HOW NEIGHBORLY CAN A CENTRALLY SYMMETRIC POLYTOPE BE?

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ABSTRACT. We show that there exist k-neighborly centrally symmetric d-dimensional polytopes with $2(n + d)$ vertices, where

$$k(d, n) = \Theta\left(\frac{d}{1 + \log((d + n)/d)}\right).$$

We also show that this bound is tight.

1. INTRODUCTION

In this paper we study the following question: **How neighborly can a centrally symmetric polytope be as a function of its dimension and the number of vertices?** Let us recall the basic definitions: A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* (cs, for short) if for every $x \in P$, $-x$ belongs to P as well. A cs polytope P is called k-*neighborly* if every set of k of its vertices, no two of which are antipodal is the vertex set of a face of P.

It is well-know that a general (*non-cs*) d-dimensional polytope with at least $d+2$ vertices can be at most $\lfloor d/2 \rfloor$-neighborly, and this bound is attained for instance by d-dimensional cyclic polytopes [15, Example 0.6]. In contrast to the general case, the neighborliness of cs polytopes appears to be quite restricted and not sufficiently understood. A cs d-polytope with at least $2(d + 2)$ vertices cannot be more than $\lfloor (d + 1)/3 \rfloor$-neighborly. The case $d = 4$ of this statement was observed by Grünbaum [7, p.116] in 1967, whereas the general case is due to McMullen and Shephard [11]. These authors have also conjectured that a cs d-polytope with $2(d+n)$ vertices cannot be more than $\lfloor (d+n-1)/(n+1) \rfloor$-neighborly for all $n \geq 3$. Their conjecture was refuted by Halsey [8] and then by Schneider [13], but only for $d \gg n$. Namely, Schneider's theorem asserts that

$$\liminf_{d \to \infty} \frac{k(d, n)}{d + n} \geq 0.2390,$$

where $k(d, n)$ denotes the largest integer k such that there exists a k-neighborly cs d-polytope with $2(n + d)$ vertices.

A particularly interesting case is when $k(d, n) = 1$. Namely, given d, how large must n be so that a cs d-polytope with $\geq 2(n + d)$ vertices cannot be even 2-neighborly? That a cs d-polytope with a sufficiently large number of vertices ($\approx (d/2)^{d/2}$) indeed cannot be 2-neighborly was verified by Burton [3]. The McMullen-Shephard conjecture posits that this already holds for $n = d - 2$, but this turns out to be incorrect, since we show that this critical n is exponential in d. No other facts on $k(d, n)$ appear to exist in the literature.

Here we compute the correct asymptotics of $k(d, n)$, thus establishing an extension of Schneider's and Burton's results. Throughout the paper we set $m := n + d$.

Theorem 1.1.

$$\frac{C_1 d}{1 + \log \frac{m}{d}} \le k(d,n) \le 1 + \frac{C_2 d}{1 + \log \frac{m}{d}},$$

where $C_1, C_2 > 0$ are absolute constants independent of d and n. In particular, there exists a cs d-polytope with $4d$ vertices that is at least $\frac{d}{400}$-neighborly.

Theorem 1.1 provides the correct asymptotic answer for the above problem and shows that the largest number of vertices in a 2-neighborly cs d-polytope is $e^{\Theta(d)}$. In fact we can say a bit more here:

Theorem 1.2. $k(d, 2^{d-1} + 1 - d) = 1$. *In other words, a 2-neighborly cs d-polytope has at most 2^d vertices.*

We prove Theorem 1.2 as a warm-up for the proof of the upper bound in Theorem 1.1. These proofs appear in Section 2. The former result is a consequence of the Danzer-Grünbaum theorem on the number of vertices of antipodal polytopes [5] (a more readily accessible source for a proof of this theorem is Aigner and Ziegler's beautiful book [1, Ch. 14]) and the observation that every 2-neighborly cs polytope is antipodal. For the latter result we establish a certain modification of the Danzer-Grünbaum argument.

The proof of the lower bound in Theorem 1.1 is based on studying the cs transforms of cs polytopes introduced in [11] and on a theorem due to Garnaev and Gluskin [6]. This theorem concerns the intersection of the m-dimensional octahedron B_1^m with any n-dimensional subspace of $\mathbb{R}^m$. The question is how close to an n-dimensional Euclidean ball such an intersection can be. We outline the necessary background on cs transforms in Section 3. The verification of the lower bound in Theorem 1.1 and the statement of the Garnaev-Gluskin theorem are provided in Section 4. The proof of Garnaev-Gluskin result and hence also of Theorem 1.1 is probabilistic in nature: it does not give an explicit construction of neighborly cs polytopes, but rather shows that they form a set of positive probability in a certain probability space. Indeed it is an interesting open question to find explicit constructions of highly neighborly cs polytopes that meet the lower bound we prove.

2. The upper bound on $k(d,n)$

The goal of this section is to verify Theorem 1.2 and the upper bound in Theorem 1.1. To this end, recall from [7, p. 420] that two vertices v and w of a d-polytope $Q \subset \mathbb{R}^d$ are an *antipodal pair* if there exist two parallel (distinct) supporting hyperplanes of Q one of which contains v and the other one contains w. A d-polytope Q is called an *antipodal polytope* if every two of its vertices form an antipodal pair. The theorem due to Danzer and Grünbaum [5] (see also [1, Ch. 14]) asserts that an antipodal d-polytope cannot have more than 2^d vertices. Theorem 1.2 is then an immediate consequence of their result and the following observation.

Lemma 2.1. *Every 2-neighborly cs d-polytope is an antipodal polytope.*

Proof: Consider two vertices v and w of a 2-neighborly cs d-polytope P. We show that they form an antipodal pair. There are two possible cases: either $v = -w$ or $v \neq -w$. In the first case, let H be any hyperplane satisfying $H \cap P = \{v\}$. (Such an H exists since v is a vertex of P). Then H is a supporting hyperplane of P that contains v, while $-H := \{x \in \mathbb{R}^d : -x \in H\}$ is a supporting hyperplane of P that contains $w = -v$. In the second case, consider the set $E = \text{conv}\{v, -w\}$. Since P

is 2-neighborly, E is an edge of P, and so there exists a hyperplane H such that $H \cap P = E$. Then H is a supporting hyperplane of P that contains v, while $-H$ is a supporting hyperplane of P that contains the edge $E' = -E = \text{conv}\{-v, w\}$, and hence also the vertex w.   □

The proof of the upper bound in Theorem 1.1 is obtained by a certain modification of the Danzer-Grünbaum argument and is based on the following simple combinatorial result. (We denote by $\binom{[m]}{s}$ the family of all s-subsets of $[m] := \{1, \ldots, m\}$.)

Lemma 2.2. *For every two integers s and m with $s \le m/2$, there exists a family $\mathcal{F} \subset \binom{[m]}{s}$ of size $\ge (Cm/s)^{s/2}$ such that $|A \cap B| \le s/2$ for every two distinct $A, B \in \mathcal{F}$. Here $C > 0$ is an absolute constant.*

Proof: We construct $\mathcal{F}$ by a simple greedy argument. Let $\mathcal{F} \subset \binom{[m]}{s}$ be an inclusion-maximal family satisfying the condition $|A \cap B| \le s/2$ for all $A, B \in \mathcal{F}$. We claim that $|\mathcal{F}| \ge (Cm/s)^{s/2}$. Indeed, for a set $A \in \mathcal{F}$ define the "forbidden collection" $A^f := \{B \in \binom{[m]}{s} \; : \; |A \cap B| > s/2\}$. Then

$$|A^f| = \sum_{k=1}^{\lceil s/2 \rceil} \binom{s}{\lfloor s/2 \rfloor + k}\binom{m-s}{\lceil s/2 \rceil - k} < \binom{m}{\lfloor s/2 \rfloor}\sum_{k=1}^{\lceil s/2 \rceil}\binom{s}{\lfloor s/2 \rfloor + k} < \binom{m}{\lfloor s/2 \rfloor}2^s,$$

and so

$$\Big|\bigcup_{A \in \mathcal{F}} A^f\Big| \le |\mathcal{F}| \cdot \binom{m}{\lfloor s/2 \rfloor}2^s \le |\mathcal{F}| \cdot 2^s\left(\frac{em}{s/3}\right)^{s/2} \le |\mathcal{F}| \cdot \left(\frac{12em}{s}\right)^{s/2}.$$

On the other hand, the maximality of $\mathcal{F}$ implies that $\mathcal{F} \cup (\bigcup_{A \in \mathcal{F}} A^f) = \binom{[m]}{s}$. Hence $|\mathcal{F}| \cdot (1 + (\frac{12em}{s})^{s/2}) \ge \binom{m}{s}$ which together with the inequality $\binom{m}{s} \ge (\frac{m}{s})^s$ yields the result.   □

We are now in a position to prove the upper bound in Theorem 1.1 asserting that $k(d, n) \le 1 + \frac{C_2 d}{1 + \log \frac{m}{d}}$ for some absolute constant $C_2 > 0$. To do so consider a cs d-polytope P on the vertex set $V = \{v_1, -v_1, \ldots, v_m, -v_m\}$ that is $2s$-neighborly. Thus $2s \le d \le m$. Let $\mathcal{F} \subset \binom{[m]}{s}$ be a family from Lemma 2.2. For each set $A \in \mathcal{F}$, define

$$P_A := P + \frac{2}{s}\sum_{i \in A} v_i \subseteq P + 2P = 3P$$

to be a translate of P, where "+" denotes Minkowski addition.

We claim that the polytopes P_A, $A \in \mathcal{F}$, have pairwise disjoint interiors, whence

$$|\mathcal{F}| \cdot \text{vol}\,(P) = \sum_{A \in \mathcal{F}} \text{vol}\,(P_A) \le \text{vol}\,(3P) = 3^d \cdot \text{vol}\,(P).$$

It follows that

$$3^d \ge |\mathcal{F}| \ge (Cm/s)^{s/2}.$$

Thus $d \cdot \log 3 \ge s/2 \cdot \log(Cm/s) \ge \Omega(s \cdot \log(m/d))$, as claimed.

We turn to show that for any two distinct $A, B \in \mathcal{F}$ the sets P_A and P_B have disjoint interiors. The symmetric difference $R := (A - B) \cup (B - A) \subseteq [m]$ has cardinality $s \le |R| \le 2s$ since $|A| = |B| = s$ and $|A \cap B| \le s/2$. Therefore, the fact that P is $2s$-neighborly implies that $\{-v_i : i \in A - B\} \cup \{v_j : j \in B - A\} \subset V$

is the vertex set, vert (F), of some proper face F of P. We want to rule out the possibility that $x + \frac{2}{s} \sum_{i \in A} v_i = y + \frac{2}{s} \sum_{i \in B} v_i$ for some $x, y \in \text{int}\,(P)$. Indeed,

$$\frac{1}{2}\left[x + \frac{2}{s}\sum_{i \in A} v_i\right] - \frac{1}{2}\left[y + \frac{2}{s}\sum_{j \in B} v_j\right]$$

$$= \frac{x-y}{2} - \frac{1}{s}\left[\sum_{i \in A-B}(-v_i) + \sum_{j \in B-A} v_j\right] = \frac{x-y}{2} - \frac{|R|}{s}\sum_{v \in \text{vert}(F)} \frac{1}{|R|}v \neq 0,$$

since $(x-y)/2$ is an interior point of P, while $\frac{|R|}{s}\sum_v \frac{1}{|R|}v$ is the $|R|/s \geq 1$-multiple of a boundary point of P, namely of the barycenter of F. The conclusion follows. $\square$

3. Centrally Symmetric transforms

Throughout the rest of the paper we denote the standard scalar product on $\mathbb{R}^n$ by $\langle -, - \rangle$. The abbreviations int, rel int, and conv stand for the interior, relative interior, and convex hull respectively.

Following [11], we define a *centrally symmetric set* (cs set, for short) as a finite spanning subset of $\mathbb{R}^d$ of the form $V = \{v_1, -v_1, \ldots, v_m, -v_m\}$. The construction described in [11] associates with a cs set $V = \{\pm v_1, \ldots, \pm v_m\} \subset \mathbb{R}^d$ another cs set $\overline{V} = \{\pm \overline{v}_1, \ldots, \pm \overline{v}_m\} \subset \mathbb{R}^{m-d} = \mathbb{R}^n$ called the *cs transform* of V. This operation possesses the following properties.

(1) Let $\overline{V} = \{\pm \overline{v}_1, \ldots, \pm \overline{v}_m\} \subset \mathbb{R}^n$ be a cs set. Then $\overline{V}$ is a cs transform of the vertex set V of a cs d-polytope P with $2m$ vertices if and only if

$$\overline{v}_i \in \text{int conv}\left\{\sum_{l \in [m]-\{i\}} \epsilon_l \overline{v}_l \ : \epsilon_l \in \{1, -1\}\right\} \quad \text{for all } i = 1, \ldots, m.$$

(2) More generally, if $V = \{\pm v_1, \ldots, \pm v_m\} \subset \mathbb{R}^d$ is the vertex set of a cs d-polytope P and $\overline{V}$ is a cs transform of V, then the set $\{\delta_i v_i : i \in I\} \subset V$ (where $\delta_i \in \{1, -1\}$, $i \in I$, are fixed signs and $I = \{i_1 < \ldots < i_k\} \subseteq [m]$) is the vertex set of a face of P if and only if

$$\sum_{i \in I} \delta_i \overline{v}_i \in \text{rel int conv}\left\{\sum_{l \in [m]-I} \epsilon_l \overline{v}_l \ : \epsilon_l \in \{1, -1\}\right\}.$$

Thus if $V = \{\pm v_1, \ldots, \pm v_m\} \subset \mathbb{R}^d$ is a cs set, then its subset $\{\delta_i v_i : i \in I\}$ (for some $\delta_i \in \{1, -1\}$ and $I = \{i_1 < \ldots < i_k\} \subseteq m$) fails to be the vertex set of a face of the cs polytope $P := \text{conv}\, V$ if and only if there exists $u \in \mathbb{R}^n$ such that

$$0 \neq \left\langle \sum_{i \in I} \delta_i \overline{v}_i, u \right\rangle \geq \left\langle \sum_{l \in [m]-I} \pm \overline{v}_l, u \right\rangle \quad \text{for all choices of signs,}$$

that is, if and only if

$$0 \neq \sum_{i \in I} \langle \delta_i \overline{v}_i, u \rangle \geq \sum_{l \in [m]-I} |\langle \overline{v}_l, u \rangle|.$$

We call a subset $\{\overline{v}_i : i \in I\}$ of $\overline{V}_+ := \{\overline{v}_1, \ldots, \overline{v}_m\} \in \mathbb{R}^n$ *dominant* if there exists $0 \neq u \in \mathbb{R}^n$ such that

$$\sum_{i \in I} |\langle \overline{v}_{i_j}, u \rangle| \geq \sum_{l \in [m]-I} |\langle \overline{v}_l, u \rangle|, \text{ or equivalently, } \sum_{i \in I} |\langle \overline{v}_i, u \rangle| \geq \frac{1}{2} \sum_{l=1}^{m} |\langle \overline{v}_l, u \rangle|.$$

Since $\sum_{i \in I} \langle \delta_i \overline{v}_i, u \rangle \leq \sum_{i \in I} |\langle \overline{v}_i, u \rangle|$ for $\delta_i = \pm 1$, and since equality is attained for a certain choice of signs, we obtain the following criterion.

Lemma 3.1. *A cs set $\overline{V} = \{\pm \overline{v}_1, \ldots, \pm \overline{v}_m\} \subset \mathbb{R}^n$ is a cs transform of the vertex set of a k-neighborly cs d-polytope with $2m = 2(n+d)$ vertices if and only if the set $\overline{V}_+ := \{\overline{v}_1, \ldots, \overline{v}_m\}$ does not contain dominant subsets of size k.*

Thus to prove lower bounds on $k(d, n)$ it suffices to construct vector configurations spanning $\mathbb{R}^n$ that do not contain small dominant subsets. This is done in the following section.

4. Vector configurations without small dominant subsets

For a vector $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ its l_1^m and l_2^m norms are defined as $\|x\|_1 := \sum_{i=1}^{m} |x_i|$ and $\|x\|_2 := \sqrt{\sum_{i=1}^{m} |x_i|^2}$, respectively. Thus the unit ball of l_1^m is the m-dimensional octahedron $B_1^m := \{x \in \mathbb{R}^m : \sum_{i=1}^{m} |x_i| \leq 1\}$, while the unit ball of l_2^m is the m-dimensional Euclidean ball $B_2^m = \{x \in \mathbb{R}^m : \sum_{i=1}^{m} x_i^2 \leq 1\}$. The theorem due to Garnaev and Gluskin [6] (see [10] for a simplified proof) quantifies the extent to which the intersection of B_1^m with an n-dimensional subspace of $\mathbb{R}^m$ can be close to B_2^n. It asserts that for any natural numbers d and n there exists a subspace L^d of $\mathbb{R}^m$ of codimension d (equivalently, of dimension $m - d = n$), such that

$$(1) \qquad \|x\|_2 \leq \widetilde{C} \sqrt{\frac{1 + \log(m/d)}{d}} \cdot \|x\|_1 \quad \text{for all } x \in L^d,$$

where $\widetilde{C}$ is an absolute constant independent of d and m. (In fact, the set of such subspaces has a positive measure in the Grassmannian manifold $G_{m-d,m}$ of all codimension d subspaces of $\mathbb{R}^m$ endowed with the normed unitary invariant measure.) In the following we refer to such subspace L^d as a Garnaev-Gluskin subspace. A weaker version of this theorem, with $(1 + \log(m/d))^{3/2}$ instead of $(1 + \log(m/d))^{1/2}$, had been shown earlier by Kašin [9].

Since $L^d \subset \mathbb{R}^m$ is an n-dimensional space, there is a linear injective map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose image is L^d. Let A be the $m \times n$ matrix representing this map, and let $\overline{v}_1, \cdots, \overline{v}_m \in \mathbb{R}^n$ be the rows of this matrix. Then for every $0 \neq u \in \mathbb{R}^n$, $T(u)$ is a non-zero element of L^d whose i-th coordinate is given by $\langle \overline{v}_i, u \rangle$ (for $i = 1, \ldots, m$). Hence for every k-element subset $\{\overline{v}_i : i \in I\}$ of $\{\overline{v}_1, \cdots, \overline{v}_m\}$ and for every $0 \neq u \in \mathbb{R}^n$, we have

$$\begin{aligned}
\sum_{i \in I} |\langle \overline{v}_i, u \rangle| &\leq \sqrt{k} \cdot \sqrt{\sum_{i \in I} \langle \overline{v}_i, u \rangle^2} && \text{(by the Cauchy-Schwarz inequality)} \\
&\leq \sqrt{k} \cdot \|T(u)\|_2 \leq \sqrt{k} \cdot \widetilde{C} \cdot \sqrt{\frac{1 + \log(m/d)}{d}} \cdot \|T(u)\|_1 && \text{(by Eq. (1))} \\
&= \sqrt{k} \cdot \widetilde{C} \cdot \sqrt{\frac{1 + \log(m/d)}{d}} \cdot \sum_{l=1}^{m} |\langle \overline{v}_l, u \rangle| < \frac{1}{2} \sum_{l=1}^{m} |\langle \overline{v}_l, u \rangle|
\end{aligned}$$

as long as $k < \frac{1}{4\hat{C}^2} \cdot \frac{d}{1+\log(m/d)}$. Therefore we infer the following result.

Lemma 4.1. *Let L^d be a Garnaev-Gluskin subspace of $\mathbb{R}^m$. Denote by $\overline{V}_+ = \{\overline{v}_1, \cdots, \overline{v}_m\} \subset \mathbb{R}^n$ the set of rows of the matrix representing a map $T : \mathbb{R}^n \to \mathbb{R}^m$ whose image is L^d. Then $\overline{V}_+$ does not contain dominant subsets of the size smaller than $\lceil \frac{1}{4C^2} \cdot \frac{d}{1+\log\frac{m}{d}} \rceil$.*

The lower bound in Theorem 1.1 asserting that for every n and d there exists a cs d-polytope with $2m = 2(n + d)$ vertices that is $\Omega(\frac{d}{1+\log\frac{m}{d}})$-neighborly is then an immediate corollary of Lemmas 3.1 and 4.1.

To obtain an estimate on the constant for the $n = d$ case (the "in particular"-part of Theorem 1.1), we use the following result essentially due to Kašin (see [2, page 21]) asserting that there exists an orthogonal transformation U of $\mathbb{R}^d$ such that

$$(2) \qquad \|x\|_2 \leq \frac{4R^2}{\sqrt{d}} \left(\|U^{-1}x\|_1 + \|x\|_1 \right) \qquad \text{for all } x \in \mathbb{R}^d.$$

Here R is the *volume ratio* of the octahedron B_1^d (the notion introduced by Szarek [14]), that is,

$$(3) \qquad R := \left(\frac{\text{vol}(\sqrt{d}B_1^d)}{\text{vol}(B_2^d)} \right)^{1/d} = \left(\frac{2^d d^{d/2}}{d!} \frac{\Gamma(d/2+1)}{\pi^{d/2}} \right)^{1/d} \leq \left(\frac{2e}{\pi} \right)^{1/2},$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Consider the $2d$-element set $\overline{V}_+ := \{e_1, \ldots, e_d, Ue_1, \ldots, Ue_d\} \subset \mathbb{R}^d$. The calculations completely analogous to those in the proof of Lemma 4.1, but using Eq. (2) instead of (1), imply that all dominant subsets of the set $\overline{V}_+$ have size of at least

$$\frac{d}{2^7 R^4} \overset{\text{by (3)}}{\geq} \frac{\pi^2}{2^9 e^2} d \geq \frac{d}{400}.$$

This fact together with Lemma 3.1 yields the second part of the Theorem 1.1, that is, existence of a $\frac{d}{400}$-neighborly cs d-polytope with $4d$ vertices.

5. Concluding remarks

We close the paper with the following remark concerning the exact value of $k(n, d)$. Let $\overline{V}_+ = \{\overline{v}_1, \ldots, \overline{v}_m\} \subset \mathbb{R}^n$ be an m-element set that spans $\mathbb{R}^n$. Denote by $A = A(\overline{V}_+)$ the $m \times n$ matrix whose rows are the elements of $\overline{V}_+$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the map represented by A, and let $L^d = L^d(\overline{V}_+)$ be the image of T. Clearly L^d has codimension d in $\mathbb{R}^m$, and every codimension d subspace of $\mathbb{R}^m$ arises this way.

Fix an integer $s \in [m]$ and consider the norm $||| - |||_s$ on $\mathbb{R}^m$ defined by

$$|||(x_1, \ldots, x_m)|||_s := \max\{\sum_{i\in\sigma} |x_i| : \sigma \subset [m], |\sigma| = s\}.$$

(E.g. $||| - |||_1$ coincides with the l_∞^m norm.) Since the i-th coordinate of $x = T(u) \in L_d$ equals $\langle \overline{v}_i, u \rangle$, it follows that $\overline{V}_+$ does not have a dominant subset of size s if and only if $|||x|||_s < \frac{1}{2}\|x\|_1$ for every $0 \neq x \in L^d$. Thus Lemma 3.1 implies that

$$(4) \quad k(d, n) = \max\{s : c_d(l_1, ||| - |||_s) < 1/2\} = \min\{s : c_d(l_1, ||| - |||_s) \geq 1/2\} - 1,$$

where $c_d = c_d(l_1, ||| - |||_s)$ is the *d-th Gelfand number,*

$$c_d := \inf_{L^d \subset \mathbb{R}^m} \sup_{x \in L_d - \{0\}} \frac{|||x|||_s}{\|x\|_1}.$$

(Here the infimum is taken over all codimension d subspaces of $\mathbb{R}^m$). Gelfand numbers have received a good deal of attention in Banach Space Theory [4, 12].

It is a notoriously difficult question to find explicit constructions for spaces that satisfy the conditions as in the work of Kašin, Garnaev and Gluskin. Perhaps it is less difficult, though, to construct highly neighborly cs polytopes. Any progress on this problem would be of interest.

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