HOW TO DRAW A GRAPH

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1. Introduction

WE use the definitions of (**11**). However, in deference to some recent attempts to unify the terminology of graph theory we replace the term 'circuit' by 'polygon', and 'degree' by 'valency'.

A graph G is *3-connected* (*nodally 3-connected*) if it is simple and non-separable and satisfies the following condition; if G is the union of two proper subgraphs H and K such that $H \cap K$ consists solely of two vertices u and v, then one of H and K is a link-graph (arc-graph) with ends u and v.

It should be noted that the union of two proper subgraphs H and K of G can be the whole of G only if each of H and K includes at least one edge or vertex not belonging to the other. In this paper we are concerned mainly with nodally 3-connected graphs, but a specialization to 3-connected graphs is made in § 12.

In § 3 we discuss conditions for a nodally 3-connected graph to be planar, and in § 5 we discuss conditions for the existence of Kuratowski subgraphs of a given graph. In §§ 6–9 we show how to obtain a convex representation of a nodally 3-connected graph, without Kuratowski subgraphs, by solving a set of linear equations. Some extensions of these results to general graphs, with a proof of Kuratowski's theorem, are given in §§ 10–11. In § 12 we discuss the representation in the plane of a pair of dual graphs, and in § 13 we draw attention to some unsolved problems.

2. Peripheral polygons

In this section we use the 'nodes' and 'branches' of a graph defined in ((**11**) § 4).

Let J be a polygon of G and let $\beta(J)$ denote the number of bridges of J in G. If $\beta(J) \leqslant 1$ we call J a *peripheral* polygon of G.

Let B be any bridge of J in a non-separable graph G. The vertices of attachment of B, which must be at least two in number, subdivide J into arc-graphs. We call these the *residual* arc-graphs of B in J. If one of these includes all the vertices of attachment of a second bridge B' of J in G we say that B' *avoids* B. Then B avoids B'.

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(2.1) *Let G be a nodally 3-connected graph. Let J be a polygon of G and B any bridge of J in G. Then either J is peripheral or J has another bridge B' which does not avoid B.*

Proof. Suppose that J is not peripheral and that every other bridge of J avoids B. Let B' be a second bridge of J. There is a residual arc-graph M of B in J which includes all the vertices of attachment of B'. Let H be the union of M and all the bridges of J other than B, having all their vertices of attachment on M. Let K be the union of all the other bridges, including B, and the complementary arc-graph of M in J. Then H and K are proper subgraphs of G whose union is G, and $H \cap K$ consists solely of the two ends of M. But neither H nor K is an arc-graph joining the ends of M. This contradicts the hypothesis that G is nodally 3-connected.

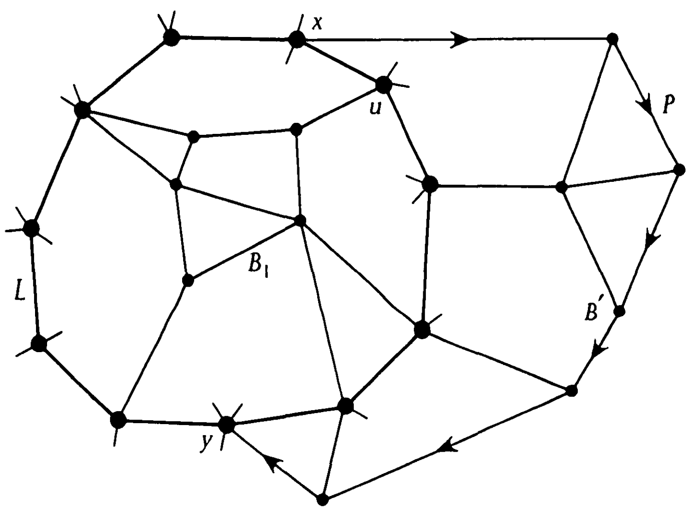

FIG. 1

(2.2) *Let G be a nodally 3-connected graph. Let K_1 be a polygon of G, B_1 a bridge of K_1 in G, C a subgraph of B_1, and L a branch of G in K_1. Then we can find a peripheral polygon J of G such that $L \subset J$ and $J \cap C \subseteq K_1 \cap C$.*

Proof. If K_1 is peripheral we can put $J = K_1$. Consider the remaining case $\beta(K_1) \geqslant 2$.

Let B' be a second bridge of K_1. By (2.1) we can choose B' so that it does not avoid B_1. Let its residual arc-graph containing L be L', with ends x and y. By ((**11**) (2.4)) we can find a simple path P from x to y in B' which avoids K_1. This construction is illustrated in Fig. 1, in which the thick lines represent K_1.

Consider the polygon $K_2 = L' \cup G(P)$. It has a bridge B_2 which contains B_1. Moreover, the only vertices of B_1 which are vertices of attachment of

B_2 are those in L'. Since B_1 does not avoid B', by (2.1), there is a vertex u of attachment of B_1 which is not on L'. Hence B_2 contains the complementary arc-graph of L' in K_1. We thus have

$$B_1 \subset B_2, \tag{1}$$

$$B_1 \cap K_2 \subset B_1 \cap K_1, \tag{2}$$

$$C \cap K_2 \subseteq C \cap K_1. \tag{3}$$

If K_2 has a second bridge we repeat the foregoing procedure with K_2 replacing K_1 and B_2 replacing B_1. By (1) the process must terminate. When it does we have a peripheral polygon K_n such that $L \subset K_n$. Moreover,

$$C \cap K_n \subseteq C \cap K_1,$$

by repeated application of (3). We may therefore put $J = K_n$.

(2.3) *Let G be a nodally 3-connected graph which is not a polygon or a link-graph, and let L be a branch of G. Then we can find two peripheral polygons J_1 and J_2 of G such that $J_1 \cap J_2 = L$.*

Proof. By ((**11**) (2.5)) we can find a polygon of G containing L. By (2.1) we can construct a peripheral polygon J_1 of G such that $L \subset J_1$. Since $G \neq J_1$, J_1 has a bridge.

Let the bridge of J_1 be B. As the ends, a and b say, of L are nodes they are vertices of attachment of B. There is a simple path P from a to b in B avoiding J_1. Let K_1 denote the polygon $L \cup G(P)$. Let C be the complementary arc-graph of L in J_1 and let B_1 be the bridge of K_1 containing C. By (2.2) there is a peripheral polygon J_2 of G such that $L \subset J_2$ and $J_2 \cap C \subseteq K_1 \cap C \subseteq L$. Hence $J_1 \cap J_2 = L$.

(2.4) *Let G be a nodally 3-connected graph, K a polygon of G, B a bridge of K in G, and L a branch of G contained in K. Let J_1 and J_2 be peripheral polygons of G such that $L \subseteq J_1 \cap J_2$ and neither $B \cap J_1$ nor $B \cap J_2$ is a subgraph of K. Then we can find a peripheral polygon J_3, distinct from J_1 and J_2, such that $L \subset J_3$.*

Proof. Write $C = (B \cap J_1) \cup (B \cap J_2)$. By (2.2) we can find a peripheral polygon J_3 of G such that $L \subset J_3$ and $J_3 \cap C \subseteq K \cap C$. The second of these properties ensures that J_3 is distinct from J_1 and J_2.

Consider the set of cycles of a connected graph G, as defined in (**11**). The rank of this set, the maximum number of cycles independent under mod-2 addition, is

$$p_1(G) = \alpha_1(G) - \alpha_0(G) + 1. \tag{4}$$

This is shown, for example, in (**5**) and (**12**). We refer to the elementary cycle associated with a peripheral polygon as a *peripheral cycle.*

(2.5) *Let G be a nodally 3-connected non-null graph. Then we can find a set of $p_1(G)$ independent peripheral cycles of G.*

Proof. Suppose we have found a set of $r < p_1(G)$ independent peripheral cycles of G (r may be zero). Let U be the set of all their linear combinations. We can find a cycle not in U. Since this cycle is a sum of elementary cycles we can find an elementary cycle $S_1 \notin U$. ((**11**) (3.2).)

Assume that S_1 is not peripheral. Let B_1 and B' be distinct bridges of $G.S_1$. By (2.1) we may suppose that B' does not avoid B_1.

Suppose first that B' has two vertices of attachment, x and y, such that each of the residual arc-graphs M_1 and M_2 of x and y in $G.S_1$ includes a vertex of attachment of B as an internal vertex. We can find a simple path P from x to y in B' avoiding $G.S_1$. Let X_i denote the polygon $M_i \cup G(P)$, $(i = 1, 2)$. The sum of the elementary cycles $E(X_1)$ and $E(X_2)$ is S_1. Hence we may assume without loss of generality that $E(X_1) \notin U$. We write $E(X_1) = S_2$. Evidently there is a bridge B_2 of $G.S_2$ in G such that $B_1 \subset B_2$.

In the remaining case it is easy to verify, first that each vertex of attachment of B' is a vertex of attachment of B_1, and then that B' and B_1 have the same vertices of attachment, three in number. Let us denote them by x, y, and z. By ((**11**) (4.3)) there is a Y-graph Y of B', with ends x, y, and z, which spans $G.S_1$. Let the arms of Y ending at x, y, and z be A_x, A_y, and A_z respectively. If L_{xy} is the residual arc-graph of B and B' in $G.S_1$ with ends x and y we denote the polygon $L_{xy} \cup A_x \cup A_y$ by X_z. We define X_x and X_y analogously. The sum of the elementary cycles $E(X_x)$, $E(X_y)$, and $E(X_z)$ is S_1. We may therefore assume without loss of generality that $E(X_x) \notin U$. We now write $E(X_x) = S_2$. Again we observe that there is a bridge B_2 of $G.S_2$ in G such that $B_1 \subset B_2$.

In either case if $G.S_2$ has a second bridge we repeat the procedure with S_2 replacing S_1 and B_2 replacing B_1. We then obtain an elementary cycle $S_3 \notin U$ such that some bridge B_3 of $G.S_3$ satisfies $B_2 \subset B_3$. Continuing in this way until the process terminates we obtain a peripheral cycle $S_n \notin U$.

We now have a set of $r+1$ independent peripheral cycles of G. If $r+1 < p_1(G)$ we repeat the operation to obtain a set of $r+2$, and so on. The theorem follows.

(2.6) *Let G be a nodally 3-connected non-null graph, with at least two edges, which is not a polygon. Suppose that no edge of G belongs to more than two distinct peripheral polygons. Then G has just $p_1(G)+1$ distinct peripheral cycles, and they constitute a planar mesh of G.*

Proof. Let **M** be the class of all peripheral cycles of G. Each edge of G belongs to just two members of **M**, by (2.3). Each non-null cycle of G

is a sum of members of $\mathbf{M}$, by (2.5). Hence $\mathbf{M}$ satisfies the conditions for a planar mesh of G given in (**11**).

It is clear that the members of $\mathbf{M}$ sum to zero. But there is no proper non-null subset of $\mathbf{M}$ whose members sum to zero, by ((**11**) (3.4)). Hence $\mathbf{M}$ has just $p_1(G)+1$ members.

(2.7) *A peripheral polygon K of a non-separable graph G belongs to every planar mesh of G.*

Proof. Suppose $\mathbf{M} = \{J_1, J_2, \ldots, J_k\}$ is a planar mesh of G not including K. Then each subclass of $\mathbf{M}$ summing to K has two or more members. It follows that the residual graphs of $G.K$, as defined in ((**11**) § 3), are proper subgraphs of G. But each has all its vertices of attachment on $G.K$, by ((**11**) (3.6)). Hence $\beta(K) \geqslant 2$, contrary to hypothesis.

(2.8) *If $\mathbf{M}$ is a planar mesh of a nodally 3-connected graph G, then each member of $\mathbf{M}$ is peripheral.*

Proof. If G is edgeless, a link-graph, or a polygon this result is trivial. Otherwise it follows from (2.3) and (2.7).

These two theorems show that a nodally 3-connected graph has at most one planar mesh. (See (**13**) (**14**).)

We can determine whether a given nodally 3-connected graph G has a planar mesh as follows. Assuming G has at least two edges and is not a polygon we construct $p_1(G)$ independent peripheral cycles by the method of (2.5). If a planar mesh exists it consists of these $p_1(G)$ cycles and their mod-2 sum.

3. Planarity

Let G be any graph. Let f be a 1–1 mapping of $V(G)$ onto a set U of $\alpha_0(G)$ distinct points of a sphere or closed plane Π. If e is any edge of G with ends v and w we choose an open arc in Π with end-points $f(v)$ and $f(w)$ and denote it by $f(e)$. If e is a loop the two end-points of $f(e)$ coincide. We now define a graph H as follows. $V(H) = U$, $E(H)$ is the set of all arcs $f(e)$, $e \in E(G)$, and the incident vertices of an edge $f(e)$ are its two end-points. We call H a *representation* of G in Π if it satisfies the following conditions.

(i) No edge of H contains any vertex of H.

(ii) If e and e' are distinct edges of G, then $f(e)$ and $f(e')$ are disjoint.

A graph G is said to be *planar* if it has a representation in Π.

Let H be a representation in Π of a planar graph G. If $K \subseteq H$ then the union of $V(K)$ and the edges of K is the *complex* $|K|$ of K. If K is a polygon of H then $|K|$ is a simple closed curve. Any residual domain of $|K|$ which does not meet $|H|$ is then called a *face* of H *bounded* by K. If B is a bridge of a polygon K in H then, by ((11) (2.4)), $|B|$ does not meet both residual domains of $|K|$. We therefore have

(3.1) *Each peripheral polygon of H bounds a face of H.*

Now distinct faces of H are clearly disjoint. Hence, by the topology of Π, no edge of H belongs to the bounding polygons of three distinct faces. But each peripheral polygon of G corresponds under f to a peripheral polygon of H. Hence, by (3.1), we have the following theorem.

(3.2) *If a graph G has three distinct peripheral polygons with a common edge, then G is non-planar.*

4. The Kuratowski graphs

A graph defined by six nodes $A_1, A_2, A_3, B_1, B_2, B_3$, and nine branches, each A_i being joined to each B_j by a single branch, is a *Kuratowski graph of Type I*. A graph defined by five nodes A_1, A_2, A_3, A_4, A_5, and ten branches, each pair of distinct nodes being joined by a single branch, is a *Kuratowski graph of Type II*. Examples of the two types are given in Fig. 2.

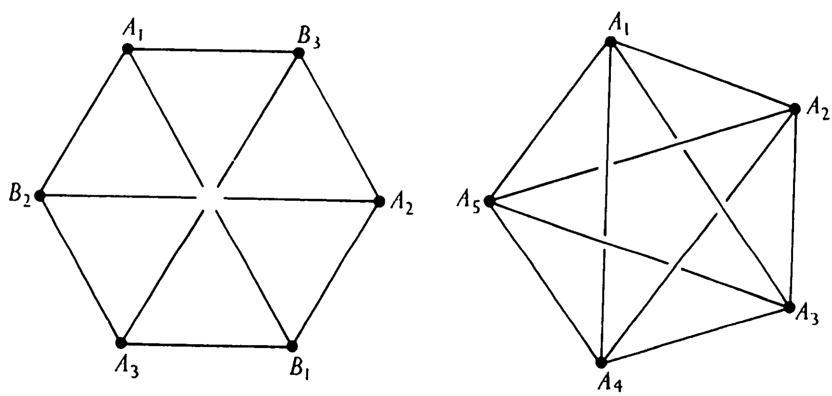

FIG. 2

(4.1) *Every Kuratowski graph is non-planar.*

Proof. This well-known result can be regarded as a consequence of (3.2). It is easily verified that in a graph of Type I each 4-branched polygon $A_i B_j A_k B_l$ is peripheral, and each branch belongs to three such polygons. In a graph of Type II each 3-branched polygon $A_i A_j A_k$ is peripheral, and there are three of them through each branch.

COROLLARY. *Any graph having a Kuratowski subgraph is non-planar.*

5. Peripheral polygons and Kuratowski subgraphs

Let J be a polygon of a graph G. Let a_1, a_2, a_3, a_4 be distinct vertices of J such that a_1 and a_3 separate a_2 from a_4 on J. Let L_{13} and L_{24} be disjoint arc-graphs of G spanning J, the ends of L_{13} being a_1 and a_3, and those of L_{24} being a_2 and a_4. Then we say that L_{13} and L_{24} are *crossing diagonals* of J.

(5.1) *Given a peripheral polygon of G with a pair of crossing diagonals we can find a Kuratowski subgraph of G of Type I.*

Proof. We use the foregoing notation, J being the peripheral polygon. Since J is peripheral, L_{13} and L_{24} have internal vertices x and y respectively. (See Fig. 3.) By ((**11**) (2.4)) there is a simple path P from x to y

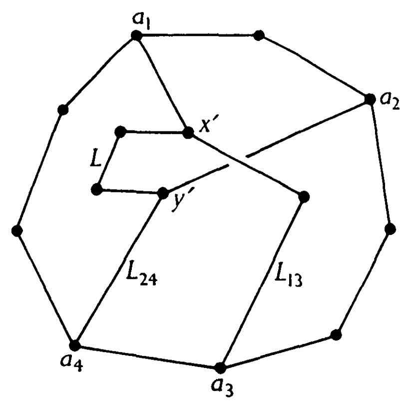

FIG. 3

in G avoiding J. Let x' be the last vertex of P in $V(L_{13})$ and y' the next vertex of P in $V(L_{24})$. Let L be the arc-graph defined by the part of P extending from x' to y'. Then $J \cup L_{13} \cup L_{24} \cup L$ is a Kuratowski graph of Type I. Its nodes are $a_1, a_2, a_3, a_4, x', y'$. Its branches are arc-graphs $a_1 a_2$, $a_2 a_3$, $a_3 a_4$, and $a_4 a_1$ in J, arc-graphs $x'a_1$ and $x'a_3$ in L_{13}, arc-graphs $y'a_2$ and $y'a_4$ in L_{24}, and L.

(5.2) *Let J be a peripheral polygon of a graph G. Let a, b, and c be distinct vertices of J. Let Y_1 and Y_2 be Y-graphs of G, each with ends a, b, and c, which span J. Suppose further that $Y_1 \cap Y_2$ consists solely of the vertices a, b, and c. Then we can find a Kuratowski subgraph of G.*

Proof. Let the centres of Y_1 and Y_2 be y_1 and y_2 respectively. There is a simple path P from y_1 to y_2 in G avoiding J. Let x_1 be the last vertex of P in $V(Y_1)$ and x_2 the next vertex of P in $V(Y_2)$. Let L be the arc-graph defined by the part of P extending from x_1 to x_2.

If $x_1 = y_1$ and $x_2 = y_2$ it is clear that the $J \cup Y_1 \cup Y_2 \cup L$ is a Kuratowski graph of Type II. We may therefore suppose, without loss of generality, that $x_1 \neq y_1$ and that x_1 is on the arm ay_1 of Y_1.

Now $Y_2 \cup L$ is a bridge of $Y_1 \cup J$ in $J \cup Y_1 \cup Y_2 \cup L$ with vertices of attachment a, b, c, and x_1. Hence, by ((**11**) (4.3)), there is a Y-graph $Y_3 \subseteq Y_2 \cup L$, with ends b, c, and x_1, which spans $Y_1 \cup J$ (Fig. 4). We denote the centre of Y_3 by y_3.

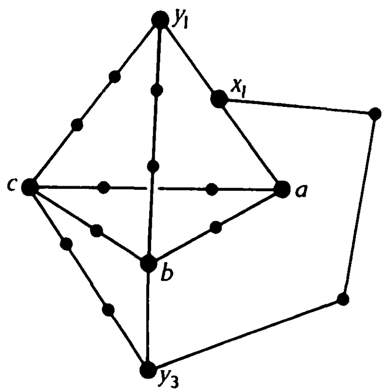

FIG. 4

We can obtain a Kuratowski graph of Type I from $J \cup Y_1 \cup Y_3$ by deleting the branch bc (but retaining its two ends).

(5.3) *Let J be a peripheral polygon of a graph G. Let a and b be distinct vertices of J, and let N be one of the residual arc-graphs of a and b in J. Let Y_1 and Y_2 be Y-graphs of G spanning J and such that $Y_1 \cap Y_2 \subset J$. Suppose that the ends of Y_i are a, b, and c_i, with c_i in $V(N)$. ($i = 1, 2$.) Then we can find a Kuratowski subgraph of G.*

Proof. If $c_1 = c_2$ this follows from (5.2). Otherwise J has a pair of crossing diagonals and we apply (5.1).

(5.4) *Let L be a branch of a graph G such that L is contained in two distinct peripheral polygons J_1 and J_2. Suppose $J_1 \cap J_2 \neq L$. Then we can find a Kuratowski subgraph of G of Type* I.

Proof. Let the ends of L be a and b. Let the other branches of G having b as an end, and contained in J_1 and J_2, be L_1 and L_2 respectively. Since J_1 and J_2 are distinct we can arrange, replacing L by another branch of G common to J_1 and J_2 if necessary, that L_1 and L_2 are distinct. Let their ends, other than b, be c_1 and c_2 respectively.

Suppose $c_1 \in V(J_2)$, so that L_1 is a bridge of J_2. Since J_2 is peripheral G then consists solely of three branches joining the nodes b and c_1. These branches are evidently L, L_1, and L_2. Hence $J_1 \cap J_2 = L$, contrary to hypothesis. We deduce that $c_1 \notin V(J_2)$. Similarly $c_2 \notin V(J_1)$.

Consider the vertices of J_2 in order, beginning with b, c_2. Let d be the next member of this sequence in $V(J_1)$. If $d = a$ we have $J_1 \cap J_2 = L$, contrary to hypothesis. Hence $d \neq a$. Moreover, b and d separate a and c_1 in J_1. Let N be the residual arc-graph of b and d in J_2 not including a (Fig. 5).

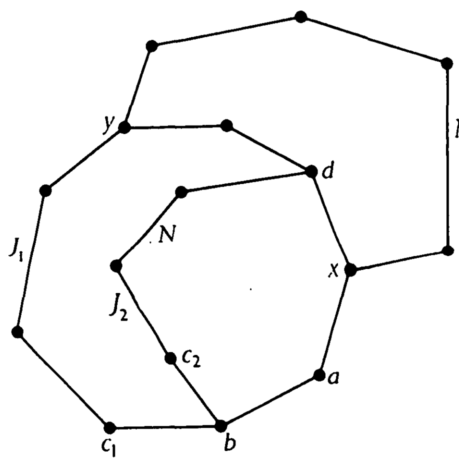

FIG. 5

Since J_2 is peripheral we can find a simple path P from a to c_1 in G avoiding J_2. Let x be the last vertex of P on the residual arc-graph of b and d in J_1 which contains a, and let y be the next vertex of P on the complementary arc-graph of J_1. Let N' be the arc-graph defined by the part of P extending from x to y.

N and N' are crossing diagonals of J_1. An application of (5.1) completes the proof.

(5.5) *Let L be a branch of a graph G common to three distinct peripheral polygons J_1, J_2, and J_3 of G. Then we can find a Kuratowski subgraph of G.*

Proof. By (5.4) we may suppose $J_1 \cap J_2 = J_2 \cap J_3 = J_3 \cap J_1 = L$. Let the ends of L be a and b. Let the complementary arc-graphs of L in J_1, J_2, and J_3 be L_1, L_2, and L_3 respectively.

Since J_1 is peripheral both L_2 and L_3 have internal vertices. Moreover, we can find internal vertices x_2 of L_2 and x_3 of L_3, and an arc-graph N_{23} of G with ends x_2 and x_3 such that N_{23} spans $J_1 \cup J_2 \cup J_3$. Similarly we can find internal vertices x_1 of L_1 and x_3' of L_3, and an arc-graph N_{13} with ends x_1 and x_3' such that N_{13} spans $J_1 \cup J_2 \cup J_3$. (See Fig. 6.)

Suppose first that N_{23} and N_{13} have a common internal vertex. Then $N_{23} \cup N_{13}$ is a bridge of $J_1 \cup J_2 \cup J_3$ in $J_1 \cup J_2 \cup J_3 \cup N_{23} \cup N_{13}$. So by ((**11**) (4.3)) there is a Y-graph $Y \subseteq N_{23} \cup N_{13}$, with ends x_1, x_2, and x_3, which spans $J_1 \cup J_2 \cup J_3$. But then $Y \cup L_1 \cup L_2 \cup L_3$ is a Kuratowski graph of Type I.

In the remaining case $L_1 \cup N_{13}$ and $L_2 \cup N_{23}$ are Y-graphs spanning J_3 and having their intersection contained in L_3. An application of (5.3) completes the proof.

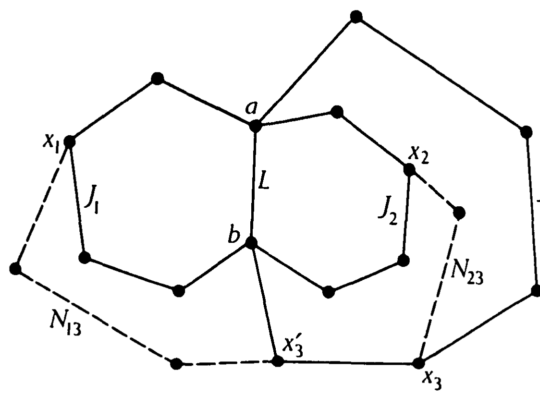

FIG. 6

6. Barycentric mappings

Let J be a peripheral polygon of a nodally 3-connected graph G having no Kuratowski subgraphs. We suppose further that there are at least three nodes of G in $V(J)$. We denote the set of nodes of G in $V(J)$ by $N(J)$, and the number of such nodes by n.

Let Q be a (geometrical) n-sided convex polygon in the Euclidean plane. Let f be a 1–1 mapping of $N(J)$ onto the set of vertices of Q such that the cyclic order of nodes in J agrees, under f, with the cyclic order of vertices of Q.

We write $m = \alpha_0(G)$ and enumerate the vertices of G as $v_1, v_2, v_3, \ldots, v_m$, so that the first n are the nodes of G in J. We extend f to the other vertices of G by the following rule. If $n < i \leqslant m$ let $A(i)$ be the set of all vertices of G *adjacent* to v_i, that is joined to v_i by an edge. For each v_j in $A(i)$ let a unit mass m_j be placed at the point $f(v_j)$. Then $f(v_i)$ is required to be the centroid of the masses m_j.

To investigate the feasibility of this requirement we set up a system of Cartesian coordinates, denoting the coordinates of $f(v_i)$, $1 \leqslant i \leqslant m$, by (x_i, y_i). Define a matrix $K(G) = \{C_{ij}\}$, $1 \leqslant (i,j) \leqslant m$, as follows. If $i \neq j$ then C_{ij} is minus the number of edges joining v_i and v_j. If $i = j$ then C_{ij} is the valency of v_i. Then the foregoing requirement specifies the coordinates x_j and y_j, for $n < j \leqslant m$, as the solutions of the equations

$$\sum_{j=1}^{m} C_{ij} x_j = 0, \tag{5}$$

$$\sum_{j=1}^{m} C_{ij} y_j = 0, \tag{6}$$

where $n < i \leqslant m$. For $1 \leqslant j \leqslant n$ the coordinates x_j and y_j are of course already known.

Let K_1 be the matrix obtained from $K(G)$ by striking out the first n rows and columns. Let G_0 be the graph obtained from G by deleting the edges of J and identifying all the vertices of J to form a single new vertex. Then, with a suitable enumeration of the vertices of G_0, we can say that K_1 is obtained from $K(G_0)$ by striking out the first row and column. Hence the determinant of K_1 is the number of subgraphs of G_0 which are trees containing all the vertices, and this number is non-zero since G_0 is connected (see for example ((**1**) § 3)).

Since $\det(K_1) \neq 0$ equations (5) and (6) have unique solutions for the unknown coordinates x_j and y_j $(n < j \leqslant m)$.

We refer to the mapping f, thus extended, as a *barycentric* mapping of G.

Choose a line l in the plane and define $\varphi(i)$, $1 \leqslant i \leqslant m$, as the perpendicular distance of $f(v_i)$ from l, counted positive on one side of l and negative on the other. We call v_i *φ-active* if there is an adjacent vertex v_j of G such that $\varphi(j) \neq \varphi(i)$. Thus all the vertices of J are φ-active.

The nodes v_i of J with the greatest value of $\varphi(i)$ are the *positive φ-poles* of G (with respect to f). The number of positive φ-poles is either 1 or 2. In the latter case the two positive φ-poles are joined by a side of Q parallel to l. In each case there is a unique vertex-graph or arc-graph $G^+ \subset J$ joining the positive φ-poles. Similarly the nodes v_i of J with the least value of $\varphi(i)$ are the *negative φ-poles* of G, and they are vertices of a subgraph G^- of G analogous to G^+.

Let P be a simple path in G. We call P a *rising* (*falling*) *φ-path* if each vertex of P other than the last corresponds to a smaller (greater) value of the function $\varphi(i)$ than does the immediately succeeding vertex.

(6.1) *Suppose that v_i, where $n < i \leqslant m$, is a φ-active vertex. Then it has adjacent vertices v_j and v_k such that $\varphi(v_j) < \varphi(v_i) < \varphi(v_k)$.*

Proof. This follows from the definition of a φ-active vertex, together with the fact that $f(v_i)$ is at the centroid of the points $f(v_j)$ such that v_i and v_j are adjacent.

(6.2) *Let v_i be a φ-active vertex. Then we can find a rising φ-path P from v_i to a positive φ-pole, and a falling φ-path P' from v_i to a negative φ-pole.*

Proof. If v_i is a positive φ-pole it defines a degenerate path which may be taken as P. If v_i is not a positive φ-pole it has an adjacent vertex v_j such that $\varphi(j) > \varphi(i)$, by (6.1). But v_j satisfies the definition of a φ-active vertex. Hence either it is a positive φ-pole or it has an adjacent vertex v_k such that $\varphi(k) > \varphi(j)$. Continuing in this way until the process terminates we obtain a sequence $v_i, v_j, v_k, \ldots$ defining a rising φ-path P from v_i to a positive φ-pole.

The path P' is constructed analogously.

7. φ-inactive vertices

We continue with the discussion of the barycentric mapping f defined in § 6.

Suppose that v_i is a φ-inactive node of G. Then $i > n$. Let Z be the subgraph of G defined by the vertices v_j such that $\varphi(j) = \varphi(i)$, and the edges which join pairs of such vertices. Let Z_1 be the edgeless subgraph of Z defined by its vertices of attachment. Let B be the bridge of Z_1 in Z having v_i as a vertex.

Since G is non-separable, B has at least two vertices of attachment. If it has only two, G is not nodally 3-connected, contrary to hypothesis. For if B was an arc-graph v_i would not be a node of G. Hence there exist three vertices of attachment, x, y, and z, of B in Z. These vertices belong to Z_1, that is they are vertices of attachment of Z. They are therefore φ-active.

By ((**11**) (4.3)) there is a Y-graph $Y \subseteq B$, with ends x, y, and z.

By (6.2) we can construct rising φ-paths P_x, P_y, and P_z from x, y, and z respectively to positive φ-poles. Write $T = G^+ \cup G(P_x) \cup G(P_y) \cup G(P_z)$. Then T is a bridge of B in $T \cup B$, with vertices of attachment x, y, and z. Hence, by ((**11**) (4.3)), there is a Y-graph $Y^+ \subseteq T$ with ends x, y, and z. Any other vertex v_t of Y^+ satisfies $\varphi(t) > \varphi(i)$.

Similarly, using falling φ-paths, we can find a Y-graph Y^- of G with ends x, y, and z such that any other vertex v_t of Y^- satisfies $\varphi(t) < \varphi(i)$.

As any vertex v_t of Y satisfies $\varphi(t) = \varphi(i)$, the intersection of any two of Y, Y^+, and Y^- consists solely of the vertices x, y, and z. Hence $Y \cup Y^+ \cup Y^-$ is a Kuratowski graph of Type I. From this contradiction we deduce

(7.1) *Every node of G is φ-active.*

(7.2) *Suppose $v_i \notin V(J)$. Then $f(v_i)$ is in the interior of Q.*

Proof. Suppose the contrary. Then we can choose l so that the interior of Q lies entirely on the negative side of l and $\varphi(i) \geqslant 0$. Then no rising φ-path can be constructed from v_i to a positive φ-pole. Hence v_i is

φ-inactive, by (6.2). It is not a node of G, by (7.1). Hence it is an internal vertex of a branch L of G. The other internal vertices of L are also φ-inactive, by (6.1). Hence every vertex v_j of L satisfies $\varphi(j) = \varphi(i)$. As the ends of L are nodes they must be two positive φ-poles. It follows that L is a bridge of J in G. Since J is peripheral, G has no nodes except the ends of L. This is contrary to assumption.

8. Peripheral polygons in a barycentric mapping

A peripheral polygon K of the graph G under discussion must have at least three vertices of attachment. For otherwise $G = H \cup K$, where $H \cap K$ consists entirely of the two nodes of G on K. Since G has at least three nodes this is contrary to the assumption that G is nodally 3-connected.

(8.1) *Let K be a peripheral polygon of G such that $V(K)$ includes just three nodes x, y, and z of G. Then $f(x)$, $f(y)$, and $f(z)$ are not collinear.*

Proof. Suppose that $f(x)$, $f(y)$, and $f(z)$ are collinear. We choose l to pass through all three of them. Then each vertex v_t of K satisfies $\varphi(t) = 0$, by (6.1).

By (6.2) and (7.1) we can construct rising φ-paths from x, y, and z to positive φ-poles. Let these paths be P_x, P_y, and P_z respectively. Write $T = G^+ \cup G(P_x) \cup G(P_y) \cup G(P_z)$. Then T is a bridge of K in $K \cup T$ with vertices of attachment x, y, and z. By ((**11**) (4.3)) we can find a Y-graph $Y \subseteq T$ with ends x, y, and z. Each vertex v_t of Y other than x, y, and z satisfies $\varphi(t) > 0$.

Similarly, using falling φ-paths, we can find a Y-graph $Y' \subseteq G$ with ends x, y, and z such that any other vertex v_t of Y' satisfies $\varphi(t) < 0$.

Applying (5.2) we find that G has a Kuratowski subgraph, contrary to assumption.

(8.2) *Let K be a peripheral polygon of G. Let v_p, v_q, v_r, and v_s be nodes of G in $V(K)$ such that v_p and v_r separate v_q and v_s in K. Then it is not true that*

$$\varphi(p) \geqslant \varphi(q) \leqslant \varphi(r) \geqslant \varphi(s) \leqslant \varphi(p). \tag{7}$$

Proof. Assume (7). Construct rising φ-paths P_p and P_r, from v_p and v_r, to positive φ-poles. ((6.2), (7.1).) In the connected graph $G(P_r) \cup G(P_p) \cup G^+$ we can find a simple path P from v_p to v_r. Let $v_{p'}$ be the last vertex of P on the residual arc-graph of v_q and v_s in K containing v_p, and let $v_{r'}$ be the next vertex of P on the residual arc-graph of v_q and v_s in K containing v_r. Let N_1 be the arc-graph defined by the part of P extending from $v_{p'}$ to $v_{r'}$. (See Fig. 7.)

Now $v_{p'}$ and $v_{r'}$ are nodes of G in $V(K)$ which separate v_q and v_s in K. Moreover,

$$\varphi(p') \geqslant \varphi(q) \leqslant \varphi(r') \geqslant \varphi(s) \leqslant \varphi(p'). \tag{8}$$

N_1 spans K, and each vertex v_j of N_1 other than $v_{p'}$ and $v_{r'}$ satisfies $\varphi(j) > \min[\varphi(p'), \varphi(r')]$.

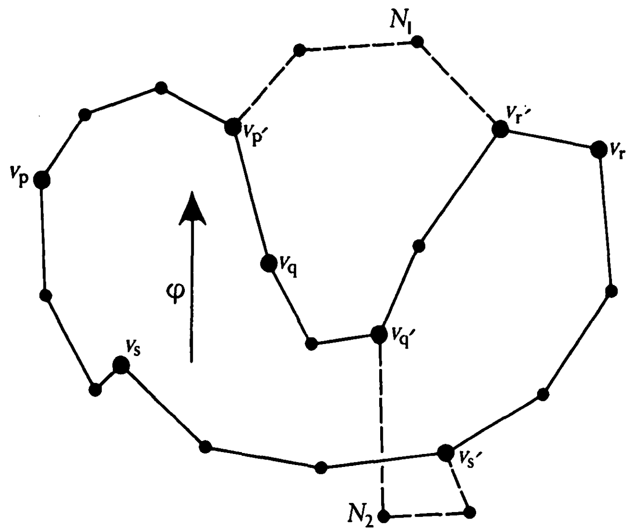

FIG. 7

A similar construction with falling φ-paths from v_q and v_s yields two nodes $v_{q'}$ and $v_{s'}$ of G in $V(K)$ which separate $v_{p'}$ and $v_{r'}$ in K and satisfy

$$\varphi(p') \geqslant \varphi(q') \leqslant \varphi(r') \geqslant \varphi(s') \leqslant \varphi(p'). \tag{9}$$

It yields also an arc-graph N_2, with ends $v_{q'}$ and $v_{s'}$, which spans K and is such that each vertex v_j of N_2 other than $v_{q'}$ and $v_{s'}$ satisfies

$$\varphi(j) < \max[\varphi(q'), \varphi(s')] \leqslant \min[\varphi(p'), \varphi(r')].$$

Now N_1 and N_2 are crossing diagonals of K. Hence G has a Kuratowski subgraph, by (5.1), contrary to hypothesis.

(8.3) *The nodes of any peripheral polygon K of G are mapped by f onto distinct points of the plane, no three of which are collinear.*

Proof. Suppose that v_p, v_q, and v_r are distinct nodes of K such that $f(v_p)$, $f(v_q)$, and $f(v_r)$ are collinear. Choose l to pass through $f(v_p)$, $f(v_q)$, and $f(v_r)$. There is a fourth node v_s of K, by (8.1), and we can adjust φ so that $\varphi(s) \leqslant 0$. We can further adjust the notation so that v_p and v_r separate v_q and v_s in K. But then (7) is true and this contradicts (8.2). We deduce that no three distinct vertices of K are mapped by f onto collinear points. This implies that any two nodes v_p and v_q of K are mapped by f onto distinct points. For K has a third node, v_r say, and if $f(v_p) = f(v_q)$ the points $f(v_p)$, $f(v_q)$, and $f(v_r)$ are collinear.

(8.4) *Let L be a branch of G having just $t \geqslant 1$ internal vertices, and let its ends be a and b. Then $f(a)$ and $f(b)$ are distinct and f maps the internal vertices onto t distinct points of the segment $f(a)f(b)$ subdividing it into $t+1$ equal parts. Moreover, the order of the vertices from a to b in L agrees with that of their images in $f(a)f(b)$.*

Proof. $f(a)$ and $f(b)$ are distinct by (2.3) and (8.3). The rest of Theorem (8.4) follows at once from the definition of a barycentric mapping.

Let e be any edge of G, with ends u and v. By (8.3) and (8.4), $f(u)$ and $f(v)$ are distinct. We denote the open segment $f(u)f(v)$ by $f(e)$.

(8.5) *Let K be any peripheral polygon of G. Then f maps the nodes of G in K onto the vertices of a (geometrical) convex polygon Q_K so that the cyclic order of nodes in K agrees with that of vertices in Q_K.*

Proof. Let the branches of G in K, taken in their cyclic order in K, be $L_1, L_2, \ldots, L_k, L_1$. By § 7, $k \geqslant 3$. Let the common end of L_i and L_{i+1} $(1 \leqslant i \leqslant k,\ L_{k+1} = L_1)$ be w_i.

The two ends of L_i are mapped by f onto distinct points of the plane (8.3). Let these determine a line l_i. This line determines two open half-planes one of which, D_i say, contains the images under f of all the other nodes of G in K, that is all the vertices w_i, by (8.2). The intersection of the closures of the half-planes D_i is a convex polygon Q_K of the kind required.

Using (8.4) we see that the images of the vertices of K are distinct points of the boundary of Q_K, and that their cyclic order on Q_K agrees with that of the vertices in K.

If $K \neq J$ we define R_K as the interior of Q_K. But we define R_J as the exterior of Q_J, that is of Q.

(8.6) *Let e be any edge of R. Then just two distinct peripheral polygons of G pass through e, and the two corresponding regions R_K lie on opposite sides of the segment $f(e)$.*

Proof. That e belongs to just two peripheral polygons K and K' of G follows from (2.3) and (5.5). If one of these is J the theorem follows from (7.2). Suppose therefore that neither K nor K' is J.

Let L be the branch of G containing e, and let v_r and v_s be its two ends. Choose l to pass through the distinct points $f(v_r)$ and $f(v_s)$, (8.3). v_r and v_s are not φ-poles, since e is not an edge of the peripheral polygon J.

Assume that R_K and $R_{K'}$ are on the same side of l, which we can suppose to be the positive side. Choosing nodes v_p and v_q of K and K' respectively, distinct from v_r and v_s, we have $\varphi(p) > 0, \varphi(q) > 0$. Construct rising φ-paths P_p and P_q, from v_p and v_q respectively, to positive φ-poles. Construct also falling φ-paths P_r and P_s, from v_r and v_s respectively, to negative φ-poles. In the connected graph $G(P_r) \cup G(P_s) \cup G^-$ we can find an arc-graph N with ends v_r and v_s. This arc-graph has at least one internal vertex, and each internal vertex v_t satisfies $\varphi(t) < 0$.

Considering the connected graph $G(P_p) \cup G(P_q) \cup G^+$ we see that there is a bridge B of the polygon $L \cup N$ in G meeting each of K and K' in at least one vertex not in $V(L)$. Hence, by (2.4), there is a third peripheral polygon through e, which is impossible by (5.5).

9. Barycentric representations

Let H be a graph whose vertices are the points $f(v)$, $v \in V(G)$, and whose edges are the open segments $f(e)$, $e \in E(G)$. The incident vertices of an edge $f(e)$ are its two end-points. Let $|H|$ be the union of $V(H)$ and the segments $f(e)$, and let S be its complementary set in the plane. For each point A of S we define $\delta(A)$ as the number of distinct peripheral polygons K of G such that $A \in R_K$. By the definition of the regions R_K, and (7.2), the function δ has the value 1 throughout the exterior of Q.

(9.1) $\delta(A) = 1$ *for each* A *in* S.

Proof. We may assume that A is in the interior of Q. Choose a point B outside Q such that the segment AB passes through no vertex of H and is parallel to no edge of H. The points of intersection of AB with edges of H subdivide AB into subsegments within each of which the function δ must be constant. But δ has the same value within any two neighbouring subsegments, by (8.6). Hence $\delta(A) = \delta(B) = 1$.

From (9.1) we deduce that no region R_K contains any point of $|H|$.

If two distinct edges e and e' of G are such that the open segments $f(e)$ and $f(e')$ have a common point P it follows that the two segments lie on a common line l. Then no peripheral polygon of G contains both e and e', by (8.3) and (8.4). Hence if A in S is sufficiently near P there are two distinct peripheral polygons K, one through e and the other through e', such that $A \in R_K$. This is contrary to (9.1). We deduce that $f(e)$ and $f(e')$ are disjoint.

From these two observations we deduce further that no edge of H contains any vertex of H, and that no two distinct vertices of G are mapped by f onto the same point of the plane.

It follows that H is a representation of G in the closed plane. We call it a *barycentric* representation of G on the convex polygon Q. It is a convex representation of G as defined in (**11**), provided we ignore the trivial distinction that the segments $f(e)$ are closed in (**11**) and open in the present paper. We sum up our results as follows.

(9.2) *Let G be a nodally 3-connected graph having no Kuratowski subgraph. Let J be a peripheral polygon of G which includes just $n \geqslant 3$ nodes of G. Let Q be an n-sided convex polygon in the Euclidean plane. Then there is a unique barycentric representation of G on Q mapping the nodes of G occurring on J onto the vertices of Q in any arbitrarily specified way preserving the cyclic order.*

(9.3) *Let G be a nodally 3-connected graph having at least one polygon. Then if G has no Kuratowski subgraph we can construct a convex representation of G.*

Proof. If G is a polygon this result is trivial. Otherwise we can find a peripheral polygon J of G, by (2.2), and since G is a non-separable there are at least two nodes of G in $V(J)$.

If there are more than two nodes of G in $V(J)$ we use (9.2). Otherwise since G is nodally 3-connected it consists of two nodes u and v joined by three distinct branches L_1, L_2, and L_3. Since G is simple we may suppose L_1 and L_2 to have internal vertices. We may now represent G by a convex polygon with one diagonal, in an obvious way.

10. Straight representations

The planar meshes and subclasses of planar meshes discussed below are sets in which a particular element may be considered to appear more than once. In a union $\mathbf{M} \cup \mathbf{N}$ the multiplicity of an element is taken to be the sum of its multiplicities in $\mathbf{M}$ and $\mathbf{N}$.

(10.1) *Let G be the union of two proper subgraphs H and K such that $H \cap K$ is either null or a vertex-graph* ((**11**) § 2). *Let $\mathbf{M}_H$ and $\mathbf{M}_K$ be planar meshes of H and K respectively. Then $\mathbf{M}_H \cup \mathbf{M}_K$ is a planar mesh of G. Moreover, any planar mesh of G can be represented in this form.*

Proof. $\mathbf{M}_H \cup \mathbf{M}_K$ satisfies condition (i) of ((**11**) § 1) for a planar mesh of G. Let J be any non-null cycle of G. Then $J \cap E(H)$ and $J \cap E(K)$ are disjoint cycles of H and K respectively. Each is either null or a sum of members of $\mathbf{M}_H$ or $\mathbf{M}_K$. Hence J is a sum of members of $\mathbf{M}_H \cup \mathbf{M}_K$. Thus $\mathbf{M}_H \cup \mathbf{M}_K$ satisfies condition (ii) of ((**11**) § 1).

Now let $\mathbf{M}$ be any planar mesh of G. Since no polygon of G can have edges in both H and K we can write $\mathbf{M} = \mathbf{M}'_H \cup \mathbf{M}'_K$, where the members of $\mathbf{M}'_H$ are contained in E_H and those of $\mathbf{M}'_K$ in E_K. It is evident that each edge of H occurring in any member of $\mathbf{M}'_H$ occurs in just two of them.

If J is a non-null cycle of H it is a sum of members of $\mathbf{M}$. But the members of $\mathbf{M}'_K$ involved in this sum add up to $J \cap E(K) = \emptyset$. Hence J is a sum of members of $\mathbf{M}'_H$. We deduce that $\mathbf{M}'_H$ is a planar mesh of H. Similarly $\mathbf{M}'_K$ is a planar mesh of K. The theorem follows.

(10.2) *Let G be the union of two proper subgraphs H and K such that $H \cap K$ consists solely of two vertices x and y. Let L_H and L_K be arc-graphs with ends x and y in H and K respectively. Let $\mathbf{M}_H$ and $\mathbf{M}_K$ be planar meshes of $H \cup L_K$ and $K \cup L_H$ respectively. Then the following propositions hold.*

(i) *We can write $\mathbf{M}_H = \{C_1, C_2, \ldots, C_h\}$ and $\mathbf{M}_K = \{D_1, D_2, \ldots, D_k\}$ so that $E(L_K) \subseteq C_1 \cap C_2$ and $E(L_H) \subseteq D_1 \cap D_2$.*

(ii) *The class*

$$\mathbf{M} = \{C_1 + D_1 + E(L_H) + E(L_K), \\ C_2 + D_2 + E(L_H) + E(L_K), C_3, \ldots, C_h, D_3, \ldots, D_k\}$$

is then a planar mesh of G.

Proof. $L_H \cup L_K$ is a polygon of both $H \cup L_K$ and $K \cup L_H$. Hence any specified edge of L_K belongs to just two members C_1 and C_2 of $\mathbf{M}_H$. Since L_K spans H it follows that $E(L_K) \subseteq C_1 \cap C_2$. A similar argument involving L_H and $\mathbf{M}_K$ completes the proof of (i).

It is clear from the construction of $\mathbf{M}$ that each edge of G occurring in any member of $\mathbf{M}$ occurs in just two of them. Moreover, each member of $\mathbf{M}$ is an elementary cycle of G. For example, $C_1 + D_1 + E(L_H) + E(L_K)$ corresponds to the union of the arc-graphs $H.(C_1 + E(L_K))$ and $K.(D_1 + E(L_H))$, which is a polygon.

Let J be any non-null cycle of G. It may happen that the number of edges of $J \cap E(H)$ incident with x, loops being counted twice, is even. Then $J \cap E(H)$ and $J \cap E(K)$ are cycles of H and K respectively. If $J \cap E(H)$ is non-null it can be represented as a sum of members of $\mathbf{M}_H$ other than C_1, since the members of $\mathbf{M}_H$ sum to zero. Then the sum cannot involve C_2 either and so $J \cap E(H)$ is a sum of members of $\mathbf{M}$. A similar argument applies to $J \cap E(K)$. It follows that J is a sum of members of $\mathbf{M}$. In the remaining case the number of edges of $(J + C_1 + D_1 + E(L_H) + E(L_K)) \cap E(H)$ incident with x is even. Hence $J + C_1 + D_1 + E(L_H) + E(L_K)$, and therefore also J, is a sum of members of $\mathbf{M}$. We deduce that $\mathbf{M}$ is a planar mesh of G.

(10.3) *Let G be the union of two proper subgraphs H and K having only a vertex v in common. Let A_H and A_K be links of H and K respectively incident with v, and let their other ends be w_H and w_K respectively. Let G' be formed from G by adjoining a new link A with ends w_H and w_K. Then if G has a planar mesh so does G'.*

Proof. Let $\mathbf{M}$ be a planar mesh of G. By (10.1), $\mathbf{M} = \mathbf{M}_H \cup \mathbf{M}_K$, where

$\mathbf{M}_H$ and $\mathbf{M}_K$ are planar meshes of H and K respectively. We note the elementary cycle $Z = \{A, A_H, A_K\}$ of G'.

If A_H belongs to no member of $\mathbf{M}_H$ we adjoin Z to $\mathbf{M}_H$ as a single extra term, denoting the resulting class by $\mathbf{N}_H$. Otherwise we delete a term, Z_H say, of $\mathbf{M}_H$ containing A_H, and adjoin one extra term $Z+Z_H$, again denoting the resulting class by $\mathbf{N}_H$. If A_H belongs to no member of $\mathbf{M}_H$ it is convenient to write $Z_H = \emptyset$. We define $\mathbf{N}_K$ and Z_K similarly, and write $\mathbf{M}' = \mathbf{N}_H \cup \mathbf{N}_K$.

The above construction ensures that each edge of G' belonging to a term of $\mathbf{M}'$ belongs to just two such terms. If X is any non-null cycle of G' then either X or $X+Z$ is a cycle X_1 of G, and $X_1 \cap E(H)$ and $X_1 \cap E(K)$ are cycles of H and K respectively. If $X_1 \cap E(H)$ is non-null it can be expressed as the sum of the members of a subset of $\mathbf{M}_H$ which does not involve every occurrence of Z_H, if $Z_H \neq \emptyset$. Hence $X_1 \cap E(H)$ is a sum of members of $\mathbf{N}_H$, and similarly $X_1 \cap E(K)$ is a sum of members of $\mathbf{N}_K$. We deduce that X is a sum of members of $\mathbf{M}'$. The theorem follows.

(10.4) *Let G be the union of two subgraphs H and L, where L is an arc-graph spanning H. Let $\mathbf{M}_H = \{C_1, C_2, C_3, \ldots, C_h\}$ be a planar mesh of H such that the ends x and y of L are vertices of $G.C_1$. Let the residual arc-graphs of x and y in $G.C_1$ be L_1 and L_2. Then $\mathbf{M} = \{E(L \cup L_1), E(L \cup L_2), C_2, \ldots, C_h\}$ is a planar mesh of G.*

Proof. Clearly $\mathbf{M}$ satisfies condition (i) of ((**11**) § 1) for a planar mesh of G. If J is any non-null cycle of G then either J or $J+E(L \cup L_1)$ is a cycle of H and therefore a sum of terms of $\mathbf{M}_H$ not including C_1. Hence J is a sum of members of $\mathbf{M}$.

We say that $\mathbf{M}$ is obtained from $\mathbf{M}_H$ by *subdividing* C_1.

(10.5) *Let G be the union of two subgraphs H and L, where L is an arc-graph spanning H. Let $\mathbf{M} = \{C_1, C_2, \ldots, C_g\}$ be a planar mesh of G such that $C_1 \cap C_2 = L$. Let the complementary arc-graphs of L in $G.C_1$ and $G.C_2$ be L_1 and L_2 respectively. Then $\mathbf{M}_H = \{E(L_1 \cup L_2), C_2, \ldots, C_g\}$ is a planar mesh of H.*

Proof. $\mathbf{M}_H$ clearly satisfies condition (i) of ((**11**) § 1). Any non-null cycle J of H is a cycle of G, and therefore a sum of members of $\mathbf{M}$ other than C_1. This sum does not involve C_2 since $J \cap E(L) = \emptyset$. Hence J is a sum of members of M_H.

(10.6) *Let G be a graph having a planar mesh* $\mathbf{M}$. *Then each subgraph of G has a planar mesh.*

Proof. Suppose we form a graph K from G by deleting a single edge A. We form a class $\mathbf{M}_K$ from $\mathbf{M}$ as follows. If A belongs to no member of $\mathbf{M}$, then $\mathbf{M}_K = \mathbf{M}$. In the remaining case A belongs to just two terms C_1 and C_2 of $\mathbf{M}$. If $C_1 = C_2$ we form $\mathbf{M}_K$ by deleting these two terms from $\mathbf{M}$. Otherwise C_1+C_2 can be expressed as a sum of disjoint elementary cycles of K, by ((**11**) (3.2)). We then form $\mathbf{M}_K$ by replacing C_1 and C_2 by this set of elementary cycles, each counted once only.

By this construction $\mathbf{M}_K$ satisfies condition (i) for a planar mesh of K. If J is any non-null cycle of K it can be expressed as a sum of terms of $\mathbf{M}$, not involving C_1 if C_1 and C_2 exist. In the latter case the sum does not involve C_2 either since $A \notin E(K)$. Hence J is a sum of members of $\mathbf{M}_K$. We deduce that $\mathbf{M}_K$ is a planar mesh of K.

Suppose $H \subseteq G$. We can remove edges from G, one by one, until we obtain a subgraph H' of G such that $E(H') = E(H)$. Then H can be obtained from H' by deleting some isolated vertices. H' has a planar mesh, by repeated application of the preceding argument if $E(H) \neq E(G)$, and H has the same planar mesh, by (10.1).

(10.7) *If a graph has a planar mesh it has no Kuratowski subgraph.*

Proof. The Kuratowski subgraph would have a planar mesh, by (10.6). This is impossible, by (2.7) and the proof of (4.1).

(10.8) *Let G be any simple graph having a planar mesh. Then by adding new links to G, with ends in $V(G)$, we can construct a nodally 3-connected graph T having a planar mesh.*

Proof. Suppose that G is the union of two proper subgraphs H and K such that $H \cap K$ is null. Choose vertices x in $V(H)$ and y in $V(K)$ and adjoin a new link A with ends x and y. The new graph G_1 is simple. Now H and K have planar meshes, by (10.1), and the link-graph $G_1 . \{A\}$ has a null planar mesh. Hence G_1 has the same planar mesh as G, by two applications of (10.1).

Continuing in this way until the process terminates we construct a simple connected graph T_1 with the same planar mesh as G.

Now suppose T_1 is the union of two proper subgraphs H and K such that $H \cap K$ is a vertex-graph. Since T_1 is connected, the common vertex v is incident with edges A_H in $E(H)$ and A_K in $E(K)$. Let the other ends of A_H and A_K be w_H and w_K respectively. Adjoining a new link A joining w_H and w_K we obtain a new simple graph G_2. This graph has a planar mesh, by (10.3).

Continuing in this way until the process terminates we obtain a simple non-separable graph T_2 such that $G \subseteq T_2$ and T_2 has a planar mesh $\mathbf{M}_2$.

Suppose that T_2 is not nodally 3-connected. Then it is a union of two proper subgraphs H and K whose intersection consists solely of two nodes x and y, and neither of which is an arc-graph with ends x and y. By

((**11**) (3.3)) there exists C_1 in $\mathbf{M}_2$ meeting both $E(H)$ and $E(K)$. There is a second member C_2 of $\mathbf{M}_2$ with this property, for otherwise the terms of $\mathbf{M}_2$ would not sum to zero.

If A_1 is an edge of T_2 with ends x and y we may assume $A_1 \in E(H)$. If a member of $\mathbf{M}_2$ meeting both $E(H)$ and $E(K)$ contains A_1 it can, being the set of edges of a polygon, contain no other edge of H. If all such members of $\mathbf{M}_2$ contained A_1 there would be just two of them, and (3.3) of (**11**) would be violated at x. We may suppose that $G.C_1$ has vertices a in $V(H)$ and b in $V(K)$ distinct from x and y. Joining a and b by a new link A we obtain a new simple graph G_3.

Now G_3 has a planar mesh, by (10.4). Proceeding in this way until the process terminates we obtain a nodally 3-connected graph T such that $G \subseteq T$ and T has a planar mesh $\mathbf{M}$.

(10.9) *If G is a simple graph having a planar mesh we can find a straight representation of G in the plane.* (See (**3**).)

Proof. First we embed G in a nodally 3-connected graph T with a planar mesh $\mathbf{M}$. This has no Kuratowski subgraph, by (10.7). We can obtain a straight representation of T, by (9.3), and this induces a straight representation of G. (If T has no polygon it has at most one edge, by ((**11**) (2.5)), and a straight representation is obviously possible.)

(10.10) *Let G be any graph having a planar mesh. Then G is planar.*

Proof. If A is an edge of G with ends u and v, possibly coincident, we can replace it by a new vertex w and two new links, A' joining u and w and A'' joining v and w. We call this process *subdividing* A. Clearly the cycles of the new graph G_1 can be derived from those of G by replacing A, wherever it occurs, by the two edges A' and A''. Accordingly, if the process is applied to the members of a planar mesh of G it yields a planar mesh of G_1.

By repeated subdivision we can convert G into a simple graph T, and by the above observations T has a planar mesh.

By (10.9) we can construct a straight representation of T. We can reverse the subdivisions in this and so obtain a representation of G.

11. Characterizations of planar graphs

The conditions for planarity established by Kuratowski ((**2**) (**6**)) and MacLane (**7**) can be derived from the foregoing results as follows.

(11.1) *Let G be any graph. Then the propositions 'G is planar', 'G has a planar mesh', and 'G has no Kuratowski subgraph' are equivalent.*

Proof. If G has a planar mesh it is planar, by (10.10). If G is planar it has no Kuratowski subgraph, by (4.1), Corollary.

If possible choose G so that it has no Kuratowski subgraph, and no planar mesh, and so that $\alpha_1(G)+\alpha_0(G)$ has the least value consistent with these conditions. Clearly G is not a polygon or a link-graph.

Suppose G is nodally 3-connected. Then it has three distinct peripheral polygons with a common edge, by (2.3) and (2.6). Hence it has a Kuratowski subgraph by (5.5), contrary to the definition of G.

Suppose that G is non-separable. Since it is not nodally 3-connected it is a union of two proper subgraphs H and K such that $H \cap K$ consists solely of two vertices x and y, and neither H nor K is an arc-graph. Since G is non-separable both H and K must be connected. Hence x and y are joined by arc-graphs $L_H \subseteq H$ and $L_K \subseteq K$. Now $H \cup L_K$ and $K \cup L_H$ are proper subgraphs of G. By the choice of G they have planar meshes. Hence G has a planar mesh, by (10.2).

From this contradiction we deduce that G is separable. It is a union of two proper subgraphs H and K such that $H \cap K$ is either null or a vertex graph. H and K have planar meshes, by the choice of G. Hence G has a planar mesh, by (10.1), a contradiction.

We deduce that any graph without a Kuratowski subgraph must have a planar mesh. The proof of the theorem is now complete.

Given a vertex a of a graph G we write $T(a)$ for the set of all links of G incident with a. Given any set V of vertices of G we define $T(V)$ as the set of all links of G having one end in V and one in $V(G)-V$. We refer to the sets $T(V)$ as the *coboundaries* of G. Evidently each coboundary is a mod-2 sum of sets of the form $T(a)$, $a \in V(G)$, and the coboundaries form a group with respect to mod-2 addition.

A graph G^* with the same edges as G, and such that the cycles of G are the coboundaries of G^*, is a *dual graph* of G.

(11.2) *A graph is planar if and only if it has a dual graph.*

Proof. Suppose that G is planar. Let $\mathbf{M} = \{C_1, C_2, \ldots, C_g\}$ be a planar mesh of G (11.1). Let the edges of G belonging to no C_i be $A_1, A_2, \ldots, A_k$. Denote the vertices of a graph G^* by $v_1, v_2, \ldots, v_g$, $w_1, w_2, \ldots, w_k$. Put $E(G^*) = E(G)$. Take A_i, $1 \leqslant i \leqslant k$, to be a loop of G^* incident with w_i. If A in $E(G^*)$ satisfies $A \in C_i \cap C_j$, $i \neq j$, we take A to be a link of G^* joining v_i and v_j. In G^* we then have $T(w_i) = \emptyset$ and $T(v_i) = C_i$. Hence the coboundaries of G^* are the linear combinations of the members of $\mathbf{M}$, that is they are the cycles of G. Thus G has a dual graph G^*.

Conversely, suppose that G has a dual graph G^*. Let the vertices a of G^* such that $T(a)$ is non-null be enumerated as $v_1, v_2, \ldots, v_g$. Consider the class $N = \{T(v_1), T(v_2), \ldots, T(v_g)\}$. Each term is a cycle of G, each edge of G belonging to a term of $\mathbf{N}$ belongs to just two of them, and each cycle of G is a sum of members of $\mathbf{N}$. We can therefore convert $\mathbf{N}$ into a planar mesh of G by replacing each $T(v_i)$ by a class of disjoint elementary cycles of G summing to $T(v_i)$. ((**11**) (3.2).) Hence G is planar, by (11.1).

We see from (11.2) that the problem of deciding whether a given graph is planar is a special case of a more general problem, that of deciding whether a given class of subsets of a collection S, closed under mod-2 addition, can be represented as the class of coboundaries of a graph. This wider problem has practical importance as well as theoretical interest. It is analysed in (**8**), (**9**), and (**10**).

The results of the present paper suggest that the more general problem could be tackled by constructing analogues of peripheral cycles. I hope to discuss such a procedure in another paper.

The theorems of (**9**) yield, on specialization, another characterization of planar graphs: a graph is non-planar if and only if it has a polygon whose bridges cannot be classified in two sets so that the members of each set avoid one another.

12. Representations of dual graphs

Let G be a non-null 3-connected graph in which the valency of each vertex is at least 3. Suppose that G has a planar mesh $\mathbf{M} = \{C_1, C_2, \dots, C_g\}$.

Let the vertices v_j and edges A_j of $G.C_i$ be, in their natural cyclic order, $v_0, A_1, v_1, A_2, \dots, v_{n-1}, A_n, v_n = v_0$. We introduce a new vertex w_i and join it to each v_j in $V(G.C_i)$ by a single new edge A_{ij}. Fig. 8 illustrates the case $i = 1$. We repeat the operation for each member of $\mathbf{M}$, arranging that the new vertices w_k are all distinct. Let us denote the resulting graph by G_0, and the class $\{w_k\}$ by W.

Now G_0 can be constructed by repeatedly subdividing faces as in (10.4). We may therefore deduce from (10.4) that G_0 is planar, and has a planar mesh $\mathbf{M}_0 = \{T_1, T_2, T_3, \dots, T_q\}$ with the following property: $G.T_i$ is a triangle in which one vertex is a w_k and the opposite edge is an edge of G.

Consider an edge A_j of G. It belongs to two members T_r and T_s of $\mathbf{M}_0$. Now A_j belongs to two distinct members of $\mathbf{M}$, by ((**11**) (3.4)), and therefore T_r and T_s correspond to distinct members of W. Accordingly $G_0.(T_r+T_s)$ is a quadrangle X_j of G_0. Of its four distinct vertices two belong to W and two are the ends of A_j in G. We define $G^\natural$ as the graph obtained from G_0 by deleting all the edges A_j of G. By (10.5) $G^\natural$ is planar and it has a planar mesh $\mathbf{M}^\natural$ whose members are the sets $E(X_j)$.

The above construction is illustrated in Fig. 8. The full lines represent G and the broken ones $G^\natural$.

(12.1) *$G^\natural$ is 3-connected.*

Proof. Suppose $G^\natural$ is the union of the two proper subgraphs H and K, where $H \cap K$ consists solely of two or fewer vertices.

Suppose both H and K have vertices of G not in $H \cap K$. Since G is 3-connected one such vertex in $V(H)$ must be joined to one in $V(K)$ by an edge A_j of G. Considering the quadrangle X_j we see that $H \cap K$ has two distinct vertices, both in W. Call them w_1 and w_2. Since the members of $\mathbf{M}^\natural$ sum to zero there is a second member of $\mathbf{M}^\natural$ having edges in both H and K. That is, another quadrangle X_k has both w_1 and w_2 as vertices. But then the two peripheral polygons $G.C_1$ and $G.C_2$ of G have two distinct branches of G in common, which is impossible, by (2.3) and (2.8).

We may now assume that $V(H) - V(H \cap K)$ includes no vertex of G. Suppose that a vertex u of $H \cap K$ is joined by an edge A to a vertex v in $V(H) - V(K)$. Then $v \in W$, and is joined only to members of $V(G)$ in $V(H \cap K)$. But since G is simple each w_i in W is joined to three or more distinct vertices of G. We deduce that any member of $E(H)$ has both ends in $H \cap K$. The theorem follows.

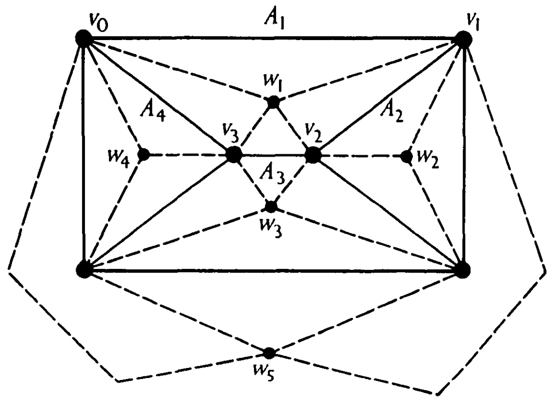

FIG. 8

It follows from (2.8) that the quadrangles X_j are peripheral polygons of $G^\natural$. By (9.2) we can construct a barycentric realization of $G^\natural$ on a 4-sided convex polygon Q in the plane. We suppose X_1 to play the part of J in §§ 5–8. In each of the convex polygons Q_{X_j}, $j > 1$, in the barycentric representation we construct the diagonal joining the two opposite vertices of the quadrangle corresponding to vertices of G. In the quadrangle Q let the vertices corresponding to vertices of G be u and v. We join these by the 'infinite segment' of the line uv outside Q. We may consider the plane to be closed by a point Ω at infinity, and say that u and v are joined by a straight segment through Ω.

Allowing the use of this infinite segment we obtain a set of diagonals giving a straight representation of G. Using the other diagonals of the quadrangles we obtain a straight representation of the dual graph G^* of G.

(There is essentially only one dual graph of G since G has only one planar mesh, by (2.6) and (2.8).) We then have simultaneous straight representations of G and G^* in which the only intersections are of corresponding edges, and two corresponding edges meet in just one point.

13. Unsolved problems

The result of § 12 raises the following questions. Can we construct simultaneous straight representations, with intersections limited as above, of G and G^* in which the residual regions of each representation are convex? Or such that corresponding edges are represented by perpendicular segments?

We might also consider representations of a planar graph on a geometrical sphere such that the vector drawn from the centre to any vertex is in the direction of the resultant of the vectors drawn to its neighbouring vertices. Does every nodally 3-connected planar graph have such a representation and if so is the representation unique for each graph?

Finally we may remark that very little is known about representations of graphs in the projective plane and higher surfaces (**4**).

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