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How the Propagation of Heat-Flux Modulations Triggers $E \times B$ Flow Pattern Formation

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We propose a novel mechanism to describe $E \times B$ flow pattern formation based upon the dynamics of propagation of heat-flux modulations. The $E \times B$ flows of interest are staircases, which are quasiregular patterns of strong, localized shear layers and profile corrugations interspersed between regions of avalanching. An analogy of staircase formation to jam formation in traffic flow is used to develop an extended model of heat avalanche dynamics. The extension includes a flux response time, during which the instantaneous heat flux relaxes to the mean heat flux, determined by symmetry constraints. The response time introduced here is the counterpart of the drivers' response time in traffic, during which drivers adjust their speed to match the background traffic flow. The finite response time causes the growth of mesoscale temperature perturbations, which evolve to form profile corrugations. The length scale associated with the maximum growth rate scales as $\Delta^2 \sim (v_{\mathrm{thi}}/\lambda T_i)\rho_i\sqrt{\chi_{\mathrm{neo}}\tau}$, where λT_i is a typical heat pulse speed, χ_{neo} is the neoclassical thermal diffusivity, and τ is the response time of the heat flux. The connection between the scale length Δ^2 and the staircase interstep scale is discussed.

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Pattern formation is a widely observed phenomenon in nonequilibrium and nonlinear systems [1]. In magnetized plasmas, $E \times B$ flow patterns are often observed to self-organize and emerge from the bath of turbulence [2,3]. A well-known example of such processes is the formation of a pattern of zonal flows [2], which are generated by drift wave turbulence by processes such as modulational instability [2], mixing of the potential vorticity of drift wave turbulence [4], etc. More recently, a new class of flow pattern, called the $E \times B$ staircases (see Fig. 1), was observed in a flux-driven full-f gyrokinetic simulation using the GYSELA code [5]. As discussed on the basis of the simulation study, $E \times B$ staircases are quasiregular steady patterns of localized shear flows and corrugated temperature profiles (see Fig. 1). The shear flows are interspaced between regions of turbulent avalanching [5–9] separated by Δ, typically in the mesoscale range $l_c < \Delta < a$. Here l_c is the turbulence correlation length and a is the system size. In the interspaced regions of extent Δ, transport is dominated by stochastic avalanches [5]. Scattering of fluctuation energy and spreading of the turbulence may occur in these regions [10–13]. The entire pattern of the localized shear layers and the regions of extent Δ is an $E \times B$ staircase, so named after potential vorticity staircases of jets in the atmosphere [14,15].

The generation of $E \times B$ staircases might not be surprising, since avalanches should mix the potential vorticity of plasmas and hence generate $E \times B$ flows [4]. However, what is remarkable here is the fact that a quasiregular pattern of flows, interspaced by stochastic regions of extent Δ, emerges from the bath of avalanches. The dynamics of this pattern formation from avalanches cannot be addressed by existing theory [6,7], as such models predict that avalanches of the system size a dominate transport. Thus, in order to describe the formation of an $E \times B$ staircase pattern, we need further development of the theory of avalanche dynamics, which explains the emergence of a particular mesoscale.

In this Letter, we propose a model to describe the formation of $E \times B$ staircases from an ensemble of heat avalanches. The model extends the basic theory of avalanche dynamics [6,7] to include a finite relaxation time, during which the heat flux relaxes to the mean value determined by symmetry constraints. The key idea for the model extension is the analogy of staircase formation to jam formation in traffic flow [16,17]. Namely, we view staircase formation as a heat flux 'jam' that causes profile corrugations (see Fig. 1), which is analogous to a traffic jam that causes corrugations in the local car density in a

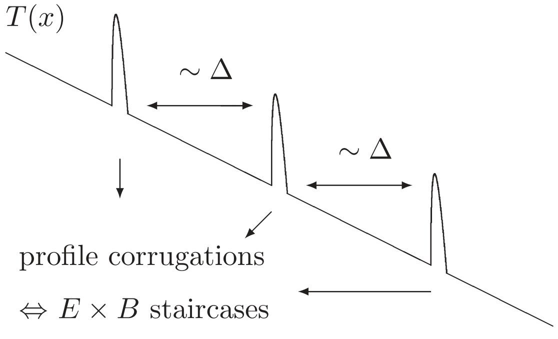

FIG. 1. $E \times B$ staircases and profile corrugations.

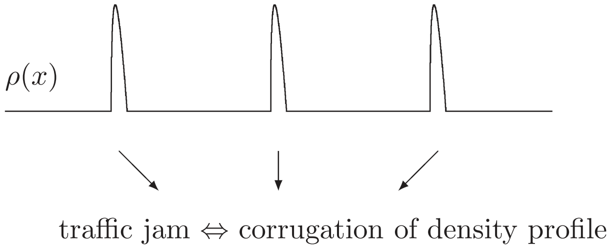

FIG. 2. Traffic jam and density corrugation.

traffic flow (see Fig. 2). In traffic jam formation, an important time scale is the drivers' response time. Since each individual car has its own instantaneous velocity, drivers often need to adjust their speed to the background traffic flow in a finite time τ. As shown in the literature [16,17], if the drivers' response time τ is too long, traffic jams can form. In traffic flow theory, 'jams' appear as quasiregular spikes in the car density, which evolve from nonlinear density waves. To model such an effect in plasmas, we are then led to introduce a finite response time τ, during which the instantaneous heat flux relaxes to the mean heat flux, determined by symmetry constraints. This approach tacitly replaces the static Fick's law with an evolutionary Guyer-Krumhansl relation [18] $\tau\partial_t Q - \chi\nabla T + Q = 0$. The latter may be systematically obtained from moments of the fluctuation entropy equation [19]. As shown in the following, the extended model equation has a mathematical structure similar to the Kuramoto-Sivashinsky equation [1], familiar from the study of pattern formation dynamics. We also show that the extended model describes a heat flux 'jam' and profile corrugation, which appears as an instability, in analogy to the clustering instability of the formation of a traffic jam. We argue that such a local amplification of heat and profile corrugations can lead to the formation of $E \times B$ staircases. We also derive the scale for the maximum growth rate and evaluate its value at a stationary state achieved via $\gamma_{\text{jam}} \sim v'_{E\times B}$. Here γ_{jam} is the maximum growth rate of the jamming instability and $v'_{E\times B}$ is the $E \times B$ shearing rate across the jam scale. The scale length Δ that gives the maximum growth rate is evaluated, and we argue that the length scale Δ falls in the mesoscale range and is comparable to the staircase step spacing.

The model equation is derived as follows. Around the marginal profile, a conserved order parameter is the temperature perturbation, and its dynamics is described by

$$\partial_t \delta T + \partial_x Q[\delta T] = 0, \tag{1}$$

where Q is the heat flux. Here, it is understood that the right-hand side vanishes, up to a source and noise. The equation is closed by employing a model for the flux $Q[\delta T]$. A useful approach for constructing $Q[\delta T]$ is to exploit the symmetry properties of heat avalanche dynamics [6,7]. These are based on the simple idea that net transport must be down the gradient in the avalanching process. In other words, blobs (local heat surpluses) propagate down the mean gradient while holes (local heat deficits) propagate up the mean gradient. This property requires $Q[\delta T]$ to satisfy joint reflection symmetry; i.e., the dynamics should be invariant under the transformation of $x \to -x$ and $\delta T \to -\delta T$. This constrains the form of $Q[\delta T]$ to be

$$Q[\delta T] = \sum_{p,q,r} \{A_{2p}(\delta T)^{2p} + B_{q,r}\partial_x^q \delta T^r + \ldots\} \tag{2}$$

with $q + r =$ even. A nontrivial, nonlinear flux is, for example,

$$Q_0[\delta T] = \frac{\lambda}{2}(\delta T)^2 - \chi_2 \partial_x \delta T + \chi_4 \partial_x^3 \delta T. \tag{3}$$

Here χ_2 is roughly comparable to the neoclassical diffusivity χ_{neo}, i.e., the diffusivity at marginality. Combining Eqs. (1) and (3) gives the Burgers equation (up to χ_4) which was derived in the former study [7].

Here, we propose an extension of the model for $Q[\delta T]$, to describe profile corrugation and $E \times B$ staircase formation. The key to the extension is an analogy between profile corrugation in heat avalanche dynamics and jam in traffic flow dynamics [16,17], as discussed above (see Table I). To proceed, it is useful to recall that traffic jam dynamics is sometimes modeled [16] as a one-dimensional 'gas dynamic' flow of the form

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{4}$$

$$\partial_t v + v\partial_x v = -\frac{1}{\tau}\left(v - V(\rho) + \frac{\nu}{\rho}\partial_x \rho\right). \tag{5}$$

Here, Eq. (4) is the continuity equation for the car density ρ, and Eq. (5) describes the dynamics of the traffic velocity, including the drivers' finite time response τ to a specified background traffic flow speed $V(\rho) - (\nu/\rho)\partial_x \rho$.

TABLE I. Comparison of heat avalanche dynamics and traffic jam dynamics.

Heat avalanche	Traffic flow
temperature $\delta T \to$ corrugation	car density perturbation $\delta\rho \to$ jam
Heat flux Q	Traffic flow v
$Q_0[\delta T]$ mean flux	$V(\delta\rho) - (\nu/\rho)\partial_x \delta\rho$ background traffic flow,
via symmetry constraints	determined empirically or by general consideration
heat flux relaxes to the mean	drivers adjust their speed to the surrounding
flux in a time τ	traffic speed in a time τ

An interesting feature of the traffic dynamics model is that the model describes jam formation. The formation of a jam is related to an instability, whose the threshold is given by $\tau > \nu/(\rho_0^2 V_0'^2)$. Here ρ_0 is an equilibrium density and $V_0' = dV/d\rho|_{\rho_0}$. Note that the instability favors a long response time; when drivers cannot promptly respond to the background traffic flow, an instability occurs and jams form. Eventually, the instability develops into a nonlinear wave in the density profile, which is termed a 'jamiton' [17]. Based on the analogy between the traffic jam and temperature corrugation by heat avalanche, we expect that such jams can be modeled in avalanche dynamics by extending the heat flux equation to

$$\partial_t Q = -\frac{1}{\tau}(Q - Q_0[\delta T]). \tag{6}$$

Namely, we include a process of relaxation of the instantaneous heat flux Q to the mean flux $Q_0[\delta T]$ in a finite time τ, as heat pulses adjust to the ambient heat flux. Equation (6), together with Eq. (1), constitutes the basic model that we consider in the rest of the Letter.

While the model equation [Eq. (6)] is derived from a heuristic argument, a more systematic derivation is possible, and is useful to gain more insight into the response time τ. One systematic approach is to consider the evolution of the two-point fluctuation phase space density correlation—i.e., $\partial_t\langle\delta f(1)\delta f(2)\rangle$—and then take its energy moment [19]. This yields the evolution equation for the turbulent heat flux correlator. Such an analysis, described in Ref. [19], yields a nonlinear version of the Guyer-Krumhansl flux-gradient relation:

$$\partial_t Q - \partial_x[D_x(\mathcal{E})\partial_x Q] = -D_y(\mathcal{E})k_y^2(Q - \chi(\mathcal{E})\partial_x\langle T\rangle). \tag{7}$$

Here, $\mathcal{E}$ is the turbulent intensity. The first term is simply the delayed response of the flux. The second term corresponds to the turbulent transport of the heat flux, which is akin to turbulence spreading and is modeled using a simple quasilinear expression $\Gamma_Q \sim -D(\mathcal{E})\partial_x Q$. The first term in the right-hand side is the turbulent eddy damping, where k_y is roughly the wavelength of the mode of the maximum growth. In the last term, $\chi(\mathcal{E}) = [D_y(\mathcal{E})k_y^2]^{-1}\langle\tilde{v}_r^2\rangle \sim \tau_c\langle\tilde{v}_r^2\rangle$ is the turbulent heat diffusivity, and this term reduces to Fick's law in the local, stationary limit by balancing against the eddy damping term. By comparing Eqs. (7) and (6), we expect that the finite relaxation time τ would be comparable to the nonlinear decorrelation time, $[k_y^2 D_y(\mathcal{E})]^{-1}$. We also note that the relaxation time τ is nonlinear, i.e., $\tau[\delta T]$, since $\tau \propto [k_y^2 D_y(\mathcal{E})]^{-1}$ and $\mathcal{E} \propto \delta T$. Finally, we remark that the extension of the flux-gradient relation to include the finite response time does not violate the second law of thermodynamics, since the increase of entropy can be guaranteed by extending the definition of the entropy production [19].

While the systematic approach has its own merit, it is not an easy task to solve Eqs. (1) and (7) simultaneously, with the nonlinear response time $\tau[\delta T]$. Here, instead, given the solid foundation for Eq. (6), we use Eq. (6) as the model for the heat flux Q, and treat τ as a parameter of the order of the turbulence correlation time τ_c. While this simplification tacitly assumes that we can neglect the dynamics of the background turbulence, such a simplification may be allowed when the background turbulence is in a stationary state.

Then, combining Eqs. (1) and (6), we obtain a single equation for δT evolution:

$$\partial_t\delta T + \lambda\delta T\partial_x\delta T = \chi_2\partial_x^2\delta T - \chi_4\partial_x^4\delta T - \tau\partial_t^2\delta T. \tag{8}$$

This equation describes the dynamics of the temperature profile, where the instantaneous heat flux relaxes toward the mean flux $Q_0[\delta T]$ in the finite time τ. Equation (8) retains avalanche dynamics, as it reduces to the Burgers equation in the limit of long wavelength and short τ, i.e., $\partial_x^4 \to 0$ and $\tau \to 0$. Here as an extension, we included τ and χ_4. As explained below, the finite response time τ allows an instability, from which the corrugation of the temperature profile develops. χ_4 is included to prevent an arbitrarily small scale structure from developing, at which point the theory breaks down.

Now, we turn to the analysis of Eq. (8). Here, the aim is to show that an instability can occur and δT grows to corrugate the temperature profile. We also derive the scale length that gives the maximum corrugation growth and compare it to the staircase width. The basic feature of the instability is nicely illustrated by a simple calculation. To see this, we consider the evolution of a perturbation $\delta T = \delta T_0 + \widetilde{\delta T}$. The dynamics is given by

$$\partial_t\widetilde{\delta T} + c_0\partial_x\widetilde{\delta T} = \chi_2\partial_x^2\widetilde{\delta T} - \chi_4\partial_x^4\widetilde{\delta T} - \tau\partial_t^2\widetilde{\delta T}, \tag{9}$$

where $c_0 \equiv \lambda\delta T_0$. If we evaluate the right-hand side in the moving frame of the initial avalanche c_0, we have $(\chi_2 - \tau c_0^2)\partial_x^2\widetilde{\delta T} - \chi_4\partial_x^4\widetilde{\delta T}$. Hence we see that if $\chi_2 - \tau c_0^2 < 0$ then we have a 'negative diffusivity,' and thus expect an instability to occur. This is analogous to zonal flow generation by a negative viscosity, which occurs during the modulational instability of drift wave turbulence [2]. Then, as zonal flows are secondary modes generated in the bath of primary drift wave turbulence, we may view the 'avalanche jamiton' as a secondary mode generated in the gas of primary avalanches. In broader contexts, both phenomena are examples of 'second sound' phenomena, which are generated in the primary gas of phonons [18]. Finally, we point out that the nonlinear dynamics of $\widetilde{\delta T}$ would be similar to that of the Kuramoto-Sivashinsky (K-S) equation [1], which consists of a quadratic nonlinear term, a negative viscosity term, and a hyperviscosity term. The K-S model is successful in reproducing many cellular patterns [1] and may be important for a nonlinear analysis of the avalanche flux jamiton. The nonlinear equation, coupled with the turbulence evolution, may be utilized for the nonlinear analysis of spikes in staircases.

The growth rate of the instability is calculated as follows. Fourier analyzing Eq. (9) gives the dispersion relation:

$$\omega_{r,k} = \pm\frac{1}{2\tau}\sqrt{\frac{r-1}{2} + 2\tau\chi_2 k^2\left(1 + \frac{\chi_4 k^2}{\chi_2}\right)}, \tag{10}$$

$$\gamma_k = -\frac{1}{2\tau} + \frac{1}{2\tau}\sqrt{\frac{r+1}{2} - 2\tau\chi_2 k^2\left(1 + \frac{\chi_4 k^2}{\chi_2}\right)}, \tag{11}$$

where $r \equiv \sqrt{\{4\tau\chi_2 k^2(1 + \chi_4 k^2/\chi_2) - 1\}^2 + 16c_0^2 k^2 \tau^2}$. The threshold for the instability ($\gamma_k > 0$) is then

$$\tau > \frac{\chi_2}{c_0^2}\left(1 + \frac{\chi_4 k^2}{\chi_2}\right). \tag{12}$$

Thus the instability occurs when the response time is sufficiently long, as in the traffic dynamics model. Note that the threshold states $c_0 > \sqrt{\chi_2/\tau}$, i.e., the initial avalanche speed has to be faster than the heat diffusion length in one relaxation time. This puts a threshold on the pulse size for growth.

We now derive the wave number that gives the maximum growth rate. This number is of interest since the scale associated with that wave number can be compared to a typical $E \times B$ staircase step width Δ_{stair}. The wave number for $\gamma_{\max}$ is obtained by solving $\partial\gamma/\partial k^2 = 0$. Seeking a solution $k^2 < \chi_2/\chi_4$, we obtain

$$k_{\max}^2 \cong \frac{\chi_2}{\chi_4}\sqrt{\frac{\chi_4 c_0^2}{4\chi_2^3}} = \frac{\lambda\delta T_0}{2\sqrt{\chi_2\chi_4}}. \tag{13}$$

Here $c_0 = \lambda\delta T_0$ is the speed of the initial avalanche. With the wave number, we can estimate the scale of the most unstable fluctuation and the maximal growth rate as $\Delta_{\max}^2 = k_{\max}^{-2}$ and $\gamma_{\max} \cong c_0/(2l_{\mathrm{diff}})$, where $l_{\mathrm{diff}} = \sqrt{\chi_2\tau} \sim \sqrt{\chi_{\mathrm{neo}}\tau}$.

Now we evaluate $\Delta_{\max}^2$ at a saturated state. Namely, once the jamming instability starts, the profile starts corrugating. Such a profile corrugation leads to the formation of $E \times B$ shear layers, which can feedback on the instability through standard shearing effects $v'_{E\times B}$ [20]. Crudely, we expect that saturation might occur when $\gamma_{\max} \sim v'_{E\times B}$. Here, $v'_{E\times B}$ is produced by corrugated profiles via a radial force balance, i.e.,

$$v'_{E\times B} \cong \frac{c}{eB}\delta T'' \cong \frac{c\delta T}{eB\Delta_{\max}^2} \sim \frac{\omega_{ci}\rho_i^2\lambda T_i}{2\sqrt{\chi_2\chi_4}}\left(\frac{\delta T}{T_i}\right)^2. \tag{14}$$

$\gamma_{\max}$ at the saturated state is obtained by taking $\delta T_0 \to \delta T$. Equating the two expressions, we find that the saturated amplitude is $\delta T/T_i \sim \{1/(v_{\mathrm{thi}}\rho_i)\}\sqrt{(\chi_4/\tau)}$. Using that result, we obtain the scale length for $\gamma_{\max}$ as

$$\frac{\Delta_{\max}^2}{\rho_i^2} \sim \frac{2v_{\mathrm{thi}}}{\lambda T_i}\sqrt{\frac{\chi_2\tau}{\rho_i^2}}. \tag{15}$$

Using typical plasma parameters $T_i \sim 1$ keV, $n \sim 10^{13}\ \mathrm{cm}^{-3}$, $B \sim 10^4$ gauss, $\epsilon_0 \sim 1/3$, and assuming $\lambda T_i \sim$ a typical pulse propagation speed $\sim 100\ \mathrm{cm}/0.1\ \mathrm{sec} \sim 10^3\ \mathrm{cm/sec}$, $\tau \sim \Delta\omega^{-1} \sim 10^{-5}$ sec, $\chi_2 \sim \chi_{\mathrm{neo}} \sim \nu_{ii}\rho_i^2/\epsilon_0^{3/2} \sim 1.4 \times 10^2\ \mathrm{cm^2/sec}$, and $l_c \sim 1.5$ cm for $k_\perp\rho_i \sim 0.2$, the length scale is qualitatively estimated to be $\Delta_{\max} \sim 12 \times l_c \sim 18$ cm. Then $\Delta_{\max}$ satisfies $l_c < \Delta_{\max} < a$, and the scale of the maximum flux 'jamiton' growth is consistent with the typical mesoscale staircase width Δ_{stair}.

In summary, we discussed the formation of $E \times B$ staircases and profile corrugation, by extending heat avalanche dynamics to include a response time for plasmas to relax the heat flux toward the mean heat flux determined by symmetry constraints. The extension was based on the analogy of the profile corrugation in plasmas to traffic jam dynamics. The finite response time allows an instability, which will occur for a long flux response time. The wave number that gives the maximum growth rate was calculated. We argued that the instability saturates when the maximum growth rate is comparable to the shearing rate exerted by the $E \times B$ staircase generated via profile corrugation. For typical plasma parameters, the scale for maximal growth rate agrees with the staircase step spacing.

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