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**How long before the end of inflation were observable perturbations produced?**

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We reconsider the issue of the number of  $e$ -foldings before the end of inflation at which observable perturbations were generated. We determine a plausible upper limit on that number for the standard cosmology which is around 60, with the expectation that the actual value will be up to 10 below this. We also note a special property of the  $\lambda\phi^4$  model which reduces the uncertainties in that case and favors a higher value, giving a fairly definite prediction of 64  $e$ -foldings for that model. We note an extreme (and highly implausible) situation where the number of  $e$ -foldings can be even higher, possibly up to 100, and discuss the shortcomings of quantifying inflation by  $e$ -foldings rather than by the change in  $aH$ . Finally, we discuss the impact of non-standard evolution between the end of inflation and the present, showing that again the expected number of  $e$ -foldings can be modified, and in some cases significantly increased.

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**I. INTRODUCTION**

With observations of perturbations in the Universe reaching a quality that seriously constrains inflationary models [1], it is timely to reexamine one of the significant uncertainties in fixing the inflationary model, the location on the inflationary potential corresponding to the observed perturbations. This is usually quantified by the number of  $e$ -foldings before the end of inflation at which our present Hubble scale equaled the Hubble scale during inflation—the epoch of horizon crossing. While in most inflation models the spectrum of perturbations generated depends only on the dynamics of the Universe around the horizon crossing, determination of the number of  $e$ -foldings requires a model of the entire history of the Universe.

Determining the appropriate number of  $e$ -foldings may shed light on the mechanism ending inflation (a goal that would also be greatly assisted by a determination of the energy scale of inflation). There are currently two popular mechanisms: steepening of the potential leading to an end of the slow-roll era, or the hybrid inflation mechanism where an instability in a second field brings inflation to an end. In the latter case, the number of  $e$ -foldings does not have great significance, but in the case of slow-roll violation, it is a significant constraint on the inflationary potential that inflation must come to an end a particular number of  $e$ -foldings after the observed perturbations were generated. It is desirable to combine this constraint with those coming from the form of the observed perturbations.

In this paper we reexamine the issue of the number of  $e$ -foldings, highlighting the sources of uncertainty. In particular, we seek to impose robust upper and lower limits on the number of  $e$ -foldings corresponding to observable perturbations, both in the case of the standard cosmological history and for models with different early evolution of the

Universe.<sup>1</sup> We are able to make some technical improvements to previous calculations, now that the standard cosmological model, featuring a low-density spatially flat universe, is firmly established. Further, we are able to investigate how the number of  $e$ -foldings is modified as one changes the properties of inflation models within the range allowed by observations.

As we were completing this paper, a paper appeared by Dodelson and Hui [2], who also consider the maximum number of  $e$ -foldings of inflation but with a less wide-ranging treatment than ours. While the original version of their paper had some discrepancies as compared to ours, they submitted a revised version of their paper simultaneously with ours which is in good agreement where the discussion overlaps.

**II. THE SIMPLEST COSMOLOGY**

Our main aim is to obtain the number of  $e$ -foldings  $N(k)$  before the end of inflation at which a comoving scale  $k$  equaled the Hubble scale  $aH$ . Normally we will focus on the scale  $k_{\text{hor}} = a_0 H_0$  which equals the present Hubble scale. Current observations are able to probe from around this scale up to  $k$  values about three orders of magnitude larger using microwave anisotropy and galaxy clustering data, and perhaps a further order of magnitude using quasar absorption line features, corresponding to a range of about 10  $e$ -foldings in total.

The number of  $e$ -foldings during inflation,  $N(k)$ , is defined by

$$e^{N(k)} \equiv \frac{a_{\text{end}}}{a_k}, \quad (1)$$

<sup>1</sup>Our results say nothing about the total number of  $e$ -foldings which may have taken place, which is expected to be much larger.

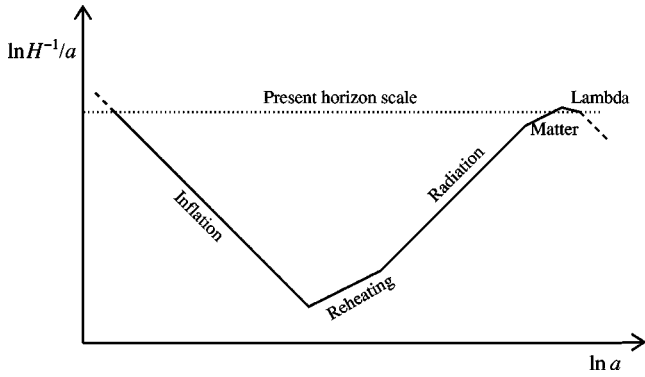


FIG. 1. A plot of  $\ln(H^{-1}/a)$  versus  $\ln a$  shows the different epochs in the  $e$ -foldings calculation. The solid curve shows the evolution from the initial horizon crossing to the present, with the dashed lines showing likely extrapolations into the past and future. The condition for inflation is that  $\ln(H^{-1}/a)$  be decreasing. Lines of constant Hubble parameter (not shown) lie at  $45^\circ$  (running top left to bottom right). The limit of exponential inflation gives a line at this angle, otherwise the inflation line is shallower. During reheating and matter domination  $H^{-1}/a \propto a^{1/2}$ , while during radiation domination  $H^{-1}/a \propto a$ . The recent domination by dark energy has initiated a new era of inflation. The horizontal dotted line indicates the present horizon scale. The number of  $e$ -foldings of inflation is the horizontal distance between the time when  $H^{-1}/a$  first crosses that value and the end of inflation.

where  $a_{\text{end}}$  is the value of the scale factor at the end of inflation and  $a_k$  is its value when the scale  $k$  equaled  $aH$  during inflation.<sup>2</sup> We will use  $N_{\text{hor}}$  to indicate  $N(a_0H_0)$ .

To determine the number of  $e$ -foldings corresponding to a scale measured in terms of the present Hubble scale, we need a complete model for the history of the Universe. At least from nucleosynthesis onwards, this is now well in place, but at earlier epochs there are considerable uncertainties. At this stage, we make the following simple assumptions for the sequence of events after inflation, considering possible alternatives in the next section. We assume that inflation is followed by a period of reheating, during which the Universe expands as matter dominated (this assumption is not true in all models—see Sec. II C). This then gives way to a period of radiation domination, which according to the standard cosmological model lasts until a redshift of a few thousand before giving way to matter domination, and then finally at a redshift below one to a cosmological constant or quintessence dominated era. We assume sudden transitions between these epochs, labelling the end of the reheating period by “reh” and the matter-radiation equality epoch by “eq.” This is illustrated in Fig. 1.

We can therefore write

$$\frac{k}{a_0H_0} = \frac{a_k H_k}{a_0H_0} = e^{-N(k)} \frac{a_{\text{end}}}{a_{\text{reh}}} \frac{a_{\text{reh}}}{a_{\text{eq}}} \frac{H_k}{H_{\text{eq}}} \frac{a_{\text{eq}} H_{\text{eq}}}{a_0H_0}. \quad (2)$$

<sup>2</sup>As discussed by Liddle, Parsons and Barrow [3], it makes more logical sense to define the amount of inflation as the ratio of  $aH$ , rather than  $a$ . More on that later; for now we follow the standard usage.

Some useful factors are (see, e.g. Ref. [4])

$$\frac{a_{\text{eq}} H_{\text{eq}}}{a_0 H_0} = 219 \Omega_0 h, \quad (3)$$

$$H_{\text{eq}} = 5.25 \times 10^6 h^3 \Omega_0^2 H_0, \quad (4)$$

$$H_0 = 1.75 \times 10^{-61} h m_{\text{pl}} \quad \text{with } h \approx 0.7. \quad (5)$$

Using the slow-roll approximation during inflation to write  $H_k^2 \approx 8\pi V_k / 3m_{\text{pl}}^2$ , we obtain

$$N(k) = -\ln \frac{k}{a_0 H_0} + \frac{1}{3} \ln \frac{\rho_{\text{reh}}}{\rho_{\text{end}}} + \frac{1}{4} \ln \frac{\rho_{\text{eq}}}{\rho_{\text{reh}}} + \ln \sqrt{\frac{8\pi V_k}{3m_{\text{pl}}^2} \frac{1}{H_{\text{eq}}}} + \ln 219 \Omega_0 h, \quad (6)$$

which agrees with Refs. [4,5] while being more precise about the prefactor. In fact, ultimately the dependence on the matter density  $\Omega_0$  will cancel out, and though a dependence on  $h$  remains this parameter is now accurately determined by observations.

### A. A plausible upper limit

The evolution of the Universe as described above is a plausible model for its entire history. Nevertheless, there are significant uncertainties in applying Eq. (6).  $V_k$  is a quantity we would hope to extract from the perturbations, but presently only upper limits exist, as the density perturbation amplitude depends on a combination of the potential and its slope, being unable to constrain either separately. Detection of primordial gravitational waves, which so far has not been achieved, is needed to break this degeneracy. We do not know how prolonged the reheating epoch might be, which is needed to determine  $\rho_{\text{reh}}$ , nor how much lower the energy density  $\rho_{\text{end}}$  at the end of inflation might be as compared to  $V_k$ .

Nevertheless, we can impose a plausible maximum on the number of  $e$ -foldings by making an assumption, namely that there is no significant drop in energy density during these last stages of inflation, so that  $V_k = \rho_{\text{end}}$ . Note, however, that this is *not* the correct way to maximize Eq. (6), a topic we return to in Sec. II D, and so is a non-trivial assumption. Having made it, the inflation line in Fig. 1 lies at  $45^\circ$ , and we can maximize the number of  $e$ -foldings by assuming that reheating is instantaneous, so that  $\rho_{\text{reh}} = \rho_{\text{end}}$ . Focussing now on the current horizon scale, this gives a maximum number of  $e$ -foldings corresponding to the horizon scale of

$$N_{\text{hor}}^{\text{max}} = \frac{1}{4} \ln \frac{\rho_{\text{eq}}}{V_{\text{hor}}} + \ln \sqrt{\frac{8\pi V_{\text{hor}}}{3m_{\text{pl}}^2} \frac{1}{H_{\text{eq}}}} + \ln 219 \Omega_0 h, \quad (7)$$

and substituting in the known quantities gives

$$N_{\text{hor}}^{\text{max}} = 68.5 + \frac{1}{4} \ln \frac{V_{\text{hor}}}{m_{\text{pl}}^4}. \quad (8)$$

The potential energy is bounded by the requirement that perturbations have the observed amplitude. For the accuracy level currently required, we can assume that the perturbations are entirely from density perturbations, whose amplitude is given in the slow-roll approximation by [4]

$$\mathcal{P}_{S,0} = \frac{8V}{3m_{\text{pl}}^4} \frac{1}{\epsilon}, \quad (9)$$

where

$$\epsilon = \frac{m_{\text{pl}}^2}{16\pi} \left( \frac{dV/d\phi}{V} \right)^2 \quad (10)$$

is the usual slow-roll parameter which observations restrict to  $\epsilon \lesssim 0.05$ . The observed perturbation amplitude on large scales is  $\mathcal{P}_{S,0} \approx 2.6 \times 10^{-9}$  [6] (ignoring a weak dependence on the precise form of the perturbations generated), giving

$$N_{\text{hor}}^{\text{max}} = 63.3 + \frac{1}{4} \ln \epsilon. \quad (11)$$

A similar formula was obtained by Dodelson and Hui [2] who additionally imposed an upper limit on  $\epsilon$  from gravitational wave limits. Note that in some models of inflation, particularly hybrid inflation models,  $\epsilon$  can be very small indeed; enough to make the last factor significant.

We have analyzed the values of  $N_{\text{hor}}^{\text{max}}$  for elements of a Monte Carlo Markov chain fit to a set of observational data, including WMAP and 2dFGRS, which generates values of  $V$  and the slow-roll parameters directly from the data as described in Ref. [7]. This confirms that for single-field inflation models the dependence on higher slow-roll parameters (via the changed normalization) is negligible and that Eq. (11) is an excellent description.

The formula we used for the perturbation amplitude assumes that there is only one dynamically important field during inflation, and may be modified if multi-field effects are important—see Ref. [8]. It would require a very large change in the perturbation amplitude to make a significant difference to Eq. (11), but if such a dramatic change is expected in a particular model, it would be necessary to recalculate the number of  $e$ -foldings specifically for that case.

We conclude that a plausible maximum number of  $e$ -foldings that can correspond to observable scales is around 62 for the standard picture of cosmological evolution following inflation. We stress that this says nothing about the total number of  $e$ -foldings that take place, which is expected to be much larger.

### B. A standard hypothesis

The assumptions made in the preceding section are not expected to hold precisely, and hence the expected number of  $e$ -foldings will be different. In this section we assess how different the number is expected to be, while remaining in the framework of the simplest cosmological history.

The two effects we need to allow for are that  $\rho_{\text{end}}$  will be less than  $V_{\text{hor}}$ , and that  $\rho_{\text{reh}}$  will be less than  $\rho_{\text{end}}$ . We can write

$$N_{\text{hor}} = N_{\text{hor}}^{\text{max}} + \frac{1}{4} \ln \frac{V_{\text{hor}}}{\rho_{\text{end}}} + \frac{1}{12} \ln \frac{\rho_{\text{reh}}}{\rho_{\text{end}}}. \quad (12)$$

The former effect is the one neglected in the previous section. Note that it *increases* the number of  $e$ -foldings required, an effect we study fully in Sec. II D. In hybrid inflation models, it is expected that there is very little reduction in the energy density during the late stages, while in slow-roll inflation models the reduction is typically one or two orders of magnitude. This term is therefore unlikely to increase  $N_{\text{hor}}$  by much more than one.

The main uncertainty resides in the final term. Reheating can easily be a prolonged process, during which the energy density drops by orders of magnitude. Indeed, in supersymmetric theories avoidance of overproduction of gravitinos requires an energy density below  $(10^{11} \text{ GeV})^4$  [9], implying a drop in energy density of around 20 orders of magnitude unless  $\epsilon$  has a tiny value. The most extreme assumption would be that reheating continues almost to nucleosynthesis, giving a lower limit at about  $(10^{-3} \text{ GeV})^4$ , though usually the electroweak scale  $(10^2 \text{ GeV})^4$  is regarded as the practical limit. Luckily the dependence has a prefactor of 1/12, so those three energy scales correspond to a reduction of  $N_{\text{hor}}$  by only 4, 15 and 11, respectively, for the case of large  $\epsilon$ . These numbers can be reduced if  $\epsilon$  is tiny as then the inflationary energy scale will be lower, but then a similar correction will be accrued from the  $\ln \epsilon$  term in Eq. (11). However, the gravitino limit may not apply in all models. In summary, a plausible value for the reduction in  $N_{\text{hor}}$  caused by reheating is 5  $e$ -foldings, with a likely range of about 5 in either direction around that.

Putting that information together, in the context of the simplest cosmology, a reasonable fiducial value for the number of  $e$ -foldings corresponding to the present Hubble scale is around 55, with an uncertainty of 5 around that. In the literature values of 50 or 60 are common, and in fact lie towards the extremes. However, we will see that, under fairly reasonable assumptions, there are several ways in which the number of  $e$ -foldings could lie outside that range, in either direction.

### C. The special case of $\lambda\phi^4$

The quartic potential  $V = \lambda\phi^4$  has been of particular interest lately as it lies in the region excluded by the WMAP analysis [1]. As the precise predictions for the spectra depend on the number of  $e$ -foldings, some care is required with models which are close to the exclusion limit, as highlighted by Barger *et al.* [10].

It turns out that for  $\lambda\phi^4$  we can be more precise, because reheating in a quartic potential has an unusual property—the expansion during the scalar field oscillations is as the radiation dominated [11], rather than the matter-dominated expansion given by oscillations in a quadratic potential. Accordingly, the duration of the epoch of reheating

no longer matters and we can take the Universe as radiation-dominated beginning at the end of inflation.<sup>3</sup> This gives

$$N_{\text{hor}}^{\text{quartic}} = N_{\text{hor}}^{\text{max}} + \frac{1}{4} \ln \frac{V_{\text{hor}}}{\rho_{\text{end}}}. \quad (13)$$

Additionally, as we have a definite model we can compute the ratio  $V_{\text{hor}}/\rho_{\text{end}}$ , which the slow-roll approximation gives as (see, e.g. Ref. [4])

$$\frac{V_{\text{hor}}}{\rho_{\text{end}}} \simeq N^2, \quad (14)$$

and the value of  $\epsilon$  which is  $1/N$ . Putting all this together gives

$$N_{\text{hor}}^{\text{quartic}} = 63.3 + \frac{1}{4} \ln N_{\text{hor}}, \quad (15)$$

whose solution is  $N_{\text{hor}}^{\text{quartic}} = 64$ . Hence under the assumptions of the simplest cosmology, the quartic potential allows an accurate specification of the number of  $e$ -foldings, the only approximation being the assumption of instantaneous transitions between epochs. The value in this model is unusually high because of the non-standard behavior during reheating and the significant reduction in  $H$  during the late stages which leads to it violating the limit of the preceding section. This large value means that the model is around the borderline of what present data allows [1,2,7,10].

#### D. Extreme cases, and a better definition of inflation

As the  $\lambda\phi^4$  case has illustrated, the plausible upper limit of Sec. II A is not as rigid as one would like, because reduction of the energy scale during inflation can play an important role. What inflation is really trying to achieve is to increase the ratio  $aH$ , and every reduction in  $H$  by a factor  $e$  then requires an extra  $e$ -folding of expansion to counter it. In terms of Fig. 1, the inflation line is shallower, and hence has a greater horizontal extent before reaching the standard post-inflationary evolution. Although the reduction in energy density shortens the evolution after inflation, it is clear from the figure [or inspection of Eq. (6)] that reducing the energy density during inflation wins, with the largest possible  $N$  being given by the smallest possible  $\rho_{\text{end}}$  (if all other parameters are unchanged) accompanied by instantaneous reheating. This again gives us Eq. (13), and in absolute generality  $\rho_{\text{end}}$  could be as late as nucleosynthesis, giving

$$N_{\text{hor}}^{\text{extreme}} \simeq 107. \quad (16)$$

<sup>3</sup>This picture may be altered if there is significant preheating [12]. However, usually it is assumed that the particles produced by preheating are rapidly converted to radiation, in which case the result as described is unchanged. If more complicated preheating phenomenology takes place (e.g. as in Ref. [13]) our results may be modified.

This is a surprisingly large value, and no plausible inflation model will generate it, but we mention it as possible in principle. To achieve such a large reduction in energy density while inflating, inflation must take place extremely close to the ‘‘coasting’’ limit of  $a\propto t$ , at which  $aH$  remains constant. In that limit, the  $e$ -foldings of inflation are very inefficient at pushing scales  $k$  outside the horizon  $aH$ . Note that such an evolution is not possible on scales with observable perturbations, as the generated spectrum would be far from scale invariant, but nothing in principle stops it occurring at the later stages.

A concrete example would be as follows. At a high energy scale, say  $V_{\text{hor}} = (10^{16} \text{ GeV})^4$ , we have a typical inflationary expansion, generating nearly scale-invariant perturbations and pushing them around 20  $e$ -foldings outside the horizon. This epoch then gives way to a fast-rolling inflationary epoch<sup>4</sup> with  $a\propto t^p$ , where  $p$  only slightly exceeds 1, with this fast-rolling epoch continuing all the way down to  $\rho_{\text{end}} = (1 \text{ GeV})^4$ . During the fast-roll era the perturbation spectrum will have sharply decreasing amplitude. As the density during this fast-roll stage is  $\rho\propto 1/a^{2/p}$ , this generates a further  $\Delta N = (2/p) \ln(V_{\text{hor}}/\rho_{\text{end}}) = 72/p \simeq 72$   $e$ -foldings. As during this evolution  $aH\propto t^{p-1} \simeq \text{const}$ , scales are not pushed further outside the Hubble radius during the fast-roll epoch, and so the perturbations generated during the slow-roll phase are correctly positioned to be those observable at the present epoch, even though nearly 100  $e$ -foldings have taken place since they were generated.

The issues raised in this section would be completely avoided had the more logical definition of the amount of inflation as the change in  $\tilde{N} \equiv \ln(aH)$  been used [3]. This definition automatically accounts for the reduction in  $H$  during inflation, and is given by

$$\tilde{N}(k) = N(k) + \ln \frac{H_{\text{end}}}{H_k} = N(k) - \frac{1}{2} \ln \frac{V_k}{\rho_{\text{end}}}. \quad (17)$$

This is sufficient to change the sign of the troublesome coefficient in Eqs. (12) and (13), thus ensuring that  $\tilde{N}$  is maximized by taking the largest possible  $\rho_{\text{end}}$  and  $\rho_{\text{reh}}$ . The plausible upper limit of Eq. (11) would then apply in general to  $\tilde{N}$ , including in the case of the quartic potential where  $\tilde{N}$  is significantly less than  $N$ .

### III. NON-STANDARD COSMOLOGIES: UPPER AND LOWER LIMITS

The previous section considered only the case of the simplest cosmology, where inflation gives way to reheating and then to the standard Hot Big Bang evolution. However, the appropriate value for  $N_{\text{hor}}$  is sensitive to modifications to that assumption, and there are no direct constraints on the evolution for most of the early history of the Universe.

In general these modifications could either increase or decrease  $N_{\text{hor}}$ . The two modifications we discuss which are

<sup>4</sup>See Ref. [14] for a general discussion of fast-roll inflation.

restricted to the period after inflation both serve to reduce the value of  $N_{\text{hor}}$ . However, we also discuss two possibilities which can raise  $N_{\text{hor}}$ , though both require modifications to the way inflation is modelled.

In this section we will neglect the possibility of a significant reduction in the energy density during the last stages of inflation, though such a reduction should be combined with the effects discussed here whenever a definite model is under discussion, and could be conveniently addressed by use of  $\tilde{N}$  in place of  $N$ .

### A. An upper limit

Although Sec. II A gives a plausible upper limit to the number of  $e$ -foldings for inflation assuming roughly constant energy density, it is still possible to raise the number further. What is needed is to replace part of the radiation-dominated era with a period where the Universe expands even more slowly. The limiting case consistent with causality is a stiff fluid dominated era where  $p=\rho$ , giving  $a \propto t^{1/3}$  and  $\rho \propto 1/a^6$ . In fact, such a period is not at all ridiculous, as this is the expansion law for a kinetic-energy dominated scalar field, and the literature contains several proposals for ending inflation by the inflaton field making a transition from potential energy domination to kinetic energy domination [15]. Further, such kinetic energy dominated periods tend to be prolonged if reheating is to proceed by gravitational particle production [11,16].

Instead of conventional reheating, we will consider a stiff fluid to dominate until an energy density  $\rho_{\text{kin}}$ , before giving way to radiation domination as before. Considering Eq. (6), the effect is to make the replacement

$$\frac{1}{3} \ln \frac{\rho_{\text{reh}}}{\rho_{\text{end}}} + \frac{1}{4} \ln \frac{\rho_{\text{eq}}}{\rho_{\text{reh}}} \rightarrow \frac{1}{6} \ln \frac{\rho_{\text{kin}}}{\rho_{\text{end}}} + \frac{1}{4} \ln \frac{\rho_{\text{eq}}}{\rho_{\text{kin}}}. \quad (18)$$

In order to find out how large this effect could be on the maximum number of  $e$ -foldings, we again take  $\rho_{\text{reh}} = \rho_{\text{end}}$  for the original scenario, while in the new scenario we take  $\rho_{\text{kin}}$  to be as small as possible. The Universe must have attained thermalized radiation domination by the time of nucleosynthesis, so the most radical modification is for the kinetic regime to end shortly before nucleosynthesis, at  $\rho_{\text{nuc}} \simeq (10^{-3} \text{ GeV})^4$ . The possible increase in  $N$  is therefore

$$N_{\text{extra}} = \frac{1}{12} \ln \frac{\rho_{\text{end}}}{\rho_{\text{nuc}}}. \quad (19)$$

As  $\rho_{\text{end}}$  could be as high as  $(10^{16} \text{ GeV})^4$ , in the most extreme case this can increase the number of  $e$ -foldings by as much as 15, as compared to the plausible maximum of Sec. II A.

In fact, stiff fluid cosmologies are constrained by the possibility of an excessive gravitational wave amplitude, which does not permit the stiff matter period to extend all the way to nucleosynthesis [17]. In practice, therefore, the increase permitted will not be as large as this calculation indicates. However, a rather detailed calculation would be required to determine the balance of reducing the inflationary energy

scale and shortening the stiff matter era which maximizes  $N$  without violating the gravitational wave constraint.

### B. Early matter domination

One possible modification to the simplest cosmology is for the long radiation-dominated epoch after reheating to be punctuated by epochs of matter domination, for example when long-lived massive particles go out of equilibrium and come to dominate the Universe before decaying. Moduli fields provide an example [18], though they are too long lived in many scenarios to be compatible with requirements.

Inserting a period of matter domination into Eq. (6) is simple, and it reduces  $N_{\text{hor}}$  by  $\Delta N = [\ln \rho_f / \rho_i] / 12$  where  $\rho_i$  and  $\rho_f$  are the densities at the beginning and end of the matter-dominated era, just as in the derivation of Eq. (12). A very prolonged period of matter domination is required to give a significant reduction.

### C. Thermal inflation

Thermal inflation was introduced in Ref. [19] as a means of solving relic abundance problems left over from the original phase of inflation. It is envisaged as one or more short periods of inflation, which are not so prolonged as to generate observable perturbations. The consequence pertinent to the present discussion is that thermal inflation corresponds to an extra stretching of the primordial perturbations, thus reducing  $N_{\text{hor}}$ .

Under the reasonable assumption that the energy density does not change significantly during thermal inflation, the effect is simply to reduce  $N_{\text{hor}}$  by the number of  $e$ -foldings  $N_{\text{thermal}}$  of thermal inflation. If thermal inflation is to achieve its purpose, this number is expected to be about 10, though there is also the possibility of multiple periods of thermal inflation.

### D. Braneworld cosmology

Another possible modification to the standard cosmology is if the Friedmann equation is modified at high energies, the archetypal example being the braneworld cosmology. For example, in the Randall-Sundrum type II model [20], at high energies we expect

$$H^2 = \frac{8\pi}{3m_{\text{Pl}}^2} \left( \rho + \frac{\rho^2}{2\lambda} \right), \quad (20)$$

where  $\lambda$  is the brane tension. A full discussion of the consequences of this is beyond the scope of this paper, but we note an interesting case where  $\lambda$  is much smaller than the energy at the end of inflation, so that the initial phase of the reheating, and possibly of the radiation-dominated era, take place during the high-energy regime  $\rho \gg \lambda$ .

Within the high-energy regime, the expansion laws corresponding to matter and radiation domination are slower than in the standard cosmology, being  $a \propto t^{1/3}$  and  $a \propto t^{1/4}$ , respectively, though the behavior of the densities as a function of the scale factor is unchanged. Slower expansion rates mean a greater change in  $aH$  relative to the change in  $a$ , which can

increase  $N_{\text{hor}}$ . However, a full calculation would have to include that inflation was taking place during the high-energy regime, which tends to force down the normalization of the potential giving rise to a particular amplitude of perturbations [21], and is beyond the scope of this paper.

#### E. An absolute minimum for $N_{\text{hor}}$

Given the uncertainties in the cosmological model, is it possible to say anything robust concerning the minimum possible value of  $N_{\text{hor}}$ ? The only guidance is that the success of primordial nucleosynthesis suggests that we should not seek to modify the standard cosmology after that epoch. As

$$\frac{a_{\text{nuc}}H_{\text{nuc}}}{a_0H_0} \approx 10^8, \quad (21)$$

we conclude that  $N_{\text{hor}}$  has a minimum of about 18  $e$ -foldings from the end of inflation. However, this extreme limit can only be realized in the unlikely case that either all the inflation really happened at such a low scale, or where repeated bouts of thermal inflation served to hold the perturbations on superhorizon scales long after they were formed.

#### IV. CONCLUSIONS

We have carried out an extensive analysis seeking to clarify the appropriate choices for the number of  $e$ -foldings from the end of inflation corresponding to observed perturbations. Assuming the simplest cosmology, we find a plausible maximum value of around 60, in good agreement with a recent paper of Dodelson and Hui [2], but noted that even fairly standard scenarios can violate it, an example being the  $\lambda\phi^4$  case which gives a higher value of 64  $e$ -foldings. That

model is also an exceptional one where a more accurate calculation is possible despite uncertainties about reheating.

In general, however, the number is sensitive both to a possible reduction in energy scale during the late stages of inflation, and to the complete cosmological evolution, and we have highlighted the effects of some plausible non-standard scenarios. In some cases, these may permit a higher maximum number of  $e$ -foldings than the standard cosmology.

Obviously the total number of  $e$ -foldings of inflation must be greater than  $N_{\text{hor}}$ , which concerns only observable scales. In almost all models of inflation it is expected to be very much greater, though these  $e$ -foldings are not accessible to observations.

In summary, for a typical inflation model it remains a sensible working hypothesis that the number of  $e$ -foldings lies between 50 and 60, where this number refers to the amount of expansion from when our present Hubble radius equalled the Hubble radius during inflation up until the end of inflation. However, if a particular model is under investigation, it may pay to attempt a more accurate calculation, at least to highlight the effect of assumptions concerning the cosmological evolution. This is particularly true if the model is expected to have a slow rate of inflation at its late stages, or to have an unusually low energy scale (corresponding to very small  $\epsilon$ ), or to have a particularly prolonged reheating period.

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