How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems

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Non-local diffusion.

The function J. Let $J : \mathbf{R}^N \to \mathbf{R}$, nonnegative, smooth with

$$\int_{\mathbf{R}^N} J(r)dr = 1.$$

Assume that is compactly supported and radially symmetric.

Non-local diffusion equation

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbf{R}^N} J(x-y)u(y,t)dy - u(x,t).$$

Non-local diffusion.

In this model, $u(x,t)$ is the density of individuals in x at time t and $J(x-y)$ is the probability distribution of jumping from y to x. Then

$$(J * u)(x,t) = \int_{\mathbf{R}^N} J(x-y)u(y,t)dy$$

is the rate at which the individuals are arriving to x from other places

$$-u(x,t) = -\int_{\mathbf{R}^N} J(y-x)u(x,t)dy$$

is the rate at which they are leaving from x to other places.

References

- P. Bates, P- Fife, X. Ren, X. Wang. Arch. Rat. Mech. Anal. (1997).
- P. Fife. Trends in nonlinear analysis. Springer, 2003.

Non-local diffusion.

The non-local equation shares some properties with the classical heat equation

$$u_t = \Delta u.$$

Properties

- Existence, uniqueness and continuous dependence on the initial data.
- Maximum and comparison principles.
- Perturbations propagate with infinite speed. If u is a nonnegative and nontrivial solution, then $u(x,t) > 0$ for every $x \in \mathbf{R}^N$ and every $t > 0$.

Remark.

There is no regularizing effect for the non-local model.

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Newmann boundary conditions.

One of the boundary conditions that has been imposed to the heat equation is the *Neumann boundary condition*, $\partial u/\partial \eta(x,t) = g(x,t)$, $x \in \partial\Omega$.

Non-local Neumann model

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + \int_{\mathbf{R}^N\setminus\Omega} G(x-y)g(y,t)dy$$

for $x \in \Omega$.

Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The last term takes into account the prescribed flux (given by the data $g(x,t)$) of individuals from outside.

Existence, uniqueness and a comparison principle

Theorem

For every $u_0 \in L^1(\Omega)$ *and* $g \in L^\infty_{loc}((0,\infty); L^1(\mathbf{R}^N \setminus \Omega))$ *there exists a unique solution* u *such that* $u \in C([0,\infty); L^1(\Omega))$ *and* $u(x,0) = u_0(x)$.
Moreover the solutions satisfy the following comparison property:

$$\text{if } u_0(x) \le v_0(x) \text{ in } \Omega, \ \text{ then } u(x,t) \le v(x,t) \text{ in } \Omega \times [0,\infty).$$

In addition the total mass in Ω *satisfies*

$$\int_\Omega u(y,t)\,dy = \int_\Omega u_0(y)dy + \int_0^t \int_\Omega \int_{\mathbf{R}^N \setminus \Omega} G(x-y)g(y,s)dydxds.$$

Asymptotic behavior

Theorem

Let $g(x,t) = h(x)$ *such that*

$$0 = \int_{\Omega} \int_{\mathbf{R}^N \setminus \Omega} G(x-y) h(y)\, dy\, dx.$$

Then there exists a unique solution φ *of the problem*

$$0 = \int_{\Omega} J(x-y)(\varphi(y) - \varphi(x))\, dy + \int_{\mathbf{R}^N \setminus \Omega} G(x-y) h(y)\, dy$$

that verifies $\int_\Omega u_0 = \int_\Omega \varphi$ *and there exists* $\beta = \beta(J,\Omega) > 0$ *such that*

$$\|u(t) - \varphi\|_{L^2(\Omega)} \le e^{-\beta t} \|u_0 - \varphi\|_{L^2(\Omega)}.$$

Asymptotic behavior

If the compatibility conditions does not hold then solutions are unbounded.
Here β_1 is given by

$$\beta_1 = \inf_{u \in L^2(\Omega), \int_\Omega u = 0} \frac{\frac{1}{2}\int_\Omega \int_\Omega J(x-y)(u(y)-u(x))^2 \, dy \, dx}{\int_\Omega (u(x))^2 \, dx}.$$

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F. Andreu, J. M. Mazon, J. D. R., J. Toledo. Preprint.

Approximations

Now, our goal is to show that the Neumann problem for the heat equation, can be approximated by suitable nonlocal Neumann problems.
More precisely, for given J we consider the rescaled kernels

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right), \quad G_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} G\left(\frac{\xi}{\varepsilon}\right)$$

with

$$C_1^{-1} = \frac{1}{2} \int_{B(0,d)} J(z) z_N^2 \, dz,$$

which is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it.

Approximations

Then, we consider the solution $u_\varepsilon(x,t)$ to

$$\begin{cases} (u_\varepsilon)_t(x,t) &= \dfrac{1}{\varepsilon^2}\displaystyle\int_\Omega J_\varepsilon(x-y)(u_\varepsilon(y,t)-u_\varepsilon(x,t))\,dy \\ & \qquad\qquad\qquad + \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbf{R}^N\setminus\Omega} G_\varepsilon(x-y)g(y,t)\,dy, \\ u_\varepsilon(x,0) &= u_0(x). \end{cases}$$

Note that the scaling of the diffusion, $1/\varepsilon^2$, is different from the scaling of the boundary flux, $1/\varepsilon$, as happens for the heat equation with Neumann boundary conditions.

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Approximations

Our first result deals with homogeneous boundary conditions, this is, $g \equiv 0$.

Theorem

Let $g = 0$ and let $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ be the solution to the heat equation with Neumann boundary conditions $\partial u/\partial \eta = 0$ and u_ε be the solution to the nonlocal model. Then,

$$\sup_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \to 0$$

as $\varepsilon \to 0$.

Note that this result holds for every G since $g \equiv 0$, and that the assumed regularity in u is guaranteed if $u_0 \in C^{2+\alpha}(\overline{\Omega})$ and $\partial u_0/\partial \eta = 0$.

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Approximations

Now let the “Neumann” kernel be given by

$$G(\xi) = C_2 J(\xi),$$

where C_2 is such that

$$\int_0^d \int_{\{z_N > s\}} J(z)\big(C_2 - z_N\big)\, dz\, ds = 0.$$

This choice of G is natural since we are considering a flux with a jumping probability that is a scalar multiple of the same jumping probability that moves things in the interior of the domain, J.

Approximations

In this case we can prove convergence with $g \neq 0$ but in a weaker sense.

Theorem

Let

$$g \in C^{1+\alpha,(1+\alpha)/2}(\overline{(\mathbf{R}^N \setminus \Omega)} \times [0,T]),$$

$$u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$$

the solution to the heat equation with Neumann boundary condition, $\partial u/\partial\eta = 0$ and u_ε be the solution to the nonlocal model. Then, for each $t \in [0,T]$

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad * - \textit{weakly in } L^\infty(\Omega)$$

as $\varepsilon \to 0$.

Approximations

Idea of why the involved scaling is correct

Let us give an heuristic idea in one space dimension, with $\Omega = (0,1)$, of why the scaling involved is the right one. We assume that

$$\int_1^{\infty} G(1-y)\,dy = -\int_{-\infty}^{0} G(-y)\,dy = \int_0^1 J(y)\,y\,dy.$$

Approximations

We have

$$
\begin{aligned}
u_t(x,t) = \; & \frac{1}{\varepsilon^2}\int_0^1 J_\varepsilon\,(x-y)\,\big(u(y,t)-u(x,t)\big)\,dy \\
& + \frac{1}{\varepsilon}\int_{-\infty}^0 G_\varepsilon\,(x-y)\,g(y,t)\,dy \\
& + \frac{1}{\varepsilon}\int_1^{+\infty} G_\varepsilon\,(x-y)\,g(y,t)\,dy := A_\varepsilon u(x,t).
\end{aligned}
$$

Approximations

If $x \in (0,1)$ a Taylor expansion gives that for any fixed smooth u and ε small enough, the right hand side $A_\varepsilon u$ becomes

$$A_\varepsilon u(x) = \frac{1}{\varepsilon^2} \int_0^1 J_\varepsilon (x-y) \left(u(y) - u(x)\right) dy$$

$$= \frac{1}{\varepsilon^3} \int_0^1 J\left(\frac{x-y}{\varepsilon}\right) \left(u(y) - u(x)\right) dy$$

$$= \frac{1}{\varepsilon^2} \int_{\mathbf{R}} J(w) \left(u(x - \varepsilon w) - u(x)\right) dw$$

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$$= \frac{1}{\varepsilon^2} \int_{\mathbf{R}} J\,(w)\,(u(x-\varepsilon w)-u(x))\,dw$$

Approximations

$$= \frac{u_x(x)}{\varepsilon} \int_{\mathbf{R}} J(w)\, w\, dw + \frac{u_{xx}(x)}{2} \int_{\mathbf{R}} J(w)\, w^2\, dw + O(\varepsilon)$$

As J is even

$$\int_{\mathbf{R}} J(w)\, w\, dw = 0$$

and hence,

$$A_\varepsilon u(x) \approx u_{xx}(x),$$

and we recover the Laplacian for $x \in (0, 1)$.

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Approximations

If $x = 0$ and ε small,

$$A_\varepsilon u(0) = \frac{1}{\varepsilon^2}\int_0^1 J_\varepsilon(-y)\,(u(y)-u(0))\,dy + \frac{1}{\varepsilon}\int_{-\infty}^0 G_\varepsilon(-y)\,g(y)\,dy$$

$$= \frac{1}{\varepsilon^3}\int_0^1 J\left(\frac{-y}{\varepsilon}\right)(u(y)-u(0))\,dy + \frac{1}{\varepsilon^2}\int_{-\infty}^0 G\left(\frac{-y}{\varepsilon}\right)g(y)\,dy$$

$$= \frac{1}{\varepsilon^2}\int_{-\infty}^0 J(w)\,(-u(-\varepsilon w)+u(0))\,dw + \frac{1}{\varepsilon}\int_0^{+\infty} G(w)\,g(-\varepsilon w)\,dw$$

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Approximations

$$= -\frac{u_x(0)}{\varepsilon}\int_{-\infty}^{0} J(w)\, w\, dw + \frac{g(0)}{\varepsilon}\int_{0}^{+\infty} G(w)\, dw + O(1)$$

$$\approx \frac{C_2}{\varepsilon}(u_x(0) + g(0)).$$

then

$$-u_x(0) = g(0)$$

and we recover the boundary condition

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Idea of the proof. General case

We set

$$w_\varepsilon = u_\varepsilon - u$$

and let $\tilde{u}$ be a $C^{2+\alpha,1+\alpha/2}$ extension of u to $\mathbf{R}^N \times [0, T]$. We define

$$L_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_\Omega J_\varepsilon(x-y)\big(v(y,t) - v(x,t)\big)dy$$

and

$$\tilde{L}_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_{\mathbf{R}^N} J_\varepsilon(x-y)\big(v(y,t) - v(x,t)\big)dy.$$

Idea of the proof. General case

Then

$$(w_\varepsilon)_t = L_\varepsilon(u_\varepsilon) - \Delta u + \frac{1}{\varepsilon}\int_{\mathbf{R}^N\setminus\Omega} G_\varepsilon(x, x-y)g(y,t)\,dy$$

$$= L_\varepsilon(w_\varepsilon) + \tilde{L}_\varepsilon(\tilde{u}) - \Delta u + \frac{1}{\varepsilon}\int_{\mathbf{R}^N\setminus\Omega} G_\varepsilon(x, x-y)g(y,t)\,dy$$

$$- \frac{1}{\varepsilon^2}\int_{\mathbf{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(y,t) - \tilde{u}(x,t)\big)\,dy.$$

Or

$$(w_\varepsilon)_t - L_\varepsilon(w_\varepsilon) = F_\varepsilon(x,t),$$

Our main task in order to prove the uniform convergence result is to get bounds on F_ε.

Idea of the proof. General case

However, the proofs of our results are much more involved than simple Taylor expansions.
This is due to the fact that for each $\varepsilon > 0$ there are points $x \in \Omega$ for which the ball in which integration takes place, $B(x, \varepsilon)$, is not contained in Ω and moreover, when working in several space dimensions, one has to take into account the geometry of the domain.

THANKS !!!