How to Quantify Deterministic and Random Influences on the Statistics of the Foreign Exchange Market

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It is shown that price changes of the U.S. dollar–German mark exchange rates upon different delay times can be regarded as a stochastic Marcovian process. Furthermore, we show how Kramers-Moyal coefficients can be estimated from the empirical data. Finally, we present an explicit Fokker-Planck equation which models very precisely the empirical probability distributions, in particular, their non-Gaussian heavy tails.

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Since high-frequency intraday data are available and easy to access, research on the dynamics of financial markets is enjoying a broad interest [1–7]. Well-founded quantitative investigations now seem to be feasible. One central issue is the understanding of the statistics of price changes which determine losses and gains. The changes of a time series of quotations $x(t)$ are commonly measured by returns $r := x(t + \Delta t)/x(t)$, logarithmic returns, or increments $\Delta x := x(t + \Delta t) - x(t)$. The identification of the underlying process leading to heavy tailed probability density function (pdf) of price changes for small Δt and the volatility clustering (see Fig. 1) is a prominent puzzle. This shape of the pdf expresses an unexpected high probability (compared to a Gaussian pdf) of large price changes which is of utmost importance for risk analysis. In a recent work [8], an analogy between the short time dynamics of the foreign exchange (FX) market and hydrodynamic turbulence has been proposed. This analogy postulates the existence of hierarchical features like a cascade process from large to small time scales. It is similar to the energy cascade in turbulence; see [9].

At least since the pioneering work of Bachelier stochastic problems of financial data are commonly treated as processes running in time t (cf. [10]). Inspired by the idea of an existing cascade process we present here a new approach, namely, we investigate how price changes on different time steps Δt are correlated. Moreover, the aim of the present paper is to present a method of how to derive the underlying mathematical model for a cascade directly from the given financial data. This method yields an estimation of an effective stochastic equation in the form of a Fokker-Planck equation (also known as a Kolmogorov equation) in the variable Δt. The solutions of this equation yield the probability distributions with high accuracy (see Fig. 1), including the heavy tailed statistics. Thus our method is not anymore based on the conventional phenomenological comparison between models and several stochastic aspects of financial data [11]. The Fokker-Planck equation provides the knowledge as to how the statistics of price changes on different delay times are correlated. This includes an autocorrelation analysis in time t. Furthermore, the analogy between financial data and turbulence can be quantified (for results on turbulent data, see [13]).

In the following we present results of the analysis of a data set, $x(t)$, which consists of 1 472 241 quotes for U.S. dollar–German mark exchange rates from the years 1992 and 1993 as used in Ref. [8]. We focus on the analysis of price changes measured as increments Δx

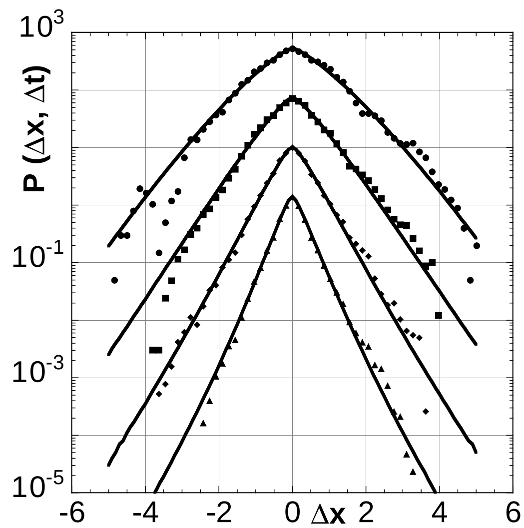

FIG. 1. Probability densities (pdf) $p(\Delta x, \Delta t)$ of the price changes $\Delta x = x(t + \Delta t) - x(t)$ for the time delays $\Delta t = 5120$, $10\,240, 20\,480, 40\,960$ s (from bottom to top). Symbols: the results obtained from the analysis of middle prices of bit-ask quotes for the U.S. dollar–German mark exchange rates from 1 October 1992 until 30 September 1993. Full lines: results form a numerical iteration of the effective Fokker-Planck equation (4),(6); the probability distribution for $\Delta t = 40\,960$ s was taken as the initial condition. The units of Δx are multiples of the standard deviation σ of Δx at $\Delta t = 40\,960$ s. The pdfs are shifted in vertical directions for convenience of presentation; thus the y axis is given in arbitrary units.

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which are given here in units of the standard deviation of Δx at $\Delta t = 40\,960$ s. In order to characterize the statistics of these price changes, price increments Δx_1, Δx_2 for delay times Δt_1, Δt_2 at the same time t are considered. The corresponding joint probability density functions

$$p(\Delta x_2, \Delta t_2; \Delta x_1, \Delta t_1) \tag{1}$$

are evaluated for various time delays $\Delta t_2 < \Delta t_1$ directly from the given data set. One example of a contour plot of these functions is exhibited in Fig. 2. If the two price changes Δx_1, Δx_2 were statistically independent, the joint pdf would factorize into a product of two probability density functions: $p(\Delta x_2, \Delta t_2; \Delta x_1, \Delta t_1) = p(\Delta x_2, \Delta t_2) p(\Delta x_1, \Delta t_1)$. The tilted form of the joint probability density clearly shows that such a factorization does not hold and that the two price changes are statistically dependent. This dependence is in accordance with observations for cross-correlation functions of the same data [5]. The tilted form of the joint pdf also confirms what is expected intuitively. If market values went up over a certain period Δt_1, then it is more likely that, on a shorter period Δt_2 within the larger one, the market went up instead of down.

To analyze these correlations in more detail, we address the question: What kind of statistical process underlies the price changes over a series of nested time delays Δt_i of decreasing duration? In general, a complete characterization of the statistical properties of the data set requires the evaluation of joint pdfs $p^N(\Delta x_1, \Delta t_1; \ldots; \Delta x_N, \Delta t_N)$ depending on N variables (for arbitrarily large N). In the case of a Markov process (a process without memory), an important simplification arises: The N-point pdf p^N is generated by a product of the conditional probabilities $p(\Delta x_{i+1}, \Delta t_{i+1} \,|\, \Delta x_i, \Delta t_i) = p(\Delta x_{i+1}, \Delta t_{i+1}; \Delta x_i, \Delta t_i)/p(\Delta x_i, \Delta t_i)$ [14], for $i = 1, \ldots, N-1$. As a necessary condition, the Chapman-Kolmogorov equation [15]

$$p(\Delta x_2, \Delta t_2 \,|\, \Delta x_1, \Delta t_1) = \int d(\Delta x_i) p(\Delta x_2, \Delta t_2 \,|\, \Delta x_i, \Delta t_i) p(\Delta x_i, \Delta t_i \,|\, \Delta x_1, \Delta t_1) \tag{2}$$

should hold for any value of Δt_i, with $\Delta t_2 < \Delta t_i < \Delta t_1$.

We checked the validity of the Chapman-Kolmogorov equation for different Δt_i triplets by comparing the directly evaluated conditional probability distributions $p(\Delta x_2, \Delta t_2 \,|\, \Delta x_1, \Delta t_1)$ with the ones calculated (p_{cal}) according to (2). In Fig. 3, the contour lines of the two corresponding pdfs for all values of Δx_1 are superimposed for the purpose of illustration. Only in the outer regions there are visible deviations, probably resulting from a finite resolution of the statistics. Cuts for some exemplarily chosen values of Δx_1 are shown in addition in Fig. 3.

As is well known, the Chapman-Kolmogorov equation yields an evolution equation for the change of the distribution functions $p(\Delta x, \Delta t \,|\, \Delta x_1, \Delta t_1)$ and $p(\Delta x, \Delta t)$ across the scales Δt [15].

For the following it is convenient (and without loss of generality) to consider a logarithmic time scale

$$\tau = \ln(40\,960 \text{ s}/\Delta t)\,. \tag{3}$$

Then, the limiting case $\Delta t \to 0$ corresponds to $\tau \to \infty$. The Chapman-Kolmogorov equation formulated in differential form yields a master equation, which can take the form of a Fokker-Planck equation (for a detailed discussion, we refer the reader to [15]):

$$\frac{d}{d\tau} p(\Delta x, \tau) = \left[-\frac{\partial}{\partial \Delta x} D^{(1)}(\Delta x, \tau) + \frac{\partial^2}{\partial \Delta x^2} D^{(2)}(\Delta x, \tau) \right] p(\Delta x, \tau)\,. \tag{4}$$

The drift and diffusion coefficients $D^{(1)}(\Delta x, \tau)$ and $D^{(2)}(\Delta x, \tau)$, respectively, can be estimated directly from the data as moments $M^{(k)}$ of the conditional probability distributions (cf. Fig. 3):

$$\begin{aligned} D^{(k)}(\Delta x, \tau) &= \frac{1}{k!} \lim_{\Delta\tau \to 0} M^{(k)}, \\ M^{(k)} &= \frac{1}{\Delta\tau} \int d\Delta x' (\Delta x' - \Delta x)^k p(\Delta x', \tau + \Delta\tau \,|\, \Delta x, \tau)\,. \end{aligned} \tag{5}$$

To determine the functional Δx dependence of $D^{(k)}$ we evaluated the moments $M^{(k)}$ for different small Δt; see Fig. 4. The coefficient $M^{(1)}$ shows a linear dependence on Δx, while $M^{(2)}$ can be approximated by a polynomial of degree two in Δx. This behavior was found for all scales τ and $\Delta\tau$. Therefore the drift term $D^{(1)}$ is well approximated by a linear function of Δx, whereas the diffusion term $D^{(2)}$ follows a function quadratic in Δx. For large values of Δx our statistics becomes poorer and thus uncertainty increases. In fact, from a careful analysis of the data, which is based on the functional dependences of $M^{(1)}$ and $M^{(2)}$ as shown in Fig. 4, we obtained the following approximation:

$$\begin{aligned} D^{(1)} &= -0.44 \Delta x\,, \\ D^{(2)} &= 0.003 \exp(-\tau/2) + 0.019(\Delta x + 0.04)^2. \end{aligned} \tag{6}$$

To perform a quantitative test of our result, we used these coefficients for a numerical solution of the

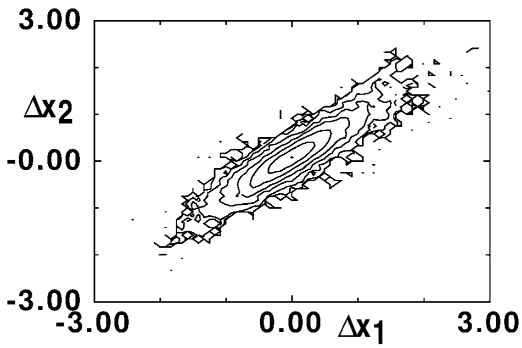

FIG. 2. Contour plot of the joint pdf $p(\Delta x_2, \Delta t_2; \Delta x_1, \Delta t_1)$ for the simultaneous occurrence of price differences $\Delta x_1(\Delta t_1)$ and $\Delta x_2(\Delta t_2)$; $\Delta t_1 = 6168$ s and $\Delta t_2 = 5120$ s. The contour lines correspond to $\log p = -1, -1.5, \dots, -4.5$. If the two price changes were statistically independent, the joint pdf would factorize into a product of two pdfs: $p(\Delta x_2, \Delta t_2; \Delta x_1, \Delta t_1) = p(\Delta x_2, \Delta t_2) p(\Delta x_1, \Delta t_1)$. The tilted form of the joint pdf provides evidence that such a factorization does not appear for small values of $|\log(\Delta t_1/\Delta t_2)|$.

Fokker-Planck equation. As an initial condition the data of the pdf for large time delays (upper curve in Fig. 1) were fitted by an empirical function represented by the full line. Figure 1 shows that the numerical solutions nicely fit the experimentally determined pdfs on smaller time differences. In contrast to the use of phenomenological supposed fitting functions (cf. [2,8,10]), this method provides the evolution process of pdfs from large time delays to smaller ones. This evolution shows how the pdfs deviate more and more from a Gaussian shape as τ increases or, respectively, Δt decreases. This definitely is a new quality in describing the hierarchical structure of such data sets. Now it becomes clear that one must not require stationary probability distributions for price differences for different Δt; on the contrary the coupling between different scales Δt via a Markov process is essential. Thus also a proper modeling of the time evolution of price differences for a fixed time delay must take into account the coupling of these quantities to price differences on different time delays.

It may be worthwhile to remark that the observed quadratic dependence of the diffusion term D^2 corresponds to the found logarithmic scaling of the intermittency parameter in [8], which was taken as an essential point to propose the analogy between turbulence and the financial market.

We remind the reader that the Fokker-Planck equation is equivalent to a Langevin equation of the form (we use the Ito interpretation [15])

$$\frac{d}{d\tau} \Delta x(\tau) = D^{(1)}(\Delta x(\tau), \tau) + \sqrt{D^{(2)}(\Delta x(\tau), \tau)}\, F(\tau), \quad (7)$$

where $F(\tau)$ is a fluctuating δ-correlated force with Gaussian statistics; here $\langle F(\tau) F(\tau') \rangle = 2\delta(\tau - \tau')$. In our approximation (6) for large τ (small Δt) the stochastic process is very close to a linear stochastic process with

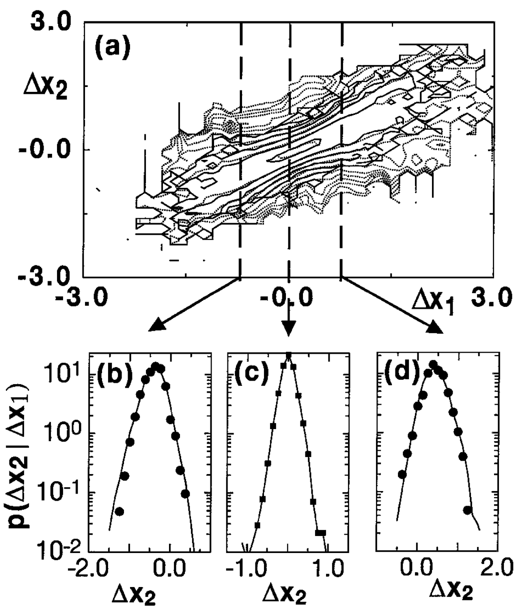

FIG. 3. (a) Contour plot of the conditional pdf $p(\Delta x_2, \Delta t_2 | \Delta x_1, \Delta t_1)$ for $\Delta t_2 = \Delta t_1/(1.2)^2$ and $t_1 = 5120$. The separation of the contour lines is the same as in Fig. 2. In order to verify the Chapman-Kolmogorov equation (2), the directly evaluated pdf (solid lines) is compared with the integrated pdf (dotted lines). Assuming a statistical error of the square root of the number of events of each bin we find that both pdfs are statistically identical. (b), (c), and (d) corresponding cuts for $\Delta x_1 = -0.6, 0.0, 0.6$. The symbols represent the directly evaluated pdfs. The solid lines are results of the integration of the Chapman-Kolmogorov equation.

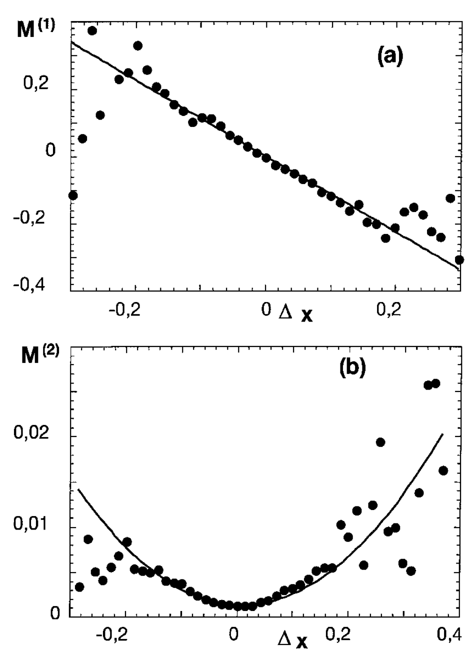

FIG. 4. Kramers-Moyal coefficients (a) $M^{(1)}$ and (b) $M^{(2)}$ estimated from the conditional pdf shown in Fig. 3. The solid curves present a linear and a quadratic fit, respectively.

multiplicative noise, known as quadratic noise Ornstein-Uhlenbeck process [16]:

$$\frac{d}{d\tau}\Delta x(\tau) = -0.44\Delta x(\tau) + \sqrt{.019}\,\Delta x(\tau)F(\tau). \quad (8)$$

(Note constant and linear perturbation terms in $D^{(2)}$ are neglected, because they affect only very small values of Δx; for the range of Δx, see Fig. 1.) For small τ (large Δt) the influence of the additive term in $D^{(2)}$, i.e., additive noise, becomes more important, which explains that for these τ values the form of the pdfs is closer to a Gaussian shape. For large τ (small Δt) the multiplicative noise dominates and the pdfs become more and more heavy tailed. In the limit $\tau \rightarrow \infty$, the pdf is given by the stationary solution of the Fokker-Planck equation, which exhibits extremely non-Gaussian behavior [16]. However, using the numerical values given in equation (6), it is easily verified that the pdfs in Fig. 1 are still far away from the stationary state.

The stochastic equation (8) yields realizations of price changes $\Delta x(\tau)$, whose ensemble averages can be described by the probability distributions $p(\Delta x, \tau)$. Thus, the Langevin equation (7) produces the possibility to simulate the price cascades for time delays from about one day down to several minutes. Furthermore, with this last presentation of our results it becomes clear that we are able to separate the deterministic and the noisy influence on the hierarchical structure of the financal data in terms of the coefficients $D^{(1)}$ and $D^{(2)}$, respectively. Furthermore, we see that the problems of heavy tailed probability densities are due to the dependence of $D^{(2)}$ on Δx, which means nothing else than the presence of multiplicative noise.

Summarizing, it is the concept of a cascade in time hierarchy that allowed us to derive the results of the present paper, which in turn quantitatively supports the initial concept of an analogy between turbulence and financial data. Furthermore, we have shown that the smooth evolution of the pdfs down along the cascade towards smaller time delays is caused by a Markov process with multiplicative noise. We have presented the explicit form of a Fokker-Planck equation with which the observed statistics can be recalculated accurately. At last we want to mention that with this result also the probability of simultaneous price changes for many different time delay times can be determined, i.e., a more or less complete statistical characterization of price differences is given. The main intention of this work was not only to give the correct stochastic equation for the U.S. dollar–German mark exchange rates of one year but to present a new promising method to analyze the statistics of financial data.

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