CC BY ND

Logic and Logical Philosophy
Volume 19 (2010), 309–328
DOI: 10.12775/LLP.2010.011

Rafał Gruszczyński
Andrzej Pietruszczak

HOW TO DEFINE A MEREOLOGICAL (COLLECTIVE) SET

Abstract. As it is indicated in the title, this paper is devoted to the problem of defining mereological (collective) sets. Starting from basic properties of sets in mathematics and differences between them and so called conglomerates in Section 1, we go on to explicate informally in Section 2 what it means to join many objects into a single entity from point of view of mereology, the theory of *part of* (*parthood*) relation. In Section 3 we present and motivate basic axioms for *part of* relation and we point to their most fundamental consequences. Next three sections are devoted to formal explication of the notion of *mereological set* (*collective set*) in terms of sums, fusions and aggregates.

We do not give proofs of all theorems. Some of them are complicated and their presentation would divert the reader's attention from the main topic of the paper. In such cases we indicate where the proofs can be found and analyzed by those who are interested.

Keywords: Mereology, mereological sum, mereological fusion, mereological aggregate, mereological set, collective set, set theory, formal ontology.

1. Two notions of set

1.1. Sets in mathematics

Sets in mathematics are collections of objects. We can have a collection of points in space, a collection of natural numbers, the collection of all triangles in $\mathbb{R}^3$ and so on. As Hao Wang writes in [8] one can create such collections in two ways: either extensionally or intensionally.

Received October 25, 2010
© 2010 by Nicolaus Copernicus University
ISSN: 1425-3305

CC BY ND

According to the extensional conception of set, having some elements at hand we can collect them to form a set and this process can be continued *ad infinitum*. This approach to sets seems to be advocated by Cantor when he writes:

> a set is collection into a whole of definite distinct objects of our intuition or of our thought.[1]

In contemporary axiomatic set theory the axioms of pairing and sum seem to be formal embodiment of this idea.

The intensional conception is Frege's favourite and according to it a set is the extension of a concept or a property. As we learn from the so-called Russell's paradox this is very prone to inconsistencies and must be limited in one way or another. The set theoretical axiom of comprehension allows us to construct sets this way but only within sets whose existence were already proved from other axioms. In contemporary mathematics Frege's idea is most reflected probably in class existence axiom according to which every condition determines some class which may happen not to be a set.

Sets constructed in one of these two ways are sometimes called *distributive*. Whichever way we choose they satisfy what we can call *the basic principle for distributive sets*:

$$\forall_x (x \text{ is an element of the set of } S\text{-es} \iff x \text{ is } S)\,, \tag{†}$$

where 'S' is a schematic letter representing the place where we can put any linguistic expression such that 'is S' forms a predicate[2]. For example:

$$\forall_x (x \text{ is an element of the set of real numbers} \iff x \text{ is a real number})\,.$$

The elements of the set of human beings are all and only human beings, similarly the elements of the set of triangles in $\mathbb{R}^3$ are all and only triangles in $\mathbb{R}^3$. We will not find heads, kidneys and hearts in the set of all mammals, since every element of this set is a mammal.

---

[1] Cited after [1], p. 15.

[2] To be precise, (†) holds for those sets that are expressible by means of some formula of a language we employ (or those whose elements are describable in this language). It is not hard to notice that all sets created in accordance with the intensional conception satisfy this property as well.

CC BY ND

Distributive sets are abstract entities, that is they do not occupy space-time. It is not problematic for sets whose elements are abstract objects, like numbers or geometrical figures. However, distributive sets are abstract even in case their elements are concrete objects. Let us repeat Quine's argument for this (see [5, pp. 114–115]). Suppose that we encounter a heap of stones. The set X of all stones in this heap is definitely different from the set Y of all atoms out of which these stones are built. It is the case since no stone is an atom and vice versa. Suppose now that X and Y are spatio-temporal objects. The only candidate for them is the heap itself. Thus we have that:

$$X = \text{the heap of stones}\,,$$
$$Y = \text{the heap of stones}\,.$$

So, using elementary properties of identity, we get that $X = Y$, which is absurd in light of what was said above. In consequence, we must admit that sets of spatio-temporal objects are abstract entities and, in consequence, all distributive sets are such.

1.2. Sets as conglomerates of objects

Suppose that what we take into account are U.S. states. We can consider the set of all states:

$$ST := \{x : \ x \text{ is a U.S. state}\}\,,$$

which according to what was said above is an abstract object. We can say a couple of things about it, as that it has exactly 50 elements, that California is among them and so on. However, let us stress it one more time, ST is abstract. On the other hand experience tells us that there is a huge spatio-temporal object that consists of the elements of ST, that is the United States of America. And we are inclined to say that elements of ST are also elements of the U.S.A. But in what sense? It cannot be the same sense as in case of set theoretical '$\in$', since in this case what we speak about is some abstract object. Here we speak about the vast region of space, that is non-abstract, physical object. Moreover, we are inclined to agree that it is true that:

$$x \text{ is a U.S. state} \Longrightarrow x \text{ is an element of the U.S.A.}$$

but false that:

$$x \text{ is an element of the U.S.A.} \Longrightarrow x \text{ is a U.S. state}\,.$$

In consequence (†) is only partially satisfied in this case and we speak about different notion of *being an element*. A moment of reflection leads us to the conclusion that what we encounter here is rather *being part of* in spatio-temporal sense than *being an element* in set theoretical meaning. So when we speak about the U.S.A. as about the set of U.S. states, we definitely do not use the notion of *distributive set* but a different one, whose extension embraces not only abstract objects but also physical ones. Let us call such sets *mereological* ('*meros*' means part in Greek) or *collective ones*.

Now, when we say that California is an element of the U.S.A., we simply say that California *is part of* the U.S.A. With this interpretation in mind we can see that it is true that:

$$x \text{ is a U.S. state} \Longrightarrow x \text{ is part of the U.S.A.}$$

while the converse implication is not true. In general, when we take into account any mereological set of S-es, then it is true that:

$$x \text{ is } S \Longrightarrow x \text{ is part of a mereological set of } S\text{-es}$$

while in general it is not true that:

$$x \text{ is part of a mereological set of } S\text{-es} \Longrightarrow x \text{ is } S.$$

For example New York City is definitely part of the U.S.A., but it is not a U.S. state.

2. Joining many objects into one

The act of construction of a mereological set may be compared to that of assembling an object from a separate ones or joining a group of objects into a single entity. Having four objects a, b, c and d at our disposal we can join them to form some «new» object x, as it is presented in Figure 1. The fact that x is an assembly or a conglomerate of a, b, c and d may be written down as:

$$x = [\![a, b, c, d\,]\!]\,.$$

CC BY ND

This of course makes sense only in case there exists such thing as an assembly of a, b, c and d and it is unique. For now we simply assume that it does and is.

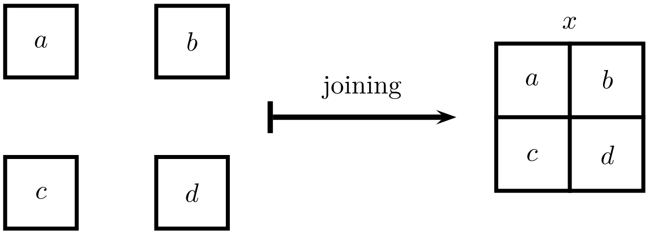

Figure 1: Separate objects a, b, c and d are joined to form one single object x.

We may now inquire what is relevant for the depicted act of joining or what are its essential features. First, we can see that every single one of objects a, b, c and d is part of x. Second, whatever part of x we take (any its fragment), it overlaps one of the four objects in question. On the other hand we can notice as well that any object which is exterior to every a, b, c and d must be exterior to x as well, and vice versa — if something is exterior to x, then it is exterior to every a, b, c and d.

It can be seen from both the figure and the above characterization, that ontological status of x may be «inherited» from that of its substrata. That is, if a, b, c, d are physical objects, then x seems to be a physical object as well. If they are abstract entities, x is undoubtedly abstract too.

All this is different from what is true within the realm of distributive sets. Having a, b, c and d, whatever their nature may be, we can (using set theoretical axioms of pairing and sum existence) form the set $\{a, b, c, d\}$. We have that:

$$y \in \{a, b, c, d\} \iff (y = a \vee y = b \vee y = c \vee y = d)$$

and in case of x we only have that:

$$(y = a \vee y = b \vee y = c \vee y = d) \Longrightarrow y \text{ is part of } x\,,$$

while the converse implication:

$$y \text{ is part of } x \Longrightarrow (y = a \vee y = b \vee y = c \vee y = d)$$

CC BY ND

is usually false. For example:

$$[\![a, b]\!] \text{ is part of } [\![a, b, c, d]\!]\,,$$

but is different from all a, b, c and d.

Another important feature distinguishing distributive and mereological (collective) sets is transitivity of *element* relation. In case of distributive sets an interpretation of '$\in$' usually is not transitive (although can be, as in transitive models of axiomatic set theory), while in case of mereological sets *parthood* is always transitive relation. If we refer to Figure 1 we can see that if we carve any piece from a, b, c or d, then this piece must be part of x. Sometimes it is argued that transitivity should not be ascribed to *parthood* but those who maintain it to be the case use 'part of' in different meaning from the one being used in mereology.

To conclude, while defining mereological sets we must remember about the following facts:

(1) only the implication from right to left in (†) is true if we interpret 'is an element' as *is part of*,

(2) parts of parts of mereological sets are their parts as well.

3. Basic properties of part of relation

We will now briefly analyze *part of* relation in order to do the groundwork for various definitions of mereological sets. However, we are not going to give up set theoretical apparatus. Quite the contrary, we will use set theory in a totaly similar way to its usage for example in group theory. Thus what we describe is *part of* relation in set theoretical setting. From now on, if not stated otherwise, 'set' used without any modifier will always mean *distributive set*.

Let M be an arbitrary non-empty set and let $\sqsubset \subseteq M \times M$. We call $\sqsubset$ *the relation of being part of* and in case $x \sqsubset y$ we say that x *is part of* y, '$x \not\sqsubset y$' is to mean $\neg\, x \sqsubset y$. *Part of* is the only primitive concept of the theory we are going to present. The theory of *part of* relation is called *mereology*.

Except for this we use standard logical constants: quantifiers '$\exists$' and '$\forall$', sentential operators '$\neg$', '$\wedge$', '$\vee$', '$\Rightarrow$' and '$\Leftrightarrow$'. For any set X, $\mathcal{P}(X)$ is its power set, while $\mathcal{P}_+(X) := \mathcal{P}(X) \setminus \{\emptyset\}$.

We assume that $\langle M, \sqsubset\rangle$ is a strict partial order, that is it satisfies the following two axioms of asymmetry and transitivity:

$$\forall_{x,y\in M}(x \sqsubset y \Rightarrow y \not\sqsubset x)\,, \tag{$\mathtt{as}_{\sqsubset}$}$$

$$\forall_{x,y,z\in M}(x \sqsubset y \wedge y \sqsubset z \Rightarrow x \sqsubset z)\,. \tag{$\mathtt{t}_{\sqsubset}$}$$

Irreflexivity of $\sqsubset$ is an immediate consequence of ($\mathtt{as}_{\sqsubset}$):

$$\forall_{x\in M}\ x \not\sqsubset x\,. \tag{$\mathtt{irr}_{\sqsubset}$}$$

As it is hard to express properties of *part of* relation just in terms of $\sqsubset$ we introduce some standard relations definable by means of the only primitive relation:

$$x \sqsubseteq y \overset{\text{df}}{\iff} x \sqsubset y \vee z = y\,, \tag{$\mathtt{df}\ \sqsubseteq$}$$

$$x \bigcirc y \overset{\text{df}}{\iff} \exists_z(z \sqsubseteq x \wedge z \sqsubseteq y)\,, \tag{$\mathtt{df}\ \bigcirc$}$$

$$x \wr y \overset{\text{df}}{\iff} \neg\exists_z(z \sqsubseteq x \wedge z \sqsubseteq y)\,. \tag{$\mathtt{df}\ \wr$}$$

In case $x \sqsubseteq y$ (resp. $x \bigcirc y$, $x \wr y$) we say that x *is ingrediens of* y (resp. x *overlaps* y, x *is exterior to* y). We will also write '$x \not\sqsubseteq y$' instead of '$\neg\, x \sqsubseteq y$'. Moreover, for every x from M we put:

$$\mathbb{P}(x) := \{y \in M : y \sqsubset x\}\,,$$
$$\mathbb{I}(x) := \{y \in M : y \sqsubseteq x\}\,,$$
$$\mathbb{O}(x) := \{y \in M : y \bigcirc x\}\,.$$

Let us point to some simple consequences of ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$) and the definitions. By ($\mathtt{df}\ \sqsubseteq$), ($\mathtt{df}\ \bigcirc$) and ($\mathtt{df}\ \wr$) — without specific axioms ($\mathtt{as}_{\sqsubset}$) and ($\mathtt{t}_{\sqsubset}$) — we obtain, respectively, that $\sqsubseteq$ and $\bigcirc$ are reflexive, $\wr$ is irreflexive, $\bigcirc$ and $\wr$ are symmetric, $\wr$ is the complement of $\bigcirc$, and $\sqsubseteq$ is included in $\bigcirc$:

$$\forall_{x\in M}\ x \sqsubseteq x\,, \tag{$\mathtt{r}_{\sqsubseteq}$}$$

$$\forall_{x\in M}\ x \bigcirc x\,, \tag{$\mathtt{r}_{\bigcirc}$}$$

$$\forall_{x,y\in M}(x \bigcirc y \iff y \bigcirc x)\,, \tag{$\mathtt{s}_{\bigcirc}$}$$

$$\forall_x\ \neg\, x \wr x\,, \tag{$\mathtt{irr}_{\wr}$}$$

$$\forall_{x,y\in M}(x \wr y \iff y \wr x)\,, \tag{$\mathtt{s}_{\wr}$}$$

$$\forall_{x,y\in M}(x \wr y \iff \neg\, x \bigcirc y). \tag{3.1}$$

$$\forall_{x,y\in M}(x \sqsubseteq y \vee y \sqsubseteq x \Longrightarrow x \bigcirc y)\,. \tag{3.2}$$

CC BY ND

From ($\mathtt{irr}_{\sqsubset}$) and ($\mathtt{df}\ \sqsubseteq$) we obtain that:

$$\forall_{x,y\in M}(x \sqsubset y \iff x \sqsubseteq y \wedge x \neq y)\,. \tag{3.3}$$

By ($\mathtt{as}_{\sqsubset}$), ($\mathtt{irr}_{\sqsubset}$) and ($\mathtt{df}\ \sqsubseteq$) we obtain that $\sqsubseteq$ is antisymmetric:

$$\forall_{x,y\in M}(x \sqsubseteq y \wedge y \sqsubseteq x \Longrightarrow x = y)\,. \tag{$\mathtt{antis}_{\sqsubseteq}$}$$

Finally, ($\mathtt{t}_{\sqsubset}$) and the definitions entail that $\sqsubseteq$ is transitive and has two following properties:

$$\forall_{x,y,z\in M}(x \sqsubseteq y \wedge y \sqsubseteq z \Longrightarrow x \sqsubseteq y)\,, \tag{$\mathtt{t}_{\sqsubseteq}$}$$

$$\forall_{x,y,z\in M}(x \sqsubseteq y \wedge z \bigcirc x \Longrightarrow z \bigcirc y)\,, \tag{3.4}$$

$$\forall_{x,y,z\in M}(x \sqsubseteq y \wedge z \wr y \Longrightarrow z \wr x)\,. \tag{3.5}$$

The third axiom we put upon $\sqsubset$ is the so-called *separation condition* known in mereology under the name *Strong Supplementation Principle*:

$$\forall_{x,y\in M}(x \not\sqsubseteq y \Rightarrow \exists_{z\in M}(z \sqsubseteq x \wedge z \wr y))\,. \tag{$\mathtt{SSP}$}$$

The so-called *Weak Supplementation Principle*:

$$\forall_{x,y\in M}(x \sqsubset y \Rightarrow \exists_{z\in M}(z \sqsubset y \wedge z \wr x))\,, \tag{$\mathtt{WSP}$}$$

which is one of the fundamental properties of parthood considered in mereology, follows from the axioms assumed.

FACT 3.1. *($\mathtt{WSP}$) follows from ($\mathtt{as}_{\sqsubset}$), ($\mathtt{SSP}$), ($\mathtt{df}\ \sqsubseteq$) and ($\mathtt{df}\ \wr$).*[3]

PROOF. Assume that $x \sqsubset y$. Hence, by ($\mathtt{as}_{\sqsubset}$) and ($\mathtt{irr}_{\sqsubset}$) we get $y \not\sqsubset x$ and $y \neq x$, which means that $y \not\sqsubseteq x$. Applying ($\mathtt{SSP}$) we get some z for which: $z \sqsubseteq y$ and $z \wr x$. But $z \neq y$. Otherwise we obtain contradiction: $y \wr x$ and $\neg\, y \wr x$, by ($\mathtt{df}\ \wr$) and ($\mathtt{df}\ \sqsubseteq$), since $x \sqsubset y$. Thus, $z \sqsubset y$. ⊣

LEMMA 3.2 ([3, 4]). *($\mathtt{irr}_{\sqsubset}$) is a consequence of ($\mathtt{WSP}$). Thus, ($\mathtt{WSP}$) and ($\mathtt{t}_{\sqsubset}$) entail ($\mathtt{as}_{\sqsubset}$).*

PROOF. Suppose towards contradiction that $x \sqsubset x$. Then, by ($\mathtt{WSP}$), for some x we have that $z \sqsubset x$ and $z \wr x$. From the first fact, by ($\mathtt{df}\ \sqsubseteq$) and ($\mathtt{df}\ \wr$), we obtain that $\neg\, z \wr x$. ⊣

[3] In [3] (p. 220) it is proved that in any structure $\langle M, \sqsubseteq\rangle$ condition ($\mathtt{WSP}$) follows from ($\mathtt{r}_{\sqsubseteq}$), ($\mathtt{antis}_{\sqsubseteq}$), (3.3), ($\mathtt{SSP}$) and ($\mathtt{df}\ \wr$). In consequence we obtain the above fact.

Fact 3.3. *It follows from* (WSP) *that unless M is not a singleton, there is no element in it which is part of all other elements, i.e. in M there is no object which is ingrediens of all members of M. To put it another way, if the set M has at least two elements, then it does not have the smallest element. Therefore*:

$$\mathrm{Card}(M) > 1 \iff \neg\,\exists_{x\in M}\forall_{y\in M}\; x \sqsubseteq y\,.$$

Definition 3.1. By *a mereological atom* we will mean an object that has no parts. Let $\mathbb{At}$ be the set of all mereological atoms, i.e.:

$$\mathbb{At} := \{a \in M :\ \mathbb{P}(a) = \emptyset\}. \qquad (\texttt{df}\ \mathbb{At})$$

Definition 3.2. By *an isolated element* with respect to $\sqsubset$ we will mean a mereological atom which is not part of any object. Let $\mathbb{Is}$ be the set of all isolated elements, i.e.:

$$\mathbb{Is} := \{a \in \mathbb{At} : \neg\,\exists_{x\in M}\; a \sqsubset x\}. \qquad (\texttt{df}\ \mathbb{Is})$$

4. Mereological (collective) sets as mereological sums

Let us start with a short analysis of the situation depicted in Figure 1. The objects we have at hand are a, b, c and d, while x is composed of them. What does it mean? As it was pointed on page 313 two intuitions are important.

- First, every one of objects a, b, c and d is part of x (the more so it is its ingrediens).
- Second, whatever ingrediens of x we take, we can see that it must overlap one of the objects a, b, c and d.

Actually, these two points together are nothing less than one of the ways in which mereological sets are defined. We will call mereological sets as characterized above *mereological sums*. Thus:

> x is *a mereological sum* of S-es if and only if every S is ingrediens of x and every ingrediens of x has some common ingrediens with at least one S.

If we engage mathematical apparatus, then we can define mereological sum as a binary relation between elements of M and its subsets: $\mathbb{Sum} \subseteq$

$M \times \mathcal{P}(M)$. And this is done in the following way:

$$z \,\mathsf{Sum}\, X \overset{\text{df}}{\iff} \forall_{x \in X}\, x \sqsubseteq z \wedge \forall_{y \in M}(y \sqsubseteq z \Rightarrow \exists_{x \in X}\, y \bigcirc x)\,. \qquad (\texttt{df}\ \mathsf{Sum})$$

By means of $\mathbb{I}$ and $\mathbb{O}$ we may write down ($\texttt{df}\ \mathsf{Sum}$) as:

$$z \,\mathsf{Sum}\, X \overset{\text{df}}{\iff} X \subseteq \mathbb{I}(z) \subseteq \bigcup\{\mathbb{O}(x) : x \in X\}\,.$$ [4]

Returning to our example from Figure 1 and taking $X = \{a, b, c, d\}$ we of course have that $x \,\mathsf{Sum}\, \{a, b, c, d\}$.

Philosophically (or ontologically) important fact about Sum is that there is no sum of the empty set. Indeed, we obtain contradiction by ($\texttt{r}_{\sqsubseteq}$) and the assumption that there is x such that $x \,\mathsf{Sum}\, \emptyset$:

$$\neg \exists_{x \in M}\, x \,\mathsf{Sum}\, \emptyset\,. \tag{4.1}$$

We can interpret (4.1) as asserting non-existence of the empty mereological (collective) set.

Let us notice, in connection with (4.1), that Sum satisfies the following condition that could be accepted as its definition:

$$\begin{aligned} z \,\mathsf{Sum}\, X \iff\ & X \neq \emptyset \wedge \\ & \forall_{x \in X}\, x \sqsubseteq z \wedge \forall_{y \in M}(y \sqsubset z \Rightarrow \exists_{x \in X}\, y \bigcirc x)\,. \end{aligned} \tag{4.2}$$

Indeed, by (4.1) we obtain "$\Rightarrow$". For "$\Leftarrow$", notice that if $X \neq \emptyset$ and $\forall_{x \in X}\, x \sqsubseteq z$, then for some $x_0 \in X$ we have that $x_0 \bigcirc z$, by ($\texttt{df}\ \sqsubseteq$) and ($\texttt{df}\ \bigcirc$). Thus $z \,\mathsf{Sum}\, X$.

With the use of $\mathbb{P}$, $\mathbb{I}$, and $\mathbb{O}$ (4.2) may be written down as:

$$z \,\mathsf{Sum}\, X \overset{\text{df}}{\iff} \emptyset \neq X \subseteq \mathbb{I}(z) \wedge \mathbb{P}(z) \subseteq \bigcup\{\mathbb{O}(x) : x \in X\}\,.$$

Some fundamental facts about Sum which follow from ($\texttt{df}\ \sqsubseteq$), ($\texttt{df}\ \bigcirc$) and ($\texttt{df}\ \mathsf{Sum}$) are:

$$\forall_{x \in M}\, x \,\mathsf{Sum}\, \{x\}\,, \tag{4.3}$$

$$\forall_{x \in M}\, x \,\mathsf{Sum}\, \mathbb{I}(x)\,, \tag{4.4}$$

$$\forall_{x \in M}(\mathbb{P}(x) \neq \emptyset \iff x \,\mathsf{Sum}\, \mathbb{P}(x)). \tag{4.5}$$

We say that z is the *greatest element* of X (with respect to $\sqsubseteq$) iff $z \in X$ and for any $x \in X$ we have that $x \sqsubseteq z$. From ($\texttt{df}\ \sqsubseteq$), ($\texttt{df}\ \bigcirc$) and ($\texttt{df}\ \mathsf{Sum}$) follows the following lemma.

[4] Of course, $\bigcup\{\mathbb{O}(x) : x \in X\} = \{y \in M : \exists_{x \in X}\, x \bigcirc y\}$.

Lemma 4.1 ([3, p. 68]). *For any* $X \in \mathcal{P}(M)$ *and* $z \in M$*:* $z \,\mathsf{Sum}\, X$ *and* $z \in X$ *iff* z *is the greatest element of* X*.*

Proof. "⇒" Immediate, by the definitions.

"⇐" Let z be the greatest element of X. Then $z \in X$ and $\forall_{x \in X}\, x \sqsubseteq z$. Moreover, if we take any $y \in M$ such that $y \sqsubseteq z$, then $y \bigcirc z$ by (3.2). ⊣

Remark 4.1. Some weaknesses of (df Sum) are hidden in the facts that it allows x either to be composed «out of itself» (4.3) or to be among elements out of which it is composed ((4.4) and Lemma (4.1)). These properties seem to be counterintuitive from point of view of joining objects (see Figure 2). We will examine these issues in Section 6.

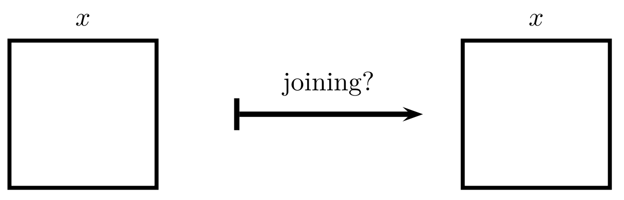

Figure 2: Object x is a mereological sum of itself.

What about existence of sums of any other subsets of the domain? Well, the axioms we assumed does not decide this problem. It may happen that there will be non-empty subsets of M with no mereological sum. The following model will illustrate such situations.

Model 1. Let $M := \{1, 2, 3, 4\}$ and $\sqsubset := \{\langle 1, 4\rangle, \langle 2, 4\rangle, \langle 3, 4\rangle\}$. Directly from (4.3), (4.4) and (4.5) we have that: $4 \,\mathsf{Sum}\, \{4\}$, $4 \,\mathsf{Sum}\, \{1, 2, 3, 4\}$ and $4 \,\mathsf{Sum}\, \{1, 2, 3\}$. However neither of the sets $\{1, 2\}$, $\{1, 3\}$ nor $\{2, 3\}$ has its mereological sum in this model. For example the only sensible candidate for a sum of $\{1, 2\}$ is 4. However it is «too large», since $3 \sqsubset 4$ but $3 \wr 1$ and $3 \wr 2$.

However, what we can demonstrate is that if there exists a sum of some given set, then it must be unique (see [3, p. 76], [4, p. 221]).

Fact 4.2. *The following property of* Sum *is a consequence of* ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$), (SSP) *and accepted definitions*:

$$\forall_{y,z \in M} \forall_{X \in \mathcal{P}(M)} (y \,\mathsf{Sum}\, X \wedge z \,\mathsf{Sum}\, X \Rightarrow y = z) . \qquad (\mathtt{fun}\ \mathsf{Sum})$$

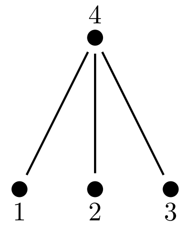

Figure 3: None of sets $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ has its mereological sum.

5. Mereological (collective) sets as mereological fusions

Mereological fusion is an alternative approach to mereological sets. As we already mentioned it on page 313 we can say that x is «built» out of some objects just in case everything that is exterior to every single one of them is exterior to x and vice versa. Equivalently we may say: everything that overlaps x also overlaps one of these objects, and vice versa — if something overlaps one the objects in question, it must overlaps x as well. This two characterizations are usually taken as explication of the notion of *mereological fusion*. We may formalize this notion in the following way:

$$z \mathbin{\mathbb{F}\mathsf{us}} X \overset{\text{df}}{\iff} \forall_{y\in M}(y \bigcirc z \iff \exists_{x\in X}\, y \bigcirc x)\,. \qquad (\mathtt{df}\ \mathbb{F}\mathsf{us})$$

By means of $\mathbb{O}$ we may write ($\mathtt{df}\ \mathbb{F}\mathsf{us}$) as:

$$z \mathbin{\mathbb{F}\mathsf{us}} X \overset{\text{df}}{\iff} \mathbb{O}(z) = \bigcup\{\mathbb{O}(x) : x \in X\}\,.$$

Moreover, directly from (3.1) we have that:

$$z \mathbin{\mathbb{F}\mathsf{us}} X \iff \forall_{y\in M}(y \wr z \iff \forall_{x\in X}\, y \wr x)\,.$$

By ($\mathtt{df}\ \mathbb{F}\mathsf{us}$), we can say about fusion the same what was said about the sum:

$$\neg\,\exists_{x\in M}\, x \mathbin{\mathbb{F}\mathsf{us}} \emptyset\,, \qquad (5.1)$$

$$\forall_{x\in M}\, x \mathbin{\mathbb{F}\mathsf{us}} \{x\}\,, \qquad (5.2)$$

$$\forall_{x\in M}\, x \mathbin{\mathbb{F}\mathsf{us}} \mathbb{I}(x)\,, \qquad (5.3)$$

$$\forall_{x\in M}(\mathbb{P}(x) \neq \emptyset \iff x \mathbin{\mathbb{F}\mathsf{us}} \mathbb{P}(x)). \qquad (5.4)$$

cc BY ND

What is the mutual dependency between $\mathbb{F}\mathrm{us}$ and $\mathbb{S}\mathrm{um}$? The following known theorems and Model 2 answer this question. In [3, p. 69], [4, Fact 2], [3, p. 114] and [4, Lemma 2] the following facts are proved.

Theorem 5.1 ([3, 4]). *Suppose that* $\langle M, \sqsubset\rangle$ *satisfies* $(\mathtt{t}_{\sqsubset})$. *Then*:

(i) *if* $\langle M, \sqsubset\rangle$ *satisfies* $(\mathtt{t}_{\sqsubset})$, *then* $\mathbb{S}\mathrm{um} \subseteq \mathbb{F}\mathrm{us}$.

(ii) $\mathbb{F}\mathrm{us} \subseteq \mathbb{S}\mathrm{um}$ *iff* $\langle M, \sqsubset\rangle$ *satisfies* $(\mathtt{t}_{\sqsubset})$ *and* $(\mathtt{SSP})$.[5]

Model 2. This model shows that $(\mathtt{SSP})$ is indispensable to prove both the uniqueness of $\mathbb{S}\mathrm{um}$ and the inclusion $\mathbb{F}\mathrm{us} \subseteq \mathbb{S}\mathrm{um}$. It is easily checked

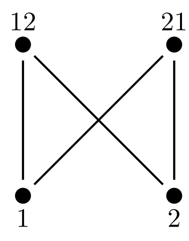

Figure 4: Both 12 and 21 are sums of $\{1, 2\}$.

that in the model from Figure 4 where $M = \{1, 2, 12, 21\}$ and

$$\sqsubset := \{\langle 1, 12\rangle, \langle 1, 21\rangle, \langle 2, 12\rangle, \langle 2, 21\rangle\}$$

it is the case that $12 \; \mathbb{S}\mathrm{um} \; \{1, 2\}$ and $21 \; \mathbb{S}\mathrm{um} \; \{1, 2\}$. Thus there exist two different sums of the same set of objects. Whether this is fault or advantage depends on the level of abstraction. If we want objects to be identified by their parts only, then this obviously is a fault. If we want them to be identified by something else as well (structure, order etc.) this may be desirable to keep things like that. Moreover, the model proves that in structures satisfying $(\mathtt{as}_{\sqsubset})$ and $(\mathtt{t}_{\sqsubset})$ it does not have to be the case that $\mathbb{F}\mathrm{us} \subseteq \mathbb{S}\mathrm{um}$: $12 \; \mathbb{F}\mathrm{us} \; \{1, 2, 21\}$ but $\neg\, 12 \; \mathbb{S}\mathrm{um} \; \{1, 2, 21\}$.

[5] In [4, Fact 2] and [4, Lemma 2] we obtain analogous facts for structures $\langle M, \sqsubseteq\rangle$ with $(\mathtt{t}_{\sqsubseteq})$ instead of $(\mathtt{t}_{\sqsubset})$. Hence we obtain the above facts.

cc BY ND

6. Mereological (collective) sets as aggregates

So far we have explained mereological (collective) sets in two ways: by means of sum and by means of fusion. However, as we already pointed to it in Remark 4.1 concerning properties of $\mathbb{Sum}$, both approaches have some faults from point of view of intuitions connected with collecting objects into something «new» (see [3, Ch. 4, §4]). First, the object which is a sum (fusion) can be among the very objects that are summed (fused). Second, the object can be a sum (fusion) of itself. Thus what we lose is a plural character of the process of fusing or summing things. This may be especially strange in case of mereological atoms, that is objects that have no parts. They are also sums or fusions of something, that is of themselves.

Because of the imperfections[6] mentioned above, we introduce one more notion, of *aggregate* of objects. What we want to achieve is the following property of aggregates: if z is an aggregate of the group of S-es, then no S is identical with z and every S is part of z.

In the light of the words above, we say that z is an aggregate of S-es iff

(1) the group of S-es has at least two members and

(2) every S is part of z (so z has at least two parts) and

(3) every part of z satisfies at least one of the following conditions:

   (a) it is an S,
   (b) it is part of some S,
   (c) some S is its part,
   (d) it has common part with some S.

Taking $X \in \mathcal{P}(M)$ the above intuitions can be expressed as follows:

$$\begin{aligned} z \mathrel{\mathbb{Agr}} X \iff & \exists_{x,y\in X}\, x \neq y \wedge \forall_{x\in X}\, x \sqsubset z \wedge \\ & \forall_{y\in M}\big(y \sqsubset z \Longrightarrow (y \in X \vee \exists_{x\in X}\, y \sqsubset x \vee \\ & \exists_{x\in X}\, x \sqsubset y \vee \exists_{x\in X}\exists_{u\in M}(u \sqsubset y \wedge u \sqsubset x))\big). \end{aligned}$$

The complexity of the above formula may be substantially reduced by noting that the consequent of the implication is equivalent to the state-

[6] We speak about imperfections from philosophical or ontological point of view, not from mathematical one. Mathematically both definitions (of sum and fusion) are, so to say, flawless.

CC BY ND

ment $\exists_{x\in X}\ x \bigcirc y$. Moreover, existence of two elements in X may be replaced by the weaker condition that X is not empty (this is due to the definition below, ($\mathtt{t}_{\sqsubset}$), (WSP) and Fact 6.7). Non-emptiness of X is quite important and it cannot be dropped without changing some essential properties of aggregates (see Section 7). So we may take the following formula as the definition of a mereological aggregate:

$$z \mathbin{\mathsf{Agr}} X \iff X \neq \emptyset \wedge \forall_{x\in X}\ x \sqsubset z \wedge \forall_{y\in M}(y \sqsubset z \Rightarrow \exists_{x\in X}\ y \bigcirc x)\,. \tag{df Agr}$$

By means of $\mathbb{P}$ and $\mathbb{O}$ it may be written down as:

$$z \mathbin{\mathsf{Agr}} X \overset{\text{df}}{\iff} \emptyset \neq X \subseteq \mathbb{P}(z) \subseteq \bigcup\{\mathbb{O}(x) : x \in X\}\,.$$

The very basic properties distinguishing Agr from Sum and Fus are expressed in the following fact, which we obtain from irreflexivity of $\sqsubset$ and (df Agr).

FACT 6.1. *Suppose $\langle M, \sqsubset\rangle$ satisfies* ($\mathtt{irr}_{\sqsubset}$). *Then*:

$$\forall_{z\in M}\forall_{X\in\mathcal{P}(M)}(z \mathbin{\mathsf{Agr}} X \Rightarrow z \notin X).$$

Consequently, we have that: $\neg\exists_{x\in M}\ x \mathbin{\mathsf{Agr}} \{x\}$ *and* $\neg\exists_{x\in M}\ x \mathbin{\mathsf{Agr}} \mathbb{I}(x)$.

As an immediate corollary from (df Agr), (df $\sqsubseteq$) and (df $\bigcirc$) we obtain the following equivalence:

$$\forall_{x\in M}(\mathbb{P}(x) \neq \emptyset \iff x \mathbin{\mathsf{Agr}} \mathbb{P}(x)). \tag{6.1}$$

By (df Agr), (df $\sqsubseteq$) and (df Sum) we have:

FACT 6.2 ([3, p. 122]). $\mathsf{Agr} \subseteq \mathsf{Sum}$.

PROOF. Suppose that $z \mathbin{\mathsf{Agr}} X$. By (df Agr) and (df $\sqsubseteq$), $\forall_{x\in X}\ x \sqsubseteq z$. Further, suppose that $y \sqsubseteq z$. If $y = z$, then any element x of X is such that $x \bigcirc y$. If $y \sqsubset z$, then by (df Agr) we obtain that $\exists_{x\in X}\ x \bigcirc y$. ⊣

Facts 4.2 and 6.2 immediately entail the following:

COROLLARY 6.3. *Let $\langle M, \sqsubset\rangle$ satisfy* ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$) *and* (SSP). *Then every subset of M can have at most one aggregate*:

$$\forall_{y,z\in M}\forall_{X\in\mathcal{P}(M)}(y \mathbin{\mathsf{Agr}} X \wedge z \mathbin{\mathsf{Agr}} X \Rightarrow y = z). \tag{fun Agr}$$

CC BY ND

Fact 6.4 ([3, p. 122]). *Suppose* $\langle M, \sqsubset\rangle$ *satisfies* ($\mathtt{irr}_{\sqsubset}$). *Then for every* $z \in M$ *and* $X \in \mathcal{P}(M)$: *if* $z \,\mathbb{Sum}\, X$ *and* $z \notin X$, *then* $z \,\mathbb{Agr}\, X$.

Proof. Suppose $z \,\mathbb{Sum}\, X$ and $z \notin X$. First, $X \neq \emptyset$, by (4.1). Second, $\forall_{x\in X}\ x \sqsubseteq z$, so $\forall_{x\in X}\ x \sqsubset z$ by ($\mathtt{irr}_{\sqsubset}$). Third, take arbitrary $y \sqsubset x$. Thus $y \sqsubseteq x$ by ($\mathtt{df}\ \sqsubseteq$) and $\exists_{x\in X}\ y \bigcirc y$ by the assumption and ($\mathtt{df}\ \mathbb{Sum}$). ⊣

From Facts 6.1, 6.2 and 6.4 we obtain:

Corollary 6.5. *If* $\langle M, \sqsubset\rangle$ *satisfies* ($\mathtt{irr}_{\sqsubset}$), *then for every* $z \in M$ *and* $X \in \mathcal{P}(M)$: $z \,\mathbb{Agr}\, X$ *iff* $z \,\mathbb{Sum}\, X$ *and* $z \notin X$.

Moreover, by Lemma 4.1 and Fact 6.4 we have:

Corollary 6.6. *Suppose* $\langle M, \sqsubset\rangle$ *satisfies* ($\mathtt{irr}_{\sqsubset}$). *Then for every* $z \in M$ *and* $X \in \mathcal{P}(M)$: *if* $z \,\mathbb{Sum}\, X$ *and* X *does not contain greatest element, then* $z \,\mathbb{Agr}\, X$.

Fact 6.7 ([3, p. 123]). *Suppose that* $\langle M, \sqsubset\rangle$ *satisfies* ($\mathtt{t}_{\sqsubset}$) *and* ($\mathtt{WSP}$). *Then if a subset of* M *has an aggregate, then it does not have the greatest element. Consequently, it has at least two members.*

Proof. Suppose towards contradiction that $z \,\mathbb{Agr}\, X$ and x^* is the greatest element of X, i.e. (a) $x^* \in X$ and (b) $\forall_{x\in X}\ x \sqsubseteq x^*$. Then, by (a) and ($\mathtt{df}\ \mathbb{Agr}^\star$), $x^* \sqsubset z$. Hence, by ($\mathtt{WSP}$), there is y such that (c) $y \sqsubset z$ and (d) $y \wr x^*$. By (c) and ($\mathtt{df}\ \mathbb{Agr}$) we have that there is $x_0 \in X$ such that $y \bigcirc x_0$. But, by (b), $x_0 \sqsubseteq x^*$. Therefore from (3.4), which we obtain by ($\mathtt{t}_{\sqsubset}$), we get $y \bigcirc x^*$, which contradicts (d). ⊣

Model 3. It is worth noticing that existence of the greatest element in X and axioms ($\mathtt{as}_{\sqsubset}$) and ($\mathtt{t}_{\sqsubset}$) (without ($\mathtt{WSP}$)) does not entail non-existence of an aggregate of X. The simple model $M := \{1,2\}$ with $\sqsubset := \{\langle 1,2\rangle\}$ proves this is the case. $X = \{1\}$ has the greatest element and $2 \,\mathbb{Agr}\, \{1\}$.

From Lemma 4.1 and Corollary 6.6 we obtain:

Theorem 6.8 ([3, p. 124]). *Let* $\langle M, \sqsubset\rangle$ *satisfies* ($\mathtt{irr}_{\sqsubset}$). *Then the following statement:*

$$\forall_{X\in\mathcal{P}_+(M)}(X \text{ does not have the greatest element} \Longrightarrow \exists_{x\in M}\ z \,\mathbb{Agr}\, X) \tag{$\exists\mathbb{Agr}$}$$

is equivalent to:

$$\forall_{X\in\mathcal{P}_+(M)}\exists_{z\in M}\ z \,\mathbb{Sum}\, X\,. \tag{$\exists\mathbb{Sum}$}$$

cc BY ND

Figure 5: $\{1\}$ has the greatest element and $2 \ \mathbb{Agr} \ \{1\}$. Notice that (WSP) is not satisfied.

Model 4. There exist structures $\langle M, \sqsubset\rangle$ which satisfy ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$) and (SSP) but no subset of the universe has its aggregate. An arbitrary structure consisted of isolated elements only will serve as a model. The only sums (fusions) that exist in such structures are sums (fusions) of the form $x \ \mathbb{Sum} \ \{x\}$ and $x \ \mathbb{Fus} \ \{x\}$. The simplest structure of this kind consist of exactly one object.

Figure 6: The simplest non-trivial structure which satisfies ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$) and (SSP), but such that $\mathbb{Agr} = \emptyset$.

Remark 6.1. As it is demonstrated in [3, p. 109] and [4, p. 228] the following collections of sentences are equivalent:

(a) ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$), ($\mathtt{fun}\ \mathbb{Sum}$) and ($\exists\mathbb{Sum}$);

(b) ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$), (SSP) and ($\exists\mathbb{Sum}$);

(c) ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$), (WSP) and ($\exists\mathbb{Sum}$).

(a) is an axiomatization of the classical mereology, which is a counterpart of the very first collection of axioms of Leśniewski's from [2]. By Theorem 6.8 the above axiomatizations are equivalent to the following ones:

(a′) ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$), ($\mathtt{fun}\ \mathbb{Sum}$) and ($\exists\mathbb{Agr}$);

(b′) ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$), (SSP) and ($\exists\mathbb{Agr}$),

(c′) ($\mathtt{as}_{\sqsubset}$), ($\mathtt{t}_{\sqsubset}$), (WSP) and ($\exists\mathbb{Agr}$).

CC BY ND

7. Sums and aggregates of the empty set

The requirement in (df $\mathbb{Agr}$) that only non-empty sets can have its aggregates is very important for the theory of mereological sets. Let us remind that in case of sums and fusions we do not have to exclude the empty set *explicite*, since the very definitions together with reflexivity of $\sqsubseteq$ guarantee that there is neither sum nor fusion of the empty set. For the aggregate relation the situation is different. However, as it was pointed in Section 4, (4.2) is equivalent to (df $\mathbb{Sum}$) in which non-emptiness is involved *explicite*.

Let us now consider the situation in which we define alternative relations by means of the following definitions:

$$z\ \mathbb{Sum}^\star\ X \overset{\text{df}}{\iff} \forall_{x\in X}\ x \sqsubseteq z \wedge \forall_{y\in M}(y \sqsubset z \Rightarrow \exists_{x\in X}\ y \bigcirc x)\,, \qquad (\text{df } \mathbb{Sum}^\star)$$

$$z\ \mathbb{Agr}^\star\ X \overset{\text{df}}{\iff} \forall_{x\in X}\ x \sqsubset z \wedge \forall_{y\in M}(y \sqsubset z \Rightarrow \exists_{x\in X}\ y \bigcirc x)\,. \qquad (\text{df } \mathbb{Agr}^\star)$$

By means of $\mathbb{P}$, $\mathbb{I}$ and $\mathbb{O}$ we may write down (df $\mathbb{Sum}^\star$) and (df $\mathbb{Agr}^\star$) as:

$$z\ \mathbb{Sum}^\star\ X \overset{\text{df}}{\iff} X \subseteq \mathbb{I}(z) \wedge \mathbb{P}(z) \subseteq \bigcup\{\mathbb{O}(x) :\ x \in X\}\,,$$

$$z\ \mathbb{Agr}^\star\ X \overset{\text{df}}{\iff} X \subseteq \mathbb{P}(z) \subseteq \bigcup\{\mathbb{O}(x) :\ x \in X\}\,.$$

Now we will show their consequences. The following fact is obvious.

Fact 7.1. *For any $z \in M$ and $X \in \mathcal{P}(M)$:*

(i) $\mathbb{Sum} \subseteq \mathbb{Sum}^\star$ *and* $\mathbb{Agr} \subseteq \mathbb{Agr}^\star$.

(ii) $z\ \mathbb{Sum}^\star\ X \wedge X \neq \emptyset \Longrightarrow z\ \mathbb{Sum}\ X$.

(iii) $z\ \mathbb{Agr}^\star\ X \wedge X \neq \emptyset \Longrightarrow z\ \mathbb{Agr}\ X$.

The definitions (df $\mathbb{Agr}^\star$), (df $\mathbb{Sum}^\star$) and (df $\mathbb{At}$) entail the following equivalences:

$$z \in \mathbb{At} \iff z\ \mathbb{Sum}^\star\ \emptyset \iff z\ \mathbb{Agr}^\star\ \emptyset\,. \qquad (7.1)$$

In consequence we obtain the following fact.

Fact 7.2. *In any structure $\langle M, \sqsubset\rangle$, solely by definitions* (df $\sqsubseteq$), (df $\bigcirc$), (df $\mathbb{Sum}$), (df $\mathbb{Sum}^\star$), (df $\mathbb{Agr}$), (df $\mathbb{Agr}^\star$) *and* (df $\mathbb{At}$), *we obtain that the following three statements are equivalent:*

(a) $\mathbb{Sum}^\star \subseteq \mathbb{Sum}$,

(b) $\mathbb{Agr}^\star \subseteq \mathbb{Agr}$,

(c) *there is no atom in M.*

cc BY ND

Model 5. From (7.1) it follows that $\mathbb{A}\mathrm{gr}^\star$ does not satisfy the counterpart of (fun $\mathbb{A}\mathrm{gr}$) in Corollary 6.3. Consider the structure from Figure 6, that is $M := \{1, 2\}$ and both 1 and 2 are isolated. In such case we have that $1 \;\mathbb{A}\mathrm{gr}^\star\; \emptyset$ and $2 \;\mathbb{A}\mathrm{gr}^\star\; \emptyset$ but $1 \neq 2$. The same can be said of course about $\mathbb{S}\mathrm{um}^\star$ and (4.2). Obviously $\langle M, \sqsubset\rangle$ satisfies (as$_\sqsubset$), (t$_\sqsubset$) and (SSP).

From (7.1), i.e. solely by definitions (df $\mathbb{S}\mathrm{um}^\star$), (df $\mathbb{A}\mathrm{gr}^\star$) and (df $\mathbb{A}\mathrm{t}$), we obtain that the following statements are equivalent:

(a) $\forall_{x,y\in M}(x \;\mathbb{S}\mathrm{um}^\star\; \emptyset \wedge y \;\mathbb{S}\mathrm{um}^\star\; \emptyset \Longrightarrow x = y)$

(b) $\forall_{x,y\in M}(x \;\mathbb{A}\mathrm{gr}^\star\; \emptyset \wedge y \;\mathbb{A}\mathrm{gr}^\star\; \emptyset \Longrightarrow x = y)$,

(c) there is at most one atom in M.

In consequence we obtain the following fact.

Fact 7.3. *In any structure $\langle M, \sqsubset\rangle$ satisfying* (as$_\sqsubset$), (t$_\sqsubset$) *and* (SSP) *the following two statements are equivalent*:

(a) $\forall_{x,y\in M}\forall_{X\in\mathcal{P}(M)}(x \;\mathbb{S}\mathrm{um}^\star\; X \wedge y \;\mathbb{S}\mathrm{um}^\star\; X \Rightarrow x = y)$,

(b) $\forall_{x,y\in M}\forall_{X\in\mathcal{P}(M)}(x \;\mathbb{A}\mathrm{gr}^\star\; X \wedge y \;\mathbb{A}\mathrm{gr}^\star\; X \Rightarrow x = y)$,

(c) *there is at most one atom in M.*

Proof. "(a) ⇒ (b)" By Fact 6.2.

"(b) ⇒ (c)" (7.1) (we use (b) for $X = \emptyset$).

"(c) ⇒ (a)" If $X \neq \emptyset$ then we use Facts 4.2 and 7.1. If $X = \emptyset$ then we refer to (7.1). ⊣

8. Conclusion

In the course of the paper we defined mereological sets in five different ways. Two of the defintions — (df $\mathbb{S}\mathrm{um}$) and (df $\mathbb{F}\mathrm{us}$) — are equivalent on the basis of axioms (as$_\sqsubset$), (t$_\sqsubset$) and (SSP). Assuming the same axioms, (df $\mathbb{A}\mathrm{gr}$) is equivalent to (df $\mathbb{S}\mathrm{um}$) and (df $\mathbb{F}\mathrm{us}$) in those cases in which a non-empty subset of the domain does not have the greatest element with respect to $\sqsubseteq$. (df $\mathbb{A}\mathrm{gr}$) seems to be that definition which is the closest to modeling the process of joining many objects into a single one. (df $\mathbb{A}\mathrm{gr}^\star$) has the same properties as (df $\mathbb{A}\mathrm{gr}$) but it allows for entities «built out of nothing», which may be problematic from philosophical point of view. The last property is shared by (df $\mathbb{S}\mathrm{um}^\star$), which is a

cc BY ND

modification of (df Sum). Apart from this, (df Sum) and (df Sum*) have the same attributes.

References

[1] Fraenkel, A., and Y. Bar-Hillel, and A. Levy, *Foundations of set theory*, second revised edition, North Holland Publishing Company, Amsterdam–London, 1973.

[2] Leśniewski, S., "On the foundations of mathematics", pages 174–382 in *Collected Works*, S. J. Surma and others, editors, number 44 in Nijhoff International Philosophy Series, Kluwer Academic Publishers, Dordrecht, 1991 (in Polish: 1927–1931).

[3] Pietruszczak, A., *Metamereologia*, Wydawnictwo Uniwersytetu Mikołaja Kopernika, Toruń, 2000.

[4] Pietruszczak, A., "Pieces of mereology", *Logic and Logical Philosophy* 14, 2 (2005): 211–234. DOI: 10.12775/LLP.2005.014

[5] Quine, W. V. O., "Logic and the reification of universals", pp. 102–129 in: *From a logical point of view*, Harvard University Press, 2003.

[6] Quine, W. V. O., *Set theory and its Logic*, The Belknap Press, Harvard, 1963.

[7] Simons, P., *Parts. A Study in Ontology*, Clarendon Press, Oxford, 1987.

[8] Wang, H., "What is logic?", *The Monist* 77, 3 (1994): 261–278.

Rafał Gruszczyński
Andrzej Pietruszczak
Department of Logic
Nicolaus Copernicus University
ul. Asnyka 2
87-100 Toruń, Poland
{gruszka, pietrusz}@uni.torun.pl