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How to Explain Ubiquity of Constant Elasticity of Substitution (CES) Production and Utility Functions Without Explicitly Postulating CES

Olga Kosheleva
The University of Texas at El Paso, olgak@utep.edu

Vladik Kreinovich
The University of Texas at El Paso, vladik@utep.edu

Thongchai Dumrongpokaphan
Chiang Mai University, tcd43@hotmail.com

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Olga Kosheleva, Vladik Kreinovich, and Thongchai Dumrongpokaphan

Abstract In many situations, the dependence of the production or utility on the corresponding factors is described by the CES (Constant Elasticity of Substitution) functions. These functions are usually explained by postulating two requirements: an economically reasonable postulate of homogeneity (that the formulas should not change if we change a measuring unit) and a less convincing CSE requirement. In this paper, we show that the CES requirement can be replaced by a more convincing requirement – that the combined effect of all the factors should not depend on the order in which we combine these factors.

1 Formulation of the Problem

CES production functions and CES utility function are ubiquitous. Most observed data about production y is well described by the *CES production function*

$$y = \left(\sum_{i=1}^{n} a_i \cdot x_i^r \right)^{1/r}, \tag{1}$$

where x_i are the numerical measures of the factors that influence production, such as amount of capital, amount of labor, etc.; see, e.g., [5, 16, 17, 23].

Olag Kosheleva
University of Texas at El Paso, 500 W. University, El Paso, Texas 79968, USA,
e-mail: olgak@utep.edu

Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso, 500 W. University, El Paso, Texas 79968, USA, e-mail: vladik@utep.edu

Thongchai Dumrongpokaphan
Department of Mathematics, Faculty of Science, Chiang Mai University, Thailand,
e-mail: tcd43@hotmail.com

A similar formula (1) describes how the person's utility y depends on different factors x_i such as amounts of different types of consumer goods, utilities of other people, etc.; see, e.g., [6, 11, 12, 28].

How this ubiquity is explained now. The current explanation for the empirical success of CES function is based on the following two requirements.

The first requirement is that the corresponding function $y = f(x_1,\ldots,x_n)$ is *homogeneous*, i.e., that:

$$f(\lambda\cdot x_1,\ldots,\lambda\cdot x_n) = \lambda\cdot f(x_1,\ldots,x_n). \tag{2}$$

This requirement makes perfect economic sense: e.g., we can describe different factors by using different monetary units, and the results should not change if we replace the original unit by a one which is λ times smaller. After this replacement, the numerical value of each factor changes from x_i to $\lambda\cdot x_i$ and y is replace by $\lambda\cdot y$. The value $f(\lambda\cdot x_1,\ldots,\lambda\cdot x_n)$ that we obtain by using the new units should thus be exactly λ times larger than the value $f(x_1,\ldots,x_n)$ obtained in the original units – and this is exactly the requirement (2).

The second requirement is that the corresponding function $f(x_1,\ldots,x_n)$ should provide *constant elasticity of substitution* (CES). The requirement is easier to explain for the case of two factors $n = 2$. In this case, this requirement deals with "substitution" situations in which we change x_1 and then change the original value x_2 to the new value $x_2(x_1)$ so that the overall production or utility remain the same.

The corresponding substitution rate can then be calculated as $s \stackrel{\text{def}}{=} \dfrac{dx_2}{dx_1}$. The substitution function $x_2(x_1)$ is explicitly defined by the equation $f(x_1,x_2(x_1)) = \text{const}$. By using the formula for the derivative of the implicit function, we can conclude that the substitution rate has the form

$$s = -\frac{f_{,1}(x_1,x_2)}{f_{,2}(x_1,x_2)},$$

where we denoted

$$f_{,1}(x_1,x_2) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_1}(x_1,x_2) \text{ and } f_{,2}(x_1,x_2) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_2}(x_1,x_2).$$

The requirement is that for each percent of the change in ratio $\dfrac{x_2}{x_1}$, we get the same constant number of percents change in s:

$$\frac{ds}{d\left(\dfrac{x_2}{x_1}\right)} = \text{const}.$$

This explanation needs strengthening. While homogeneity is a reasonable requirement, the above CES condition sounds somewhat too mathematical to be fully convincing for economists.

To explain the ubiquity of CSE production and utility functions, it is therefore desirable to come up with additional – hopefully, more convincing – arguments in favor of these functions. This is what we intend to do in this paper.

2 Main Idea Behind a New Explanation

Main idea. In our explanation, we will use the fact that in most practical situations, we combine several factors. We can combine these factors in different order:

- For example, we can first combine the effects of capital and labor into a single characteristic that describes the joint even of both factors, and then combine it with other factors.
- Alternatively, we can first combine capital with other factors, and only then combine the resulting combined factor with labor, etc.

The result should not depend on the order in which we perform these combinations.

What we do in this paper. In this paper, we show that this idea implies the CES functions. Thus, we indeed get a new explanation for the ubiquity of CES production and utility functions.

3 Derivation of the CES Functions from the Above Idea

Towards formalizing our idea. Let us denote a function that combines factors i and j into a single quantity x_{ij} by $f_{i,j}(x_i,x_j)$. Similarly, let us denote a function that combines the values x_{ij} and $x_{k\ell}$ into a single quantity $x_{ijk\ell}$ by $f_{ij,k\ell}(x_{ij},x_{k\ell})$. In these terms, the requirement that the resulting values do not depend on the order implies, e.g., that we always have

$$f_{12,34}(f_{1,2}(x_1,x_2),f_{3,4}(x_3,x_4)) = f_{13,24}(f_{1,3}(x_1,x_3),f_{2,4}(x_2,x_4)). \tag{3}$$

Additional requirement. In both production and utility situations, for each i and j, the combination function $f_{i,j}(x_i,x_j)$ is an increasing function of both variables x_i and x_j. It is reasonable to require that it is continuous, and then when one of the factors tends to infinity, the result also tends to infinity. Under these reasonable assumptions, the combination functions tends out to be *invertible* in the following sense:

Definition 1. *A function* $f : A \times B \to C$ *is called* invertible *if the following two conditions are satisfied:*

- *for every* $a \in A$ *and for every* $c \in C$*, there exists a unique value* $b \in B$ *for which* $c = f(a,b)$*;*
- *for every* $b \in B$ *and for every* $c \in C$*, there exists a unique value* $a \in A$ *for which* $f(a,b) = c$*.*

Comment. In mathematics, functions invertible in the sense of Definition 1 are called *generalized quasigroups*; see, e.g., [4].

Let us now formalize the above requirement.

Definition 2. *Let* X_i*,* X_{ij}*, and* X *be sets, where* $i = 1,2,3,4$*. We say that invertible operations* $f_{i,j} : X_i \times X_j \to X_{ij}$ *and* $f_{ij,k\ell} : X_{ij} \times X_{k\ell} \to X$ *(for different i, j, k, and* ℓ*)* satisfy the generalized associativity requirement *if for all* $x_i \in X_i$*, we have*

$$f_{12,34}(f_{1,2}(x_1,x_2), f_{3,4}(x_3,x_4)) = f_{13,24}(f_{1,3}(x_1,x_3), f_{2,4}(x_2,x_4)). \qquad (3)$$

Comment. In mathematical terms, this requirement is known as *generalized mediality* [4].

Groups and Abelian groups: reminder. To describe operations that satisfy the generalized associativity requirement, we need to recall that a set G with an associative operation $g(a,b)$ and a unit element e (for which $g(a,e) = g(e,a) = a$) is called a *group* if every element is invertible, i.e., if for every a, there exists an a' for which $g(a,a') = e$. A group in which the operation $g(a,b)$ is commutative is known as *Abelian*.

Proposition. [2, 3, 4, 25, 26, 27] *For every set of invertible operations that satisfy the generalized associativity requirement, there exists an Abelian group G and 1-1 mappings* $r_i : X_i \to G$*,* $r_{ij} : X_{ij} \to G$ *and* $r_X : X \to G$ *for which, for all* $x_i \in X_i$ *and* $x_{ij} \in X_{ij}$*, we have*

$$f_{ij}(x_i,x_j) = r_{ij}^{-1}(g(r_i(x_i), r_j(x_j))) \text{ and}$$

$$f_{ij,kl}(x_{ij},x_{k\ell}) = r_X^{-1}(g(r_{ij}(x_{ij}), r_{k\ell}(x_{k\ell}))).$$

Discussion. All continuous 1-D Abelian groups with order-preserving operations are isomorphic to the additive group of real numbers, with $g(a,b) = a+b$. Thus, we can conclude that all combining operations have the form

$$f_{ij}(x_i,x_j) = r_{ij}^{-1}(r_i(x_i) + r_j(x_j)), \qquad (4)$$

i.e., equivalently, $f_{ij}(x_i,x_j) = y$ means that

$$r_{ij}(y) = r_i(x_i) + r_j(x_j). \qquad (5)$$

Let us use homogeneity: result. We will now prove that homogeneity leads exactly to the desired CES combinations. This will give us the desired new explanation of the ubiquity of the CES operations.

Homogeneity leads to CES operations: proof. Homogeneity means that if the relation (5) holds for some values x_i, x_j, and y, then, for every λ, a similar relation holds for re-scaled values $\lambda \cdot x_i$, $\lambda \cdot x_j$, and $\lambda \cdot y$, i.e.:

$$r_{ij}(\lambda \cdot y) = r_i(\lambda \cdot x_i) + r_j(\lambda \cdot x_j).$$

To utilize this requirement, let us use the idea of substitution: for each possible value $x'_i = x_i + \Delta x_i$, let us find the corresponding value $x'_j = x_j + \Delta x_j$ for which the right-hand side of the formula (5) remains the same – and thus, the combined value y remains the same:

$$r_i(x'_1) + r_j(x'_j) = r_i(x_i + \Delta x_i) + r_j(x_j + \Delta x_j) = r_i(x_i) + r_j(x_j). \quad (6)$$

In general, the substitute value x'_j is a function of x'_i: $x'_j = x'_j(x'_i)$. When $\Delta x_i = 0$, i.e., when $x'_i = x_i$, we clearly have $x'_j = x_j$, so $\Delta x_j = 0$. For small Δx_i, we get $y'_j = y_j + k \cdot \Delta x_i + o(\Delta x_i)$, where $k \stackrel{\text{def}}{=} \dfrac{dx'_j}{dx'_i}$, so $\Delta x_j = k \cdot \Delta x_i + o(\Delta x_i)$ for some real number k.

Here, $r_i(x_i + \Delta x_i) = r_i(x_i) + r'_i(x_i) \cdot \Delta x_i + o(\Delta x_i)$, where, as usual, f' denotes the derivative. Similarly,

$$r_j(x_j + \Delta x_j) = r_j(x_j + k \cdot \Delta x_i + o(\Delta x_i)) = r_j(x_j) + k \cdot r'_j(x_j) \cdot \Delta x_i + o(\Delta x_i).$$

Thus, the condition (6) takes the form

$$r_i(x_i) + r_j(x_j) + (r'_i(x_i) + k \cdot r'_j(x_j)) \cdot \Delta x_i + o(\Delta x_i) = r_i(x_i) + r_j(x_j).$$

Subtracting the right-hand side from the both sides, dividing both sides of the resulting equation by Δx_i, and tending Δx_i to 0, we conclude that

$$r'_i(x_i) + k \cdot r'_j(x_j) = 0,$$

i.e., that

$$k = -\frac{r'_i(x_i)}{r'_j(x_j)}. \quad (7)$$

Homogeneity means, in particular, that if now apply the combination function r_{ij} to the values

$$\lambda \cdot x'_i = \lambda \cdot x_i + \lambda \cdot \Delta x_i$$

and

$$\lambda \cdot x'_j = \lambda \cdot x_j + \lambda \cdot k \cdot \Delta x_i + o(\Delta x_i),$$

then we should get the value $\lambda \cdot y$. So:

$$r_i(\lambda \cdot x_i + \lambda \cdot \Delta x_i) + r_j(\lambda \cdot x_j + \lambda \cdot k \cdot \Delta x_i + o(\Delta x_i)) = r_i(\lambda \cdot x_i) + r_j(\lambda \cdot x_j). \quad (8)$$

For small Δx_i, we have

$$r_i(\lambda \cdot x_i + \lambda \cdot \Delta x_i) = r(\lambda \cdot x_i) + \lambda \cdot \Delta x_i \cdot r_i'(\lambda \cdot x_i) + o(\Delta x_i),$$

where f' denote a derivative, and similarly,

$$r_j(\lambda \cdot x_j + \lambda \cdot k \cdot \Delta x_i + o(\Delta x_1)) = r(\lambda \cdot x_2) + \lambda \cdot k \cdot \Delta x_i \cdot r_j'(\lambda \cdot x_j) + o(\Delta x_i).$$

Substituting these expressions into the formula (8), we conclude that

$$r_i(\lambda \cdot x_i) + \lambda \cdot \Delta x_i \cdot r'(\lambda \cdot x_i) + r_j(\lambda \cdot x_j) + \lambda \cdot k \cdot r_j'(\lambda \cdot x_j) \cdot \Delta x_i + o(\Delta x_i) =$$

$$r_i(\lambda \cdot x_i) + r_j(\lambda \cdot x_j).$$

Subtracting the right-hand side from the left-hand side, dividing the result by Δx_i and tending Δx_i to 0, we conclude that

$$r'(\lambda \cdot x_i) + k \cdot r_j'(\lambda \cdot x_j) = 0,$$

i.e., in view of the formula (7), that

$$r'(\lambda \cdot x_i) - \frac{r_i'(x_i)}{r_j'(x_j)} \cdot r_j'(\lambda \cdot x_j) = 0.$$

Moving the second term to the right-hand side and dividing both sides by $r_i'(x_i)$, we conclude that

$$\frac{r_i'(\lambda \cdot x_i)}{r_i'(x_i)} = \frac{r_j'(\lambda \cdot x_j)}{r_j'(x_j)}.$$

The right-hand side of this formula does not depend on x_i at all, thus, the left-hand side also does not depend on x_i, it only depends on λ:

$$\frac{r_i'(\lambda \cdot x_i)}{r_i'(x_i)} = c(\lambda)$$

for some function $c(\lambda)$. Thus, the derivative $R_i(x_i) \stackrel{\text{def}}{=} r_i'(x_i)$ satisfies the functional equation

$$R_i(\lambda \cdot x_i) = R_i(x_i) \cdot c(\lambda)$$

for all λ and x_i.

It is know that every continuous solution to this equation has the form $r_i'(x_i) = R_i(x_i) = A_i \cdot x_i^{\alpha_i}$ for some A_i and α_i; see, e.g., [4]. For differentiable functions, this can be easily proven if we differentiate both sides of this equation by c and take $c = 1$. Then, we get $x_i \cdot \dfrac{dR_i}{dc_i} = c \cdot R_i$. Separating variables, we get

$$\frac{dR_i}{R_i} = c \cdot \frac{dx_i}{x_i}.$$

Integration leads to $\ln(R_i) = c \cdot \ln(x_i) + C_1$ and thus, to the desired formula.

Integrating the above expression for $r_i'(x_i)$, we get $r_i(x_i) = a_i \cdot x_i^{\beta_i} + C_i$ and similarly, $r_j(x_j) = a_j \cdot x_j^{\beta_j} + C_j$. One can easily check that homogeneity implies that $\beta_i = \beta_j$ and $C_i + C_j = 0$, so the sum $r_i(x_i) + r_j(x_j)$ takes the form $a_i \cdot x_i^r + a_j \cdot x_j^r$.

By considering a similar substitution between x_i and y (in which x_j remains intact), we conclude that $r_{ij}(y) = \text{const} \cdot y^r$, so we indeed get the desired formula $r_{ij}(x_i, x_j) = (a_i \cdot x_i^r + a_j \cdot x_j^r)^{1/r}$. By using similar formulas to combine x_{ij} with x_k, etc., we get the desired CES combination function.

4 Possible Application to Copulas

What is a copula: a brief reminder. Specifically, a 1-D probability distribution of a random variable X can be described by its *cumulative distribution function* (cdf) $F_X(x) \stackrel{\text{def}}{=} \text{Prob}(X \le x)$. A 2-D distribution of a random vector (X,Y) can be similarly described by its 2-D cdf $F_{XY}(x,y) = \text{Prob}(X \le x \,\&\, Y \le y)$.

It turns out that we can always describe $F(x,y)$ as

$$F_{XY}(x,y) = C_{XY}(F_X(x), F_Y(y))$$

for an appropriate function $C_{XY} : [0,1 \times [0,1] \to [0,1]$ known as a *copula*; see, e.g., [20, 22].

For a joint distribution of several random variables X, Y, ..., Z, we can similarly write

$$F_{XY\ldots Z}(x,y,\ldots,z) \stackrel{\text{def}}{=} \text{Prob}(X \le x \,\&\, Y \le y \,\&\, \ldots \,\&\, Z \le z) =$$
$$C_{XY\ldots Z}(F_X(x), F_Y(y), \ldots, F_Z(z))$$

for an appropriate multi-D copula $C_{XY\ldots Z}$.

Vine copulas. When we have many ($n \gg 1$) random variables, then to exactly describe their joint distribution, we need to describe a general function of n variables. Even if we use two values for each variable, we get 2^n combinations, which for large n can be astronomically large. Thus, a reasonable idea is to approximate the multi-D distribution.

A reasonable way to approximate is to use 2-D copulas. For example, to describe a joint distribution of three variables X, Y, and Z, we first describe the joint distribution of X and Y as $F_{XY}(x,y) = C_{XY}(F_X(x), F_Y(y))$, and then use an appropriate copula $C_{XY,Z}$ to combine it with $F_Z(z)$:

$$F_{XYZ}(x,y,z) \approx C_{XY,Z}(F_{XY}(x,y), F_Z(z)) = C_{XY,Z}(C_{XY}(F_X(x), F_Y(y), F_Z(z)).$$

Such an approximation, when copulas are applied to one another like a vine, are known as *vine copulas*; see, e.g., [1, 7, 8, 10, 13, 14, 15, 18, 19, 21, 24].

Natural analogue of associativity. It is reasonable to require that the result of the vine copula approximation should not depend on the order in which we combine the variables. In particular, for four random variables X, Y, Z, and T, we should get the same result in the following two situations:

- if we first combine X with Y, Z and T, and then combine the two results; or
- if we first combine X with Z, Y with T, and then combine the two results.

Thus, we require that for all possible real numbers x, y, z, and t, we get

$$C_{XY,ZT}(C_{XY}(F_X(x),F_Y(y)),C_{ZT}(F_Z(z),F_T(t))) =$$

$$C_{XZ,YT}(C_{XZ}(F_X(x),F_Z(z)),C_{YT}(F_Y(y),F_T(t))).$$

If we denote $a = F_X(x)$, $b = F_Y(y)$, $c = F_Z(z)$, and $d = F_T(t)$, we conclude that for every a, b, c, and d, we have

$$C_{XY,ZT}(C_{XY}(a,b),C_{ZT}(c,d)) = C_{XZ,YT}(C_{XZ}(a,c),C_{YT}(b,d)).$$

This is exactly the generalized associativity requirement. Thus, if we extend copulas to invertible operations, then we can conclude that copulas can be *re-scaled to associative operations* – in the sense of the above Proposition.

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