

**INSTITUTE OF MATHEMATICAL ECONOMICS**



WORKING PAPERS

No. 339

**ZiF Research Group**

on

***PROCEDURAL APPROACHES TO CONFLICT RESOLUTION***

October 2001 - August 2002

Fascicle 11

**How to Cope with Division Problems under  
Interval Uncertainty of Claims?**

by

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August 2002

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# How to cope with division problems under interval uncertainty of claims?\*

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August 2002

## Abstract

The paper deals with division situations where individual claims can vary within closed intervals. Uncertainty of claims is removed

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\*This paper was written while the authors were research fellows at the ZiF (Bielefeld) for the project "Procedural Approaches to Conflict Resolution", 2002. We thank our hosts for their hospitality.

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by compromising in a consistent way the upper and lower bounds of the claim intervals. Deterministic division problems with compromise claims are then considered and classical division rules from the bankruptcy literature are used to generate several procedures leading to efficient and reasonable rules for division problems under interval uncertainty of claims.

*Keywords:* Claims; Division problems; Interval uncertainty; Rules

## 1 Introduction

Division problems where claimants are facing uncertainty regarding their claims arise from many economic situations. We concentrate here on situations where a certain amount of money has to be divided among claimants who can merely indicate the range of their claims in the form of a closed interval, and the available amount is smaller than the aggregated lower claim. Funds's allocation of a firm among its divisions (cf. Pulido et al. (2002a, b)), taxation problems (cf. Young (1988)), priority problems (cf. Moulin (2000)), distribution of delay costs of a joint project among the agents involved (cf. Branzei et al. (2002)), various disputes including those generated by inheritance (cf. O'Neill (1982)) or by cooperation in joint projects based on restricted willingness to pay of agents (cf. Tijs and Branzei (2002)) fit into this framework.

We conquer interval uncertainty of claims by compromising, in a consistent way, the upper and lower bounds of the claim intervals, and by tackling deterministic division problems based on compromise claims. Several procedures which yield families of efficient and reasonable rules are described.

Building blocks for the introduced families of parametric solutions are one-point solutions generated by rules for classical division problems.

Three of the most well known rules, namely the proportional rule, the constrained equal awards rule, and the constrained equal losses rule, are used in our examples in the next sections. The reader is referred to Herrero and Villar (2001) for understanding their characterizing properties and getting insight into types of situations in which one of these rules is more suitable than others.

The outline of the paper is as follows. In Section 2 we formally introduce the family of division problems under interval uncertainty of claims, and define compromise claims. The model of a bankruptcy problem and the three appealing well known division rules are briefly presented. Then it is indicated how compromise claims can be used to generate uncertainty-free division problems related to a division problem under interval uncertainty of claims and how rules for deterministic division problems yield efficient and reasonable rules for the division problem affected by uncertainty. Section 3 introduces and studies two families of rules. Rules in one family are based on averaging solutions generated by compromise claims, while rules in the other family are based on spreading the available amount over compromise claims. For each family it is shown that the rules are efficient and reasonable. A transparent rule which is a particular case of averaging is motivated by the wishes of the claimants. Section 4 deals with multi-stage rules obtained by aggregating shares allocated to claimants in successive stages. The case of a two-stage rule is exemplified. We conclude in Section 5 with remarks on axiomatic characterization and existing literature on division rules under interval uncertainty.

## 2 Division problems under interval uncertainty of claims

Let  $N = \{1, \dots, n\}$  be the set of claimants among which an estate  $E$  has to be divided, where each claimant  $i \in N$  faces uncertainty regarding his claim. We denote by  $I_i = [a_i, b_i]$  the claim interval of claimant  $i$ , where  $a_i$  is the lower bound of the claim interval, while  $b_i$  is the upper bound.

Let  $\mathfrak{I}$  be the family of closed intervals in  $\mathbb{R}_+$  and  $\mathfrak{I}^N$  be the set of all vectors of the form  $I = (I_1, \dots, I_n)$ . A *division problem under interval uncertainty* is defined as a pair  $(E, I)$ , where  $0 < E \leq \sum_{i \in N} a_i$ . We denote by  $\mathfrak{D}^N$  the set of all division problems of the form  $(E, I)$ .

Note that if all claim intervals  $I_i, i \in N$  are degenerated intervals, i.e.  $I_i = [a_i, a_i]$ , the problem  $(E, I)$  coincides with the classical bankruptcy problem  $(E, a)$  with  $a = (a_1, \dots, a_n)$  and  $0 < E \leq \sum_{i \in N} a_i$ . Moreover, all division problems on  $N$  with sharp claims w.r.t. the available amount  $E$ , of the form  $(E, d)$  with  $d = (d_1, \dots, d_n)$  and  $0 < E \leq \sum_{i \in N} a_i \leq \sum_{i \in N} d_i$  appear as particular cases of a problem  $(E, I) \in \mathfrak{D}^N$ . In the following we use the notation  $\mathcal{D}^N$  to refer to the family of classical division problems related to a division problem under interval uncertainty of claims.

A *rule* for division problems under interval uncertainty of claims is a function  $\varphi : \mathfrak{D}^N \rightarrow \mathbb{R}^N$  specifying for each problem  $(E, I) \in \mathfrak{D}^N$  and  $i \in N$  the feasible payoff  $\varphi_i(E, I) \in \mathbb{R}$ . A rule  $\varphi$  is

(i) *efficient* if

$$\sum_{i \in N} \varphi_i(E, I) = E \text{ for all } (E, I) \in \mathfrak{D}^N;$$

(ii) *reasonable* if

$$\varphi_i(E, I) \in [0, b_i] \text{ for all } (E, I) \in \mathfrak{SD}^N \text{ and each } i \in N.$$

An efficient rule allocates shares to claimants so that the total available amount  $E$  is cleared. A reasonable rule gives each claimant a feasible (non-negative) amount which is smaller than the upper bound of the corresponding claim interval.

In Sections 3 and 4 we will provide procedures for generating efficient and reasonable solutions based on the selection of a suitable rule  $f$  for a classical division problem  $(E, d) \in \mathcal{D}^N$ . For the rest of the paper we will assume that  $f : \mathcal{D}^N \rightarrow \mathfrak{R}^N$  is continuous w.r.t. the claim vector and satisfies the following two properties:

(i) *efficiency*, i.e.

$$\sum_{i \in N} f_i(E, d) = E \text{ for all } (E, d) \in \mathcal{D}^N,$$

and

(ii) *reasonability*, i.e.

$$f_i(E, d) \in [0, d_i] \text{ for all } (E, d) \in \mathcal{D}^N \text{ and each } i \in N.$$

To exemplify our procedures we use the proportional rule (*PROP*), the constrained equal awards rule (*CEA*), and the constrained equal losses rule (*CEL*). For a classical division problem  $(E, d) \in \mathcal{D}^N$  these rules are defined as follows (cf. Herrero and Villar (2001)):

(i) The  $i$ -th coordinate of *PROP*  $(E, d)$  is given by

$$PROP_i(E, d) = \frac{E}{\sum_{i \in N} d_i} d_i \text{ for } i = 1, \dots, n.$$

According to this rule, the amount  $E$  is divided among the claimants proportionally to their individual claims.

(ii) The  $i$ -th coordinate of  $CEA(E, d)$  is given by

$$CEA_i(E, d) = \min \{d_i, \alpha\} \text{ for } i = 1, \dots, n,$$

where  $\alpha$  solves  $\sum_{i \in N} \min \{d_i, \alpha\} = E$ . The idea here is that every claimant receives the same amount as long as this does not exceed his claim.

(iii) The  $i$ -th coordinate of  $CEL(E, d)$  is given by

$$CEL_i(E, d) = \max \{0, d_i - \beta\} \text{ for } i = 1, \dots, n,$$

where  $\beta$  solves  $\sum_{i \in N} \max \{0, d_i - \beta\} = E$ . Here the difference between the aggregate claim and the estate is distributed equally. Since for some claimants the corresponding amount might be negative, the rule respects the fact that no claimant ends up with a negative payoff.

To each division problem under interval uncertainty of claims one can associate a set of uncertainty-free problems in  $\mathcal{D}^N$  based on the idea to compromise uniformly the interval claims by weighting the upper bound with  $t \in [0, 1]$  and the lower bound with  $(1 - t)$ .

Let  $I = (I_1, \dots, I_n)$  be the vector of interval claims in the problem  $(E, I) \in \mathfrak{SD}^N$  and  $t \in [0, 1]$ . We define the  $t$ -compromise claim  $c^t = (c_1^t, \dots, c_n^t)$  by

$$c_i^t = tb_i + (1 - t)a_i \text{ for each } i \in N. \quad (1)$$

Given the amount  $E$ , for each  $t$ -compromise claim  $c^t$ , we can consider the deterministic division problem  $(E, c^t) \in \mathcal{D}^N$ , which we call the  $t$ -compromise problem. Applying a rule  $f$  to  $(E, c^t)$  yields a solution for the problem  $(E, I) \in \mathfrak{SD}^N$ . We define the  $t$ -compromise solution of  $(E, I)$  based on  $f$  as  $\varphi^{f,t}(E, I) = f(E, c^t)$ .

**Remark 1** Note that the vector  $I = (I_1, \dots, I_n)$  of claim intervals generates a hypercube  $\prod_{i \in N} I_i$  in  $\mathfrak{R}_+^N$ . Of course, each point  $z$  in it can be considered as a compromise claim. However, we will concentrate mainly on the  $t$ -compromise claims defined by (1), which lie on the diagonal through the lower claim point  $a$  and the upper claim point  $b$  of this hypercube.

### 3 One-stage solutions based on compromise claims

In this section two families of solutions based on compromise claims are introduced. One is based on averaging  $t$ -compromise solutions, and the other one is based on spreading the available amount over  $t$ -compromise claims.

#### 3.1 Averaging $t$ -compromise solutions

Let  $\mu$  be a probability measure on  $([0, 1], \mathcal{B})$  where  $\mathcal{B}$  is the  $\sigma$ -algebra of Borel subsets of  $[0, 1]$ . Let  $f$  be a rule for classical division problems. Then the  $\mu$ -compromise rule based on  $f$ ,  $\varphi^{f, \mu}$ , is defined by

$$\varphi_i^{f, \mu}(E, I) := \int_0^1 f_i(E, c^t) d\mu(t) = \int_0^1 \varphi_i^{f, t}(E, I) d\mu(t)$$

for each  $(E, I) \in \mathfrak{SD}^N$  and each  $i \in N$ .

**Proposition 2** Let  $f$  and  $\mu$  be as above. Then the rule  $\varphi^{f, \mu}$  is efficient and reasonable.

**Proof.** To prove that  $\varphi^{f, \mu}$  is efficient take  $(E, I) \in \mathfrak{SD}^N$ . Then we have

$$\sum_{i \in N} \varphi_i^{f, \mu}(E, I) = \int_0^1 \sum_{i \in N} f_i(E, c^t) d\mu(t) = \int_0^1 E d\mu(t) = E.$$



The reasonability of  $\varphi^{f,\mu}$  follows from

$$0 \leq f_i(E, c^t) \leq c_i^t \leq b_i$$

by integrating over  $[0, 1]$  and using the monotonicity property of integrals. ■

**Example 3** Let  $\delta_a$  be the Dirac measure on  $([0, 1], \mathcal{B})$  with  $a \in [0, 1]$  as atom, i.e. for all  $A \in \mathcal{B}$ ,  $\delta_a(A) = 1$  if  $a \in A$ ,  $\delta_a(A) = 0$  otherwise. Take  $\mu = \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{\frac{1}{2}} + \frac{1}{3}\delta_1$ . Then

$$\varphi_i^{f,\mu}(E, I) = \frac{1}{3} \left( f_i(E, c^0) + f_i(E, c^{\frac{1}{2}}) + f_i(E, c^1) \right) \text{ for each } i \in N.$$

**Example 4** Let  $\mu$  be the Lebesgue measure  $\lambda$ ,  $f = CEA$ ,  $E = 8$ ,  $I_1 = [3, 10]$ ,  $I_2 = [8, 10]$ . Then

$$\begin{aligned} \varphi^{CEA,t}(E, I) &= CEA(E, c^t) = CEA(8, (3 + 7t, 8 + 2t)) \\ &= \begin{cases} (3 + 7t, 5 - 7t) & \text{if } t \in [0, \frac{1}{7}], \\ (4, 4) & \text{if } t \in (\frac{1}{7}, 1]. \end{cases} \end{aligned}$$

$$\begin{aligned} &\varphi^{CEA,\lambda}(E, I) \\ &= \left( \int_0^1 CEA_1(8, (3 + 7t, 8 + 2t)) dt, \int_0^1 CEA_2(8, (3 + 7t, 8 + 2t)) dt \right) \\ &= \left( 3\frac{13}{14}, 4\frac{1}{14} \right). \end{aligned}$$

A transparent procedure leading to solutions taking explicitly into account the wishes of the claimants is presented in the following.

Each claimant proposes a value  $t \in [0, 1]$  for compromising claims. Let  $t_i \in [0, 1]$  be the value proposed by agent  $i \in N$ . Then the set of uncertainty-free division problems  $(E, c^{t_i})$ ,  $i \in N$  is considered, where  $c^{t_i} = (c_1^{t_i}, \dots, c_n^{t_i})$ , with  $c_j^{t_i} = t_i b_j + (1 - t_i) a_j$  for  $j = 1, \dots, n$ , is the vector of sharp claims

corresponding to the wish of claimant  $i$ . By averaging the solutions of these deterministic problems which are obtained using a rule  $f$ , one obtains the solution  $\varphi^{f,\mu}$  of the division problem under interval uncertainty of claims where  $\mu = \frac{1}{n} \sum_{i \in N} \delta_{t_i}$  because

$$\frac{1}{n} \sum_{i \in N} f(E, c^{t_i}) = \frac{1}{n} \sum_{i \in N} \varphi^{f,t_i}(E, I) = \varphi^{f,\mu}(E, I).$$

If the claimants express their joint wishes by delivering the same value  $\tilde{t} \in [0, 1]$  then only one deterministic problem, namely  $(E, c^{\tilde{t}})$ , has to be solved and this corresponds to the rule  $\varphi^{f,\mu}$  with  $\mu = \delta_{\tilde{t}}$ .

**Remark 5** *In the procedure above the claimants deliver values  $t_1, \dots, t_n$  to generate compromise claims. We can also design a procedure where the claimants deliver directly compromise claims  $z^1, \dots, z^n$  from the hypercube introduced in Remark 1 and then divide  $E$  w.r.t. a rule  $f$  applied to the classical division problems  $(E, z^i) \in \mathcal{D}^N$ ,  $i \in N$  as follows:*

$$\varphi^{f,(z^1, \dots, z^n)}(E, I) = \frac{1}{n} \sum_{i \in N} f(E, z^i).$$

### 3.2 Spreading $E$ over compromise claims

Let  $\mu$  be a probability measure on  $\langle [0, 1], \mathcal{B} \rangle$  where  $\mathcal{B}$  is the  $\sigma$ -algebra of Borel subsets of  $[0, 1]$ . Let  $f$  be a rule for classical division problems.

Let  $\sigma : [0, 1] \rightarrow \mathfrak{R}_+$  be a  $\mu$ -integrable (spread) function with  $\int_0^1 \sigma(t) d\mu(t) = 1$ . Let  $\mathfrak{SD}^N(\sigma)$  be the subset of  $\mathfrak{SD}^N$  consisting of  $(E, I)$  such that  $(\sigma(t)E, c^t) \in \mathcal{D}^N$  for each  $t \in [0, 1]$ .

Then, we can define a rule  $\varphi^{f,\mu,\sigma} : \mathfrak{SD}^N(\sigma) \rightarrow \mathfrak{R}^N$  based on  $f$  as follows:

$$\varphi_i^{f,\mu,\sigma}(E, I) := \int_0^1 f_i(\sigma(t)E, c^t) d\mu(t)$$

for each  $(E, I) \in \mathfrak{SD}^N(\sigma)$  and each  $i \in N$ .

Note that by taking  $\mu = \delta_s$  and

$$\sigma(t) = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}$$

we obtain  $\varphi^{f, \mu, \sigma} = \varphi^{f, s}$ .

If we take  $\sigma(t) = 1$  for all  $t \in [0, 1]$ , then  $\varphi^{f, \mu, \sigma} = \varphi^{f, \mu}$ .

**Proposition 6** *Let  $f, \mu, \sigma$  be as above. Then  $\varphi^{f, \mu, \sigma} : \mathfrak{SD}^N(\sigma) \rightarrow \mathfrak{R}^N$  is efficient and reasonable.*

**Proof.** Take  $(E, I) \in \mathfrak{SD}^N(\sigma)$ . Then the efficiency of  $\varphi^{f, \mu, \sigma}$  follows from

$$\sum_{i \in N} \varphi_i^{f, \mu, \sigma}(E, I) = \int_0^1 \sum_{i \in N} f_i(\sigma(t) E, c^t) d\mu(t) = \int_0^1 \sigma(t) E d\mu(t) = E.$$

For the reasonability of  $\varphi^{f, \mu, \sigma}$  note that from

$$0 \leq f_i(\sigma(t) E, c^t) \leq c_i^t \leq b_i$$

follows that

$$0 \leq \varphi_i^{f, \mu, \sigma}(E, I) = \int_0^1 f_i(\sigma(t) E, c^t) d\mu(t) \leq \int_0^1 b_i d\mu(t) = b_i.$$

■

**Example 7** *Let  $f = \text{CEL}$  for two-person division situations. Suppose  $(E, I) \in \mathfrak{SD}^{\{1,2\}}$  is such that  $E = 9$ ,  $I_1 = [6, 10]$ ,  $I_2 = [12, 20]$ . Let  $\delta_a$  and  $\lambda$  be as in Examples 3 and 4, respectively. Take  $\mu = \frac{1}{3}\delta_0 + \frac{2}{3}\lambda$ ,  $\sigma(0) = 2$ , and  $\sigma(t) = \frac{1}{2}$  for  $t \in (0, 1]$ . Then  $\int_0^1 \sigma(t) E d\mu(t) = E$  and  $c^t = (6 + 4t, 12 + 8t)$  for  $t \in [0, 1]$ . So,  $(\sigma(t) E, c^t) \in \mathcal{D}^{\{1,2\}}$  for each  $t \in [0, 1]$ . Further*

$$\text{CEL}(\sigma(0) E, c^0) = \text{CEL}(18, (6, 12)) = (6, 12),$$

$$\text{CEL}(\sigma(t) E, c^t) = \text{CEL}\left(4\frac{1}{2}, (6 + 4t, 12 + 8t)\right) = \left(0, 4\frac{1}{2}\right) \text{ for } t \in (0, 1].$$

Then

$$\begin{aligned}
\varphi^{CEL, \frac{1}{3}\delta_0 + \frac{2}{3}\lambda, \sigma}(E, I) &= \frac{1}{3}CEL(18, (6, 12)) + \\
&\quad \frac{2}{3} \int_0^1 CEL\left(4\frac{1}{2}, (6 + 4t, 12 + 8t)\right) d\lambda(t) \\
&= \frac{1}{3}(6, 12) + \frac{2}{3}\left(0, 4\frac{1}{2}\right) = (2, 7).
\end{aligned}$$

## 4 Multi-stage solutions based on compromise claims in adjusted claim intervals

In this section we present a family of multi-stage solutions which are again based on a rule for a classical division problem. At each stage of the solution a part of the available amount is divided among the claimants and then the claim intervals are adjusted. Note that at each step of the procedure given below the corresponding uncertainty-free division problem is well defined.

Let  $k$  be a positive integer and  $E$  be the available amount in the division problem  $(E, I) \in \mathfrak{SD}^N$ . We can see the amount  $E$  as a budget of a firm that has to be allocated to its divisions during a fixed number of periods. Based on this interpretation, our idea is to take a sequence  $\langle E_1, \dots, E_k \rangle$  with  $\sum_{r=1}^k E_r = E$  and a sequence  $\langle t_1, \dots, t_k \rangle$  of numbers in  $[0, 1]$ , and to divide at each step  $r \in \{1, \dots, k\}$  the amount  $E_r$ , according to the compromise claim vector  $c^{tr} = t_r b^r + (1 - t_r) a^r$ , where  $a^1 = a$ ,  $b^1 = b$ , and  $a^r$  and  $b^r$  for  $r = 2, \dots, k$  are defined as follows:

$$a^r = \max(0, a^{r-1} - f(E_{r-1}, c^{t_{r-1}})), \quad b^r = b^{r-1} - f(E_{r-1}, c^{t_{r-1}}). \quad (2)$$

Then as a result we obtain the aggregate payoff vector  $\sum_{r=1}^k f(E_r, c^{tr})$ , which can be denoted by  $\varphi^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle}(E, I)$ .

**Proposition 8** Let  $f, k, \langle E_1, \dots, E_k \rangle, \langle t_1, \dots, t_k \rangle$  be as above. Then the rule  $\varphi^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle}$  is efficient and reasonable.

**Proof.** Take  $(E, I) \in \mathfrak{SD}^N$ . Then the efficiency of  $\varphi^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle}$  follows by noticing that

$$\sum_{i=1}^n \varphi_i^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle} (E, I) = \sum_{i=1}^n \sum_{r=1}^k f_i(E_r, c^{tr}) = \sum_{r=1}^k E_r = E.$$

For the reasonability of  $\varphi^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle}$  note that for each  $i \in N$  by (2) and by the reasonability of  $f$  we have

$$\begin{aligned} f_i(E_k, c^{tk}) &\leq b_i^k = b_i^{k-1} - f_i(E_{k-1}, c^{t_{k-1}}) \\ &= b_i^{k-2} - f_i(E_{k-2}, c^{t_{k-2}}) - f_i(E_{k-1}, c^{t_{k-1}}) \\ &= \dots = b_i^1 - f_i(E_{k-1}, c^{t_{k-1}}) - \dots - f_i(E_1, c^{t_1}) \\ &= b_i^1 - \sum_{r \in \{1, \dots, k-1\}} f_i(E_r, c^{tr}), \end{aligned}$$

which implies

$$\varphi_i^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle} (E, I) = \sum_{r \in \{1, \dots, k\}} f_i(E_r, c^{tr}) \leq b_i^1 = b_i.$$

Further, by the reasonability of  $f$  we have also  $f_i(E_r, c^{tr}) \geq 0$  for each  $r \in \{1, \dots, k\}$  and  $i \in N$ . Hence,

$$\varphi_i^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle} (E, I) = \sum_{r \in \{1, \dots, k\}} f_i(E_r, c^{tr}) \geq 0.$$

We conclude that  $\varphi_i^{f, \langle (t_1, E_1), \dots, (t_k, E_k) \rangle} (E, I) \in [0, b_i]$  for each  $i \in N$ . ■

**Example 9** Let  $f = PROP$ ,  $(E, I) = (20, ([16, 20], [4, 10]))$ ,  $\langle E_1, E_2 \rangle =$

$\langle \frac{1}{2}E, \frac{1}{2}E \rangle$ , and  $\langle t_1, t_2 \rangle = \langle 0, 1 \rangle$ . Then

$$\begin{aligned}
\varphi^{f, (0, \frac{1}{2}E), (1, \frac{1}{2}E)}(E, I) &= PROP\left(\frac{1}{2}E, c^0\right) + PROP\left(\frac{1}{2}E, c^1\right) \\
&= PROP(10, (16, 4)) + \\
&\quad PROP(10, ((20, 10) - PROP(10, (16, 4)))) \\
&= (8, 2) + PROP(10, (12, 8)) \\
&= (8, 2) + (6, 4) = (14, 6).
\end{aligned}$$

## 5 Final remarks

In this paper we focus on division problems where individual claims can vary within closed intervals, and conquer interval uncertainty by considering uncertainty-free problems where rules for classical division problems are helpful. Since axiomatic characterizations of classical division rules for deterministic bankruptcy problems can be found in the literature (cf. Young (1987), Dagan (1996), Herrero et al. (1999), Chun (1988)), the study of the introduced families of parametrized solutions from an axiomatic point of view is not undertaken. It turns out that all proposed procedures in the present paper yield efficient and reasonable solutions to division problems under interval uncertainty of claims. Of course, other procedures leading to efficient and reasonable solutions could be considered.

It is interesting to compare our results with the inspiring result of Yager and Kreinovich (2001). Translated in our terminology, they study a situation in which each claimant  $i \in N$  has an interval of possible weights  $[a_i, b_i] \subseteq [0, 1]$  and the problem is to assign to each  $i \in N$  a certain weight  $w_i \in [a_i, b_i]$  where  $\sum_{i \in N} w_i = 1$  and  $\sum_{i \in N} a_i \leq 1 \leq \sum_{i \in N} b_i$ . By using axioms of anonymity, merge and continuity they find a unique solution; then the available amount

$E$  is divided proportionally w.r.t. this solution. For a different interpretation of the lower bounds of the corresponding claim vectors and an analysis of the related problem the reader is referred to Pulido et al. (2002b).

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