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How Violent Are Fast Controls?

by

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ABSTRACT:

Consider a controllable system: $\dot{x} = \mathbf{A}x + \mathbf{B}u$, with $x(0) = 0$. Given any time $T > 0$ there is then a control operator $\mathbf{C}_T : \xi \mapsto u(\cdot)$ giving the (unique) minimum norm control such that $x(T) = \xi$. We show that $\|\mathbf{C}_T\| \sim \gamma T^{-(K+1/2)}$ where γ is computable from $\mathbf{A}, \mathbf{B}$ and K is the minimal exponent giving the rank condition.

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1. Introduction

Consider a linear control system

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \quad , \quad x(0) = 0 \tag{1.1}$$

with $\mathbf{A}, \mathbf{B}$ constant matrices ($n \times n$ and $n \times m$, respectively, so $x(\cdot)$ is $I\!R^n$-valued and the control $u(\cdot)$ is $I\!R^m$-valued). Assuming this is controllable, we know that for each terminal time $T > 0$ and each target $\xi \in I\!R^n$ there exist controls $u(\cdot)$ giving $x(T) = \xi$ and, indeed, that there is unique such control

$$u_{opt} = u_{opt}(\cdot; T, \xi) \in L^2([0,T] \to I\!R^m) =: \mathcal{U} = \mathcal{U}_T \tag{1.2}$$

minimizing $\|u\|$. Of course the norm to be minimized is that of $\mathcal{U} = L^2([0,T] \to I\!R^m)$ so $\|u_{opt}\|^2$ gives the least control energy needed to reach the target ξ at time T.

It is to be expected that more violent control would be needed as the time T available becomes shorter[3]. Our object in this paper is to give a precise (asymptotic) answer to the question of the title. Since the optimal control u_{opt} is given by a linear operator

$$\mathbf{C}_T : \xi \mapsto u_{opt}(\cdot; T, \xi) : I\!R^n \longrightarrow \mathcal{U} = \mathcal{U}_T \ , \tag{1.3}$$

the principal result can be stated as

$$\|\mathbf{C}_T\| \sim \gamma T^{-(K+1/2)} \qquad \text{as } \ T \to 0 \tag{1.4}$$

where K is the minimal exponent giving the well known *rank condition* for controllability:

$$rank \ [\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^K\mathbf{B}] = n \tag{1.5}$$

and $\gamma \neq 0$ is also computable from $\mathbf{A}, \mathbf{B}$.

It is worth noting that this also estimates sensitivity for observation of the adjoint problem. If one can observe $y := \mathbf{B}^* z$ for z satisfying $\dot{z} = -\mathbf{A}^* z$, then one easily sees that one recovers the state through $z(T) = \mathbf{C}_T^* y(\cdot)$. This means that the uncertainty in the recovered state due to noise or measurement error $e(\cdot)$ in the observation is estimated by $\|\mathbf{C}_T^* e(\cdot)\| \leq \|\mathbf{C}_T\| \ \|e(\cdot)\|$. The same result (1.4) shows how sensitivity to error increases as the observation time shrinks. Plausibly, we might anticipate that the expected noise energy is proportional to times i.e.,

$$Exp[\int_0^T |e|^2 dt] \sim \sigma^2 T, \ \text{ so } \ Exp[\|e(\cdot)\|] \sim \sigma T^{1/2}.$$

[3] The uniqueness of u_{opt} and the linearity of the map: $\xi \mapsto u_{opt}$ follow from the Hilbert space projection theorem under more general conditions than here. From uniqueness it follows that $\|u_{opt}\|$ is strictly decreasing in T for each $\xi \neq 0$.

The sensitivity estimate then becomes

$$\|\text{expected uncertainty in } z(T)\| \sim \gamma\sigma T^{-K} \quad \text{as } T \to 0. \tag{1.6}$$

The formula (2.3) is classical but it is interesting to observe historically that the question of the title seems to have been considered first for distributed parameter systems[4] although it was posed for the present finite dimensional case at least as far back as 1975 [3].

2. Formulation

Treatment of (1.1) is expressible in terms of the matrix exponential, given by the convergent series

$$e^{s\mathbf{A}} := \sum_0^\infty (s^k/k!)\mathbf{A}^k. \tag{2.1}$$

The solution x of (1) is then given by

$$x(t) = \int_0^t e^{(t-s)\mathbf{A}}\mathbf{B}u(s)ds$$

so, in particular, one has $x(T) = \mathbf{V}u(\cdot)$ where

$$\mathbf{V} = \mathbf{V}_T : \mathcal{U} \to \mathbb{R}^n : u(\cdot) \mapsto \int_0^T e^{(T-s)\mathbf{A}}\mathbf{B}\, u(s)ds. \tag{2.2}$$

A standard argument shows $\|u(\cdot)\|_{\mathcal{U}}$ is minimized, subject to the condition $\mathbf{V}u = \xi$, by taking $u \in \mathcal{R}(V^*)$ whence

$$u_{opt}(\cdot; T, \xi) = \mathbf{V}^*\omega \ \text{ with, necessarily, } \ \mathbf{V}\mathbf{V}^*\omega = \xi;$$

so one has $\mathbf{C}_T : \xi \mapsto u_{opt}$ given by

$$\mathbf{C}_T = \mathbf{V}_T^*(\mathbf{V}_T\mathbf{V}_T^*)^{-1} \tag{2.3}$$

where controllability gives, as we see, invertibility of $\mathbf{V}_T\mathbf{V}_T^* =: \mathbf{W}_T = \mathbf{W}$. We easily see that $\mathbf{V}^* : \mathbb{R}^n \to \mathcal{U}$ is given by

$$[\mathbf{V}^*\omega](t) = [e^{(T-t)\mathbf{A}}\mathbf{B}]^*\omega \qquad \text{for } t \in [0,T] \tag{2.4}$$

so the $n \times n$ matrix $\mathbf{W} := \mathbf{V}\,\mathbf{V}^*$ is given by

$$\mathbf{W} = \mathbf{W}_T = \int_0^T e^{s\mathbf{A}}\mathbf{B}[e^{s\mathbf{A}}B]^* ds. \tag{2.5}$$

[4] One has $\log \|\mathbf{C}_T\| = \mathcal{O}(1/T)$ (sharply) for the known infinite dimensional cases[4], [1], [2].

Clearly, $\mathbf{W}$ is self-adjoint and (at least) semidefinite from its form. We have the identity

$$\begin{aligned}(2.6)\qquad \|u_{opt}\|^2 &= \|\mathbf{C}_T\xi\|^2 = \langle \mathbf{V}^*\mathbf{W}^{-1}\xi,\ \mathbf{V}^*\mathbf{W}^{-1}\xi\rangle \\ &= (\mathbf{W}^{-1}\xi)\cdot(\mathbf{V}\,\mathbf{V}^*\mathbf{W}^{-1}\xi) = (\mathbf{W}_T^{-1}\xi)\cdot\xi\end{aligned}$$

which makes it clear that our object must be to compute $\mathbf{W}_T^{-1}$ asymptotically.

The key to our approach is the invertibility of

$$(2.7)\qquad \mathbf{Q} := \lim_{T\to 0} T^{-(2K+1)}\mathbf{\Gamma}_T\mathbf{W}_T\mathbf{\Gamma}_T,$$

using a suitable family of operators $\mathbf{\Gamma} = \mathbf{\Gamma}_T$ such that

$$(2.8)\qquad \mathbf{\Gamma}_T \text{ invertible for } T \neq 0\,, \qquad \mathbf{\Gamma}_T = \mathbf{\Gamma}_0 + \mathcal{O}(T);$$

see (2.12), below.

Given the matrices $\mathbf{A}, \mathbf{B}$ we consider the nested sequence $(\mathcal{S}_0, \mathcal{S}_1, \ldots)$ of subspaces of $\mathbb{R}^n$ given recursively by

$$\begin{aligned}(2.9)\qquad \mathcal{S}_k &= \mathcal{S}_{k-1} + \mathcal{R}(\mathbf{A}^k\mathbf{B}) \qquad \text{with } \mathcal{S}_{-1} := \{0\}, \\ \mathcal{S}_0 &= \mathcal{R}(\mathbf{B}),\ \ \mathcal{S}_1 = \mathcal{R}(\mathbf{B}) + \mathcal{R}(\mathbf{A}\,\mathbf{B}), \ldots\end{aligned}$$

so each $\mathcal{S}_k$ is the column space (range) of the composite matrix $[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^k\mathbf{B}]$. The assumption of controllability means that $\mathcal{S}_K = \mathbb{R}^n$ for large enough K (i.e., (1.5)) and we fix K as the *minimal* exponent/index giving this.

For each k $\;(0 \le k \le K)$ we can find the orthogonal complement of $\mathcal{S}_{k-1}$ in $\mathcal{S}_k$ and let $\mathbf{E}_k$ be the orthogonal projection on this subspace. This gives the important fact that

$$(2.10)\qquad \mathbf{E}_k\mathbf{A}^j\mathbf{B} = \mathbf{0} \ \text{ for } \ j < k \le K$$

since $j < k$ gives $\mathcal{R}(\mathbf{A}^j\mathbf{B}) \subset \mathcal{S}_{k-1} \subset \mathcal{N}(\mathbf{E}_k)$. We observe, although we do not need the fact, that

$$m \ge \dim\mathcal{R}(\mathbf{B}) = \dim\mathcal{R}(\mathbf{E}_0) \ge \dim\mathcal{R}(\mathbf{E}_1) \ge \ldots \ge \dim\mathcal{R}(\mathbf{E}_K);$$

we *will* need the fact that $\mathcal{S}_{K-1} \neq \mathcal{S}_K = \mathbb{R}^n$ by the definition of K so $\dim\mathcal{R}(\mathbf{E}_K) \neq 0$ and $\mathbf{E}_K \neq \mathbf{0}$. The construction of $\{\mathbf{E}_k\}$ gives a direct sum decomposition

$$\begin{aligned}\mathbf{1} &= \mathbf{E}_0 + \ldots + \mathbf{E}_K\,, \qquad\qquad \mathbb{R}^n = \oplus_0^K\mathcal{R}(\mathbf{E}_k), \\ (2.11)\qquad \mathcal{S}_k &= \mathcal{R}(\mathbf{E}_0) \oplus \ldots \oplus \mathcal{R}(\mathbf{E}_k) \ \text{ for } \ k = 0, \ldots, K.\end{aligned}$$

Thus, introducing

$$(2.12)\qquad \mathbf{\Gamma} = \mathbf{\Gamma}_T := \sum_0^K k!\,T^{K-k}\mathbf{E}_k$$

we see that (2.8) holds with $\mathbf{\Gamma}_0 = K!\,\mathbf{E}_K \neq \mathbf{0}$.

3. Principal Computation

Our object in this section is to obtain (2.7), with $\mathbf{\Gamma}_T$ as in (2.12), computing $\mathbf{Q}$ and showing it is invertible.

The integral expression (2.5) gives, on substituting $s = T\sigma$,

$$T^{-(2K+1)}\mathbf{\Gamma W \Gamma} = \int_0^1 [T^{-K}\mathbf{\Gamma} e^{T\sigma\mathbf{A}}\mathbf{B}][T^{-K}\mathbf{\Gamma} e^{T\sigma\mathbf{A}}\mathbf{B}]^* d\sigma.$$

Using (2.12) and (2.1), we have (for $T > 0$)

$$\begin{aligned} T^{-K}\mathbf{\Gamma} e^{T\sigma\mathbf{A}}\mathbf{B} &= \sum_{k=0}^{K} k! T^{-k}\mathbf{E}_k \quad \sum_{j=0}^{\infty} \frac{\sigma^j}{j!} T^j \mathbf{A}^j \mathbf{B} \\ &= \sum_{k=0}^{K}\sum_{j=0}^{\infty} \frac{k!\sigma^j}{j!} T^{j-k}\mathbf{E}_k\mathbf{A}^j\mathbf{B}. \end{aligned}$$

By (2.10), the terms with $j < k$ vanish so no negative powers of T actually appear on the right; we then split the sum into the terms with $j = k$ and those with $j \geq k+1$ for which we set $i = j - (k+1) = 0, 1, \ldots$. Thus,

$$\begin{aligned} T^{-K}\mathbf{\Gamma} e^{T\sigma\mathbf{A}}\mathbf{B} &= \sum_{k=0}^{K} \sigma^k \mathbf{E}_k \mathbf{A}^k \mathbf{B} \\ &\quad + T\left[\sum_{k=0}^{K}\sum_{i=0}^{\infty} \frac{k!\sigma^{i+k+1}T^i}{(i+k+1)!}\mathbf{E}_k\mathbf{A}^j\mathbf{B}\right] \\ &= \mathbf{P}(\sigma) + T\mathbf{R}_1(T,\sigma). \end{aligned} \tag{3.1}$$

Restricting our attention to $T \leq 1$, which is certainly permissible as we are only interested in the limit $T \to 0$, an easy estimation gives the uniform bound

$$\begin{aligned} \|\mathbf{R}_1(T,\sigma)\| &\leq \sum_{k=0}^{K}\sum_{i=0}^{\infty} \|\mathbf{A}\|^{i+k+1}\|\mathbf{B}\|/(i+1)! \\ &= (1 + \ldots + \|\mathbf{A}\|^K)\|\mathbf{B}\|(e^{\|\mathbf{A}\|} - 1) \end{aligned}$$

since $(i+k+1)! \geq (i+1)!\,k!$. Hence, (3.1) gives[5]

$$T^{-K}\mathbf{\Gamma}\, e^{T\sigma\mathbf{A}}\mathbf{B} = \mathbf{P}(\sigma) + \mathcal{O}(T)$$

and

$$\begin{aligned} &(T^{-K}\mathbf{\Gamma} e^{T\sigma\mathbf{A}}\mathbf{B})\;(T^{-K}\mathbf{\Gamma} e^{T\sigma\mathbf{A}}\mathbf{B})^* \\ &= [\mathbf{P}(\sigma) + T\mathbf{R}_1(T,\sigma)]\;[\mathbf{P}(\sigma) + T\mathbf{R}_1(T,\sigma)]^* \\ &= \mathbf{P}(\sigma)\mathbf{P}^*(\sigma) + T\mathbf{R}_2(T,\sigma), \end{aligned} \tag{3.2}$$

[5]Note that our estimation of $\mathbf{R}_1$ precisely legitimates the use of the $\mathcal{O}(T)$ notation.

with $\mathbf{R}_2(T,\sigma) := (\mathbf{R}_1\mathbf{P}^* + \mathbf{P}\mathbf{R}_1^* + T\mathbf{R}_1\mathbf{R}_1^*)$ uniformly bounded. Thus, integrating,

$$T^{-(2K+1)}\mathbf{\Gamma}\mathbf{W}\mathbf{\Gamma} = \mathbf{Q} + \mathcal{O}(T) \tag{3.3}$$

with $\mathcal{O}(T) = T\int \mathbf{R}_2 d\sigma =: T\mathbf{R}_3(T)$ and

$$\begin{aligned}\mathbf{Q} &:= \int_0^1 \mathbf{P}(\sigma)\mathbf{P}^*(\sigma)d\sigma \\ &= \sum_{j,k=0}^{K} (j+k+1)^{-1}\mathbf{E}_j\mathbf{A}^j\mathbf{B}\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k.\end{aligned} \tag{3.4}$$

We must show that $\mathbf{Q}$ is invertible.

Lemma : $\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi = 0 \Longrightarrow \mathbf{E}_k\xi = 0$.

PROOF : For any $\xi \in I\!R^n$ we have $\mathbf{E}_k\xi \in \mathcal{S}_k := \mathcal{S}_{k-1} + \mathcal{R}(\mathbf{A}^k\mathbf{B})$ by definition so we may write

$$\mathbf{E}_k\xi = \mathbf{A}^k\mathbf{B}\eta + \xi' \qquad (\xi' \in \mathcal{S}_{k-1})$$

for some $\eta \in I\!R^m$. Then, assuming $\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi = 0$, we would have

$$\begin{aligned}\|\mathbf{E}_k\xi\|^2 &= (\mathbf{A}^k\mathbf{B}\eta + \xi')\cdot(\mathbf{E}_k\xi) \\ &= \eta\cdot(\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi) + (\mathbf{E}_k\xi')\cdot\xi = 0\end{aligned}$$

since $\mathbf{E}_k\xi' = 0$ for $\xi' \in \mathcal{S}_{k-1}$. □

From (3.4) we see that

$$\xi\cdot\mathbf{Q}\xi = \int_0^1 \xi\cdot[\mathbf{P}(\sigma)\mathbf{P}^*(\sigma)\xi]d\tau = \int_0^1 \|\mathbf{P}^*(\sigma)\xi\|^2 d\tau$$

so $\mathbf{Q}\xi = 0$ only if $\mathbf{P}^*(\sigma)\xi \equiv 0$.

From the definition of $\mathbf{P}(\cdot)$, this would mean that each term $\sigma^k\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi$ would have to vanish and, by the Lemma, this would imply $\mathbf{E}_k\xi = 0$ for each k. Hence, from (2.11), $\mathbf{Q}\xi = 0$ would give $\xi = 0$. We have thus shown that $\mathbf{Q}\xi = 0$ only for $\xi = 0$; for an $n\times n$ matrix $\mathbf{Q}$, this ensures invertibility.

4. Results

We must draw the desired conclusions from (3.3).

It is clear from the bound on $\mathbf{R}_1(T,\sigma)$ and the obvious fact that $\mathbf{P}(\sigma)$ is bounded uniformly on $[0,1]$ that $\mathbf{R}_2(T,\sigma)$ is uniformly bounded so $\mathbf{R}_3(T)$ is uniformly bounded – say, $\|\mathbf{R}_3(T)\| \le M_3$ for $0 \le T \le 1$. Restricting attention to $T \le 1/2M_3\|\mathbf{Q}^{-1}\| =: \tau$, we have

$$\begin{aligned}(\mathbf{Q}+T\mathbf{R}_3)^{-1} &= \mathbf{Q}^{-1}(\mathbf{1}+T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1} \\ &= \mathbf{Q}^{-1} + T\mathbf{R}_4(T)\end{aligned}$$

with

$$\begin{aligned}\|\mathbf{R}_4\| &\le \|\mathbf{Q}^{-1}\|\,\|(\mathbf{1}+T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1}-\mathbf{1}\|/T\\ &= \|\mathbf{Q}^{-1}\|\,\|\mathbf{Q}^{-1}\mathbf{R}_3(\mathbf{1}+T\mathbf{Q}^{-1}\mathbf{R}_3)^{-1}\|\\ &\le \|\mathbf{Q}^{-1}\|\,\|\mathbf{Q}^{-1}\mathbf{R}_3\|/(1-T\|\mathbf{Q}^{-1}\mathbf{R}_3\|)\\ &\le 2M_3\|\mathbf{Q}^{-1}\|^2 \qquad \text{for } 0\le T\le\tau,\end{aligned}$$

Now, inverting each side of (3.3) is legitimate for $T \neq 0$ and gives

$$\begin{aligned}T^{2K+1}\mathbf{\Gamma}_T^{-1}\mathbf{W}_T\mathbf{\Gamma}_T^{-1} &= \mathbf{Q}^{-1}+T\mathbf{R}_4,\\ T^{2K+1}\mathbf{W}_T^{-1} &= \mathbf{\Gamma}_T(\mathbf{Q}^{-1}+T\mathbf{R}_4)\mathbf{\Gamma}_T\\ &= \mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0+T\mathbf{R}_5(T)\end{aligned}\tag{4.1}$$

with, obviously, $\|\mathbf{R}_5(T)\|$ uniformly bounded on $0 \le T \le \tau$. In particular, this proves (independently of the standard controllability arguments) the invertibility of $\mathbf{W}_T$ — at least for small $T > 0$ and so *a fortiori* for all $T > 0$ by the non-negativity of the integrand in (2.5).

From (2.6) and the positivity of $\mathbf{W}, \mathbf{W}^{-1}$ we then have

$$\begin{aligned}\|\mathbf{C}_T\|^2 &:= \max\{\|\mathbf{C}_T\xi\|^2 : \|\xi\|=1\}\\ &= \max\{\xi\cdot\mathbf{W}_T^{-1}\xi : \|\xi\|=1\} = \|\mathbf{W}_T^{-1}\|\\ &= T^{-(2K+1)}[\|\mathbf{\Gamma}_0\mathbf{Q}^{-1}\mathbf{\Gamma}_0\| + \mathcal{O}(T)].\end{aligned}\tag{4.2}$$

This, of course, is just (1.4) with, from (2.12),

$$\gamma := \|\mathbf{\Gamma}_0\mathbf{Q}^{-1}\Gamma_0\|^{1/2} = K!\|\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\|^{1/2}\tag{4.3}$$

once one show $\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K \neq \mathbf{0}$ so $\gamma \neq 0$. Note that the positivity of $\mathbf{Q}$, hence of $\mathbf{Q}^{-1}$, gives

$$\begin{aligned}\|\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\| &= \max\{\xi\cdot(\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\xi) : \|\xi\|=1\}\\ &= \max\{\xi\cdot\mathbf{Q}^{-1}\xi : \|\xi\|=1,\ \xi=\mathbf{E}_K\xi\}\\ &= \|\mathbf{Q}^{-1}|_{\mathcal{R}(\mathbf{E}_K)}\|.\end{aligned}$$

Since $\mathcal{R}(\mathbf{E}_K) \neq \{0\}$ by the minimality of K, this is clearly non-zero.

At this point we work out in somewhat greater detail the case of scalar control ($m = 1$). The $n \times 1$ matrix $\mathbf{B}$ is now just a vector and we set

$$\beta_0 = \mathbf{B}, \quad \beta_k = \mathbf{A}^k\beta_0 \qquad \text{for } k = 0,\ldots,n-1.\tag{4.4}$$

Note that controllability gives $K = n - 1$ in this case so, for scalar control, our result (4.2) becomes

$$\|\mathbf{C}_T\| \sim \gamma T^{-(n+1/2)} + \mathcal{O}(T^{-(n-1/2)}).\tag{4.5}$$

To compute γ here, note first that $(\beta_0, \ldots, \beta_{n-1})$ is a basis for $I\!R^n$ and let $(\varepsilon_0, \ldots, \varepsilon_{n-1})$ be the orthonormal basis obtained from that by the Gram-Schmidt procedure so $\mathbf{E}_k : \xi \mapsto (\xi \cdot \varepsilon_k)\varepsilon_k$ and $\mathbf{B}^*\mathbf{A}^{*k}\mathbf{E}_k\xi$ becomes $(\beta_k \cdot \varepsilon_k)\ (\xi_k \cdot \varepsilon_k)$. Then (3.4) becomes

$$\mathbf{Q}\xi = \sum_{j=0}^{K-1}\left[\sum_{k=0}^{K-1}\frac{(\beta_j \cdot \varepsilon_j)(\beta_k \cdot \varepsilon_k)}{j+k+1}(\xi \cdot \varepsilon_k)\right]\varepsilon_j. \tag{4.6}$$

This shows that, *re-written in terms of the orthonormal basis* $(\varepsilon_0, \ldots, \varepsilon_{n-1})$, the new matrix for $\mathbf{Q}$ is just $\mathbf{DHD}$ where $\mathbf{D} :=$ diag $[\beta_j \cdot \varepsilon_j]$ and $\mathbf{H}$ is the $n \times n$ Hilbert matrix. Then

$$\begin{aligned}\|\mathbf{E}_K\mathbf{Q}^{-1}\mathbf{E}_K\| &= \varepsilon_K \cdot \mathbf{Q}^{-1}\varepsilon_K \\ &= [\text{lower right corner element of } \mathbf{D}^{-1}\mathbf{H}^{-1}\mathbf{D}^{-1}] \\ &= (\beta_{n-1} \cdot \varepsilon_{n-1})^2\ [\text{lower right corner element of } \mathbf{H}^{-1}]\end{aligned}$$

whence

$$\gamma = C_n(\beta_{n-1} \cdot \varepsilon_{n-1}) \tag{4.7}$$

with $C_n := (n-1)!$ [lower right corner element of the inverse of the $\mathbf{H}^{-1}]^{1/2}$. The coefficient C_n grows extremely rapidly with n but, of course, is fixed for any given dimensionality. Thus, $\beta_{n-1} \cdot \varepsilon_{n-1}$ provides the only dependence on the *particular* system (1.1); it is just the norm of the component of $\mathbf{A}^{n-1}\beta_0$ orthogonal to $span\{\beta_0, \ldots, \mathbf{A}^{n-2}\beta_0\}$.

Returning to the general case, we now consider the asymptotics for a particular target ξ (rather than the 'worst case' treatment above). We have, from (2.6),

$$x(T; u(\cdot)) = \xi \Longrightarrow \|u(\cdot)\| \geq \|\mathbf{C}_T\xi\| = (\xi \cdot \mathbf{W}_T^{-1}\xi)^{1/2}.$$

From (4.1) we have

$$\|\mathbf{C}_T\xi\| = T^{-(K+1/2)}K!(\xi_K \cdot \mathbf{Q}^{-1}\xi_K)^{1/2} + \mathcal{O}(T^{-(K-1/2)})$$

where we have abbreviated $\xi_K := \mathbf{E}_K\xi$. We may write this, assuming[6] $\xi_K \neq 0$, as

$$\|\mathbf{C}_T\xi\| \sim (K!\|\mathbf{Q}^{-1/2}\xi_K\|)T^{-(K+1/2)}, \tag{4.8}$$

which gives the same asymptotic growth rate for (almost all) targets.

[6] This is 'almost always' true — it fails only when ξ happens to lie exactly in the (proper) subspace $\mathcal{N}(\mathbf{E}_K)$ in which case one has slower blowup. Even in that case slight perturbations would, almost inevitably, give *some* component in $\mathcal{R}(\mathbf{E}_K)$ so this analysis would dominate.

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