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How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities

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Introduction and notations.

We make a local study of the projection onto convex sets in real Hilbert space. Let H be a real Hilbert space, $K\subset H$ a closed convex subset, the projection operator onto K will be denoted by P. For every $u\in K$, we set $S_K(u) = \bigcup_{\lambda>0}\lambda(K-u)$, $\Pi_K(u)=\overline{S_K(u)}$. If $f\in H$, $[f]$=vector space generated by f. If K is a cone with vertex 0, then $K^{\perp}=\{v\in H, \forall f\in K, \langle f, v\rangle\leqq 0\}$. In particular, for $f\in H$, $[f]^{\perp}=\{v\in H, \langle f, v\rangle=0\}$. For K a cone with vertex 0, and $u\in K$, we have

$$S_K(u)=K+[u],\ \Pi_K(u)=\overline{K+[u]}.$$

Finally, for K an arbitrary closed convex set, and $v\in H$, we define

$$\Sigma_K(v)=\Pi_K(Pv)\cap[v-Pv]^{\perp}.$$

In § 1, we prove under reasonable hypotheses a theorem which shows the role played by the "curvature of the boundary" of K near Pv, for the conical differentiability of P at v. After giving some zoology from geometry or integration, we restrict our attention to the case where $\forall v\in H$, $S_K(Pv)\cap[v-Pv]^{\perp}$ is dense in $\Sigma_K(Pv)$. A convex set that satisfies this property will be called polyhedric. We get the following

THEOREM. *If K is polyhedric, $\forall v\in H$, $\forall z\in H$, then the curve $t\rightarrow P(v+tz)$ is strongly right-differentiable at 0, with a derivative $\gamma=\mathrm{Proj}_{\Sigma_K(v)}(z)$.*

In § 2, we assume that H is a lattice, with respect to a closed positive cone K. Then K is a polyhedric set under the simple hypothesis that $x\rightarrow x^+=\sup\{x, 0\}$ is a bounded map. If $f:[0, T[\rightarrow H$ is right-differentiable, then by setting $u(t)=\mathrm{Proj}_K(f(t))$, the preceding theorem gives

$$\forall t\in[0, T[,\ \frac{d^+u}{dt}=\mathrm{Proj}_{\Sigma(f(t))}\left(\frac{d^+f}{dt}\right).$$

Let us now assume the stronger condition:

$$\forall x \in H, \ \langle x^+, x^- \rangle \leqq 0 .$$

Then, under the hypothesis that $\forall t, \frac{d^+f}{dt} \in K^\perp$, we get

$$\frac{d^+u}{dt} = \mathrm{Proj}_{\Pi_K(u(t))}\left(\frac{d^+f}{dt}\right) \in -K .$$

In § 3, we give some applications to variational inequalities by using the above results in the two cases:

$$\begin{cases} H=H_0^1(\Omega), & K=\{x\in H, \ x\geqq 0\} \\ H=H^1(\Omega), & K=\{x\in H, \ x_{|\partial\Omega}\geqq 0\} , \end{cases}$$

Ω being an open subset of R^N, with sufficiently regular boundary. We obtain results that were proved in [2] by completely different methods. For a generalization in another direction, see [3].

I. Some general facts about the projection onto convex sets.

Let K and P be as in the introduction, v and z two elements of H. We set

$$\gamma(t) = \frac{P(v+tz)-Pv}{t} .$$

Since P is a contraction, $|\gamma(t)| \leqq |z|$, $\forall t>0$.

PROPOSITION 1. *Let γ be a weak limit-point of $\gamma(t)$ as $t \to 0$. Then,*

$$\begin{cases} \gamma \in \Sigma_K(v), \ \langle \gamma, z-\gamma \rangle \geqq 0 \\ \forall w \in S_K(Pv) \cap [v-Pv]^\perp, \ \langle z-\gamma, w \rangle \leqq 0 . \end{cases}$$

PROOF. Since $P(v+tz)=t\gamma(t)+Pv$, we have

$$\langle v+tz-(t\gamma(t)+Pv), Pv-(t\gamma(t)+Pv)\rangle \leqq 0$$

$$\Rightarrow t^2\langle \gamma(t), \gamma(t)-z\rangle \leqq t\langle v-Pv, \gamma(t)\rangle = \langle v-Pv, P(v+tz)-Pv\rangle \leqq 0 .$$

Dividing by t^2, then using the weak lower semi-continuity of the norm, we obtain $\langle \gamma, \gamma-z\rangle \leqq 0$ for each weak limit-point γ. Moreover, $0 \geqq \langle v-Pv, \gamma(t)\rangle \geqq t\langle \gamma(t), \gamma(t)-z\rangle$. Since $\gamma(t)$ is bounded, we deduce $\langle v-Pv, \gamma\rangle = 0$. In any case we have $\gamma \in \Pi_K(Pv)$, so we conclude that $\gamma \in \Sigma_K(v)$. Let us now assume $\gamma(t_n) \rightharpoonup \gamma$, with $t_n \to 0$ as $n \to +\infty$, and set $\delta_n = \gamma(t_n) - \gamma$. If we consider $u \in K$ such that $\langle v-Pv, u-Pv\rangle = 0$, then by $\langle v-Pv, \gamma\rangle = 0$, the inequality

$$\langle v-Pv+t_n(z-\gamma)-t_n\delta_n, u-Pv-t_n\gamma-t_n\delta_n\rangle \leqq 0$$

implies

$$\langle z-\gamma,\ u-Pv\rangle \leqq \langle v-Pv,\ \delta_n\rangle + \langle \delta_n,\ u-Pv\rangle + Ct_n\,,$$

where C is a finite constant. Hence as $n\to+\infty$, we get

$$\langle z-\gamma,\ u-Pv\rangle \leqq 0\,.$$

Now if $w\in S_K(Pv)\cap[v-Pv]^{\perp}$, we have $w=\lambda(u-Pv)$ for some $\lambda>0$ and some $u\in K$. Since $\langle u-Pv,\ v-Pv\rangle=0$, we can apply the previous result and obtain $\langle z-\gamma,\ w\rangle\leqq 0$.

THEOREM 1. *Let $K\subset H$ be a closed convex set. We fix $v\in H$ and $w\in\sum_K(v)$. We assume that there exists a bounded linear self-adjoint operator L on H such that*

$$\begin{cases} L\circ \mathrm{Proj}_{\Sigma_K(v)}=\mathrm{Proj}_{\Sigma_K(v)}\circ L\,, \\ P(v+tw)=Pv+tL^2w+o(t) \qquad (t>0)\,. \end{cases}$$

Then, for any $z\in H$ such that $\mathrm{Proj}_{\Sigma_K(v)}(z)=w$, we have $P(v+tz)=Pv+tL^2w+o(t)$.

PROOF. Let us first verify the following property:

$$\forall z\in H,\ \forall w'\in(\Sigma_K(v))^{\perp},\ \limsup_{t\to 0^+}\left\langle \frac{P(v+tz)-Pv}{t},\ w'\right\rangle\leqq 0\,.$$

We use Proposition 1. If $\left\langle \frac{P(v+t_nz)-Pv}{t_n},\ w'\right\rangle \geqq \alpha>0$ for $t_n\to 0$ and large n, there are a subsequence t_{n_k} and $\gamma\in H$ such that $\frac{P(v+t_{n_k}z)-Pv}{t_{n_k}}\to\gamma$. Then $\langle\gamma,\ w'\rangle\geqq\alpha>0$, which is contradictory with the two facts: $\gamma\in\Sigma_K(v)$ and $w'\in(\Sigma_K(v))^{\perp}$. Now take $z=w+w'$ with $w'\in(\Sigma_K(v))$ and $\langle w,\ w'\rangle=0$. Then we have

$$|P(v+tz)-P(v+tw)|^2\leqq\langle tw',\ P(v+tz)-P(v+tw)\rangle$$

$$=\langle tw',\ Pv-P(v+tw)\rangle+\langle tw',\ P(v+tz)-Pv\rangle\,.$$

But

$$\langle tw',\ Pv-P(v+tw)\rangle=-t^2\langle w',\ L^2w\rangle-\langle tw',\ o(t)\rangle\,.$$

Since L is linear and commutes with $P_{\Sigma_K(v)}$, it also commutes with $P_{(\Sigma_K(v))^{\perp}}=I-P_{\Sigma_K(v)}$. Thus $\langle Lw,\ Lw'\rangle=0$. Dividing by t^2, and using the above results, we get $\limsup_{t\to 0^+}\left|\frac{P(v+tz)-P(v+tw)}{t}\right|^2\leqq 0$, and the conclusion follows.

Before we describe some consequences of this theorem for the polyhedric cones of functional analysis, let us illustrate it by some examples.

EXAMPLE 1. $K=\{u\in H,\ |u|\leqq 1\}$.

For $v\in K$, and $w\in\sum_K(v)$, let us study $P(v+tw)$

——If $|v|<1$, then $P(v+tw)=v+tw$ for small values of t.

——If $|v|=1$, then $\Sigma_K(v)=\{w\in H,\ \langle v,\ w\rangle\leqq 0\}$. So $|v+tw|\leqq(1+t^2|w|^2)^{1/2}=$

$1+o(t)$. And we have $P(v+tw)=\dfrac{v+tw}{\sup\{1, |v+tw|\}}=v+tw+o(t)$.

——If $|v|>1$, then by $Pv=\dfrac{v}{|v|}$ we have $\Sigma_K(v)=[v]^{\perp}$. So if $w\in\Sigma_K(v)$, $|v+tw|=(|v|^2+t^2|w|^2)^{1/2}=|v|+o(t)$. And then $P(v+tw)=Pv+\dfrac{1}{|v|}tw+o(t)$.

Using Theorem 1 with $L=\dfrac{1}{\sup\{1, |v|\}}Id$, we get that $\forall v\in H$, $\forall z\in H$, $P(v+tz)$ is strongly right-differentiable at $t=0$, with derivative

$$\frac{1}{\sup\{1, |v|\}}\operatorname{Proj}_{\Sigma_K(v)}(z).$$

EXAMPLE 2. $H=R^N$ and K is a compact convex subset with C^2 boundary. We may assume that K has interior points. Let v be a point outside K. $\Pi_K(Pv)$ is a closed half-space in H, the dual cone of the half-line generated by $v-Pv$. We may assume for convenience $Pv=0$ and $\Sigma_K(v)=R^{N-1}$.

In a neighbourhood of 0, the boundary of K may be represented by the equation: $x_N=-\varphi(x_1, \cdots, x_{N-1})$ where φ is a C^2 convex function, defined in a neighbourhood of 0 in R^{N-1}, such that $\varphi(0)=0$, $D\varphi(0)=0$. w being an element of $R^{N-1}=\Sigma_K(v)$, the projection of $v+tw$ onto K is the element $(x(t), -\varphi(x(t)))$ of R^N for which $\operatorname{Min}_{R^{N-1}}((|v-Pv|+\varphi(x))^2+|x-tw|^2)$ is achieved.

It is easily seen that this condition implies that

$$|v-Pv|D^2\varphi(0)(x(t))+x(t)=tw+o(t),$$

and we notice that $D^2\varphi(0): R^{N-1}\to R^{N-1}$ is a positive self-adjoint linear operator. Setting $L=(I+|v-Pv|D^2\varphi(0))^{-1/2}\circ\operatorname{Proj}_{\Sigma_K(v)}$, we deduce from Theorem 1 that P is differentiable at v, with differential

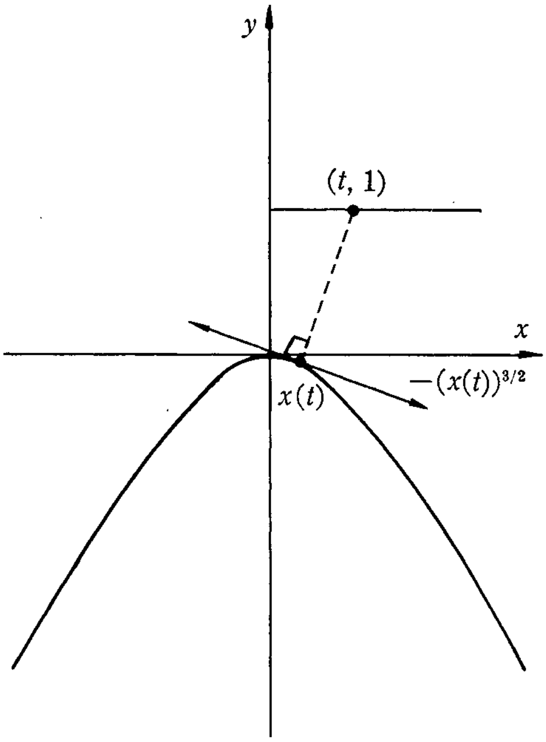
$$dP(v)=(I+|v-Pv|D^2\varphi(0))^{-1}\circ\operatorname{Proj}_{\Sigma_K(v)}.$$

This formula has a simple geometric interpretation: $v\in H\backslash K$ being chosen, we find φ after a suitable change of coordinates. The different eigenvalues $(\alpha_1, \alpha_2, \cdots, \alpha_k)$ of $D^2\varphi(0)$ are exactly the principal curvature numbers for δK at Pv. If we denote by $P_r x$ the projection of $x\in H$ onto the eigen-space corresponding to the eigenvalue α_r, the differential of P at v is given by the formula:

$$dP(v)=\sum_{r=1}^{k}\frac{P_r}{1+|v-Pv|\alpha_r}.$$

EXAMPLE 3. $H=R^2$, $K=\{(x, y)\in H, y\leqq-|x|^{3/2}\}$. The boundary of K is C^1, but not C^2. We consider the point $v=(0, 1)$.

The convex set K being invariant by symmetry with respect to y-axis, it is enough to consider $w=(1, 0)$. Then $P(t, 1)=(x(t), -(x(t))^{3/2})$ with $x(t)\geqq 0$. From the equation $(t-x(t))-\dfrac{3}{2}(x(t))^{1/2}(1+(x(t))^{3/2})=0$, we deduce first $x(t)=0(t^2)$,



and then $x(t) \cong \frac{4}{9} t^2$. So by Theorem 1 (with $L=0$), for any $z \in H$ the curve $P(q+tz)$ starts for $t=0$ with a speed equal to 0.

Let us say that P is semi-differentiable at v if there exists a map $dP(v)$: $H \to H$ positively homogeneous of degree one such that, for $t<0$, $P(v+tz)-Pv = tdP(v)(z)+o(t)$, $\forall z \in H$.

EXAMPLE 4. It we take the product of a finite number of regular convex sets such as in Example 2, we get in the general case a manifold with boundary and corners (the most simple case being a square in R^2). For such a convex set, one can prove that P is semi-differentiable at every point.

It is also possible to take infinite products. As a particular case, let (Ω, μ) be a positively measured space, $\mathcal{H}$ a real Hilbert space. We set

$$H = L^2(\Omega, \mathcal{H}),$$

$$K = \{v \in H, |v(x)| \leqq 1, \ \mu.\text{a.e. in } \Omega\}.$$

It is obvious that

$$\Sigma_K(v) = \{w \in H, w(x) \in \boldsymbol{C}(x), \ \mu.\text{a.e. in } \Omega\}$$

where $\boldsymbol{C}(x)$ is a convex cone, μ. a. e. defined by

$$\boldsymbol{C}(x) = \{z \in \mathcal{H}, \langle z, v(x) \rangle \leqq 0\} \qquad \text{if } |v(x)| = 1,$$

$$\boldsymbol{C}(x) = \{z \in \mathcal{H}, \langle z, v(x) \rangle = 0\} \qquad \text{if } |v(x)| > 1.$$

So $P_{\Sigma_K(v)}$ is commuting with all transformations of the type

$$(Tz)(x)=\lambda(x)z(x), \quad \lambda\in L^\infty(\Omega, R), \quad \lambda(x)\geqq 0, \quad \mu. \text{ a. e.}$$

Applying Theorem 1 with $(Lw)(x)=\left(\frac{1}{\sup\{1, |v(x)|\}}\right)^{1/2} w(x)$, we get the result that P is semi-differentiable at every point $v\in H$, and $\forall v\in H$, $\forall z\in H$, $(dP(v))(z)(x) =\frac{1}{\sup\{1, |v(x)|\}}(P_{\Sigma_K(v)}(z))(x)$, μ. a. e. in Ω.

THEOREM 2. *We suppose now that* $\forall v\in H$, $S_K(Pv)\cap[v-Pv]^\perp$ *is dense in* $\Pi_K(Pv)\cap[v-Pv]^\perp$. *Then, for any* $f:[0, T[\to H$ *right-differentiable at every point,* $u(t)=P(f(t))$ *is right-differentiable, and*

$$\frac{d^+u}{dt}=\mathrm{Proj}_{\Pi_K(u(t))\cap[f(t)-u(t)]^\perp}\left(\frac{d^+f}{dt}\right).$$

PROOF. Since P is a contraction, it is enough to prove the result at $t=0$ for a curve $g(t)=v+tz$. Thus Theorem 2 will be a consequence of the Theorem 1 applied with $L=Id$ if we prove

LEMMA 1. *Let* K *and* v *be as in Theorem* 1. *Then, for any* $w\in \overline{S_K(Pv)\cap[v-Pv]^\perp}$ *we have* $P(v+tw)=Pv+tw+o(t)$.

PROOF. We have to prove that

$$\lim_{t\searrow 0}\left(\frac{P(v+tw)-Pv}{t}-w\right)=0.$$

First if $w\in S_K(Pv)\cap[v-Pv]^\perp$, then $Pv+tw\in K$ for small t. Moreover, for any $u\in K$ we have

$$\langle v+tw-(Pv+tw), u-(Pv+tw)\rangle=\langle v-Pv, u-Pv\rangle-t\langle v-Pv, w\rangle.$$

And then, by virtue of $w\in[v-Pv]^\perp$, this expression is $\leqq 0$, by the definition of Pv. So $P(v+tw)-Pv=tw$ for t small enough, and the result is true for $w\in S_K(Pv)\cap[v-Pv]^\perp$. Since the maps: $w\to\frac{P(v+tw)-Pv}{t}-w$ are uniformly lipschitzian for $t>0$, the result is true by density for $w\in\overline{S_K(Pv)\cap[v-Pv]^\perp}$.

REMARK 1. It is also possible to deduce simply the result of Theorem 2 from Proposition 1 and the estimate $\langle\gamma(t), \gamma(t)-z\rangle\leqq 0$.

II. Study of the projection onto the positive cone of a Hilbert lattice.

Let X, Y be two real Banach spaces, and $T: X\to Y$ a positively homogeneous map, of degree 1. The equivalence of the three following assertions will be used later:

i) T is continuous at 0, from X to Y with the weak topology of Y.

ii) T is continuous at 0, from X to Y.

iii) $\exists M<+\infty: \forall x\in X$, $\|Tx\|_Y\leqq M\|x\|_X$.

It is enough to see that i) $\Rightarrow$ iii). If iii) is not satisfied, there exists a sequence $\{x_n\}$ of vectors in X, such that

$$\|x_n\|_X \leqq 1, \qquad \|Tx_n\|_Y \geqq n^2.$$

We set $y_n = \frac{x_n}{n}$. Then $y_n \to 0$, $\|Ty_n\|_Y = \frac{1}{n}\|Tx_n\|_Y \geqq n$, so Ty_n has no limit point for the weak topology of Y.

THEOREM 3. *Let X, Y be two Banach spaces. We assume that Y is a reflexive Banach lattice, and that $K=\{y \in Y, y \geqq 0\}$ is closed. Let $T: X \to Y$ be positively homogeneous of degree 1, and $\forall x_1, x_2$, $T(x_1+x_2) \leqq Tx_1 + Tx_2$. Then,*

a) *T is continuous from X to Y with its weak topology if and only if it is continuous at 0.*

b) *If we have for any $x \in X$, $\|Tx\|_Y = \|x\|_X$, and Y is uniformly convex, then T is continuous from X to Y with the strong topology.*

PROOF. T being continuous at 0, there exists M such that

$$\forall x \in X, \quad \|Tx\|_Y \leqq M\|x\|_X.$$

So if $x_n \to x$, Tx_n is bounded in Y. Let y be a weak limit-point for the sequence Tx_n. We have

$$Tx_n - Tx + T(x - x_n) \in K \quad \text{and} \quad Tx - Tx_n + T(x_n - x) \in K.$$

Since T is continuous at 0, as $n \to +\infty$, $T(x-x_n) \to 0$ and $T(x_n - x) \to 0$. Since K is weakly closed, passing to the limit we have $y - Tx \in K$ and $Tx - y \in K$ and hence $y = Tx$.

COROLLARY 1. *Let H be a Hilbert lattice, with $K=\{x \in H, x \geqq 0\}$ closed.*

a) *The map $x \to x^+ = \sup\{x, 0\}$ is continuous from H to H with the weak topology, if and only if it is bounded. As a particular case, this is the case if we have*

$$\forall x \in H, \quad \langle x^+, x^- \rangle \leqq 0.$$

b) *If we have for any $x \in H$, $\langle x^+, x^- \rangle = 0$, then the previous mapping is continuous from H to H.*

PROOF. We consider $Tx = x^+ + x^-$. Since we have $\forall x \in H$, $|Tx|^2 - |x|^2 = 4\langle x^+, x^- \rangle$, we observe that $|Tx| \leqq |x| \Leftrightarrow \langle x^+, x^- \rangle \leqq 0$, and $|Tx| = |x| \Leftrightarrow \langle x^+, x^- \rangle = 0$. Using $x^+ = \frac{1}{2}(x + Tx)$, Corollary 1 appears as an immediate consequence of Theorem 3.

COROLLARY 2. *H and K being as in the beginning of Corollary 1, we assume that for any $x \in H$, $|x^+| \leqq M|x|$. Also let $u \in K$ and $h \in (\Pi_K(u))^\perp$. Then $\overline{S_K(u) \cap [h]^\perp} = \Pi_K(u) \cap [h]^\perp$.*

PROOF. First, $-\phi \leqq u$, if $\phi \in K - u$. Since $u \geqq 0$, $u \geqq \sup\{-\phi, 0\} = \phi^-$, so $\phi^- \in -(K-u)$. Since the positive part is positively homogeneous of degree 1, and continuous from H to the weak topology of H, the condition $\phi\in\Pi_K(u)\cap[h]^{\perp}$ implies $\phi^-\in K\cap-\Pi_K(u)\subset[h]^{\perp}$. Then $\phi^+=\phi+\phi^-\in K\cap[h]^{\perp}\subset S_K(u)\cap[h]^{\perp}$. Let now $\phi_n\to\phi^-$ with $\phi_n\in-S_k(u)$. Then we have $\phi_n^+\to\phi^-$, and $(-\phi_n)^-=\phi_n^+\in -S_K(u)\cap K\subset-S_K(u)\cap[h]^{\perp}$, so that $\varphi_n=\phi^+-\phi_n^+\in S_K(u)\cap[h]^{\perp}$, and $\varphi_n\to\phi$.

COROLLARY 3. *Let H be a Hilbert lattice as in Corollary* 1. *We suppose now that for any $x\in H$, $\langle x^+, x^-\rangle\leq 0$. Then, for any $u\in K$ we have* $\mathrm{Proj}_{\Pi_K(u)}(K^{\perp})\subset-K$.

We shall use

LEMMA 2. *Let C_1, C_2 be two closed convex subsets of H. If $\exists\varepsilon>0$ such that $\forall x\in C_1$, $\forall y\in C_2$, $\langle x, y\rangle\geq(\varepsilon-1)|x||y|$, then the sum C_1+C_2 is closed.*

PROOF. We may suppose $\varepsilon\leq 1$. Then,

$$\forall x\in C_1, \forall y\in C_2,\ |x+y|^2\geq|x|^2+2(\varepsilon-1)|x||y|+|y|^2\geq\varepsilon(|x|^2+|y|^2).$$

Let $z=\lim_{n\to+\infty}(x_n+y_n)$, $x_n\in C_1$, $y_n\in C_2$. Then $|x_n|$ and $|y_n|$ are bounded in H. If $x_{n_p}\rightharpoonup x$, $y_{n_p}\rightharpoonup y$, $z=x+y\in C_1+C_2$.

PROOF OF COROLLARY 3. It is equivalent to prove

$$K^{\perp}\subset(\Pi_K(u))^{\perp}+\Pi_K(u)\cap-K.$$

By Lemma 2, with $\varepsilon=1$, the sum $(\Pi_K(u))^{\perp}+\Pi_K(u)\cap-K$ is closed. So, by duality we have to check $\Pi_K(u)\cap(\Pi_K(u)\cap-K)^{\perp}\subset K$. If $w\in\Pi_K(u)$, $w^-\in K\cap-\Pi_K(u)$. If moreover $w\in(\Pi_K(u)\cap-K)^{\perp}$, we get $\langle w^-, w\rangle\geq 0$. Then,

$$|w^-|^2=\langle w^+-w, w^-\rangle=\langle w^+, w^-\rangle-\langle w, w^-\rangle\leq 0$$
$$\Rightarrow w^-=0,\quad\text{or } w\in K.$$

Let H be a Hilbert lattice, such that $K=\{x\in H, x\geq 0\}$ is closed, and let M be a constant such that for any $x\in H$, $|x^+|\leq M|x|$.

For g: $[0, T[\to H$, we consider the solution $u(t)$ of the variational inequality:

$$\begin{cases} u(t)\in K \\ \forall v\in K,\ \langle g(t)-u(t), v-u(t)\rangle\leq 0. \end{cases}$$

Then we have

THEOREM 4. *If g is right-differentiable, then u is also right-differentiable. Its derivative $\frac{d^+u}{dt}$ is given by the variational system*

$$(S)\quad \begin{cases} \dfrac{d^+u}{dt}\in\Pi_K(u(t))\cap[g(t)-u(t)]^{\perp}, \\ \textit{for all } w\in K\cap[g(t)-u(t)]^{\perp}, \left\langle\dfrac{d^+u}{dt}, w-u(t)\right\rangle\geq\left\langle\dfrac{d^+g}{dt}, w-u(t)\right\rangle, \\ \left|\dfrac{d^+u}{dt}\right|^2=\left\langle\dfrac{d^+g}{dt}, \dfrac{d^+u}{dt}\right\rangle. \end{cases}$$

PROOF. Let P be the projection operator onto K. First we notice that if $v \in H$, then

$$\Pi_K(Pv) \cap [v-Pv]^{\perp} = \overline{(K+[Pv]) \cap [v-Pv]^{\perp}} \quad \text{(Corollary 2)}$$

$$= \overline{K \cap [v-Pv]^{\perp} + [Pv]} \qquad (\text{since } \langle v, v-Pv \rangle = 0)$$

$$= \Pi_{K \cap [v-Pv]^{\perp}}(Pv) .$$

$g(t)$ being right-differentiable, we know from Corollary 2 and Theorem 2 that $u(t)$ is also right-differentiable, and $\frac{d^+u}{dt}$ is characterized by the relations:

$$\frac{d^+u}{dt} \in \Pi_K(u(t)) \cap [g(t)-u(t)]^{\perp} ,$$

$$\forall w_1 \in \Pi_K(u(t)) \cap [g(t)-u(t)]^{\perp}, \left\langle \frac{d^+u}{dt}, w_1 - \frac{d^+u}{dt} \right\rangle \geqq \left\langle \frac{d^+g}{dt}, w_1 - \frac{d^+u}{dt} \right\rangle .$$

Setting $w_1 = 0$, and then $w_1 = 2\frac{d^+u}{dt}$, we get $\left| \frac{d^+u}{dt} \right|^2 = \left\langle \frac{d^+g}{dt}, \frac{d^+u}{dt} \right\rangle$. In the residual inequality $\left\langle \frac{d^+u}{dt}, w_1 \right\rangle \geqq \left\langle \frac{d^+g}{dt}, w_1 \right\rangle$, it is necessary and sufficient to substitute $w_1 = w - u(t)$, $w \in K \cap [g(t)-u(t)]^{\perp}$.

COROLLARY 4. *We add the hypothesis*

$$\forall x \in H, \qquad \langle x^+, x^- \rangle \leqq 0 .$$

If we have $\frac{d^+g}{dt} \in K^{\perp}$ *for any* $t \in [0, T[$, *the system* (S) *can be replaced by the simpler one:*

$$(S_0) \qquad \begin{cases} \dfrac{d^+u}{dt} \in \Pi_K(u(t)) \cap -K \\ \forall w \in K, \left\langle \dfrac{d^+u}{dt}, w-u(t) \right\rangle \geqq \left\langle \dfrac{d^+g}{dt}, w-u(t) \right\rangle \\ \qquad \left| \dfrac{d^+u}{dt} \right|^2 = \left\langle \dfrac{d^+g}{dt}, \dfrac{d^+u}{dt} \right\rangle . \end{cases}$$

PROOF. By Corollary 3, if $\frac{d^+g}{dt} \in K$, we get

$$\mathrm{Proj}_{\Pi_K(u(t))}\left(\frac{d^+g}{dt} \right) \in \Pi_K(u(t)) \cap -K \subset [g(t)-u(t)]^{\perp} \cap -K .$$

So

$$\mathrm{Proj}_{\Pi_K(u(t))}\left(\frac{d^+g}{dt} \right) = \mathrm{Proj}_{\Sigma_K(g(t))}\left(\frac{d^+g}{dt} \right) = \frac{d^+u}{dt} \in -K .$$

III. Applications to variational inequalities.

First we recall the essential results about capacity theory in Dirichlet spaces. For more details, c. f. A. Ancona [1].

Let $(X, \mathcal{A}, \xi)$ be a positively measured topological space with its borelian

σ-algebra. We assume that X is locally compact, admitting a countable compact covering, and, for any $\mathcal{K}$ compact $\subset X$, $\xi(\mathcal{K})<+\infty$. We consider a vector subspace H of $L^2(X)$, with a hilbertian scalar product denoted by $((\ ))_H$ with the following properties i)$\sim$vi).

i) The inclusion of H into $L^2(X)$ is continuous,

ii) H is a sublattice of $L^2(X)$ for the order defined by $K=\{u\in H, u\geqq 0$, ξ. a. e. in $X\}$,

iii) $\forall x\in H$, $((x^+, x^-))_H\leqq 0$.

iv) Let $\mathcal{C}(X)$ be the space of continuous functions with compact support in X.

$$Z=\mathcal{C}(X)\cap H \text{ is dense in } H \text{ and in } \mathcal{C}(X).$$

v) $\forall\mathcal{K}$ compact $\subset X$, $\forall V$ a neighbourhood of $\mathcal{K}$ in X, $\exists f\in Z$ such that $f\leqq 1$ in X, $\operatorname{supp}(f)\subset V$, $f=1$ in $\mathcal{K}$. Using the Hahn-Banach theorem, one can deduce from v) the following.

vi) If $\mu\in H^*$ such that $\langle\mu, f\rangle\geqq 0$, $\forall f\in K\cap Z$, then there exists a nonnegative measure $\tilde{\mu}$ such that $\tilde{\mu}|_Z=\mu|_Z$. By iv) this measure is unique.

For $x\in H$, we set $Tx=x^+ + x^-$. The norm in H^* is denoted by $\|\ \|_{H^*}$. Denoting by $\mathcal{M}(X)$ the space of Radon measures in X, we can draw the following inclusion diagram

$$\begin{array}{ccccccc} & H & & & \mathcal{M}(X) & & \\ \nearrow & & \searrow & \nearrow & & \searrow & \\ Z & & & L^2(X) & & & Z^*=(H\cap\mathcal{C}(X))^*. \\ \searrow & & \nearrow & \searrow & & \nearrow & \\ & \mathcal{C}(X) & & & H^* & & \end{array}$$

DEFINITION 1. Let $A\in\mathcal{A}$, f a measurable function such that its ξ. a. e. equivalence class is in H, we set

$f\geqq\lambda$ in A in the sense of H ($f\geqq_H\lambda$ in A)

$\Leftrightarrow \exists f_n\in H$, $f_n\underset{H}{\to} f$ as $n\to+\infty$, and $f_n\geqq\lambda$, ξ. a. e. in a neighbourhood of A.

Our aim is to define pointwise the elements of H, in a manner which is both more precise than ξ. a. e. equivalence, and compatible with the inequality of Definition 1. For that we define an adequate notion of magnitude for measurable sets.

DEFINITION 2. For $A\in\mathcal{A}$, we introduce the closed convex set

$$\Gamma_A=\{u\in H, u\underset{H}{\geqq} 1 \text{ in } A\},$$

and then

$$\operatorname{cap}(A)=\begin{cases} \inf_{u\in\Gamma_A}\|u\|^2_H & \text{if } \Gamma_A\neq\emptyset \\ +\infty & \text{if } \Gamma_A=\emptyset. \end{cases}$$

PROPOSITION 2. *Let $\mathcal{K}$ be a compact set $\subset X$. Then $f \geqq_H \lambda$ in $\mathcal{K} \Leftrightarrow \exists f_n \in Z$, $f_n \geqq \lambda$ in $\mathcal{K}$, $f_n \underset{H}{\rightarrow} f$.*

PROOF. a) Let us first suppose that $f_n \rightarrow f$, $f_n \in Z$, $f_n \geqq \lambda$ in $\mathcal{K}$. We choose $h \in Z$ such that $h=1$ in $\mathcal{K}$. Then $g_n = f_n + \frac{1}{n} h$ converges to f as $n \rightarrow +\infty$, and $g_n \geqq 1$ in a neighbourhood of $\mathcal{K}$.

b) As a particular case, let us first suppose $f \geqq_H 0$ in $\mathcal{K}$. We introduce $C_1 = \{f \in H, f \geqq_H 0 \text{ in } \mathcal{K}\}$ and $C_2 = \{f \in Z, f \geqq 0 \text{ in } \mathcal{K}\}$.

To prove $\bar{C}_2 \supset C_1$, we have just to check that if $\mu \in H^*$ is nonnegative on C_2, it is also nonnegative on C_1. But if μ is nonnegative on C_2, it is trivially nonnegative on $K \cap Z$. So by vi), we can identify (in Z^*) μ with a nonnegative measure $\tilde{\mu}$ in X. By iv), μ is nonnegative on C_2 if and only if $\operatorname{supp}(\tilde{\mu}) \subset \mathcal{K}$. Since μ is nondecreasing in H ordered by K, to prove that μ is nonnegative on C_1, it is sufficient to check that $\mu(u)=0$, $\forall u \in K$ such that $u=0$, ξ. a. e in a neighbourhood of $\mathcal{K}$. (Because $u = \lim (u_n^+ - u_n^-)$, with $u_n \in H$, $u_n \geqq 0$, ξ. a. e in a neighbourhood of $\mathcal{K}$). Let $\psi_n \in Z$ such that ψ_n converges to u. Then $\psi_n^+ \rightarrow u^+ = u$. So we get a subsequence n_k and numbers $\alpha_{p,n_k} \geqq 0$ for $1 \leqq p \leqq n_k$ such that $\sum_{p=1}^{n_k} \alpha_{p,n_k} \psi_p^+ = \varphi_k \underset{n \rightarrow +\infty}{\longrightarrow} u$. And $\varphi_k \in K \cap Z$. By hypothesis, $u=0$, ξ. a. e in an open $V \supset \mathcal{K}$. We consider $h \in K \cap Z$ such that $\operatorname{supp}(h) \subset V$, $h=1$ in $\mathcal{K}$. And for any $l \in N$, $\inf \{\varphi_k, l \cdot h\} \rightarrow 0$ as $k \rightarrow +\infty$, hence also

$$\langle \mu, \inf \{\varphi_k, l \cdot h\} \rangle_{H,H^*} \longrightarrow 0 \quad \text{as} \quad k \rightarrow +\infty .$$

On the other hand, $\tilde{\mu}(T(\varphi_k - \varphi_{k+r})) = \langle \mu, T(\varphi_k - \varphi_{k+r}) \rangle_{H,H^*} \leqq \|\mu\|_{H^*} \|T(\varphi_k - \varphi_{k+r})\|_H \leqq \|\mu\|_{H^*} \|\varphi_k - \varphi_{k+r}\|_H$ by iii). So φ_k, being a Cauchy sequence in $L^1(\tilde{\mu})$, converges to ζ in $L^1(\tilde{\mu})$. By Fatou lemma: $\tilde{\mu}(\zeta) = \tilde{\mu}(\liminf_{h,l \rightarrow +\infty} (\inf \{\varphi_k, l \cdot h\})) \leqq 0$. And so $\tilde{\mu}(\varphi_k) \rightarrow 0$ as $k \rightarrow +\infty$. Since $\varphi_k \underset{H}{\rightarrow} u$ and $\mu \in H^*$, we get $\mu(u)=0$. In the general case, we consider $h \in K \cap Z$, $h \leqq 1$, $h=1$ in $\mathcal{K}$. Then $f - \lambda h \geqq_H 0$ in $\mathcal{K} \Rightarrow \exists \varphi_n \in Z$, $\varphi_n \geqq 0$ in $\mathcal{K}$, $\|\varphi_n - (f - \lambda h)\| \leqq \frac{1}{n}$. Then $\psi_n = \lambda h + \varphi_n \geqq \lambda$ in $\mathcal{K}$, and $\|\psi_n - f\| \leqq \frac{1}{n}$.

COROLLARY 5. $\forall A \in \mathcal{A}$, $\operatorname{cap}(A)=0 \Rightarrow \forall \mu \in H^* \cap \mathcal{M}^+(X)$, $\mu(A)=0$.

First if $\mathcal{K}$ is a compact subset of X, such that $\operatorname{cap}(\mathcal{K})=0$, then $\forall \varepsilon > 0$, $\exists \varphi \in Z: \varphi \geqq 1$ in $\mathcal{K}$ and $\|\varphi\|_H < \varepsilon$. Hence $\tilde{\mu}(\mathcal{K}) \leqq \tilde{\mu}(\varphi) = \langle \mu, \varphi \rangle_{H,H^*} \leqq \|\mu\|_{H^*} \varepsilon$, $\forall \varepsilon > 0$. Now if A is only measurable, we consider a sequence of compact sets $\mathcal{K}_n$ such that $\bigcup_{n \geqq 0} \mathcal{K}_n = X$. Setting $A_n = \mathcal{K}_n \cap A$, we have $\operatorname{cap}(A_n) = 0$, and $\mu(A_n) = \sup \{\mu(\mathcal{K}) : \mathcal{K} \subset A_n, \mathcal{K} \text{ compact}\} = 0$.

PROPOSITION 3. *Let $(A_n)_{n \in N}$ be measurable sets $\subset X$. Then $\operatorname{cap}(\bigcup_{n \geqq 0} A_n) \leqq \sum_{n \geqq 0} \operatorname{cap}(A_n)$.*

LEMMA 3. *$\forall A \subset X$, $B \subset X$ measurable sets, $\operatorname{cap}(A \cup B) \leqq \operatorname{cap} A + \operatorname{cap} B$.*

It is enough to prove that if $(u, v) \in H \times H$, then

$$\|\sup\{u, v\}\|^2 \leqq \|u\|^2 + \|v\|^2 .$$

First, if $w \in H$, then we have

$$\|u+w^+\|^2 = \|u\|^2 + 2((u, w^+)) + \|w^+\|^2$$
$$= \|u\|^2 + 2((u+w^+, w^+)) - \|w^+\|^2 \leqq \|u\|^2 + 2((u+w, w^+)) - \|w^+\|^2$$
$$= \|u\|^2 + \|w+u\|^2 - \|u+w-w^+\|^2 \leqq \|u\|^2 + \|w+u\|^2 .$$

Setting $w=v-u$, we have $\sup\{u, v\}=u+w^+$, and $v=w+u$.

LEMMA 4. *Let $(\Gamma_n)_{n\in N}$ be a sequence of closed convex sets in H. Then if Γ_n is decreasing,*

$$\operatorname{dist}(x, \bigcap_{n\geqq 0} \Gamma_n) = \sup_{n\geqq 0} \operatorname{dist}(x, \Gamma_n) .$$

The proof of Lemma 4 is trivial.

Now to deduce Proposition 3, we set $B_n = \bigcup_{0\leqq i\leqq n} A_i$, and $\Gamma_n = \Gamma_{B_n}$ (c. f. Def. 2). Then $\Gamma_{\bigcup_{n\geqq 0} A_n} = \bigcap_{n\geqq 0} \Gamma_n$ (c.f. [1], Prop. 6). And

$$\operatorname{cap}(\bigcup_{n\geqq 0} A_n) = [\operatorname{dist}(0, \bigcap_{n\geqq 0} \Gamma_n)]^2 = \sup_{n\geqq 0} \operatorname{cap}(B_n)$$
$$\leqq \sup_{n\geqq 0} (\sum_{i=0}^{n} \operatorname{cap}(A_i)) = \sum_{n\geqq 0} \operatorname{cap}(A_n) .$$

As a particular case, $\operatorname{cap}(A)=0$ and $\operatorname{cap}(B)=0$ imply $\operatorname{cap}(A\cup B)=0$. We shall say that a property is true quasi-everywhere in X (q. e.) if it is true in $X\backslash A$, with $\operatorname{cap}(A)=0$. So if P and Q are true q. e. in X, the property "P and Q" is also q. e. in X.

DEFINITION 4. Let f be a measurable function: $X\to R$. We shall say that f is quasi-continuous if there exists a nonincreasing sequence $\{\omega_n\}$ of open sets in X such that $\lim_{n\to+\infty} \operatorname{cap}(\omega_n)=0$, and $f|X\backslash\omega_n$ continuous, $\forall n$.

PROPOSITION 4. *If f is quasi-continuous and $f\geqq 0$, ξ. a. e, then* $\operatorname{cap}(\{f<0\}) =0$.

PROOF. We set $A=\{x\in X, f(x)>0\}$. Since $A\backslash\omega_n$ is an open subset of $X\backslash\omega_n$, $A\cup\omega_n$ is open in X. Since $\xi(A)=0$, if $\varphi_n\in H$, $\varphi_n\geqq 1$, ξ. a. e in ω_n, we have $\varphi_n\geqq 1$, ξ. a. e in $A\cup\omega_n$, a neighbourhood of A. So $\operatorname{cap}(A\cup\omega_n)\leqq\operatorname{cap}(\omega_n)$.

COROLLARY 6. *If f_1 and f_2 are two quasi-continuous functions having the same ξ. a. e equivalence class,* $\operatorname{cap}(\{f_1\neq f_2\})=0$.

We give now the fundamental results of this theory.

THEOREM 5. *If $x\in H$, it has a q. e defined quasi-continuous representative. Moreover, for any $\tilde{x}$ quasi-continuous representative for x, $\exists\{f_n\}$ such that $\forall n\in N$, $f_n\in Z$, f_n converging to $\tilde{x}$ pointwise q. e in X, and in the norm of the space H.*

By Corollary 6, two quasi-continuous representatives are equal q. e. So it suffices to see that there exists a quasi-continuous representative for x, which is a limit, pointwise q. e and in H of a sequence $f_n \in Z$. We consider a sequence $f_n \in Z$ such that

$$f_n \underset{H}{\longrightarrow} x\,,\ \sum_{k=1}^{\infty} 4^k \|f_{k+1}-f_k\|^2 < +\infty\,.$$

If $\omega_k = \{\zeta \in X, |(f_{k+1}-f_k)(\zeta)| > 2^{-k}\}$, $\operatorname{cap}(\omega_k) \leqq 4^k \|f_{k+1}-f_k\|^2$. Setting $\omega'_n = \bigcup_n^{\infty} \omega_k$, we have $\operatorname{cap}(\omega'_n) \leqq \sum_n^{\infty} 4^k \|f_{k+1}-f_k\|^2$. f_n converges simply outside of $\bigcap_1^{\infty} \omega'_n$, uniformly on each $X \backslash \omega'_n$. The limit function $\tilde{x}$ (we set $\tilde{x}=0$ in $\bigcap_1^{\infty} \omega'_n$) has the desired properties.

THEOREM 6. *We consider $x \in H$, with quasi-continuous representative $\tilde{x}$. If $\nu \in H^* \cap \mathcal{M}^+(X)$, $\tilde{x}$ is measurable for ν, ν. a. e defined, $\tilde{x} \in L^1(\nu)$, and* $\int_X \tilde{x} d\nu = \langle x, \nu \rangle_{H,H^*}$.

By Theorem 5 and Corollary 5, $\tilde{x}$ is ν. a. e. defined. Moreover, let $f_n \in Z$ be such that $f_n \to \tilde{x}$ in H, and q. e in X. Then f_n is a Cauchy sequence in $L^1(\nu)$ and H: its limit in the two spaces is equal to $\tilde{x}$. Since the equality to check is true by the definition of the $f_n \in Z$, the proof is done. Now if we consider $x \in H$, it will be understood that we consider a quasi-continuous representative. The important properties contained in the following remark will be used later, but not proved, c. f. [1] (the proof relies on potential-theoretic tools).

REMARK 2. a) If $A \in \mathcal{A}$ and $f \in H$, then we have $f \geqq_H \lambda$ on $A \Leftrightarrow \operatorname{cap}(\{f<\lambda\} \cap A) = 0$.

b) If $f_n \in H$ and $f_n \to f$ in H, then there exists a subsequence f_{n_k} such that $f_{n_k}(x) \to f(x)$ q. e. in$X$.

We now apply Theorem 4 to differentiation in variational inequalities. Let $\Omega \subset R^N$ an open bounded set, with smooth boundary Γ.

EXAMPLE 5. $H = H_0^1(\Omega)$, with the scalar product:

$$(u, v)_{H_0^1} = \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v\, dx\,.$$

Let $f : [0, T[\to H^{-1}(\Omega)$ be a right-differentiable curve. We may consider the one-parameter depending variational problem

$$\begin{cases} u(t) \geqq 0\,, \qquad \Delta u(t) \leqq -f(t) \\ (\Delta u(t) + f(t)) u(t) = 0 \end{cases}$$

in the following sense: $u(t)$ is chosen quasi-continuous for the H_0^1-capacity (defined later), $\forall t > 0$, $-(\Delta u(t) + f(t)) = \nu(t)$ is a nonnegative distribution in Ω, hence an element of $\mathcal{M}^+(\Omega) \cap H^{-1}(\Omega)$ and $\forall t \in [0, T[$, $u(t) = 0$, $\nu(t)$. a. e. in Ω. This problem can be represented by the variational inequality:

$$
(1) \qquad \begin{cases} u(t)\in K=\{v\in H_0^1,\ v\geqq 0 \text{ a.e. in } \Omega\} \\ \forall v\in K, \int_\Omega \operatorname{grad} u(t)\cdot \operatorname{grad}(v-u(t))dx \geqq \int_\Omega f(t)(v-u(t))dx \end{cases}
$$

and introducing $g(t)=(-\Delta)^{-1}f(t)$, it becomes

$$
\begin{cases} u(t)\in K \\ \forall v\in K, (u(t), v-u(t))_{H_0^1} \geqq (g(t), v-u(t))_{H_0^1} . \end{cases}
$$

First we observe that H is a lattice for the order given by K. If $u\in H_0^1(\Omega)$, $\operatorname{grad} u^+\cdot \operatorname{grad} u^-=0$, a.e $\Rightarrow (u^+, u^-)_H=0$. The properties i), iv) and v) are trivially checked, so setting $X=\Omega$, $H=H_0^1(\Omega)$, we can apply the capacity theory.

LEMMA 5. $u\in K \Rightarrow \Pi_K(u)=\{w\in H, w\geqq 0$ *q. e in* $\{u=0\}\}$.

The inclusion $\Pi_K(u)\subset \{w\in H, w\geqq 0$ q. e in $\{u=0\}\}$ is a consequence of Remark 2, b). On the other hand, $[\Pi_K(u)]^\perp=K^\perp\cap[u]^\perp=\{z\in H, -\Delta z=\nu\in\mathcal{M}^+(\Omega)$, $\nu(\{u>0\})=0\}$ so that $(\Pi_K(u))^\perp\subset\{w\in H, w\geqq 0$ q. e in $\{u=0\}\}^\perp$ by Theorem 6. And $\{w\in H, w\geqq 0$ q. e in $\{u=0\}\}=\Pi_K(u)$ since $\Pi_K(u)$ is closed.

COROLLARY 7. *The solution $u(t)$ of problem* (1) *is right-differentiable, the derivative* $\dfrac{d^+u}{dt}$ *is given by*

$$
(1.1) \qquad \begin{cases} \dfrac{d^+u}{dt}\geqq 0 \ q.e \ in \ \{u(t)=0\}, \qquad \nu(t)\Big(\Big\{\dfrac{d^+u}{dt}>0\Big\}\Big)=0 \\ \forall w\in K_t, \Big(\dfrac{d^+u}{dt}, w-u(t)\Big)_{H_0^1}\geqq \int_\Omega \dfrac{d^+f}{dt}(w-u(t))dx \\ \int_\Omega \Big|\operatorname{grad}\Big(\dfrac{d^+u}{dt}\Big)\Big|^2 dx=\int_\Omega \dfrac{d^+u}{dt}\cdot\dfrac{d^+f}{dt}dx \end{cases}
$$

where $\nu(t)=\Delta(g(t)-u(t))$, *and*

$$
K_t=\{w\in H_0^1(\Omega), \quad w\geqq 0 \text{ and } \nu(t)\ (w>0)=0\} .
$$

PROOF. We show that if $\zeta\geqq 0$ q. e in $\{u(t)=0\}$, $(\zeta, g(t)-u(t))_{H_0^1}=0\Leftrightarrow \nu(t)(\zeta>0)=0$. By Theorem 6, $\zeta\in L^1(\nu(t))$, and

$$
\int_\Omega \zeta d\nu(t)=\langle\zeta, \Delta(g(t)-u(t))\rangle_{H,H^*}=-(\zeta, g(t)-u(t))_H .
$$

Since $(u(t), g(t)-u(t))=0$, we have $\nu(t)(u(t)>0)=0$. Thus $(\zeta, g(t)-u(t))=-\int_{\{u(t)=0\}}\zeta d\nu_{(t)}$, and we are done.

REMARK 3. We could deduce from Corollary 7 the study of t-differentiation for the variational problem

$$
\begin{cases} u(t)\geqq \psi(t) \\ \Delta u(t)\leqq -f(t), \end{cases} \qquad (\Delta u(t)+f(t))(u(t)-\psi(t))=0 .
$$

COROLLARY 8. *Under the additional hypothesis:* $\frac{d^+f}{dt} \leqq 0, \forall t \in [0, T[$, *the derivative* $\frac{d^+u}{dt}$ *satisfies the system*

$$(1.2) \quad \begin{cases} \frac{d^+u}{dt} \leqq 0, \quad \frac{d^+u}{dt} = 0 \quad q.e \ in \ \{u(t)=0\}, \\ \forall w \in K, \int_\Omega \operatorname{grad}\left(\frac{d^+u}{dt}\right) \cdot \operatorname{grad}(w-u(t))dx \geqq \int_\Omega \frac{d^+f}{dt}(w-u(t))dx, \\ \int_\Omega \left|\operatorname{grad}\left(\frac{d^+u}{dt}\right)\right|^2 dx = \int_\Omega \frac{d^+f}{dt} \cdot \frac{d^+u}{dt} dx, \end{cases}$$

which is equivalent to

$$\begin{cases} \frac{d^+u}{dt} \leqq 0, \quad \frac{d^+u}{dt} = 0 \quad q.e \ in \ \{u(t)=0\}, \\ \Delta \frac{d^+u}{dt} + \frac{d^+f}{dt} = -\mu(t), \quad \mu(t) \in \mathcal{M}^+(\Omega), \\ \mu(t) \ (u(t)>0) = 0. \end{cases}$$

EXAMPLE 6. $\mathcal{H} = H^1(\Omega)$ with the scalar product $(u, v) = \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v \cdot dx + \int_\Omega uvdx$

$$\boldsymbol{K} = \{u \in \mathcal{H}, u|_\Gamma \geqq 0, \text{ a.e on } \Gamma\}.$$

Let $f: [0, T[\to L^2(\Omega)$ and $\varphi: [0, T] \to L^2(\Omega)$ be two right-differentiable functions. We consider the t-depending variational inequality

$$(2) \quad \begin{cases} u(t) \in \boldsymbol{K}, \quad \forall t \in [0, T[, \\ \forall v \in \boldsymbol{K}, \quad (u(t), v-u(t)) \geqq \int_\Omega f(t)(v-u(t))dx + \int_\Gamma \varphi(t)(v-u(t))d\Gamma. \end{cases}$$

We notice that by the "trace theorem", the mapping:

$$w \in H \to \int_\Omega f(t)w\,dx + \int_\Gamma \varphi(t)w|_\Gamma d\Gamma$$

is continuous for every t, and so represented by an element of $(H^1(\Omega))^*$.

We introduce $H = H^{1/2}(\Gamma)$, $K = \{w \in H, w \geqq 0\}$, and for $u \in H$ we consider the solution $\check{u}$ of the equation $\check{u} \in \mathcal{H}$, $-\Delta\check{u} + \check{u} = 0$, $\check{u}|_\Gamma = u$. We define the scalar product on H by the $((u, v))_H = (\check{u}, \check{v})_H$. H is a Hilbert sublattice of $L^2(\Gamma)$ for the ordering given by K since we have $\sup\{u, v\} \in H$, $\forall (u, v) \in H \times H$. On the other hand, it can be regarded as a subspace of $\mathcal{H}$, by means of the map:

$$u \longrightarrow \check{u}.$$

We now follow an idea of Sylverstein [4] to show that

$$\forall u \in H,\ ((u^+, u^-))_H \leqq 0\,.$$

If $u \in H$, we set $Tu = u^+ + u^-$. And for $x \in \mathcal{H}$, $\mathcal{C}x = x^+ + x^-$, x^+ being the positive part for the natural ordering on $H^1(\Omega)$.

First we remark that for any $v \in \mathcal{K}$ such that $v|_\Gamma = 0$, and for any $w \in H$, we have $(\check{w}, v) = 0$. Then for $u \in H$, we have $\check{T}u|_\Gamma = \mathcal{C}\check{u}|_\Gamma = Tu$, taking $w = Tu$, $v = \check{T}u - \mathcal{C}\check{u}$, we also have $|\check{T}u|^2 - |\mathcal{C}\check{u}|^2 = (2\check{T}u, \check{T}u - \mathcal{C}\check{u}) - |\check{T}u - \mathcal{C}\check{u}|^2 = -|\check{T}u - \mathcal{C}\check{u}|^2 \leqq 0$. Hence, $\|Tu\|_H = |\check{T}u| \leqq |\mathcal{C}\check{u}| = |\check{u}| = \|u\|_H$, $\forall u \in H$. The properties i), iv) and v) are easy to check, setting $X = \Gamma$, H as above, we may apply the capacity theory. The inequality (2) admits the pointwise formulation:

$$\begin{cases} u(t) \geqq 0, \text{ a.e on } \Gamma \\ -\Delta u(t) + u(t) = f(t) \text{ in } \Omega \\ \dfrac{\partial u(t)}{\partial n} \geqq \varphi(t) \text{ on } \Gamma \text{ and } \Big(\dfrac{\partial u(t)}{\partial n} - \varphi(t)\Big) u(t) = 0 \text{ on } \Gamma\,, \end{cases}$$

where $u|_\Gamma$ is chosen quasi-continuous for the $H^{1/2}(\Gamma)$-capacity, in the sense that $\dfrac{\partial u(t)}{\partial n} - \varphi(t)$ is a positive measure on Γ, and setting $\nu(t) = \dfrac{\partial u(t)}{\partial n} - \varphi(t)$, $\nu(t)(u(t) > 0) = 0$. For $u \in \mathcal{K}$, we set $u_1 = u|_\Gamma$, $u_2 = \mathrm{Proj}_{H_0^1(\Omega)}(u)$. Introducing the solution $\psi(t)$ of the problem

$$\begin{cases} -\Delta \psi(t) + \psi(t) = 0 \\ \dfrac{\partial \psi(t)}{\partial n} = \varphi(t) \text{ on } \Gamma \end{cases}$$

and setting $g(t) = \psi(t)|_\Gamma$, the system (2) is equivalent to

$$\begin{cases} -\Delta u_2(t) + u_2(t) = f(t)\,, \\ u_1(t) \in K\,, \\ \forall v \in K,\ ((u_1(t), v - u_1(t))) \geqq ((g(t), v - u_1(t)))\,. \end{cases}$$

Setting $\nu(t) = \dfrac{\partial u(t)}{\partial n} - \varphi(t)$ (a nonnegative measure on Γ) and

$$K_{(t)} = \{v \in H, v|_\Gamma \geqq 0, \text{ q. e}, \nu(t)(v|_\Gamma > 0) = 0\}\,,$$

we get immediately from Theorem 4 the following

COROLLARY 9. *$u(t)$ is right-differentiable, and $\dfrac{d^+u}{dt}$ is the solution of the variational problem*

$$
(2.1)\quad\begin{cases}\frac{d^+u}{dt}|_\Gamma \geqq 0,\ q.e\ \ in\ \Gamma\cap\{u(t)=0\},\ \nu(t)\Big(\frac{d^+u}{dt}|_\Gamma>0\Big)=0,\\ \forall w\in K_{(t)},\ \Big(\frac{d^+u}{dt},\ w-u(t)\Big)_{H^1(\Omega)}\geqq\int_\Omega \frac{d^+f}{dt}(w-u(t))dx+\int_\Gamma \frac{d^+\varphi}{dt}(w-u(t))d\Gamma,\\ \int_\Omega\left|\frac{d^+u}{dt}\right|^2dx+\int_\Omega\left|\mathrm{grad}\Big(\frac{d^+u}{dt}\Big)\right|^2dx=\int_\Omega\frac{d^+f}{dt}\cdot\frac{d^+u}{dt}dx+\int_\Gamma\frac{d^+\varphi}{dt}\cdot\frac{d^+u}{dt}d\Gamma.\end{cases}
$$

COROLLARY 10. *Under the additional hypotheses that* $\forall t,\ \frac{d^+f}{dt}\in\mathcal{M}^-(\Omega)$, $\frac{d^+\varphi}{dt}\in\mathcal{M}^-(\Gamma)$, *the derivative* $\frac{d^+u}{dt}$ *satisfies the system*

$$
(2.2)\quad\begin{cases}\frac{d^+u}{dt}\leqq 0,\quad \frac{d^+u}{dt}=0\quad q.e\ \ in\ \Gamma\cap\{u(t)=0\},\\ \forall w\in K,\ \Big(\frac{d^+u}{dt},\ w-u(t)\Big)_{H^1(\Omega)}\geqq\int_\Omega \frac{d^+f}{dt}(w-u(t))dx+\int_\Omega \frac{d^+\varphi}{dt}(w-u(t))d\Gamma,\\ \int_\Omega\left|\frac{d^+u}{dt}\right|^2dx+\int_\Omega\left|\mathrm{grad}\Big(\frac{d^+u}{dt}\Big)\right|^2dx=\int_\Omega\frac{d^+f}{dt}\cdot\frac{d^+u}{dt}dx+\int_\Gamma\frac{d^+\varphi}{dt}\cdot\frac{d^+u}{dt}d\Gamma.\end{cases}
$$

So $\frac{\partial}{\partial n}\Big(\frac{d^+u}{dt}\Big)-\frac{d^+\varphi}{dt}=\mu(t)$, *with* $\mu(t)\in\mathcal{M}^+(\Gamma)$, $\mu(t)(u(t)>0)=0$.

SKETCH OF PROOF. In $\mathcal{K}$, $\frac{d^+g}{dt}\in\boldsymbol{K}^\perp$ because $\frac{\partial}{\partial n}\Big(\frac{d^+g}{dt}\Big)=\frac{d^+\varphi}{dt}\leqq 0$. So $\frac{d^+\mu_1}{dt}\in -K$, and the inequality for μ_1 gets simpler. Moreover, we have $\frac{d^+u}{dt}|_\Gamma\leqq 0$, and $(-\Delta+\hat{I})\Big(\frac{d^+u}{dt}\Big)\leqq 0$ in Ω. So $\frac{d^+u}{dt}\leqq 0$ by the max. principle. The rest of deduction is a purely algebraic matter.

Corollaries 8 and 10 are results of H. Brézis, who got them from direct functional arguments. The idea of Lemma 1 is due to F. Mignot ([3]).

References

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