k : \mathbb{N}^\mathbb{N} \times 2^\mathbb{N} \rightrightarrows \{0, 1\}$ be computable multivalued functions witnessing this. In other words, $k(p, q) = \Omega(p)$ for every q such that $q \in \mathbf{Path}_2(T)$ for some $T \in h(p)$ (on such pairs k is single-valued).

For $n \in \mathbb{N}$, let $p_n = (\bar{0}[n] * 1)^\frown \bar{0} \in \mathbb{N}^\mathbb{N}$, where $\bar{0}$ is the only argument on which Ω takes value 0. Clearly, $\lim_n p_n = \bar{0}$, and, since h has a computable, and hence continuous, realizer, there exist $T \in h(\bar{0})$ and a sequence of infinite trees (T_n) with $T_n \in h(p_n)$ such that $\lim T_n = T$.

For any $n \in \mathbb{N}$, let $q_n \in \mathbf{Path}_2(T_n)$ so that $k(p_n, q_n) = \Omega(p_n) = 1$. Since $2^{\mathbb{N}}$ is compact we may assume that $\lim_n q_n = q$ for some $q \in 2^{\mathbb{N}}$. For every m, if n is sufficiently large, $q[m] = q_n[m]$ and $T_n \cap 2^m = T \cap 2^m$, and hence $q[m] \in T$. Thus $q \in \mathbf{Path}_2(T)$.

Again, k has a continuous realizer and we should have $\lim_n k(p_n, q_n) = k(\bar{0}, q) = \Omega(\bar{0}) = 0$, which is impossible since $k(p_n, q_n) = 1$ for all n. □

Corollary 6.11 $\mathbf{Sep} <_c C_1$.

Proof Straightforward from Lemma 6.4 and Theorem 6.10, since $\Omega \leqslant_c C_1$. □

6.2 Iterating Sep-computable functions We now show that iterating **Sep**-computable functions does not increase the degree of incomputability. Thus the situation is quite different from the case of the C_is, where $C_i \cong_c \underbrace{C_1 \circ C_1 \circ \cdots \circ C_1}_{i \text{ times}}$ and hence (recalling that $C_i <_c C_j$ when $i < j$) $C_i <_c C_i \circ C_i$ for every $i > 0$.

First, we deal with $\mathbf{Path}_2$. Actually, since the application of $\mathbf{Path}_2$ to itself is meaningless, we use a computable function to transform the output of $\mathbf{Path}_2$ into an infinite tree that is given as input to another application of $\mathbf{Path}_2$.

Lemma 6.12 *Let $f : \subseteq 2^{\mathbb{N}} \to \mathbf{InfTr}_2$ be computable. Then*

$$\mathbf{Path}_2 \circ f \circ \mathbf{Path}_2 \leqslant_c \mathbf{Path}_2 .$$

Proof For any $p \in \mathbb{N}^{\mathbb{N}}$, let $p_0, p_1 \in \mathbb{N}^{\mathbb{N}}$ be such that $p = p_0 \oplus p_1$. Define analogously t_0 and t_1 when $t \in 2^{<\mathbb{N}}$. The maps $p \mapsto p_0$ and $p \mapsto p_1$ are obviously computable.

Given any $T \in \mathbf{InfTr}_2$ we will computably define $\widetilde{T} \in \mathbf{InfTr}_2$ such that if $T \in \mathrm{dom}(\mathbf{Path}_2 \circ f \circ \mathbf{Path}_2)$ and $p \in \mathbf{Path}_2(\widetilde{T})$ we have $p_0 \in \mathbf{Path}_2(T)$ and $p_1 \in (\mathbf{Path}_2 \circ f)(p_0)$. This suffices to prove $\mathbf{Path}_2 \circ f \circ \mathbf{Path}_2 \leqslant_c \mathbf{Path}_2$ (in the notation of Definition 4.4, $T \mapsto \widetilde{T}$ and $(T, p) \mapsto p_1$ play the role of h and k, respectively).

Let $\widehat{f} : 2^{<\mathbb{N}} \to \mathcal{P}(2^{<\mathbb{N}})$ be the computable function defined as follows. $\widehat{f}(t)$ is the set of all $s \in 2^{<\mathbb{N}}$ such that after $|t|$ steps (when at most the first $|t|$ bits of input have been used) in the computation of $f(t^\frown q)$ (for any $q \in 2^{\mathbb{N}}$) no $v \sqsubseteq s$ has been marked as not belonging to the output tree.

Notice that $\widehat{f}(t)$ is a tree and $t \sqsubseteq u$ implies $\widehat{f}(t) \supseteq \widehat{f}(u)$. Moreover, for all $p \in \mathrm{dom}(f)$ and $n \in \mathbb{N}$, $f(p) \subseteq \widehat{f}(p[n])$. Since we have that if $s \notin f(p)$ then $s \notin \widehat{f}(p[n])$ for some n, $f(p) = \bigcap_n \widehat{f}(p[n])$ for every $p \in \mathrm{dom}(f)$. Thus we can view $\widehat{f}$ as an approximation of f from above.

Let

$$\widetilde{T} = \{\, t \in 2^{<\mathbb{N}} \mid t_0 \in T \wedge t_1 \in \widehat{f}(t_0) \,\},$$

so that the map $T \mapsto \widetilde{T}$ is computable. Using the properties of $\widehat{f}$ mentioned above, it is immediate to check that $\widetilde{T}$ is a tree. Moreover, if $T \in \mathrm{dom}(\mathbf{Path}_2 \circ f \circ \mathbf{Path}_2) = \{\, T \in \mathbf{InfTr}_2 \mid \mathbf{Path}_2(T) \subseteq \mathrm{dom}(f) \,\}$ then $\widetilde{T} \in \mathbf{InfTr}_2$. In fact, if $q \in \mathbf{Path}_2(T)$ then $f(q) \in \mathbf{InfTr}_2$ and if $u \in f(q)$ has length n then the sequence $t \in 2^{2n}$ such that $t_0 = q[n]$ and $t_1 = u$ belongs to $\widetilde{T}$.

If $p \in \mathbf{Path}_2(\widetilde{T})$, then $p_0 \in \mathbf{Path}_2(T)$ is immediate. If $p_0 \in \mathrm{dom}(f)$ and $p_1 \notin \mathbf{Path}_2(f(p_0))$ then there exist m such that $p_1[m] \notin f(p_0)$ and hence n such that $p_1[m] \notin \widehat{f}(p_0[n])$. We may assume $m \leq n$, which implies $p_1[n] \notin \widehat{f}(p_0[n])$, contradicting $p[2n] \in \widetilde{T}$. Thus $p_1 \in \mathbf{Path}_2(f(p_0))$. □

In a similar way, one can prove the following lemma.

Lemma 6.13 *Let X and Y be represented spaces. Suppose that $h : \subseteq X \rightrightarrows \mathbf{InfTr}_2$, $i : \subseteq X \times 2^{\mathbb{N}} \rightrightarrows \mathbf{InfTr}_2$, and $j : \subseteq X \times 2^{\mathbb{N}} \rightrightarrows Y$ are computable. Let $\ell : \subseteq X \rightrightarrows Y$ be defined by*

$$\ell(x) = \bigcup \{\, j(x, q) \mid \exists p \in (\mathbf{Path}_2 \circ h)(x)\, q \in (\mathbf{Path}_2 \circ i)(x, p)\,\}.$$

Then $\ell \leqslant_c \mathbf{Path}_2$.

Theorem 6.14 *Let $f : \subseteq X \rightrightarrows Y$ and $g : \subseteq Y \rightrightarrows Z$ be* **Sep***-computable multivalued functions between represented spaces. Then $g \circ f : \subseteq X \rightrightarrows Z$ is* **Sep***-computable.*

Proof By Theorem 6.7 we have $f, g \leqslant_c \mathbf{Path}_2$ and there exist computable $h : \subseteq X \rightrightarrows \mathbf{InfTr}_2$, $k : \subseteq X \times 2^{\mathbb{N}} \rightrightarrows Y$, $h' : \subseteq Y \rightrightarrows \mathbf{InfTr}_2$, $k' : \subseteq Y \times 2^{\mathbb{N}} \rightrightarrows Z$ witnessing this as in Definition 4.4. Therefore, $k'(k(x, p), q) \subseteq (g \circ f)(x)$ for all $x \in \mathrm{dom}(g \circ f)$, $p \in (\mathbf{Path}_2 \circ h)(x)$, and $q \in (\mathbf{Path}_2 \circ h' \circ k)(x, p)$.

To use Lemma 6.13 we need to identify the functions involved. h (in the notation of Lemma 6.13) is h, i is $(x, p) \mapsto (h' \circ k)(x, p)$, and j is

$$(x, q) \mapsto \{\, k'(k(x, p), q) \mid p \in (\mathbf{Path}_2 \circ h)(x) \wedge q \in (\mathbf{Path}_2 \circ h' \circ k)(x, p)\,\}.$$

Then the function ℓ of Lemma 6.13 is such that $\mathrm{dom}(g \circ f) \subseteq \mathrm{dom}(\ell)$ and $\ell(x) \subseteq (g \circ f)(x)$ for every $x \in \mathrm{dom}(g \circ f)$. Thus $g \circ f \leqslant_c \ell$ by Lemma 4.8 and $\ell \leqslant_c \mathbf{Path}_2$ by Lemma 6.13. Hence $g \circ f \leqslant_c \mathbf{Path}_2$. Theorem 6.7 now implies that $g \circ f$ is **Sep**-computable. □

7 Banach Spaces in Computable Analysis

7.1 Effective Banach spaces To deal with Banach spaces in the context of computable analysis we need to give definitions which are analogous to the ones given in Section 2 for metric spaces.

Definition 7.1 (Effective Banach space) An *effective Banach space* is a triple $(X, \|\ \|, e)$ such that

1. X is a Banach space with norm $\|\ \|$;
2. $e : \mathbb{N} \to X$ is a *fundamental sequence*, that is, a sequence whose linear span is dense in X;
3. (X, d, a_e) is an effective metric space, where $d(x, y) = \|x - y\|$ and $a_e(s) = \sum_{i<|s|} a_{\mathbb{Q}}(s(i)) \cdot e(i)$ for $s \in \mathbb{N}^{<\mathbb{N}}$.

We will always assume that X is nontrivial, that is, that $\|e(i)\| \neq 0$ for some $i \in \mathbb{N}$.

Notice that an effective Banach space is separable.

The domain of the multivalued function corresponding to the Hahn-Banach Theorem consists of all effective Banach spaces. If this is interpreted naïvely, we would need a method to code any possible effective Banach space. Clearly, there are "too many" such spaces to allow a well-defined single-valued representation and, since the collection of all effective Banach spaces is not even a set, even a multi-representation approach (in the sense of [9]) is questionable.

We can overcome this problem by considering a set which contains all effective Banach spaces up to isomorphism. For this set we can then define a single-valued representation. (Notice that this approach is very close to the one used in reverse mathematics, where it is customary to represent mathematical objects by "codes.") We will adapt Weihrauch's notion of constructive metric completion (see [21]) to the case of effective Banach spaces.

7.2 Constructive Banach completions For every $s \in \mathbb{N}^{<\mathbb{N}}$, let

$$c_s = \sum_{i<|s|} a_{\mathbb{Q}}(s(i)) \cdot i,$$

where we are viewing the right-hand side as a formal linear combination of elements of $\mathbb{N}$ with scalars in $\mathbb{Q}$. Let $C = \{ c_s \mid s \in \mathbb{N}^{<\mathbb{N}} \}$.

We define sum on C and scalar multiplication of an element of C by an element of $\mathbb{Q}$ in the obvious way. A *noted pseudonormed space* is then a pair $N = (C, \| \ \|)$ such that $\| \ \| : C \to \mathbb{R}$ is a pseudonorm on C; that is,

1. $\|c_s\| = 0$ whenever $s(i) = 0$ for all $i < |s|$ (recall that $a_{\mathbb{Q}}(0) = 0$);
2. $\|c_s + c_t\| \leq \|c_s\| + \|c_t\|$, for all $s, t \in \mathbb{N}^{<\mathbb{N}}$;
3. $\|\alpha \cdot c_s\| = |\alpha| \cdot \|c_s\|$ for all $s \in \mathbb{N}^{<\mathbb{N}}$ and $\alpha \in \mathbb{Q}$.

Again, we assume that $\|c_s\| \neq 0$ for some $s \in \mathbb{N}^{<\mathbb{N}}$. The pseudonorm $\| \ \|$ defines a pseudometric d over C as usual by $d(c_s, c_t) = \|c_s - c_t\|$.

We now build the constructive Banach completion of N as a particular effective Banach space. Let $\widehat{C}$ be the set of all Cauchy sequences of elements of C which satisfy the usual effective requirement:

$$\widehat{C} = \{ (c_{s_i}) \mid \forall j \, \forall i < j \, d(c_{s_i}, c_{s_j}) < 2^{-i} \}.$$

Define an equivalence relation $\sim$ on $\widehat{C}$ by

$$(c_{s_i}) \sim (c_{t_i}) \iff \lim d(c_{s_i}, c_{t_i}) = 0,$$

and notice that this condition is equivalent to $\forall i \, d(c_{s_i}, c_{t_i}) \leq 2^{-(i-1)}$. We denote by $[c_{s_i}]_{i\in\mathbb{N}}$ the $\sim$-equivalence class of (c_{s_i}). We introduce then the linear operations on $\widehat{C}/{\sim}$ by

$$[c_{s_i}]_{i\in\mathbb{N}} + [c_{t_i}]_{i\in\mathbb{N}} = [c_{s_{i+1}} + c_{t_{i+1}}]_{i\in\mathbb{N}},$$
$$a \cdot [c_{s_i}]_{i\in\mathbb{N}} = [a_{\mathbb{Q}}(n_{k+i}) \cdot c_{s_{k+i}}]_{i\in\mathbb{N}},$$

where $a \in \mathbb{R}$, $(a_{\mathbb{Q}}(n_i))$ is a Cauchy sequence effectively converging to a, and k is such that $|a_{\mathbb{Q}}(n_0)| + \|c_{s_0}\| + 2 < 2^k$. We leave to the reader checking that these definitions are meaningful and make $\widehat{C}/{\sim}$ a vector space (some of the details are spelled out in [19, p. 75]).

We further define

$$\|[c_{s_i}]_{i\in\mathbb{N}}\|_{\widehat{C}/\sim} = \lim \|c_{s_i}\|,$$

and one can check that $\widehat{C}/{\sim}$ is a Banach space. Notice that $d_{\widehat{C}/\sim}([c_{s_i}]_{i\in\mathbb{N}}, [c_{t_i}]_{i\in\mathbb{N}}) = \lim d(c_{s_i}, c_{t_i})$.

Define $e : \mathbb{N} \to \widehat{C}/{\sim}$ by $e(n) = [c_{\bar{0}[n]*1}]_{i\in\mathbb{N}}$ (recall that $a_{\mathbb{Q}}(0) = 0$ and $a_{\mathbb{Q}}(1) = 1$) where on the right-hand side we have a constant sequence. The triple $(\widehat{C}/{\sim}, \| \ \|_{\widehat{C}/\sim}, e)$ is an effective Banach space, the *constructive Banach completion* of the noted pseudonormed space N.

The function $c_s \mapsto [c_s]_{i\in\mathbb{N}}$ maps C into $\widehat{C}/\sim$ respecting the vector operations and the (pseudo)norm. Therefore, we can view C as the linearly closed dense subspace of $\widehat{C}/\sim$ generated by the fundamental sequence e.

Definition 7.2 (The space of all effective Banach spaces) Let $\mathfrak{B}$ be the set of all constructive Banach completions. This set contains all effective Banach spaces up to isomorphism, and we consider it as the space of all effective Banach spaces.

Consider now the second countable T_0-topology on $\mathfrak{B}$ with subbasis given by the sets of the form

$$U_{\langle i,s,t,j\rangle} = \{\, (\widehat{C}/\sim, \|\ \|_{\widehat{C}/\sim}, e) \mid a_{\mathbb{Q}}(i) < d_{\widehat{C}/\sim}(a_e(s), a_e(t)) < a_{\mathbb{Q}}(j) \,\}.$$

This topology on $\mathfrak{B}$ is associated with the standard representation $\delta_{\mathfrak{B}} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathfrak{B}$ defined by $\delta_{\mathfrak{B}}(p) = (\widehat{C}/\sim, \|\ \|_{\widehat{C}/\sim}, e)$ if and only if p enumerates the set

$$\{\, \langle i,s,t,j\rangle \mid (\widehat{C}/\sim, \|\ \|_{\widehat{C}/\sim}, e) \in U_{\langle i,s,t,j\rangle} \,\}.$$

We often write $(X, \|\ \|, e) \in \mathfrak{B}$, or simply $X \in \mathfrak{B}$, in place of $(\widehat{C}/\sim, \|\ \|_{\widehat{C}/\sim}, e) \in \mathfrak{B}$, but we always understand that the construction of X as a constructive Banach completion uniquely determines both the norm and the fundamental sequence.

An element in $\mathfrak{B}$ with a computable name is a computable Banach space in the sense of [2]. Since we view $\mathfrak{B}$ as the space of all effective Banach spaces, we view the subset of its computable elements as the set of all computable Banach spaces.

7.3 Representations of closed and compact sets and of linear bounded functions

We recall some representations of closed and compact subsets of metric spaces which have been widely used in the literature (see, e.g., [5]).

Definition 7.3 (Representations of closed sets) For an effective metric space X we denote by $\mathcal{A}_+(X)$ and $\mathcal{A}_-(X)$ the hyperspace of closed subsets of X viewed, respectively, with representations ψ_+^X and ψ_-^X, where

(i) $\psi_+^X(p) = A$ if and only if $p_i \in \mathrm{dom}(\delta_X)$ for all $i \in \mathbb{N}$ (where $p_i(j) = p(\langle i,j\rangle))$ and $A = \overline{\{\, \delta_X(p_i) \mid i \in \mathbb{N} \,\}}$;
(ii) $\psi_-^X(p) = A$ if and only if $X \setminus A = \bigcup B^X_{p(i)}$ (recall that $\{B^X_n\}$ enumerates all rational open balls in X).

In the reverse mathematics literature, the elements of $\mathcal{A}_+(X)$ and of $\mathcal{A}_-(X)$ are called, respectively, separably closed sets and closed sets.

Definition 7.4 (Representations of compact sets) For an effective metric space X we denote by $\mathcal{K}(X)$ and $\mathcal{K}_-(X)$ the hyperspace of compact subsets of X viewed, respectively, with representations κ^X and κ_-^X, where

(i) $\kappa^X(p) = K$ if and only if p enumerates
$$\{\, s \mid K \subseteq \textstyle\bigcup_{i<|s|} B^X_{s(i)} \wedge \forall i < |s|\; K \cap B^X_{s(i)} \neq \varnothing \,\};$$
(ii) $\kappa_-^X(p) = K$ if and only if p enumerates
$$\{\, s \mid K \subseteq \textstyle\bigcup_{i<|s|} B^X_{s(i)} \,\}.$$

We are now in a position to define the domain of the multivalued function corresponding to the separable Hahn-Banach theorem. In doing so, we borrow an idea from [22]: to denote a partial continuous function with closed domain f we employ a realizer of f and a name for $\mathrm{dom}(f)$ with respect to a representation of the hyperspace of closed sets. Moreover, we further generalize and consider closed subsets of arbitrary elements of $\mathfrak{B}$.

Definition 7.5 (Space of partial linear bounded functionals) Let $\mathcal{PF}$ be the set of all quadruples (X, A, f, r) (usually written $f_{(X,A,r)}$) such that

(i) $X \in \mathfrak{B}$;
(ii) A is a closed linear subspace of X;
(iii) $f : A \to \mathbb{R}$ is linear and bounded;
(iv) $r = \|f\| \in \mathbb{R}$ (recall that the norm $\|f\|$ is defined by
$$\|f\| = \sup\{\, |f(x)| \mid x \in A \wedge \|x\| = 1\,\}).$$

The representation of $\mathcal{PF}$ is defined by $\delta_{\mathcal{PF}}(p) = f_{(X,A,r)}$ if and only if

(i) $\delta_{\mathfrak{B}}(p_0) = X$;
(ii) $\psi_+^X(p_1) = A$;
(iii) $\eta(p_2)$ is a realizer of f;
(iv) $\delta_{\mathbb{R}}(p_3) = r$

(the p_is were defined in Definition 7.3).

8 The Hahn-Banach Theorem

8.1 The multivalued function HB We now come to the question of the computational complexity of the Hahn-Banach Theorem. We start by giving the formal definition of the Hahn-Banach multivalued function.

Definition 8.1 (Hahn-Banach multivalued function) Let $\mathbf{HB} : \subseteq \mathcal{PF} \rightrightarrows \mathcal{PF}$ be the multivalued function with $\mathrm{dom}(\mathbf{HB}) = \{f_{(X,A,1)} \in \mathcal{PF}\}$ defined by

$$\mathbf{HB}(f_{(X,A,1)}) = \{\, g_{(X,X,1)} \mid g \restriction A = f\,\}.$$

For any computable normed space X, and in particular for any computable Banach space, Brattka [2] first proves a computable version of the Banach-Alaoglu Theorem. Then he shows that for any computable Banach space there is a $\mathbf{\Sigma}^0_2$-computable multivalued function that maps f to the extensions g which satisfy the requirements of the Hahn-Banach Theorem (although the notion of $\mathbf{\Sigma}^0_2$-computable multivalued function is not explicitly used in [2]). We will use the same ideas to show that **HB** is **Sep**-computable, but some fundamental modifications are necessary.

First, we point out that Brattka's proof is not uniform, since it breaks up into two cases, depending on whether the dimension of the normed space X is finite or infinite. Even for countable vector spaces over $\mathbb{Q}$, the function establishing whether the space is finite-dimensional is not computable, and indeed not even **Sep**-computable.[2] Since we are interested in evaluating the complexity of a multivalued function which takes in input any possible effective Banach space, we need to get rid of this dichotomy. We will thus give a uniform structure to Brattka's proof, also simplifying some steps along the way.

8.2 Selecting points in closed subsets of compact sets Brattka's proof uses a multivalued choice function on compact sets to select elements in the set $H(f)$ of all extensions of f (this is the $\mathbf{\Sigma}^0_2$-computable step in that proof). Actually, in this approach one needs to consider $H(f)$ as a *compact* subset of a compact space $\widehat{X}$. We do not need this step, since the simpler property of being *closed* in the compact set

$\widehat{X}$ is enough to apply a selection multivalued function which is **Sep**-computable by Theorem 8.3 below.

Although not necessary to our main goal, in Theorem 8.3 we also formulate a general condition of **Sep**-completeness for this selection problem. To achieve this, we recall the following notion already used in [3; 4].

Definition 8.2 (Richness) A computable metric space X is *rich*, or *computably uncountable*, if there is a computable injective map $\imath : 2^{\mathbb{N}} \hookrightarrow X$.

It is known that if $\imath$ is as above then also its partial inverse $\imath^{-1}$ is computable, and thus $\imath$ is a computable embedding.

Theorem 8.3 *For a computable metric space (X, d, a), let $\mathbf{Sel}_{\mathcal{K}(X)} : \subseteq \mathcal{K}(X) \times \mathcal{A}_-(X) \rightrightarrows X$ be the multivalued function with domain $\{ (K, A) \mid \varnothing \neq A \subseteq K \}$ and*

$$\mathbf{Sel}_{\mathcal{K}(X)}(K, A) = A$$

(where on the left-hand side A is a member of the hyperspace of the closed subsets of X, while on the right-hand side is a set of points). Thus, $\mathbf{Sel}_{\mathcal{K}(X)}$ is the multivalued function which selects a point from a nonempty closed subset of a compact subset of X. Then

*(1) $\mathbf{Sel}_{\mathcal{K}(X)}$ is **Sep**-computable;*
*(2) if X is rich then $\mathbf{Sel}_{\mathcal{K}(X)}$ is **Sep**-complete.*

Proof (1) Given $K \in \mathcal{K}(X)$ we can uniformly obtain $q \in \mathbb{N}^{\mathbb{N}}$ and an infinite sequence of finite sequences $(\left\langle x^n_j \right\rangle_{j<q(n)})$ of elements of X such that for every $n \in \mathbb{N}$ we have $K \subseteq \bigcup_{j<q(n)} B^X(x^n_j; 2^{-n})$. For $A \in \mathcal{A}_-(X)$ such that $\varnothing \neq A \subseteq K$, we can uniformly obtain sequences (b_i) in $\mathrm{ran}(a)$ and (α_i) in $\mathbb{Q}$ such that $X \setminus A = \bigcup_{i\in\mathbb{N}} B^X(b_i; \alpha_i)$. We select an element of A by approximating points which do not belong to any $B^X(b_i; \alpha_i)$. More precisely, we construct a tree $T = T(K, A) \subseteq \mathbb{N}^{<\mathbb{N}}$ by letting $s \in T$ if and only if

1. $\forall n < |s|\, s(n) < q(n)$,
2. $\forall n, i, k < |s|\, d(x^n_{s(n)}, x^i_{s(i)})_{[k]} \leq 2^{-n} + 2^{-i} + 2^{-k}$,
3. $\forall n, i, k < |s|\, d(x^n_{s(n)}, b_i)_{[k]} \geq \alpha_i - 2^{-n} - 2^{-k}$,

where for $a \in \mathbb{R}$, $a_{[k]}$ is a rational approximation within 2^{-k} of a.

Notice that, since $A \neq \varnothing$, $(T, q) \in \mathbf{InfTr}_{\mathrm{B}}$. For all $p \in \mathbf{Path}_{\mathrm{B}}(T, q)$ we have that $x = \lim x^n_{p(n)}$ exists, is computable from p, and does not belong to any $B^X(b_i; \alpha_i)$. Hence $x \in A$. This gives $\mathbf{Sel}_{\mathcal{K}(X)} \leqslant_c \mathbf{Path}_{\mathrm{B}}$. By Lemma 6.9 we have $\mathbf{Sel}_{\mathcal{K}(X)} \leqslant_c \mathbf{Sep}$.

(2) By Theorem 6.7 it suffices to show $\mathbf{Path}_2 \leqslant_c \mathbf{Sel}_{\mathcal{K}(X)}$ when X is rich. First, we show $\mathbf{Path}_2 \leqslant_c \mathbf{Sel}_{\mathcal{K}(2^{\mathbb{N}})}$. For $T \in \mathbf{InfTr}_2$ define

$$A_T = 2^{\mathbb{N}} \setminus \bigcup \{ B^{2^{\mathbb{N}}}(t^\frown \bar{0}; 2^{-(|t|-1)}) \mid t \notin T \} \subseteq 2^{\mathbb{N}}.$$

Since $B^{2^{\mathbb{N}}}(t^\frown \bar{0}; 2^{-(|t|-1)}) = \{ p \in 2^{\mathbb{N}} \mid t \sqsubseteq p \}$, we have $\mathbf{Path}_2(T) = \mathbf{Sel}_{\mathcal{K}(2^{\mathbb{N}})}(2^{\mathbb{N}}, A_T)$. Since the map $T \mapsto (2^{\mathbb{N}}, A_T)$ from $\mathbf{InfTr}_2$ to $\mathcal{K}(2^{\mathbb{N}}) \times \mathcal{A}_-(2^{\mathbb{N}})$ is computable, $\mathbf{Path}_2 \leqslant_c \mathbf{Sel}_{\mathcal{K}(2^{\mathbb{N}})}$.

If X is rich, let $\imath : 2^{\mathbb{N}} \hookrightarrow X$ be a computable injection. As observed in [3; 4], $\mathrm{ran}(\imath) \in \mathcal{K}(X)$. By the proof of the Embedding Theorem of [3; 4], the map from $\mathcal{A}_-(2^{\mathbb{N}})$ to $\mathcal{A}_-(X)$ which sends A to $\imath(A)$ is computable. Hence for every $A \in \mathcal{A}_-(2^{\mathbb{N}})$ we have

$$(\imath^{-1} \circ \mathbf{Sel}_{\mathcal{K}(X)})(\imath(2^{\mathbb{N}}), \imath(A)) = \mathbf{Sel}_{\mathcal{K}(2^{\mathbb{N}})}(2^{\mathbb{N}}, A).$$

Using the notation of the first part of this proof, we have

$$(\imath^{-1} \circ \mathbf{Sel}_{\mathcal{K}(X)})(\imath(2^{\mathbb{N}}), \imath(A_T)) = \mathbf{Path}_2(T)$$

for every $T \in \mathbf{InfTr}_2$. This shows $\mathbf{Path}_2 \leqslant_c \mathbf{Sel}_{\mathcal{K}(X)}$. □

8.3 Proof of HB $\leqslant_c$ Sep Brattka's proof uses the Effective Independence Lemma of Pour-El and Richards [16, p. 142]. To make Brattka's argument uniform we need a uniform version of that result.

Lemma 8.4 (Uniform Effective Independence Lemma) *For all $(X, \|\ \|, e) \in \mathfrak{B}$, there exists $q \in \mathbb{N}^{\mathbb{N}}$ such that, letting $R = \{\, j > 0 \mid q(j) = q(0)\,\}$, q restricted to $\mathbb{N} \setminus R$ is one-to-one and $\{\,(e \circ q)(j) \mid j \in \mathbb{N} \setminus R\,\}$ is a (possibly finite) linearly independent set whose linear span is dense in X. Let $\zeta : \mathfrak{B} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be the multivalued function such that $\zeta(X, \|\ \|, e)$ is the set of all q satisfying the condition above. Then ζ is computable.*

Proof We prove at once both statements of the lemma by defining a computable realizer for ζ. To this end we construct, uniformly in a name for $X \in \mathfrak{B}$, q by stages. We will also keep track of R by letting $R_n = \{\, 0 < j \leq n \mid q(j) = q(0)\,\}$. Let N be such that $\|e(N)\| \neq 0$.

At stage 0 we let $q(0) = N$ and $R_0 = \varnothing$.

At stage $n+1$ we suppose to have defined $q(0), \ldots, q(n)$ and $R_n \subseteq \{1, \ldots, n\}$ such that

1. $q(j) = q(0) = N$ for all $j \in R_n$;
2. $T_n = \{\,(e \circ q)(j) \mid j \leq n \wedge j \notin R_n\,\}$ is linearly independent.

We let $T_n = \{v_1, \ldots, v_k\}$ (obviously $k \leq n$).

For every $i \leq n+1$ we run a test, described below, which stops after a finite amount of time with answer either (a) or (b). If the answer is (a) then we are sure that $T_n \cup \{e(i)\}$ is linearly independent. If the answer is (b) then $e(i)$ can be approximated within $2^{-(n+1)}$ by a rational linear combination of elements of T_n. Therefore, if for some i the answer is (b) at every stage $\geq i$, then actually $e(i)$ belongs to the closure of the linear span of $T = \bigcup_{n\in\mathbb{N}} T_n$.

The test is based on the following fact, proved in [16, p. 143]. For $m, \ell \in \mathbb{N}$, let $S_{m,\ell}$ be the set of all $\langle \beta_1, \ldots, \beta_\ell \rangle \in \mathbb{Q}^\ell$ such that the denominators of $\beta_1, \ldots, \beta_\ell$ are 2^m and $1 \leq |\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_\ell|^2 \leq 4$. (The $S_{m,\ell}$s are finite and can be uniformly computably enumerated in m and ℓ.) Pour-El and Richards prove that a finite subset $\{w_1, \ldots, w_\ell\}$ of a Banach space is linearly independent if and only if for some $m \geq 2\ell$.

$$\min\{\, \|\beta_1 w_1 + \cdots + \beta_\ell w_\ell\| \mid \langle \beta_1, \ldots, \beta_\ell \rangle \in S_{m,\ell}\,\} > 2^{-m} \cdot (\|w_1\| + \cdots + \|w_\ell\|).$$

Given $i \leq n+1$ the test alternatively searches

(a) for $m \geq 2(k+1)$ such that

$$\min\{\, \|\beta_1 v_1 + \cdots + \beta_k v_k + \beta_{k+1} e(i)\| \mid \langle \beta_1, \ldots, \beta_{k+1} \rangle \in S_{m,k+1}\,\} > 2^{-m} \cdot (\|v_1\| + \cdots + \|v_k\| + \|e(i)\|),$$

(b) and for $\gamma_1, \dots, \gamma_k \in \mathbb{Q}$ such that

$$\|e(i) - (\gamma_1 v_1 + \cdots + \gamma_k v_k)\| < 2^{-(n+1)}.$$

By the fact mentioned above, at least one of the two searches succeeds, and the test will answer (a) or (b) according to the first one succeeding.

If for some $i \le n+1$ the answer is (a), we pick the least i with this property and set $q(n+1) = i$, so that $R_{n+1} = R_n$ ($i \ne N$ because $e(N) \in T_n$). Otherwise, if for all $i \le n+1$ the answer is (b), then $q(n+1) = N$ and hence $R_{n+1} = R_n \cup \{n+1\}$. It is straightforward to check that $\{ (e \circ q)(j) \mid j \in \mathbb{N} \setminus R \}$ is linearly independent and dense in X. □

The main feature of Lemma 8.4 is that we can uniformly find a sequence of linearly independent vectors whose linear span is dense in X by allowing repetitions of the single element $(e \circ q)(0)$ and forgetting all occurrences of this element after the first.

Definition 8.5 ($\mathfrak{B}^+$) Let $\mathfrak{B}^+$ be the graph of the computable multivalued function ζ of Lemma 8.4. In other words,

$$\mathfrak{B}^+ = \{ ((X, \|\ \|, e), q) \in \mathfrak{B} \times \mathbb{N}^{\mathbb{N}} \mid q \in \zeta(X, \|\ \|, e) \}.$$

When we write $X^+ \in \mathfrak{B}^+$ we mean that $X \in \mathfrak{B}$ and $X^+ = (X, q)$ for some $q \in \zeta(X)$.

Using Lemma 8.4 we obtain a uniform proof of Lemma 3 in [2].

Definition 8.6 (Identity problem) For an effective Banach space $(X, \|\ \|, e)$ the identity problem for $(X, \|\ \|, e)$ is the set

$$I(X, \|\ \|, e) = \{ (s, t) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}} \mid a_e(s) = a_e(t) \}.$$

Lemma 8.7 (Identity problem lemma) *Given $((X, \|\ \|, e), q) \in \mathfrak{B}^+$, let $e' = e \circ q$.*

(1) The function $((X, \|\ \|, e), q) \mapsto e'$ is computable.
(2) $id : (X, \|\ \|, e) \to (X, \|\ \|, e')$ and its inverse are uniformly computable in $((X, \|\ \|, e), q) \in \mathfrak{B}^+$.
(3) The function which associates to $((X, \|\ \|, e), q) \in \mathfrak{B}^+$ the characteristic function of $I(X, \|\ \|, e')$ is computable.

Proof (1) is obvious.

(2) For id it is enough to show how to uniformly compute, for any $i \in \mathbb{N}$, a $p \in \mathbb{N}^{\mathbb{N}}$ such that $((a_{e'} \circ p)(j))$ is a Cauchy sequence converging effectively to $e(i)$. Let $R = \{ j > 0 \mid q(j) = q(0) \}$ as in Lemma 8.4. The definition of p is by stages. Before stage n we have defined $p[j_n]$ with $j_n \le n$ and at that stage we possibly define $p(j_n)$ as follows. For each $s \le n$ check whether

$$d\left(e(i), \textstyle\sum_{k<|s|, k \notin R} a_{\mathbb{Q}}(s(k)) \cdot e'(k)\right)_{[j_n+2]} < 2^{-(j_n+2)} \qquad (*)$$

where, as in the proof of Theorem 8.3, for $a \in \mathbb{R}$, $a_{[k]}$ is a rational approximation within 2^{-k} of a. If (*) holds for some $s \le n$, let $p(j_n) = s$ (so that $j_{n+1} = j_n + 1$). If (*) fails for all $s \le n$, do nothing; that is, let $j_{n+1} = j_n$. Since e' is a fundamental sequence, we have $\lim j_n = \infty$ so that $p(j)$ is defined for every j. It is straightforward to check that p has the desired property. The uniform computability of id^{-1} is immediate, since $e'(n) = e(m)$ whenever $q(n) = m$.

(3) Given $((X, \| \ \|, e), q) \in \mathfrak{B}^+$, let $R^* = R \cup \{0\} = \{ j \mid q(j) = q(0) \}$. To check whether $(s, t) \in I(X, \| \ \|, e')$ recall that $a_{e'}(s) = \sum_{i<|s|} a_{\mathbb{Q}}(s(i)) \cdot (e \circ q)(i)$ and similarly for $a_{e'}(t)$. Assuming $|s| \leq |t|$ we have that $a_{e'}(s) = a_{e'}(t)$ is equivalent to the conjunction of the following conditions:

1. $\forall i \leq |s| (i \notin R^* \to s(i) = t(i))$;
2. $\forall i \leq |t| (i \geq |s| \wedge i \notin R^* \to a_{\mathbb{Q}}(t(i)) = 0)$;
3. $\sum_{i \leq |s|, i \in R^*} a_{\mathbb{Q}}(s(i)) = \sum_{i \leq |t|, i \in R^*} a_{\mathbb{Q}}(t(i))$.

Since each of these conditions is computable in R^*, and hence in q, this equivalence completes the proof. □

We now consider the space $\mathbb{R}^{\mathbb{N}}$ equipped with the (slightly nonstandard) metric

$$d((x_n), (y_n)) = \sup \left\{ \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \mid n \in \mathbb{N} \right\}.$$

If $a(s) = (a_{\mathbb{Q}}(s(0)), \ldots, a_{\mathbb{Q}}(s(|s| - 1)), 0, 0 \ldots)$ then it is easy to check that $(\mathbb{R}^{\mathbb{N}}, d, a)$ is a computable metric space.

The main reason for using d instead of the standard textbook metric for $\mathbb{R}^{\mathbb{N}}$ (defined by a series rather than a sup) is that the open balls with respect to d are of the form $I_0 \times \cdots \times I_n \times \mathbb{R} \times \mathbb{R} \times \cdots$, where each $I_i \subseteq \mathbb{R}$ is an open interval. Of course, both metrics are compatible with the product topology of $\mathbb{R}^{\mathbb{N}}$.

Lemma 8.8 *The function which maps every $(x_n) \in \mathbb{R}^{\mathbb{N}}$ to the compact space $\prod_{n \in \mathbb{N}} [-|x_n|, |x_n|] \in \mathcal{K}(\mathbb{R}^{\mathbb{N}})$ is computable.*

Proof The proof of this lemma essentially consists in checking that the proof of [2, Lemma 4] is uniform. In doing so, we spell out a few more details of the proof.

To simplify the notation, let $Y_{(x_n)} = \prod_{n \in \mathbb{N}} [-|x_n|, |x_n|]$. By [5, Theorems 3.7, 3.8 and Proposition 4.2.2], it suffices to show that the function $(x_n) \mapsto Y_{(x_n)}$ is computable when $Y_{(x_n)}$ is viewed as an element of $\mathcal{A}_+(\mathbb{R}^{\mathbb{N}})$ and of $\mathcal{K}_-(\mathbb{R}^{\mathbb{N}})$. We first deal with $\mathcal{A}_+(\mathbb{R}^{\mathbb{N}})$. Define a computable $\rho : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$\rho((x_n), (y_n)) = (\max\{-|x_n|, \min\{y_n, |x_n|\}\}).$$

Then $\{ \rho((x_n), a(s)) \mid s \in \mathbb{N}^{<\mathbb{N}} \}$ is dense in $Y_{(x_n)}$. This shows that the function mapping $(x_n) \in \mathbb{R}^{\mathbb{N}}$ to $Y_{(x_n)} \in \mathcal{A}_+(\mathbb{R}^{\mathbb{N}})$ is computable.

To compute $Y_{(x_n)}$ as an element of $\mathcal{K}_-(\mathbb{R}^{\mathbb{N}})$, we need to show that we can enumerate a list of all finite coverings of $Y_{(x_n)}$ consisting of rational open balls. In other words, we want to show that the set of all finite sets $\{B_0, \ldots, B_k\}$ of rational open balls in $\mathbb{R}^{\mathbb{N}}$ such that $Y_{(x_n)} \subseteq \bigcup_{i=0}^k B_i$ is recursively enumerable in (x_n). By our choice of the metric, each B_i is of the form $(\alpha_0^i, \beta_0^i) \times \cdots \times (\alpha_{m_i}^i, \beta_{m_i}^i) \times \mathbb{R} \times \mathbb{R} \times \cdots$ with $m_i \in \mathbb{N}$ and $\alpha_0^i, \beta_0^i, \ldots, \alpha_{m_i}^i, \beta_{m_i}^i \in \mathbb{Q}$. Let $m = \max\{ m_i \mid i \leq k \}$. Now notice that $Y_{(x_n)} \nsubseteq \bigcup_{i=0}^k B_i$ is equivalent to the existence of $\gamma_0, \ldots, \gamma_m \in \mathbb{Q}$ such that $\gamma_n \in \{ \alpha_n^i, \beta_n^i \mid i \leq k \}$ for each $n \leq m$ and

$$(\gamma_0, \ldots, \gamma_m, 0, 0, \ldots) \in Y_{(x_n)} \setminus \bigcup_{i=0}^k B_i.$$

Hence we need to check the co-recursively enumerable in (x_n) condition on finitely many $(m + 1)$-tuples. □

We recall that the Banach-Alaoglu Theorem states that the closed unit ball of the dual space of a normed vector space is compact in the weak* topology. The next theorem is a uniform version of Theorem 6 in [2]. The idea here is that we uniformly embed the closed unit ball of the dual space of an element of $\mathfrak{B}$ onto a closed subset of a compact subset of $\mathbb{R}^{\mathbb{N}}$. Moreover, this is done taking into account the change of fundamental sequence provided by Lemma 8.7. In the statement of the theorem the reader should keep in mind that ϕ restricted to fixed $(X, q) \in \mathfrak{B}^+$ is this embedding and χ computes its inverse, taking in input also the norm of the functional.

Theorem 8.9 (Uniform Computable Banach-Alaoglu Theorem) *Let $\phi : \subseteq \mathcal{PF} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the function with*

$$\mathrm{dom}(\phi) = \{ (g_{(X,X,r)}, q) \mid r \leq 1 \wedge (X, q) \in \mathfrak{B}^+ \}$$

defined by

$$\phi(g_{(X,X,r)}, q) = ((g \circ a_{e'})(n)),$$

where $e' = e \circ q$ as in Lemma 8.7.

(1) ϕ is computable and $\phi(g_{(X,X,r)}, q) = \phi(g'_{(X,X,r')}, q)$ implies $g = g'$ and $r = r'$.

(2) There exist computable functions $\widehat{\ } : \mathfrak{B}^+ \to \mathcal{K}(\mathbb{R}^{\mathbb{N}})$ and $\widetilde{\ } : \mathfrak{B}^+ \to \mathcal{A}_-(\mathbb{R}^{\mathbb{N}})$ such that $\widetilde{X^+} \subseteq \widehat{X^+}$ and $\phi(g_{(X,X,r)}, q) \in \widetilde{(X, q)}$.

(3) There exists a computable $\chi : \subseteq \mathbb{R}^{\mathbb{N}} \times \mathfrak{B}^+ \times \mathbb{R} \to \mathcal{PF} \times \mathbb{N}^{\mathbb{N}}$ such that

$$\mathrm{dom}(\chi) = \left\{ \left((a_n), X^+, r\right) \mid (a_n) \in \widetilde{X^+} \wedge r = \sup\{ \tfrac{|a_n|}{\|a_{e'}(n)\|} \mid n \in \mathbb{N} \} \right\},$$

and we have always $\chi((a_n), (X, q), r) = (g_{(X,X,r)}, q)$ for some function g such that $\phi(g_{(X,X,r)}, q) = (a_n)$.

Proof (1) is obvious.

(2) For $X^+ = (X, q)$ and $e' = e \circ q$, define

$$\widehat{X^+} = \prod_{n \in \mathbb{N}} [-\|a_{e'}(n)\|, \|a_{e'}(n)\|],$$

and let $\widetilde{X^+}$ be the set of all $(a_n) \in \widehat{X^+}$ such that

$$\forall \alpha, \beta \in \mathbb{Q}\, \forall i, j, n \in \mathbb{N} (a_{e'}(n) = \alpha a_{e'}(i) + \beta a_{e'}(j) \implies a_n = \alpha a_i + \beta a_j).$$

By Lemma 8.8, $\widehat{X^+} \in \mathcal{K}(\mathbb{R}^{\mathbb{N}})$ and $\widehat{\ }$ is computable. To show that $\widetilde{X^+} \in \mathcal{A}_-(\mathbb{R}^{\mathbb{N}})$, notice that, given α, β, i, j, we can compute k such that $\alpha a_{e'}(i) + \beta a_{e'}(j) = a_{e'}(k)$. Thus $a_{e'}(n) = \alpha a_{e'}(i) + \beta a_{e'}(j)$ is equivalent to $(n, k) \in I(X, \| \ \|, e')$. By Lemma 8.7, we can compute from X^+ the characteristic function of $I(X, \| \ \|, e')$ and thus check whether the latter condition holds. It is now obvious that $\widetilde{X^+} \in \mathcal{A}_-(\mathbb{R}^{\mathbb{N}})$ and that $\widetilde{\ }$ is computable. It is also obvious that $\phi(g_{(X,X,r)}, q) \in \widetilde{X^+}$.

(3) Let $(((a_n), X^+, r)) \in \mathrm{dom}(\chi)$ and notice that $r \leq 1$. We need to compute $g : X \to \mathbb{R}$ linear and bounded such that $\|g\| = r$ and $g(a_{e'}(n)) = a_n$. Given $x \in X$ to compute $g(x)$ within 2^{-k} it suffices to find n such that $\|x - a_{e'}(n)\| < 2^{-k}$. Then

$$|g(x) - a_n| = |g(x) - g(a_{e'}(n))| \leq r \cdot \|x - a_{e'}(n)\| < 2^{-k}$$

and we can use a_n as an approximation of $g(x)$. □

The next lemma is the uniform version of Theorem 5 of [2].

Lemma 8.10 *Let $H :\subseteq \mathcal{PF} \times \mathbb{N}^{\mathbb{N}} \to \mathcal{A}_-(\mathbb{R}^{\mathbb{N}})$ be the function with*

$$\mathrm{dom}(H) = \{ (f_{(X,A,r)}, q) \mid (X, q) \in \mathfrak{B}^+ \wedge r = 1 \}$$

defined by

$$H(f_{(X,A,1)}, q) = \{ \phi(g_{(X,X,1)}, q) \mid g \restriction A = f \}.$$

Then H is computable.

Proof Given $(f_{(X,A,1)}, q) \in \mathrm{dom}(H)$, let $X^+ = (X, q)$. We can compute $e' = e \circ q$ and $\{ y_i \mid i \in \mathbb{N} \}$, a dense subset of $A \in \mathcal{A}_+(X)$. Notice that $(a_n) \in H(f_{(X,A,1)}, q)$ if and only if $(a_n) \in \widetilde{X^+}$ and

$$\forall n, i \, |f(y_i) - a_n| \leq \|y_i - a_{e'}(n)\|.$$

Therefore, $H(f_{(X,A,1)}, q) \in \mathcal{A}_-(\mathbb{R}^{\mathbb{N}})$. The computability of H is immediate. □

Finally we can prove the first half of our main result.

Theorem 8.11 **HB** *is* **Sep**-*computable.*

Proof Given $f_{(X,A,1)} \in \mathcal{PF}$ by Lemma 8.4, we can compute $q \in \zeta(X)$ so that $X^+ = (X, q) \in \mathfrak{B}^+$. By Theorem 8.9 and Lemma 8.10, we can compute $\widehat{X^+} \in \mathcal{K}(\mathbb{R}^{\mathbb{N}})$ and $C = H(f_{(X,A,1)}, q) \in \mathcal{A}_-(\mathbb{R}^{\mathbb{N}})$ so that $C \subseteq \widehat{X^+}$. Notice that $C \neq \varnothing$ because the Hahn-Banach Theorem holds. We can thus apply the multivalued function $\mathbf{Sel}_{\mathcal{K}(\mathbb{R}^{\mathbb{N}})}$ defined in Theorem 8.3 to the pair $(\widehat{X^+}, C)$ and select a point $(a_n) \in C$. Then

$$\chi((a_n), X^+, 1) = (g_{(X,X,1)}, q) \text{ for some } g_{(X,X,1)} \in \mathbf{HB}(f_{(X,A,1)}).$$

We have thus shown $\mathbf{HB} \leqslant_c \mathbf{Sel}_{\mathcal{K}(\mathbb{R}^{\mathbb{N}})}$. Since $\mathbf{Sel}_{\mathcal{K}(\mathbb{R}^{\mathbb{N}})}$ is **Sep**-computable by Theorem 8.3.1, this completes the proof. □

8.4 Proof of Sep $\leqslant_c$ HB The proof of the other half of our main result is obtained by adapting the proof of Theorem IV.9.4 in [19].

Theorem 8.12 **Sep** $\leqslant_c$ **HB**.

Proof Let $p, q \in \mathbb{N}^{\mathbb{N}}$ be such that $\mathrm{ran}(p) \cap \mathrm{ran}(q) = \varnothing$. We will use p and q to compute $f_{(X,A,1)} \in \mathcal{PF}$ so that from any element of $\mathbf{HB}(f_{(X,A,1)})$ we can compute an element of $\mathbf{Sep}(p, q)$.

In particular, X is a constructive Banach completion and, following the construction in Subsection 7.2, we need to define a pseudonorm on the set C of all formal linear combinations of elements of $\mathbb{N}$ with scalars in $\mathbb{Q}$. To define this pseudonorm (which depends on p and q) we identify elements of $\mathbb{N}$ with finite sequences of elements of $\mathbb{Q}^2$ as follows: $2n$ and $2n+1$ are identified, respectively, with the sequences $\langle (0,0), \ldots, (0,0), (1,0) \rangle$ and $\langle (0,0), \ldots, (0,0), (0,1) \rangle$ of length $n+1$. With this identification, C is viewed as the set Q_2 of all finite sequences of elements of $\mathbb{Q}^2$. We will therefore define the pseudonorm on Q_2. Let

$$\delta_n = \begin{cases} 2^{-k} & \text{if } k = \mu i(p(i) = n) \\ -2^{-k} & \text{if } k = \mu i(q(i) = n) \\ 0 & \text{otherwise.} \end{cases}$$

δ_n is computable as a real number on the input (p, q, n). For $(\alpha, \beta) \in \mathbb{Q}^2$, let

$$\|(\alpha, \beta)\|_n = \begin{cases} \max\{|\frac{1-\delta_n}{1+\delta_n}\alpha + \beta|, |\alpha - \beta|\} & \text{if } \delta_n > 0 \\ \\ \max\{|\frac{1+\delta_n}{1-\delta_n}\alpha - \beta|, |\alpha + \beta|\} & \text{if } \delta_n < 0 \\ \\ \max\{|\alpha + \beta|, |\alpha - \beta|\} & \text{if } \delta_n = 0. \end{cases}$$

$\|(\alpha, \beta)\|_n$ is computable as a real number on input (p, q, α, β, n). Notice that $\|(\alpha, 0)\|_n = \|(0, \alpha)\|_n = |\alpha|$ for all α and n.

We can now define the pseudonorm on Q_2 by

$$\| \langle(\alpha_i, \beta_i)\rangle_{i<k} \| = \sum_{i<k} 2^{-i-1} \cdot \|(\alpha_i, \beta_i)\|_i.$$

This noted pseudonormed space generates the constructive Banach completion $X = X(p, q) \in \mathfrak{B}$. (Intuitively, X is the ℓ_1-sum of a sequence of 2-dimensional Banach spaces with slightly different metrics.) As usual, we view Q_2 as a subset of X. Let

$$A = \overline{\{\langle(\alpha_i, 0)\rangle_{i<n}\}} \in \mathcal{A}_+(X)$$

and define $f : A \to \mathbb{R}$ by setting

$$f(\langle(\alpha_i, 0)\rangle_{i<n}) = \sum_{i<n} 2^{-i-1}\alpha_i$$

and extending by continuity. The function f is linear on A and is a bounded linear functional with $\|f\| \leq 1$, since

$$\begin{aligned} |f(\langle(\alpha_i, 0)\rangle_{i<n})| &= |\sum_{i<n} 2^{-i-1}\alpha_i| \\ &\leq \sum_{i<n} 2^{-i-1}|\alpha_i| \\ &= \sum_{i<n} 2^{-i-1}\|(\alpha_i, 0)\|_i \\ &= \| \langle(\alpha_i, 0)\rangle_{i<n} \|. \end{aligned}$$

Moreover, $\| \langle(2, 0)\rangle \| = 1$ and $f(\langle(2, 0)\rangle) = 1$, which shows that $\|f\| = 1$. By evaluation and type conversion, one can compute a realizer of f. Therefore, $f_{(X,A,1)} \in \mathcal{PF}$ has been computed from (p, q) and, moreover, we have $f_{(X,A,1)} \in \operatorname{dom}(\mathbf{HB})$. Applying **HB** we obtain $g_{(X,X,1)} \in \mathcal{PF}$ with $g \restriction A = f$.

For any $n \in \mathbb{N}$, let $z_n \in Q_2$ be the sequence $\langle(0, 0), \ldots, (0, 0), (0, 1)\rangle$ of length $n + 1$. Then $|g(z_n)| \leq \|z_n\| = 2^{-n-1}$.

If $n \in \operatorname{ran}(p)$, then $\delta_n > 0$ and notice that, for $w_n = \langle(0, 0), \ldots, (0, 0), (1 + \delta_n, 0)\rangle$ of length $n + 1$, we have

$$\begin{aligned} |f(w_n) + \delta_n g(z_n)| &= |g(w_n) + \delta_n g(z_n)| \\ &= |g(w_n + \delta_n z_n)| \\ &\leq \|w_n + \delta_n z_n\| \\ &= \| \langle(0, 0), \ldots, (0, 0), (1 + \delta_n, \delta_n)\rangle \|. \end{aligned}$$

Since $\delta_n > 0$,

$$\|(1 + \delta_n, \delta_n)\|_n = \max\{|\tfrac{(1-\delta_n)}{(1+\delta_n)}(1 + \delta_n) + \delta_n|, |1 + \delta_n - \delta_n|\} = 1,$$

and so $\| \langle (0,0),\dots,(0,0),(1+\delta_n,\delta_n)\rangle \| = 2^{-n-1}$. We deduce that $|2^{-n-1}(1+\delta_n)+\delta_n g(z_n)| = |2^{-n-1}+\delta_n(2^{-n-1}+g(z_n))| \le 2^{-n-1}$. Therefore, $\delta_n(2^{-n-1}+g(z_n)) \le 0$ and so $g(z_n) \le -2^{-n-1}$. Since $|g(z_n)| \le 2^{-n-1}$ then $g(z_n) = -2^{-n-1}$.

Similarly, if $n \in \mathrm{ran}(q)$ (and thus $\delta_n < 0$) we obtain $g(z_n) = 2^{-n-1}$ by considering

$$|2^{-n-1}(1-\delta_n)+\delta_n g(z_n)| = |2^{-n-1}+\delta_n(g(z_n)-2^{-n-1})| \le 2^{-n-1}.$$

To compute an element of $\mathbf{Sep}(p,q)$, given n, look for the approximation of $g(z_n)$ within 2^{-n-2} and check if it is positive or not. This shows that from any g such that $g_{(X,X,1)} \in \mathbf{HB}(f_{(X,A,1)})$ we can uniformly compute an element of $\mathbf{Sep}(p,q)$. □

Notes

1. Brattka's notion of realizer reducibility, as well its generalization to the case of multivalued functions (Lemma 4.5.(ii)), are particular cases of Wadge's reducibility for sets of functions as defined in [21, Definition 8.2.5].

2. To see this, let $Q = \{ w \in \mathbb{Q}^{<\mathbb{N}} \mid |w| = 0 \vee w(|w|-1) \neq 0 \}$. We view Q as a vector space over $\mathbb{Q}$ in the obvious way, and let $\mathrm{Vect}_{\mathbb{Q}} = \{ V \subseteq Q \mid V \text{ is a vector space} \}$. Let $\mathrm{Dim} : \mathrm{Vect}_{\mathbb{Q}} \to 2$ be defined by
$$\mathrm{Dim}(V) = \begin{cases} 0 & \text{if } \dim(V) = \infty; \\ 1 & \text{if } \dim(V) < \infty. \end{cases}$$
Define the computable function $V : \mathbb{N}^{\mathbb{N}} \to \mathrm{Vect}_{\mathbb{Q}}$ by $V(p) = \{w \in Q \mid \forall i < |w|\ p(i) = 0\}$. Then $\mathrm{Dim} \circ V = \Omega$ and thus $\Omega \leqslant_c \mathrm{Dim}$.

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Acknowledgments

We would like to thank Vasco Brattka for profitable discussion on the subject. An extended abstract of this paper appears in the Proceedings of the Fifth International Conference on Computability and Complexity in Analysis (CCA 2008), *Electronic Notes in Theoretical Computer Science* 221, 85–102, 2008, doi:10.1016/j.entcs.2008.12.009

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