Working Paper Series

How Well Do Linear Approximation Methods Work? Results for Suboptimal Dynamic Equilibria

WP 90-11

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How Well Do Linear Approximation Methods Work? Results for Suboptimal Dynamic Equilibria

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revision: November 1990

*We benefit from numerous discussions with Robert G. King. All errors remain ours, however.

1. Introduction

Real business cycle models have recently been applied to settings in which equilibria are suboptimal.[1] In most models the solutions are approximated using some type of linearization with little attention being given to the accuracy of the approximation. There exists some work (see, for example, Christiano (1989) and Rebelo and Rouwenhorst (1989)) concerning the accuracy of linear and log-linear approximations in models where competitive equilibria are optimal but none that we are aware of for suboptimal cases. Generally, the approximations in optimal environments are quite good when the variance of the forcing process is small but deteriorate as the variance increases. Since accuracy depends on the variance of the underlying shocks, it is natural to ask whether similar approximation methods would perform as well in suboptimal settings where there may be a great deal more variation in the shocks. For example, effective tax rates are quite variable over time as are monetary shocks.

In this paper we investigate three different approximation methods in the context of a neoclassical model with a production tax and compare their solutions with solutions obtained from a discrete state space solution to the Euler equations of the model. In using discrete state space methods we adopt a very fine grid on capital and trust that the solution is arbitrarily close to the true solution. For cases where a closed form solution exists, the state space solution was roughly within two hundredths of one percent of the true solution.

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[1]For example, Baxter and King (1989) analyze an equilibrium business cycle model with a production externality. Cooley and Hansen (1989), and Eichumbaum and Singleton (1987) examine the implications of a cash-in-advance constraint in a real business cycle model. Models concerning fiscal policies are analyzed by Aiyagari, Christiano, and Eichumbaum (1990), Baxter and King (1990), Braun (1988), Mao (1990), McGrattan (1988), and Wynne (1987). King, Plosser and Rebelo (1988b) give other examples in which equilibria are suboptimal.

The three approximation methods examined are the linear and log-linear approximations analyzed by Christiano and the log-linear approximation of King, Plosser, and Rebelo (KPR) (1988a). We also consider a variant of the KPR procedure which linearizes the system around the level (instead of log) of the steady state. The approximate solutions are compared along a number of dimensions in order to understand the strengths and weaknesses of each. First we compare the policy functions for output, capital, consumption, investment, labor hours, and real interest rates and analyze them both visually and by a probability weighted difference in the values of the functions. This latter metric indicates how important the differences in the policy functions are for generating impulse responses and autocovariance functions. We then extend the comparison to the frequency domain and examine the coherence of the data generated from the approximate solutions with those generated from the true solutions. Doing so allows us to see how well the solutions match up by frequency.

These experiments produce a number of interesting results. First none of the approximation methods dominate the others. Generally they are all fairly accurate at solving for output, the capital stock, and labor hours. The KPR procedure does a good job in terms of consumption and real interest rates, but is not very accurate in terms of investment.[2] Therefore, if one were primarily interested in analyzing the effect of taxes on investment, their procedure is not recommended. On the other hand the two approximation methods studied by Christiano replicate investment quite accurately, but perform much more poorly in terms of consumption and real interest rates. These procedures might, therefore, be inappropriate for analyzing the welfare effects of tax policies.

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[2]The alternative KPR procedure, which linearizes the system around the level of the steady state, approximates the investment function quite accurately.

The paper is organized as follows. In section 2 we briefly describe the model and the approximation methods. Section 3 presents some empirical evidence regarding the behavior of tax rates in the U.S. In section 4 we compare the policy functions while in section 5 we extend comparisons to the frequency domain. Section 6 contains a brief summary and conclusions.

2. **The Model**

A. General Description

The economy is composed of three entities: firms, individuals, and the government. As in Brock and Mirman (1972), Coleman (1989), and Dotsey (1990) firms face an inelastic supply of capital K_t from individuals and combine it with labor N_t to produce output, Y_t, via a constant return to scale production technology $F(K_t, N_t)$. Output is taxed at the rate τ_t. The tax rate is given by an exogenous statistical process $\tau_t = (1-\rho)\tau^* + \rho\tau_{t-1} + e_t$, where e_t is an i.i.d. random variable and τ^* is the unconditional mean of the tax rate. We impose the condition $|\rho| < 1$ so that the tax process is stationary. With perfect competition firms pay each factor its after-tax marginal product. Formally,

$$r_t = (1-\tau_t)F_k(K_t, N_t), \tag{1a}$$

$$w_t = (1-\tau_t)F_n(K_t, N_t), \tag{1b}$$

where F_k and F_n are the marginal product of capital and labor, respectively, r_t is the after-tax rental rate on capital, and w_t is the after-tax real wage rate.

Individuals maximize expected lifetime utility subject to a set of intertemporal budget constraints. Specifically, the individual solves the following problem:

$$\max_{\{\bar{c}_t, \bar{l}_t, \bar{n}_t, \bar{k}_{t+1}\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(\bar{c}_t, \bar{l}_t) \right], \quad 0 < \beta < 1,$$

subject to

$$\bar{c}_t + \bar{k}_{t+1} \le (1-\delta)\bar{k}_t + r_t\bar{k}_t + w_t\bar{n}_t + Tr_t \quad \text{for all } t,$$

$$\bar{l}_t + \bar{n}_t = 1 \quad \text{for all } t,$$

$\bar{k}_0$ is given,

where $\bar{c}_t$ is consumption, $\bar{l}_t$ is leisure, $\bar{k}_t$ is capital, and Tr_t is per capita lump sum rebates of all tax proceeds which equal $\tau_t F(K_t, N_t)$. The attached bars indicate variables at the individual level. In this problem, the agent takes the pricing function w_t and r_t as given as well as the economy-wide capital K_t and labor N_t.

The information set at time t over which expectations are conditioned is assumed to contain current and past values of all decision variables and prices as well as the current tax rate. Agents, however, do not know future tax rates. It is assumed that capital depreciates at the rate $0 \le \delta \le 1$ and agents are endowed with one unit of time each period.

The government randomly taxes production and redistributes all taxes via a lump sum rebate. We, therefore, concentrate on the compensated effects of taxes. Otherwise $(1-\tau_t)$ could be interpreted as a multiplicative technology shock and the model would correspond to those already examined by Christiano (1989) and Rebelo and Rouwenhorst (1989). Equilibrium is achieved when aggregate demand equals aggregate supply:

$$Y_t = F(K_t, N_t) = C_t + K_{t+1} - (1-\delta)K_t \tag{2}$$

with $\bar{k}_t = K_t$ and $\bar{n}_t = N_t$.

Using the first-order conditions for the individual, the equilibrium conditions and the conditions describing competitive factor payments, the solution to the competitive equilibrium of the model economy involves finding policy functions $K_{t+1} = G(K_t, \tau_t)$ and $N_t = H(K_t, \tau_t)$ that solve the following Euler equations:

$$u_c[F(K_t,N_t) + (1-\delta)K_t - K_{t+1}, 1-N_t] = \beta\, E_t\Big[u_c[F(K_{t+1},N_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, 1-N_{t+1}] \times [(1-\tau_{t+1})F_k(K_{t+1},N_{t+1}) + (1-\delta)] \Big] \tag{3a}$$

and

$$\frac{u_l[F(K_t,N_t) + (1-\delta)K_t - K_{t+1}, 1-N_t]}{u_c[F(K_t,N_t) + (1-\delta)K_t - K_{t+1}, 1-N_t]} = (1-\tau_t)F_n(K_t,N_t), \tag{3b}$$

where u_c and u_l denote the marginal utility of consumption and leisure, respectively. Equation (3a) states that in equilibrium the individual is indifferent between consuming one extra unit of goods today and investing it in the form of capital and consuming tomorrow. Equation (3b) states that the rate of substitution of consumption for leisure in a given period must equal the cost of leisure, which is the after-tax wage rate. Essentially we are trying to find the functions $G(K_t, \tau_t)$ and $H(K_t, \tau_t)$ that are fixed points of the system (3a) and (3b). Coleman (1989) analyzes the restrictions on preferences and technology that guarantee existence and uniqueness of solutions to problems of this type.

Throughout the paper we employ a Cobb-Douglas production function:

$$Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}, \quad 0 < \alpha < 1, \tag{4}$$

and a constant relative risk aversion (CRRA) utility function:

$$u(C_t, L_t) = \begin{cases} \frac{1}{1-\sigma}\left\{ \left[C_t^{\theta} L_t^{1-\theta} \right]^{1-\sigma} - 1 \right\}, & \text{if } \sigma > 0 \text{ and } \sigma \neq 1, \\ \theta \ln C_t + (1-\theta) \ln L_t, & \text{if } \sigma = 1, \end{cases} \tag{5}$$

where $0 < \theta < 1$. Note that the CRRA utility function is unbounded from below and does not meet one of Coleman's assumptions. However, the discrete state space algorithm we use shows no dependence on the starting values and always converges to the same solution, indicating that unique solutions are obtained.

B. Various approximations

The rest of the section discusses various approximation methods for solving the dynamic equilibrium of the above model. These methods include the linear and log-linear approximations analyzed by Christiano and the linear and log-linear approximations of King, Plosser, and Rebelo. The discrete state space method is also discussed.

linear approximation (LIN)

Under the LIN approximation a first order Taylor expansion of the system (3a)-(3b) is taken around the nonstochastic steady state values of capital, labor hours, and one minus the mean of the tax rate.[3] This procedure yields a system that is linear in the levels of K_t, N_t, and $(1-\tau_t)$. It should be pointed out that in an optimal setting where there is no tax, this procedure is identical to the conventional quadratic-linear approximation where the constraints of the maximization problem are linearized while the objective is approximated by a quadratic function. This is so because both methods yield identical Euler equations. Christiano (1989) examined the performance of the quadratic-linear approximation in an optimal setting.

The policy functions for capital and labor hours that solve the linear system are (see Appendix A)

$$K_{t+1} = G_{lin}(K_t, 1-\tau_t) = A_{10} + A_{11} K_t + A_{12} (1-\tau_t), \tag{6a}$$

$$N_t = H_{lin}(K_t, 1-\tau_t) = A_{20} + A_{21} K_t + A_{22} (1-\tau_t), \tag{6b}$$

[3] The linearization could also be taken around the mean of the tax rate, τ^*. However, our experience indicates that the performance of the approximation using this procedure is worse than using $1-\tau^*$.

where the coefficients A_{ij} (i = 1,2, j = 0,1,2) are complicated functions of the underlying parameters of the economy and the coefficient A_{11} is less than one in absolute value.

Using these two decision rules, the policy functions governing output, consumption, investment, and the real interest rate, R_t, can be determined. They are

$$Y_t = F(K_t,\ H_{lin}(K_t,\ 1-\tau_t)), \tag{7}$$

$$C_t = F(K_t,\ H_{lin}(K_t,\ 1-\tau_t)) + (1-\delta)K_t - G_{lin}(K_t,\ 1-\tau_t), \tag{8}$$

$$I_t = G_{lin}(K_t,\ 1-\tau_t) - (1-\delta)K_t, \tag{9}$$

$$R_t = \frac{u_c[C_t, 1-H_{lin}(K_t, 1-\tau_t)]}{\beta\ E_t\{u_c[C_{t+1}, 1-H_{lin}(K_{t+1}, 1-\tau_{t+1})]\}} - 1. \tag{10}$$

The conditional expectations of the marginal utility of consumption, needed to compute the real interest rate, can be calculated using the probability structure implied by the tax process.

log-linear approximation (LOG)

The LOG procedure is similar to the LIN procedure except that the system is linearized around the logarithm of the nonstochastic steady state values of capital, labor and one minus the tax rate. Again, in an optimal setting this procedure is identical to the quadratic-linear approximation mentioned earlier. Specifically, let $k_t = \ln K_t$, $n_t = \ln N_t$ and $\phi_t = \ln(1-\tau_t)$, then the decision rules for k_{t+1} and n_t are given by (see Appendix A)

$$k_{t+1} = G_{log}(k_t,\ \phi_t) = B_{10} + B_{11}\ k_t + B_{12}\ \phi_t, \tag{11a}$$

$$n_t = H_{log}(k_t,\ \phi_t) = B_{20} + B_{21}\ k_t + B_{22}\ \phi_t, \tag{11b}$$

where the coefficients B_{ij} (i = 1,2, j = 0,1,2) are complicated functions of the underlying parameters of the economy and the coefficient B_{11} is less than one in absolute value.

The policy functions for output, consumption and investment are

$$Y_t = F(K_t, \exp[H_{log}(k_t, \phi_t)]), \tag{12}$$

$$C_t = F(K_t, \exp[H_{log}(k_t, \phi_t)]) + (1-\delta)K_t - \exp[G_{log}(k_t, \phi_t)], \tag{13}$$

$$I_t = \exp[G_{log}(k_t, \phi_t)] - (1-\delta)K_t. \tag{14}$$

The derivation of the real interest rate proceeds from equation (10).

The King, Plosser, and Rebelo approximation (KPR)

A common feature of the LIN procedure and the LOG procedure is that both approximate the solutions for capital and labor around the nonstochastic steady state (either in level or in log), and then use the law of motion for capital and the equilibrium condition to construct the implied functions for investment and consumption. While this procedure ensures that the market clearing condition is satisfied over the entire state space, it does not necessarily imply a more accurate approximation. By construction, the policy functions for capital and labor are linear, but consumption, investment, output and the real interest rate are not linear in the state variables.[4]

As in the LOG procedure, the linearization under the KPR approximation is taken around the logarithm of the nonstochastic steady state. But the KPR approximation yields a system of linear decision rules, including capital and labor as well as other variables that are of interest. This is achieved by linearizing not only the first-order conditions (3a) and (3b), but also the equilibrium condition (2) and other relevant identities. Thus, for example, the decision rules for output and investment are obtained by linearizing the production function and the law of motion for capital, and the decision rule

[4] Note that the investment function is linear under the LIN procedure.

for consumption is obtained by linearizing the market-clearing condition. By construction the decision rules for capital and labor will be identical to those under the LOG procedure.[5] Since the solutions are linear in the state of the system, the KPR procedure permits fast and easy calculation of impulse responses and theoretical moments. This advantage, however, is achieved at the cost that the equilibrium condition is not strictly maintained. Details of the KPR approximation can be found in King, Plosser and Rebelo (1988a) and the accompanying technical appendix.

There is an alternative version of the KPR approximation that turns out to be important. This alternative procedure linearizes the system around the level (instead of log) of the steady state as in the LIN approximation. The resulting policy functions for capital and labor (and investment) will be identical to those under the LIN procedure. As will be seen later the chief difference between the two KPR approximations is the investment function. In the following we label the original KPR procedure as KPR1 and the alternative version as KPR2.

C. Discrete State Space Method (DSS)

The basic idea of the discrete state space method is to approximate the policy functions on a finite number of grid points over the state space. The policy functions are constructed so that the first-order conditions and the equilibrium conditions of the underlying economic problem are satisfied over the discrete space. This method is based on a technique proposed by Coleman (1989) who established existence and uniqueness of fixed points to problems of this type. Technical details of our numerical algorithm are presented in Appendix B. Here we briefly sketch the procedure.[6]

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[5]Since the production function is Cobb-Douglas in our example, the decision rule for output is also identical under the two approximations.

[6]Our procedure is somewhat different from that of Coleman. In particular, we

The numerical policy functions for capital and labor can be obtained by iterating on (3a) and (3b) using the conventional successive approximation technique. Starting from an initial guess for capital (usually, a constant function) we first used (3b) to solve the implied function for labor, which is substituted into (3a) to obtain an equation that depends only on capital. The resulting equation is then used to solve a new function for capital over the discretized space. This function is taken to be the new starting point for the next iteration. The process continues until the policy function for capital converges. Once the optimal solution for capital is obtained, other variables can be easily derived.

In general, the accuracy of the DSS method depends on the number of grid points being used. In this paper we adopt 2500 points for the capital stock. These grids are defined over the ergodic set of capital, which is smaller than its feasible set.[7] In our numerical procedure, we approximate the set by first solving the problem over the feasible set using coarse grids. The implied ergodic set is then used in the second run (or the third run if necessary) to define a new range for capital. The process continues until the number of grids contained in the ergodic set exceeds 90 percent of the number of grid points being used. This procedure yields a very smooth policy function that is close to the true policy function. For example, for cases where a closed form solution exists (see Dotsey (1990)), the numerical policy functions are within two hundredths of one percent of the true solutions.[8]

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restrict the range of the policy function for capital to lie on capital grids while Coleman does not.

[7]Intuitively, the ergodic set of capital is a set of numbers $\{K | \underline{K} \leq K \leq \bar{K}\}$ such that its complement has probability zero. This means that once capital falls into this set, it stays there forever and never moves out. In our example, since the function $G(K, \tau)$ is increasing in K and decreasing in $\underline{\tau}$, the lower bound of the ergodic set, $\underline{K}$, is the stationary point of $G(K, \bar{\tau})$ where $\bar{\tau}$ is the highest tax rate, and the upper bound $\bar{K}$ is the stationary point of $G(K, \underline{\tau})$ where $\underline{\tau}$ is the lowest tax rate.

[8]The metric we use to measure the distance between the approximate solution

3. The Tax Process

As mentioned earlier, the tax process in our model economy is assumed to follow a first order autoregressive process. In this section we provide some empirical evidence on the time series behavior of tax rates in the U.S. This information will be used to formulate a finite state Markov chain for the tax rate in order to implement the discrete state space solution algorithm.

The data we used to characterize the behavior of tax rates in the U.S. are the effective tax rates on corporate investment compiled by Auerbach and Hines (1988). According to one of their measures, the effective tax rates vary from 0.036 to 0.605 in the period of 1953 to 1985, with mean and variance equal to 0.4 and 0.02, respectively.[9] A simple regression reveals the following result:

$$\tau_t = \underset{(0.06)}{0.06} + \underset{(0.13)}{0.82}\, \tau_{t-1}, \qquad R^2 = 0.55,\ SE = 0.1,\ DW = 2.05$$

where the standard errors of estimates are indicated in parentheses.

Given the empirical evidence, we choose a five state Markov chain to approximate the tax process with the first order autocorrelation coefficient, ρ, set to 0.8. The approximation method we used was proposed by Rebelo and Rouwenhorst (1989). This approximation generates a family of one parameter Markov processes that have a binomial stationary distribution. The single parameter of this family, π, determines the AR(1) coefficient of the process, which is $2\pi-1$, but does not influence other moments of the distribution. We

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and the true solution is the absolute percentage deviations weighted by the stationary probability of capital and tax. Later we also use an alternative metric to measure the distance.

[9] Auerbach and Hines computed several tax rates measures, taking into account the *firm's expectations of the tax reform.* The measure we employed is based on the assumption of variable reform probability with a low adjustment cost of investment. This measure is less persistent than other measures. We drop the 1986 observation, which has -45% effective tax rate.

briefly summarize their method below.

Let M be the number of states of the process. The transition matrix of a two-state process, Π_2, is given by

$$\Pi_2 = \begin{bmatrix} \pi & 1-\pi \\ 1-\pi & \pi \end{bmatrix}.$$

The recursive procedure to determine the transition probability of a M-state process, Π_M, is to, first, compute a matrix P_j $(j = 3,\cdots,M)$ according to

$$P_j = \pi\begin{bmatrix} \Pi_{j-1} & \underline{0} \\ \underline{0}' & 0 \end{bmatrix} + (1-\pi)\begin{bmatrix} \underline{0} & \Pi_{j-1} \\ 0 & \underline{0}' \end{bmatrix} + (1-\pi)\begin{bmatrix} \underline{0}' & 0 \\ \Pi_{j-1} & \underline{0} \end{bmatrix} + \pi\begin{bmatrix} 0 & \underline{0}' \\ \underline{0} & \Pi_{j-1} \end{bmatrix},$$

where $\underline{0}$ is a $(j-1)\times 1$ vector of zeros. To obtain the transition matrix Π_j, divide P_j by two except the top and bottom rows to restore the requirement that conditional probabilities sum to one.[10]

In addition to the parameter ρ, we wish also to limit other moments of the tax process, especially, its mean, variance and the range over which it varies over the sample period. However, given the number of states being used, one can choose only two out of the three parameters. Our experiments indicate that the performance of various approximations is not sensitive to the specification of mean, so we choose to limit the variance and the range. Specifically, the tax rate is assumed to take five equal-distance values over 0.06 and 0.63 with variance equal to 0.02. The mean of the tax rate implied by this choice is 0.35.

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[10]With $\rho = 0.8$, the transition matrix of a five-state process is given by

$$\begin{bmatrix} .6561 & .2916 & .0486 & .0036 & .0001 \\ .0729 & .6804 & .2214 & .0244 & .0009 \\ .0081 & .1476 & .6886 & .1476 & .0081 \\ .0009 & .0244 & .2214 & .6804 & .0729 \\ .0001 & .0036 & .0486 & .2916 & .6561 \end{bmatrix}$$

4. Comparison of Policy Functions

The parameterization used in comparing the various approximation methods with the discrete state space procedure is to let $\alpha = 0.3$, $\delta = 0.05$, $\beta = 0.96$ and $\theta = 0.3$. The benchmark value for σ is set to 1. Later we also consider the cases of $\sigma = 2$ and 10 in order to understand the effects of increasing the curvature of the utility function. The parameterizations chosen are well within the values used in the real business cycles literature and imply a steady state labor supply of 0.18 and a real interest rate of 4.2 percent.

A. General Characteristics of the Performance of Various Approximations

We first compare the relative accuracy of the approximations by looking at the difference between the policy functions. To make the graphs of the policy functions readable we only display them for two interior points of the tax rate (i.e., $\tau = 0.21$ and 0.49). This range covers about 87 percent of the simulated tax rates.

Figures (1a), (1b), (1c) and (1d) contain graphs of the policy functions ($\sigma = 1$) for the four approximation methods, plotted against those produced by the DSS procedure. These functions are plotted over the approximate ergodic set of capital which contains more than 2300 grid points. As can be seen, all four approximations produce policy functions for output, capital, and labor hours that are close to those generated by the DSS method. Regarding consumption, the KPR1 and DSS policy functions are almost identical. The KPR2 procedure (i.e., the one that linearizes around the level of the steady state) also performs relatively well. This is not true of either the LIN or LOG approximations. In fact, the policy function for consumption under these two procedures is not monotonic in the tax rate, which is counter intuitive. The LOG and LIN (and KPR2) approximations match the policy function for investment reasonably well, but here the KPR1 procedure performs poorly.[11] In particular, the KPR1 solution exhibits large deviations from the DSS solution near the boundaries of the ergodic set. The market clearing condition is apparently violated. Note also that the KPR1 solution prevents investment from becoming negative when tax rates are high.[12]

Regarding the real interest rate, both KPR1 and KPR2 appear to generate good approximations while LOG does not match at all. Since the real interest rate depends on the marginal utility of consumption, a poor approximation to consumption under the LOG procedure translates into a poor performance of the real interest rate. The LIN solution appears to perform better than the LOG solution, but it is still worse than the KPR approximations.

This simple visual inspection while indicative of the relative strengths and weaknesses of the various approximation procedures can also be somewhat misleading. The greatest departures from the DSS solutions usually occur near the boundaries of the ergodic set. If these areas of the state space are only reached with low probability then the deviations of the approximate solutions may not be very important. To investigate the importance of the observed disparities in the approximate solutions from those generated by the DSS procedure we weighted the differences by the stationary (unconditional) probability of landing at the particular point in the state space. This limiting distribution is approximated by a frequency count of capital and taxes calculated from a sample path of 2000 observations generated by the DSS procedure. This process is repeated 100 times in order to compute the average errors. The results of this experiment are given in Table 1a where all numbers represent the probability weighted percentage errors in the

[11] Recall that the KPR2 procedure generates the same investment function as the LIN procedure.

[12] This result is due to the fact that the KPR1 approximation expresses all variables in terms of the log (except the real interest rate), implying that the level must be positive.

policy functions. The estimated standard deviation of the mean is indicated in the parenthesis.

Looking at panel A (σ = 1), it is clear that the policy functions for investment and the real interest rate are the most difficult functions to approximate. The two KPR procedures, which appear to perform well in terms of the real interest rate, generate a weighted error of more than 50 percent from the DSS solution. Also, even though the LOG approximation appears to perform poorly with respect to consumption the weighted differences are not very large. In order to illustrate more clearly the relative merits of each approximation scheme a symbol is utilized to indicate improvement (+) or deterioration (-) as one moves from the LOG or the LIN approximation to the corresponding KPR approximations. In general, KPR1 is better with respect to consumption and the interest rate but worse in terms of investment. This result is consistent with the figures presented before. Note, however, that the performance of consumption deteriorates as one switches from LIN to KPR2.

One final observation, which deserves closer examinations, concerns the performance of the investment function when σ = 1. Contrary to the visual impression from the graphs of the policy functions, the KPR2 (or LIN) approximation generates a surprisingly larger error for investment than does the KPR1 approximation. This result is in part due to the metric we use. However, it also results from the fact that the KPR2 approximation is particularly vulnerable to some areas of the state space that generate small values for investment. In order to see this more clearly we present in Table 1b some detailed information on the errors of investment, concentrating on two areas of the state space that have positive probability. It is clear from this table that although the KPR2 investment is more accurate than the KPR1 investment in most regions of the state space, its performance worsens dramatically when the true investment is small in absolute value. Even though this area of the state space is reached with low probability, such outliers will dominate the weighted errors. As the value of σ increases this anomaly disappears.

The above discussion suggests that the metric we used is sensitive to the scale of the DSS solutions. In order to isolate this problem we employed the mean absolute error, which is scale independent, to measure the distance between the approximate solutions and the DSS solutions. This distance was divided by the "length" of the DSS solution in order to normalize the unit.[13] The results using this metric are summarized in Table 1c. Again, the figures reported are the average mean absolute errors based on 100 simulations. It is clear from this table that except for investment the relative accuracy of the four approximations is the same as before. Here, both LIN and LOG are more accurate than KPR1 in terms of investment, which is consistent with the graphs of the policy functions. It appears that on average KPR2 has a better performance than the other approximations according to this metric.

B. Performance on the Time Domain

To further see how important the discrepancies in the policy functions are, we compared the correlation matrices generated by each procedure for the case of $\sigma = 1$. A Monte Carlo simulation which involves 100 repetitions of the model economy was conducted. The results of this experiment are given in Table 2 where all figures reported are the average correlation coefficients. As expected the discrepancies between correlation coefficients generated by LIN and LOG involving interest rates are noticeable, while those involving

[13]For example, the mean absolute error of the policy function for capital is

$$\sum_{K,\tau}\{Pr[K,\tau]\times|G_i(K,\tau)-G(K,\tau)|\} \;/\; \sum_{K,\tau}Pr[K,\tau]\times|G(K,\tau)|,$$

where $Pr[K,\tau]$ is the joint probability of K and τ, $G(K,\tau)$ is the DSS solution and $G_i(K,\tau)$ (i = KPR1, KPR2, LOG, LIN) is the approximate solution.

investment under KPR1 are also fairly large. The correlation coefficients generated by KPR2 are well within 10 percent of those generated by the DSS solutions,[14] so by this standard KPR2 has the best performance.

One aspect suggested by the graphs of the policy functions is that the various approximations to the policy functions should be closer to those generated by the DSS procedure as the ergodic set of capital becomes smaller. To check this conjecture we reduced the variance of the tax process from 0.02 to 0.002 and reexamined the weighted differences. They generally fall by at least one order of magnitude and there are no significant differences in the correlation coefficients. These last results agree strongly with those of Christiano (1989) who found that the accuracy of the approximations was sensitive to the variance of the technology shock.

C. A Statistical Test: The Den Haan-Marcet Statistic

One final comparison between the various approximation procedures was performed. It involves checking whether the approximate solutions satisfy the orthogonality conditions implied by the Euler equations. We employ a procedure suggested by Den Haan and Marcet (1989) to perform such a test.[15]

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[14] In addition to the sample moments we also computed the theoretical moments implied by the KPR2 solutions. The theoretical correlations among endogenous variables are almost identical to the figures in Table 2.

[15] Let ε_{t+1} be the Euler equation residual, which in our example is given by

$$\varepsilon_{t+1} = \beta \left[(C_{t+1}/C_t)^{-\sigma} (1 - \delta + r_{t+1}) (w_{t+1}/w_t)^{(1-\theta)(\sigma-1)} \right] - 1.$$

This expression is obtained by substituting into (3a) the factor payments and the labor/leisure decision implied by (3b). The Den Haan-Marcet statistic for testing the orthogonality condition $E_t[\varepsilon_{t+1}] = 0$ is given by $m = \tilde{\underline{\gamma}}'[\Sigma \underline{z}_t'\underline{z}_t][\Sigma \underline{z}_t'\underline{z}_t\varepsilon_{t+1}^2]^{-1}[\Sigma \underline{z}_t\underline{z}_t']\tilde{\underline{\gamma}}$, where $\tilde{\underline{\gamma}}$ is a vector of the OLS estimates in a regression of ε_{t+1} on an "instrumental" vector $\underline{z}_t$, which includes variables contained in the information set at time t. This statistic has an asymptotic chisquared distribution with degree of freedom equal to the dimension of the

The results are summarized in Table 3. It is clear that for all cases under consideration none of the approximation methods satisfy the Euler equations. The probability values (i.e., the tail area) of the test statistic are close to zero and very significant. The statistics for the DSS solutions, however, are small, indicating that the orthogonality conditions are satisfied by the DSS solutions.

D. Increasing the Curvature of Utility Function

We mentioned earlier that the accuracy of approximations is sensitive to the variance of shocks. We now examine another type of perturbation which has important implications on the relative accuracy of linear approximations. Specifically, we are interested in the accuracy results with respect to the parameter governing relative risk aversion. The results reported in Table 1a and 1c show that as the value of σ rises the solutions for quantities under each of the approximation schemes become closer to those generated by the DSS procedure. The reason for this result is that as agents become more risk averse they try harder to smooth consumption and leisure and this tendency to smooth becomes the dominant aspect of behavior. Essentially as the σ value gets large all solution procedures tend to be driven by this motive and converge to the same quantity solutions. In fact, our experiments indicate that the variances of all the quantities decrease as the σ value increases.[16]

Although quantities become more accurate the performance of the interest rate actually deteriorates as the σ value increases. This result is driven

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vector $\underline{z}_t$. The value of m should be "small" if the orthogonality condition is satisfied. In our test, the vector $\underline{z}_t$ is chosen to include a constant and 2 lagged values of consumption growth (C_{t+1}/C_t), the rental rate r_{t+1} and the wage growth (w_{t+1}/w_t).

[16]For example, as σ increases from 1 to 10, the variances of consumption and capital decrease from 0.040 and 0.19 to 0.035 and 0.15, respectively. Other quantities also exhibit declining variability as the σ value increases.

by two potentially offsetting forces. Recall that the real interest rate is a function of the marginal utility of consumption, the magnitude of which is in part controlled by the parameter σ and in part controlled by the point at which the function is evaluated. Given the assumed utility function, the (ex post) real interest rate, which can be approximated by $r_t^e \cong \ln(1+r_t^e)$, will involve a term that captures the growth of consumption multiplied by σ. Thus, the accuracy of the interest rate will depend on the performance of consumption as well as the value of σ. It is clear that although consumption becomes closer to the true solution as the value of σ gets large, this gain may be outweighed by the higher σ value which makes the utility function more concave.

E. A "Nearly" Permanent Tax Process

The policy functions displayed in figures (1a)-(1d) indicate that the various approximations perform relatively well around the stationary point of the capital stock. That is, for any given tax rate the approximate solutions are close to the DSS solutions around the corresponding stationary value of capital. This observation suggests that the four approximations might yield relatively more accurate solutions if tax shocks are permanent. To check this conjecture we increased the value of ρ from 0.8 to 0.999. Although the shocks are not permanent in this case, they are highly persistent.

In Table 4 we report the approximation errors for the case of $\sigma = 1$. It is clear that, regardless of the metric used, all the approximate solutions exhibit substantial improvement over those generated by a less persistent tax shock. Again, this result stems from the fact that for a highly persistent tax shock the realized values for capital will be heavily concentrated on the surrounding area of the stationary points, and as pointed out before, all the approximation procedures are fairly accurate around these stationary points.

In figure 1e we plot the frequency distribution of the capital stock for ρ = 0.8 and 0.999, respectively. As can be seen, while the distribution has the usual bell shape when $\rho = 0.8$, it degenerates to three mass points when ρ = 0.999.[17]

One notable difference when $\rho = 0.999$ is that LOG performs better than KPR1 in terms of consumption, which should be interpreted with caution. This result implies only that the LOG approximation is more accurate than the KPR1 procedure around the stationary point of capital. It says nothing about the fit of the policy function over the entire state space. In fact, the policy function for consumption under LOG in this case is similar to the graph shown in figure 1b. Thus, if one were interested in the response of consumption to tax shocks, the LOG procedure will still yield a misleading transition path even though the stationary point is closer to the true solution.

5. Frequency Domain Comparison

In this section we extend the comparison to the frequency domain and examine the coherence function between realizations generated by the discrete state space solutions and those produced by the various approximations. This analysis allows us to compare the goodness of fit by frequency.

The coherence between the DSS solutions and the approximate solutions were estimated using a standard time series method. For each variable, a sample path of 1000 observations were generated from each solution procedure. These data were then used to estimate the coherence over the frequency $[0,\pi]$ (i.e., cycles per period). A window of width 50 was utilized to smooth the

[17] The distribution of capital is simulated from a sample path containing 2000 observations, starting from the mean tax rate at the initial date. Because of the high persistence in shocks the realizations never visit either of the two boundary points of the tax support. If more observations were generated, then the realized tax rates might cover the entire space, in which case the distribution of capital will concentrate on five stationary points.

spectral and cospectral densities needed to compute the coherence function.

The results of this experiment are presented in figure 2, which are consistent with the material reported in the previous section. Recall that LOG and KPR1 produce identical solutions for capital, labor, and output while LIN and KPR2 produce identical solutions for capital, labor, and investment. Therefore, some of the curves overlap in the figure. In general, all four approximations have a coherence of greater than 99 percent with respect to output, capital, and labor hours. Some systematic differences do exist, but their magnitudes are trivial. The two KPR methods approximate interest rates and consumption quite well while both LIN and LOG generally account for about 50 percent of consumption and the real interest rate generated by the DSS procedure. Notice that both LIN and LOG perform poorly with respect to these two variables at high frequencies (short cycles). Regarding investment, both LIN and LOG show a high degree of coherence while KPR1's coherence with DSS is noticeably lower. On balance, our experiment suggests that KPR2 seems to have the best performance in this regard.

6. Summary and Conclusions

In this paper we have investigated several approximation methods widely being used in the literature to solve for dynamic equilibria. These methods generally work quite well in models where the underlying forcing process has small variance. However, for models in which dynamics are driven by fiscal or monetary shocks, which have exhibited substantial variations over time, these approximation methods may not be very accurate. The findings of this paper generally support this conjecture.

Although our example is somewhat limited in scope, we believe that our experiments provide some useful guidelines on how to select an approximation scheme given the model that is being studied. In general, the approximation methods we investigated are all fairly accurate in terms of capital, output, and labor hours, but fail to some extent with regard to other variables. The linear KPR approximation (KPR2) performs relatively well on all dimensions compared to the other approximation procedures. The log-linear KPR approximation (KPR1) is not very accurate in terms of investment, so if one were primarily interested in the effects of taxes on investment, this procedure is not recommended. Both the LIN and LOG approximations perform poorly in terms of consumption and the real interest rate, so these procedures might not be appropriate for analyzing the welfare implications of tax policies.

One of the disadvantages frequently raised by many researchers regarding the discrete state space method is that it is costly to implement. This is certainly true for models in which the state space has large dimensions. But if the problem on hand is relatively small and one is not certain about the accuracy of the linear approximations, then the discrete state space method is an inexpensive and feasible procedure. There are many "tricks" that can be used to speed up the computation as well as to improve the accuracy. Our implementation of the algorithm takes an average of 6 to 10 CPU minutes to solve the problem on a standard mainframe computer. Considering the accuracy gain one might achieve, this cost is tolerable.

Appendix A

This appendix sketches the procedures for deriving the policy functions for capital and labor hours under both the LIN and LOG approximations.

LIN Procedure

To conserve on notation, rewrite the Euler equations (3a) and (3b) as follows:

$$g(K_t, K_{t+1}, N_t) = \beta E_t[h(K_{t+1}, K_{t+2}, N_{t+1}, 1-\tau_{t+1})], \tag{A1}$$

$$q(K_t, K_{t+1}, N_t) = w(K_t, N_t, 1-\tau_t), \tag{A2}$$

where

$$g(K_t,K_{t+1},N_t) = u_c[F(K_t,N_t) + (1-\delta)K_t - K_{t+1}, 1-N_t],$$

$$h(K_{t+1},K_{t+2},N_{t+1},1-\tau_{t+1}) = u_c[F(K_{t+1},K_{t+1}) + (1-\delta)K_{t+1} - K_{t+2}, 1-N_{t+1}] \times [(1-\tau_{t+1})F_k(K_{t+1},N_{t+1}) + (1-\delta)],$$

$$q(K_t,K_{t+1},N_t) = \frac{u_l[F(K_t,N_t) + (1-\delta)K_t - K_{t+1}, 1-N_t]}{u_c[F(K_t,N_t) + (1-\delta)K_t - K_{t+1}, 1-N_t]},$$

$$w(K_t,N_t,1-\tau_t) = (1-\tau_t)F_n(K_t,N_t).$$

Let K* and N* be the nonstochastic steady state values of capital and hours worked and τ^* the unconditional mean of τ_t. With the Cobb-Douglas technology (4) and the CRRA preferences (5), it can be shown that at the steady state,

$$N^* = \frac{(1-\alpha)(1-\tau^*)}{(1-\theta)(1-\delta A)/\theta + (1-\alpha)(1-\tau^*)} \quad \text{and} \quad K^* = A^{1/(1-\alpha)} N^*,$$

where $A = \alpha\beta(1-\tau^*)/[1-\beta(1-\delta)]$. For subsequent derivations, it is convenient to define $\tilde{K}_t = K_t - K^*$, $\tilde{N}_t = \tilde{N}_t - N^*$ and $1 - \tilde{\tau}_t = (1-\tau_t) - (1-\tau^*)$ for all t.

Now, taking a first order Taylor's approximation to equations (A1) and (A2) around K*, N* and $(1-\tau^*)$ yields the following linear system:

$$g_1^* \tilde{K}_t + g_2^* \tilde{K}_{t+1} + g_3^* \tilde{N}_t =$$
$$\beta E_t[h_1^* \tilde{K}_{t+1} + h_2^* \tilde{K}_{t+2} + h_3^* \tilde{N}_{t+1} + h_4^* (1-\tilde{\tau}_{t+1})], \tag{A3}$$

$$q_1^* \tilde{K}_t + q_2^* \tilde{K}_{t+1} + q_3^* \tilde{N}_t = w_1^* \tilde{K}_t + w_2^* \tilde{N}_t + w_3^* (1-\tilde{\tau}_t), \tag{A4}$$

where g_i^*, h_i^*, q_i^*, and w_i^* are the partial derivatives of the corresponding functions with respect to the i^{th} argument evaluated at K*, N* and τ^*. In deriving (A3) and (A4), use has been made of the fact that $g^* = \beta h^*$ and $q^* = w^*$ at the steady state.

Equations (A3) and (A4) jointly determine capital $\tilde{K}_{t+1}$ and labor $\tilde{N}_t$ for the given states $\tilde{K}_t$ and $\tilde{\tau}_t$. To solve the system, observe from (A4) that

$$\tilde{N}_t = p_1^* \tilde{K}_t - p_2^* \tilde{K}_{t+1} + p_3^* (1-\tilde{\tau}_t), \tag{A5}$$

where $p_1^* = (w_1^* - q_1^*)/(q_3^* - w_2^*)$, $p_2^* = q_2^*/(q_3^* - w_2^*)$ and $p_3^* = w_3^*/(q_3^* - w_2^*)$. Now, updating (A5), taking conditional expectations and substituting $\tilde{N}_t$ and the resulting $E_t(\tilde{N}_{t+1})$ into equation (A3) produces an "expectational" second order difference equation in $\tilde{K}_t$:

$$E_t(\tilde{K}_{t+2}) + a \tilde{K}_{t+1} + b \tilde{K}_t = c (1-\tilde{\tau}_t), \tag{A6}$$

where

$$a = [\beta(h_1^* + h_3^* p_1^*) + g_3^* p_2^* - g_2^*] / \beta(h_2^* - h_3^* p_2^*),$$
$$b = -(g_1^* + g_3^* p_1^*) / \beta(h_2^* - h_3^* p_2^*),$$
$$c = [g_3^* p_3^* - \beta(h_4^* + h_3^* p_3^*)\rho] / \beta(h_2^* - h_3^* p_2^*).$$

Using standard methods (e.g., Sargent (1979)), the above difference equation yields a solution for $\tilde{K}_{t+1}$ as follows:

$$\tilde{K}_{t+1} = \lambda_1 \tilde{K}_t + \frac{c}{\lambda_1 + a + \rho} (1-\tilde{\tau}_t), \tag{A7}$$

where λ_1 is the "stable" root (i.e., $|\lambda_1| < 1$) of the characteristic equation

$\lambda^2 + a\lambda + b = 0.$[18] Substituting (A7) into equation (A5) produces

$$\tilde{N}_t = (p_1^* - p_2^* \lambda_1) \tilde{K}_t + (p_3^* - \frac{c}{\lambda_1 + a + \rho} P_2^*) (1-\tilde{\tau}_t). \tag{A8}$$

Rewriting (A7) and (A8) in terms of K_{t+1} and N_t yields the policy functions (6a) and (6b) in the text.

LOG Procedure

The LOG procedure is identical to the LIN procedure except that the linearization is taken around the logarithm of the steady state values of K_t, N_t and $(1-\tau_t)$. Specifically, let $k_t = \ln K_t$, $n_t = \ln N_t$ and $\phi_t = \ln(1-\tau_t)$, then the Euler equations that correspond to (A1) and (A2) can be expressed as follows:

$$\bar{g}(k_t, k_{t+1}, n_t) = \beta E_t[\bar{h}(k_{t+1}, k_{t+2}, n_{t+1}, \phi_{t+1})], \tag{A9}$$

$$\bar{q}(k_t, k_{t+1}, n_t) = \bar{w}(k_t, n_t, \phi_t), \tag{A10}$$

where the functions $\bar{g}$, $\bar{h}$, $\bar{q}$ and $\bar{w}$ are defined analogously except that capital letters are replaced by their exponential representations (e.g., $K_t = \exp(k_t)$ etc.) and $(1-\tau_t) = \exp(\phi_t)$. We use different notation for these functions to emphasize the fact that the points at which these functions are evaluated are different from those under the LIN approximation.

Define $\tilde{k}_t = k_t - k^*$, $\tilde{n}_t = n_t - n^*$ and $\tilde{\phi}_t = \phi_t - \phi^*$ and following the same procedure as before, we obtain a similar difference equation in $\tilde{k}_t$:

$$E_t(\tilde{k}_{t+2}) + \bar{a}\, \tilde{k}_{t+1} + \bar{b}\, \tilde{k}_t = \bar{c}\, \tilde{\phi}_t, \tag{A11}$$

where

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[18]With the Cobb-Douglas production function and the CRRA utility function, it can be shown that the coefficient b in equation (A6) is greater than one and that one of the roots of the characteristic equation is unstable and the other is stable. This property is also discussed in King, Plosser and Rebelo (1988a). Choosing the stable root amounts to imposing a transversality condition for which an equilibrium solution must satisfy.

$$\bar{a} = [\beta(\bar{h}_1^* + \bar{h}_3^* \bar{p}_1^*) + \bar{g}_3^* \bar{p}_2^* - \bar{g}_2^*] \;/\; \beta(\bar{h}_2^* - \bar{h}_3^* \bar{p}_2^*),$$
$$\bar{b} = -(\bar{g}_1^* + \bar{g}_3^* \bar{p}_1^*) \;/\; \beta(\bar{h}_2^* - \bar{h}_3^* \bar{p}_2^*),$$
$$\bar{c} = [\bar{g}_3^* \bar{p}_3^* - \beta(\bar{h}_4^* + \bar{h}_3^* \bar{p}_3^*)\rho] \;/\; \beta(\bar{h}_2^* - \bar{h}_3^* \bar{p}_2^*),$$

and $\bar{p}_1^* = (\bar{w}_1^* - \bar{q}_1^*)/(\bar{q}_3^* - \bar{w}_2^*)$, $\bar{p}_2^* = \bar{q}_2^*/(\bar{q}_3^* - \bar{w}_2^*)$ and $\bar{p}_3^* = \bar{w}_3^*/(\bar{q}_3^* - \bar{w}_2^*)$. Since the utility function and the production functions are evaluated at different points, the values of the coefficients $\bar{a}$, $\bar{b}$ and $\bar{c}$ are different from those of equation (A6).

The capital stock $\tilde{k}_{t+1}$ that solves equation (A11) is

$$\tilde{k}_{t+1} = \bar{\lambda}_1 \tilde{k}_t + \frac{\bar{c}}{\bar{\lambda}_1 + \bar{a} + \rho} \tilde{\phi}_t. \tag{A12}$$

Again, $\bar{\lambda}_1$ is the stable root of the characteristic equation $\lambda^2 + \bar{a}\lambda + \bar{b} = 0$. The corresponding policy function for $\tilde{n}_t$ is

$$\tilde{n}_t = (\bar{p}_1^* - \bar{p}_2^* \bar{\lambda}_1) \tilde{k}_t + (\bar{p}_3^* - \frac{\bar{c}}{\bar{\lambda}_1 + \bar{a} + \rho} \bar{P}_2^*) \tilde{\phi}_t. \tag{A13}$$

Rewriting (A12) and (A13) in terms of k_{t+1} and n_t yields the policy functions (11a) and (11b) in the text.

Appendix B

In this appendix we describe the numerical method we used to solve the problem in the text. Mathematical background for this method can be found in Coleman (1989) (see, also, Baxter, Crucini and Rouwenhorst (1990)).

Let $K' = G(K,\tau)$ and $N = H(K,\tau)$ be the policy functions that solve (3a) and (3b) where for notational brevity the unprimed variables denote variables at time t and single primes denote variables at time t+1. To solve for G and H, we first obtain from (3b) a solution for labor, denoted $\tilde{H}(K,\tau,K')$, for any given triple of K, τ and K'.[19] The value of this function is used first to eliminate labor from (3a) and second, to avoid unnecessary discretization of the labor space. Once the optimal solution for capital is determined, the labor decision can be uncovered by the relation $H(K,\tau) \equiv \tilde{H}[K,\tau,G(K,\tau)]$. To solve for $G(K,\tau)$ we introduce a mapping, T, which operates on the function G and is defined recursively by (3a). Formally, this mapping is given by

$$u_c\Big[F[K,H(K,\tau)] + (1-\delta)K - TG(K,\tau),\ 1-H(K,\tau)\Big] =$$

$$\beta E\Big\{u_c\Big[F\{TG(K,\tau),H[TG(K,\tau),\tau']\}+(1-\delta)TG(K,\tau)-G[TG(K,\tau),\tau'],\ 1-H[TG(K,\tau),\tau']\Big] \times$$

$$\Big[(1-\tau')F_k\{TG(K,\tau),H[TG(K,\tau),\tau']\} + (1-\delta)\Big]\Big\}. \qquad \text{(B1)}$$

Equation (B1) is obtained from (3a) with capital and labor replaced by the appropriate functions. For example, the end-of-period capital at time t+1 is given by $G[TG(K,\tau),\tau']$. Note that TG is an argument of G. The function H, which appears also in (B1), should be interpreted as the function $\tilde{H}$ evaluated at appropriate points (e.g., $H(K',\tau') = \tilde{H}[TG(K,\tau),\tau',G(TG(K,\tau),\tau')]$, etc.). The expectation on the right side of equation (B1) is taken over values of τ' conditional on τ. In section 3, we describe a method to approximate the conditional distribution of τ.

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[19]Note that (3b) is an algebraic equation, which can be easily solved using standard methods such as the Newton-Raphson procedure.

The operator defined by (B1) maps a function G to another function TG. The fixed point of this mapping (G = TG), which exists under regularity conditions (see Coleman (1989a)), is the equilibrium solution for capital. The DSS procedure we use approximates this function by iterating on (B1) using the standard successive approximation algorithm. Starting from a given function, say, G^0, this algorithm first searches a new function $G^1 = TG^0$ that solves equation (B1) for all possible pairs of K and τ. The values of G are restricted to lie on the fixed grid points of capital. If G^1 is not equal to G^0, then the function G^1 is taken as the new starting point and the process continues. This procedure generates a sequence of policy functions $\{G^j\}$ that uniformly converges to the fixed point of the functional equation (B1) (see Coleman (1989)). Let $\varepsilon^j = \sup|G^j(K,\tau)-G^{j-1}(K,\tau)|$ be the posterior error at step j. Then the iteration stops when ε^j is a small number, at which point $G^j(K,\tau)$ is taken to be a numerical approximation to the policy function for capital. Associated with G^j is a policy function H^j for labor, obtained by substituting G^j into the function $\tilde{H}$. Once capital and labor are determined, other quantities and prices can be derived.

There are a number of ways to improve the efficiency of the algorithm. For example, starting from a zero function, the sequence of functions $\{G^j\}$ generated by (B1) is monotonic and increasing (see Coleman (1989)), so it is not necessary to evaluate both sides of (B1) over the entire state space in order to find the new policy function. The monotonicity of the functional operator defined by (B1) greatly speeds up the computational algorithm.

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Table 1a

Probability-Weighted Percentage Errors in Policy Functions
(Tax Process: ρ = 0.8)

	Log-Linear Approximation			Linear Approximation		
	LOG		KPR1	LIN		KPR2
			Panel A: σ = 1			
Y	1.15 (0.00)	[0]	1.15 (0.00)	1.95 (0.00)	[+]	1.34 (0.01)
K	1.43 (0.00)	[0]	1.43 (0.00)	1.51 (0.01)	[0]	1.51 (0.01)
N	1.64 (0.00)	[0]	1.64 (0.00)	2.77 (0.01)	[0]	2.77 (0.01)
C	2.96 (0.01)	[+]	1.35 (0.00)	1.98 (0.01)	[-]	2.99 (0.02)
I	118.86 (6.05)	[-]	137.78 (7.81)	252.80(43.92)	[0]	252.80(43.42)
R	383.16(18.94)	[+]	76.24 (3.53)	141.27 (6.11)	[+]	52.28 (3.08)
			Panel B: σ = 2			
Y	1.10 (0.00)	[0]	1.10 (0.00)	1.25 (0.00)	[0]	1.26 (0.01)
K	1.17 (0.00)	[0]	1.17 (0.00)	1.05 (0.01)	[0]	1.05 (0.01)
N	1.58 (0.01)	[0]	1.58 (0.01)	1.78 (0.00)	[0]	1.78 (0.00)
C	2.41 (0.01)	[+]	1.06 (0.00)	1.50 (0.01)	[-]	2.48 (0.01)
I	86.09 (5.62)	[-]	110.74 (3.23)	73.08 (1.81)	[0]	73.08 (1.81)
R	494.28(63.28)	[+]	100.79(11.55)	159.61(16.85)	[+]	100.20(14.01)
			Panel C: σ = 10			
Y	0.97 (0.01)	[0]	0.97 (0.00)	0.96 (0.01)	[-]	1.48 (0.01)
K	0.75 (0.00)	[0]	0.75 (0.00)	0.64 (0.00)	[0]	0.64 (0.01)
N	1.40 (0.01)	[0]	1.40 (0.01)	1.37 (0.01)	[0]	1.37 (0.00)
C	1.44 (0.00)	[+]	0.69 (0.00)	0.92 (0.00)	[-]	1.89 (0.01)
I	21.42 (0.48)	[-]	22.02 (0.55)	14.92 (0.62)	[0]	14.92 (0.62)
R	420.49(36.76)	[+]	159.41(14.78)	512.39(47.42)	[+]	87.07 (7.80)

Note: 1. The percentage error is calculated according to:

$$100 \times \sum_{K,\tau}\{\Pr[K,\tau]\times|[G_i(K,\tau)-G(K,\tau)]/G(K,\tau)|\},$$

where $\Pr[K,\tau]$ is the stationary probability of K and τ, $G_i(K,\tau)$ is the approximate solution and $G(K,\tau)$ is the DSS solution.

2. Figures reported are the average percentage errors calculated from 100 simulations of the model. Each simulation generates from the DSS procedure 2000 observations of K and τ which are used to approximate the joint distribution of K and τ. The estimated standard deviation of the mean of percentage errors is indicated in the parenthesis.

Table 1b

Detail Information on the percentage Errors of Investment

State		DSS	KPR1		KPR2	
τ (1)	K (2)	Investment (3)	I (4)	% error (5)	I (6)	% error (7)
			Panel A			
0.49	0.6426	-0.0277	0.0026	0.0546	-0.0304	0.0047
0.49	0.6567	-0.0295	0.0025	0.0542	-0.0318	0.0038
0.49	0.6662	-0.0311	0.0024	0.0538	-0.0328	0.0028
			Panel B			
0.63	0.1422	0.0016	0.0047	0.0937	-0.0230	0.7602
0.63	0.1497	-0.000001	0.0042	418.9136	-0.0238	2380.1079
0.63	0.1567	-0.0011	0.0038	0.2158	-0.0245	1.0173

Note: The first two columns indicate the state of the economy and the third column lists the values of the DSS investment for the corresponding states. Columns (4) to (6) list the values of investment and the probability weighted errors for KPR1 and KPR2, respectively.

Table 1c

Probability-Weighted Absolute Errors in Policy Functions
(Tax Process: ρ = 0.8)

	Log-Linear Approximation			Linear Approximation		
	LOG		KPR1	LIN		KPR2
			Panel A: σ = 1			
Y	.011 (.000)	[0]	.011 (.000)	.020 (.000)	[+]	.012 (.000)
K	.013 (.000)	[0]	.013 (.000)	.013 (.000)	[0]	.013 (.000)
N	.015 (.000)	[0]	.015 (.000)	.024 (.000)	[0]	.024 (.000)
C	.029 (.000)	[+]	.013 (.000)	.019 (.000)	[-]	.029 (.000)
I	.201 (.000)	[-]	.619 (.002)	.192 (.000)	[0]	.192 (.000)
R	.334 (.001)	[+]	.092 (.000)	.277 (.001)	[+]	.125 (.000)
			Panel B: σ = 2			
Y	.010 (.000)	[0]	.010 (.000)	.014 (.000)	[+]	.011 (.000)
K	.011 (.000)	[0]	.011 (.000)	.010 (.000)	[0]	.010 (.000)
N	.014 (.000)	[0]	.014 (.000)	.017 (.000)	[0]	.017 (.000)
C	.023 (.000)	[+]	.010 (.000)	.014 (.000)	[-]	.024 (.000)
I	.182 (.000)	[-]	.399 (.001)	.153 (.000)	[0]	.153 (.000)
R	.298 (.001)	[+]	.092 (.000)	.249 (.001)	[+]	.109 (.000)
			Panel C: σ = 10			
Y	.009 (.000)	[0]	.009 (.000)	.010 (.000)	[-]	.013 (.000)
K	.007 (.000)	[0]	.007 (.000)	.006 (.000)	[0]	.006 (.000)
N	.012 (.000)	[0]	.012 (.000)	.012 (.000)	[0]	.012 (.000)
C	.014 (.000)	[+]	.007 (.000)	.009 (.000)	[-]	.018 (.000)
I	.135 (.000)	[-]	.156 (.000)	.114 (.000)	[0]	.114 (.000)
R	.374 (.001)	[+]	.112 (.000)	.446 (.001)	[+]	.111 (.000)

Note: 1. The absolute error is calculated according to:

$$\Sigma_{K,\tau}\{Pr[K,\tau]\times|G_i(K,\tau)-G(K,\tau)|\} \;/\; \Sigma_{K,\tau}Pr[K,\tau]\times|G(K,\tau)|,$$

where $Pr[K,\tau]$ is the stationary probability of K and τ, $G_i(K,\tau)$ is the approximate solution and $G(K,\tau)$ is the DSS solution.

2. Figures reported are the average squared errors calculated from 100 simulations of the model. Each simulation generates from the DSS procedure 2000 observations of K and τ which are used to approximate the joint distribution of K and τ. The estimated standard deviation of the mean of absolute errors is indicated in the parenthesis.

Table 2

Correlation Matrices: $\sigma = 1$

DSS

	τ	Y	K	N	C	I	R
τ	1.000						
Y	-0.985 (0.000)	1.000					
K	-0.712 (0.002)	0.816 (0.001)	1.000				
N	-0.975 (0.000)	0.923 (0.000)	0.541 (0.002)	1.000			
C	-0.747 (0.001)	0.846 (0.001)	0.993 (0.000)	0.584 (0.001)	1.000		
I	-0.901 (0.001)	0.822 (0.001)	0.348 (0.001)	0.970 (0.000)	0.392 (0.001)	1.000	
R	-0.510 (0.005)	0.358 (0.005)	-0.210 (0.003)	0.683 (0.004)	-0.173 (0.003)	0.803 (0.002)	1.000

LOG

	τ	Y	K	N	C	I	R
τ	1.000						
Y	-0.984 (0.000)	1.000					
K	-0.709 (0.002)	0.821 (0.001)	1.000				
N	-0.970 (0.000)	0.912 (0.000)	0.522 (0.002)	1.000			
C	-0.730 (0.002)	0.828 (0.002)	0.971 (0.000)	0.560 (0.002)	1.000		
I	-0.876 (0.001)	0.800 (0.001)	0.345 (0.001)	0.939 (0.001)	0.325** (0.002)	1.000	
R	-0.406** (0.006)	0.308* (0.006)	-0.094*** (0.005)	0.503** (0.006)	-0.196* (0.005)	0.729 (0.004)	1.000

KPR1

	τ	Y	K	N	C	I	R
τ	1.000						
Y	-0.984 (0.000)	1.000					
K	-0.709 (0.002)	0.821 (0.001)	1.000				
N	-0.970 (0.000)	0.912 (0.000)	0.522 (0.002)	1.000			
C	-0.741 (0.001)	0.845 (0.001)	0.993 (0.000)	0.558 (0.002)	1.000		
I	-0.627*** (0.005)	0.537*** (0.005)	0.130*** (0.004)	0.745** (0.004)	0.158*** (0.004)	1.000	
R	-0.529 (0.004)	0.374 (0.004)	-0.196 (0.002)	0.706 (0.003)	-0.164 (0.002)	0.651** (0.003)	1.000

Table 2 (continued)

LIN

	τ	Y	K	N	C	I	R
τ	1.000						
Y	-0.977	1.000					
	(0.000)						
K	-0.719	0.831	1.000				
	(0.002)	(0.002)					
N	-0.974	0.910	0.544	1.000			
	(0.000)	(0.001)	(0.002)				
C	-0.726	0.851	0.986	0.558	1.000		
	(0.002)	(0.001)	(0.000)	(0.002)			
I	-0.901	0.799	0.346	0.975	0.364	1.000	
	(0.001)	(0.001)	(0.001)	(0.000)	(0.002)		
R	-0.359**	0.200***	-0.129***	0.474***	-0.185*	0.564**	1.000
	(0.008)	(0.007)	(0.004)	(0.009)	(0.005)	(0.011)	

KPR2

	τ	Y	K	N	C	I	R
τ	1.000						
Y	-0.986	1.000					
	(0.000)						
K	-0.719	0.825	1.000				
	(0.002)	(0.001)					
N	-0.974	0.923	0.544	1.000			
	(0.000)	(0.000)	(0.002)				
C	-0.743	0.845	0.999	0.574	1.000		
	(0.001)	(0.001)	(0.000)	(0.001)			
I	-0.901	0.815	0.346	0.975	0.380	1.000	
	(0.001)	(0.001)	(0.001)	(0.000)	(0.001)		
R	-0.535	0.386	-0.202	0.711	-0.167	0.848	1.000
	(0.004)	(0.004)	(0.002)	(0.003)	(0.002)	(0.002)	

Note:

1. Figures reported are the average of 100 repetitions; each repetition contains 1000 observations generated by each solution procedure.
2. The estimated standard deviation of the mean of correlation coefficients is indicated in the parenthesis.
3. * = % difference ≥ 10 % and ≤ 20 %.
   ** = % difference > 20 % and ≤ 30 %.
   *** = % difference > 30 %.

Table 3

The Den Haan-Marcet statistic for testing the orthogonality conditions

σ	DSS	LOG	KPR1	LIN	KPR2
1	9.13 (0.24)	60 (0.00)	237 (0.00)	27 (0.00)	19 (0.01)
2	8.33 (0.30)	51 (0.00)	263 (0.00)	46 (0.00)	88 (0.00)
10	6.94 (0.43)	55 (0.00)	99 (0.00)	30 (0.00)	65 (0.00)

Note: The statistic (degree of freedom = 7) is computed based on a realization of 1000 observations, generated by each solution procedure. The probability value (i.e., the tail area) of the test statistic is indicated in the parenthesis. Test results are the same for smaller sample size and larger degree of freedom (i.e., more lags used to form instruments).

Table 4

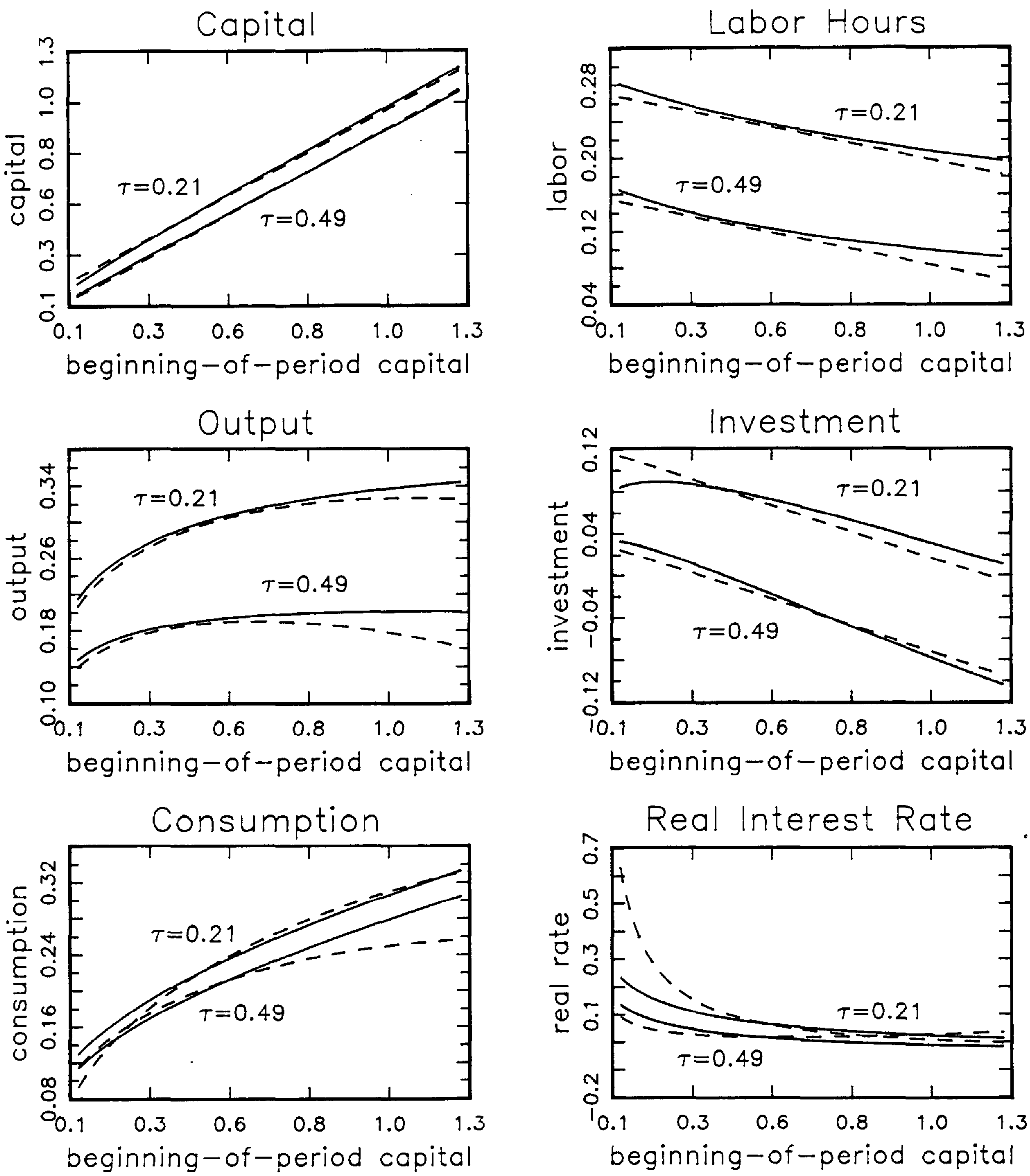
Probability-Weighted Errors in Policy Functions (σ = 1)
(Tax Process: ρ = 0.999)

	Log-Linear Approximation			Linear Approximation		
	LOG		KPR1	LIN		KPR2
	Panel A: Percentage Errors					
Y	.190 (.013)	[0]	.190 (.013)	.803 (.082)	[+]	.191 (.015)
K	.076 (.004)	[0]	.076 (.004)	1.616 (.240)	[0]	1.616 (.240)
N	.272 (.018)	[0]	.272 (.018)	1.141 (.116)	[0]	1.141 (.116)
C	.138 (.010)	[-]	.390 (.036)	1.839 (.222)	[-]	2.735 (.300)
I	2.328 (.100)	[-]	4.510 (.275)	34.869(5.042)	[0]	34.869(5.042)
R	1.331 (.058)	[+]	.825 (.041)	14.811(2.033)	[+]	8.647 (.714)
	Panel B: Absolute Errors					
Y	.002 (.000)	[0]	.002 (.000)	.008 (.001)	[+]	.002 (.000)
K	.001 (.000)	[0]	.001 (.000)	.011 (.002)	[0]	.011 (.002)
N	.003 (.000)	[0]	.003 (.000)	.011 (.001)	[0]	.011 (.001)
C	.001 (.000)	[-]	.004 (.000)	.016 (.002)	[-]	.024 (0.02)
I	.017 (.001)	[-]	.023 (.001)	.231 (.032)	[0]	.231 (.032)
R	.011 (.000)	[+]	.007 (.000)	.147 (.020)	[+]	.086 (.007)

Note: Figures reported are the average errors calculated from 100 simulations of the model. Each simulation generates from the DSS procedure 2000 observations of K and τ which are used to approximate the joint distribution of K and τ. The estimated standard deviation of the mean errors is indicated in the parenthesis.

Figure 1a: Policy Functions (DSS vs. LIN)

(parameters: σ = 1, variance of τ = 0.02)



DSS: ———— LIN: - - -

Figure 1b: Policy Functions (DSS vs. LOG)

(parameters: σ = 1, variance of τ = 0.02)

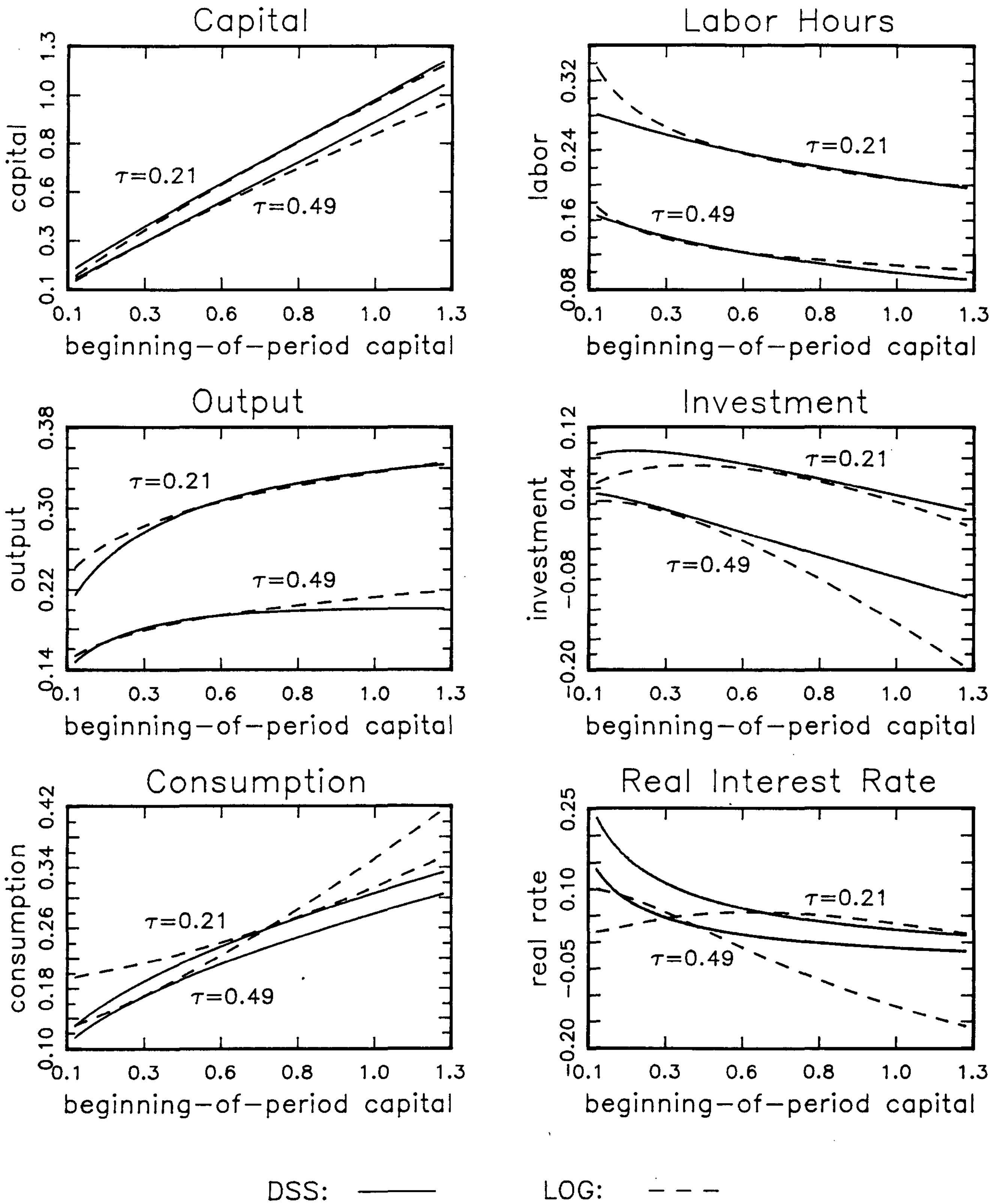
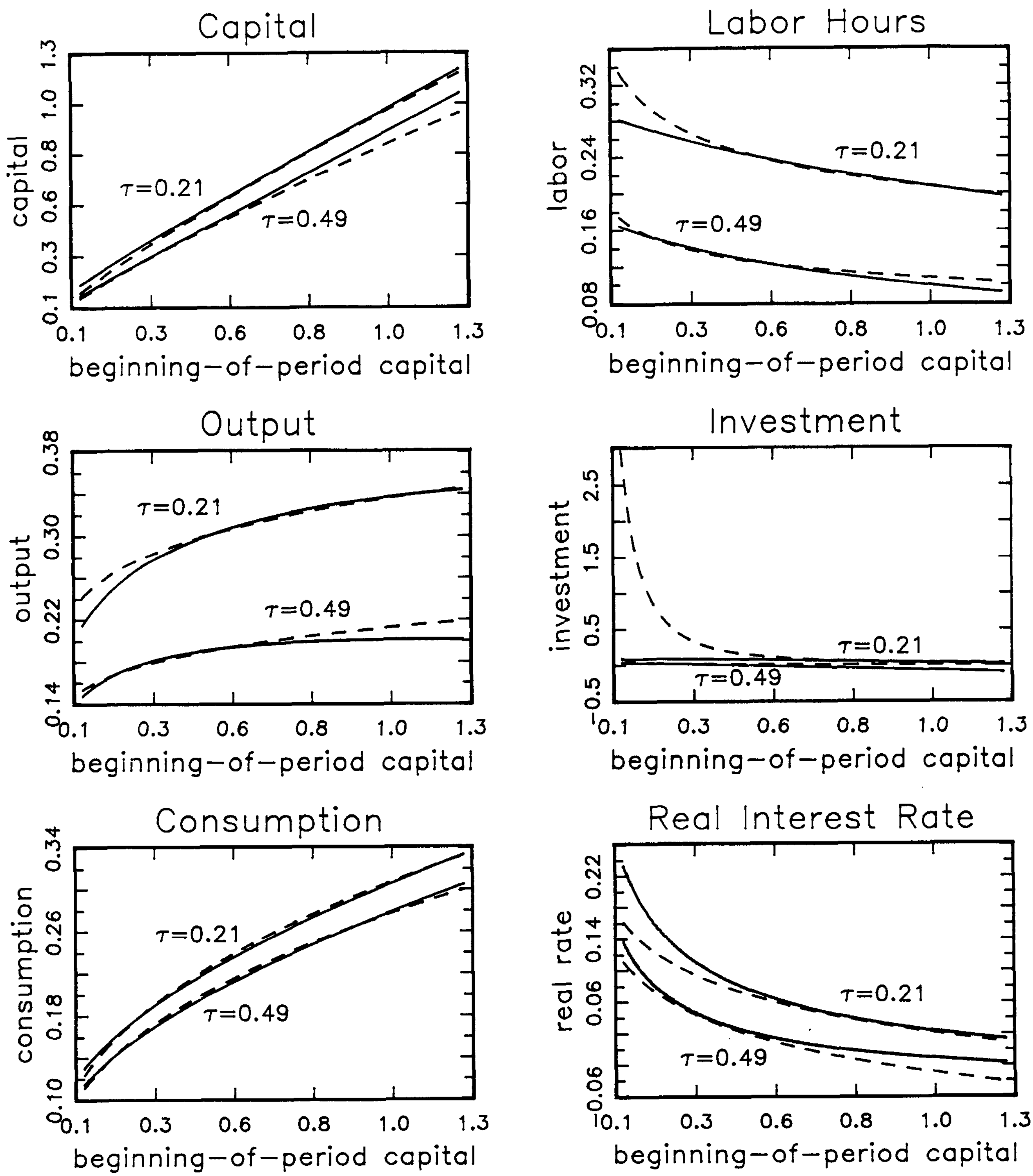


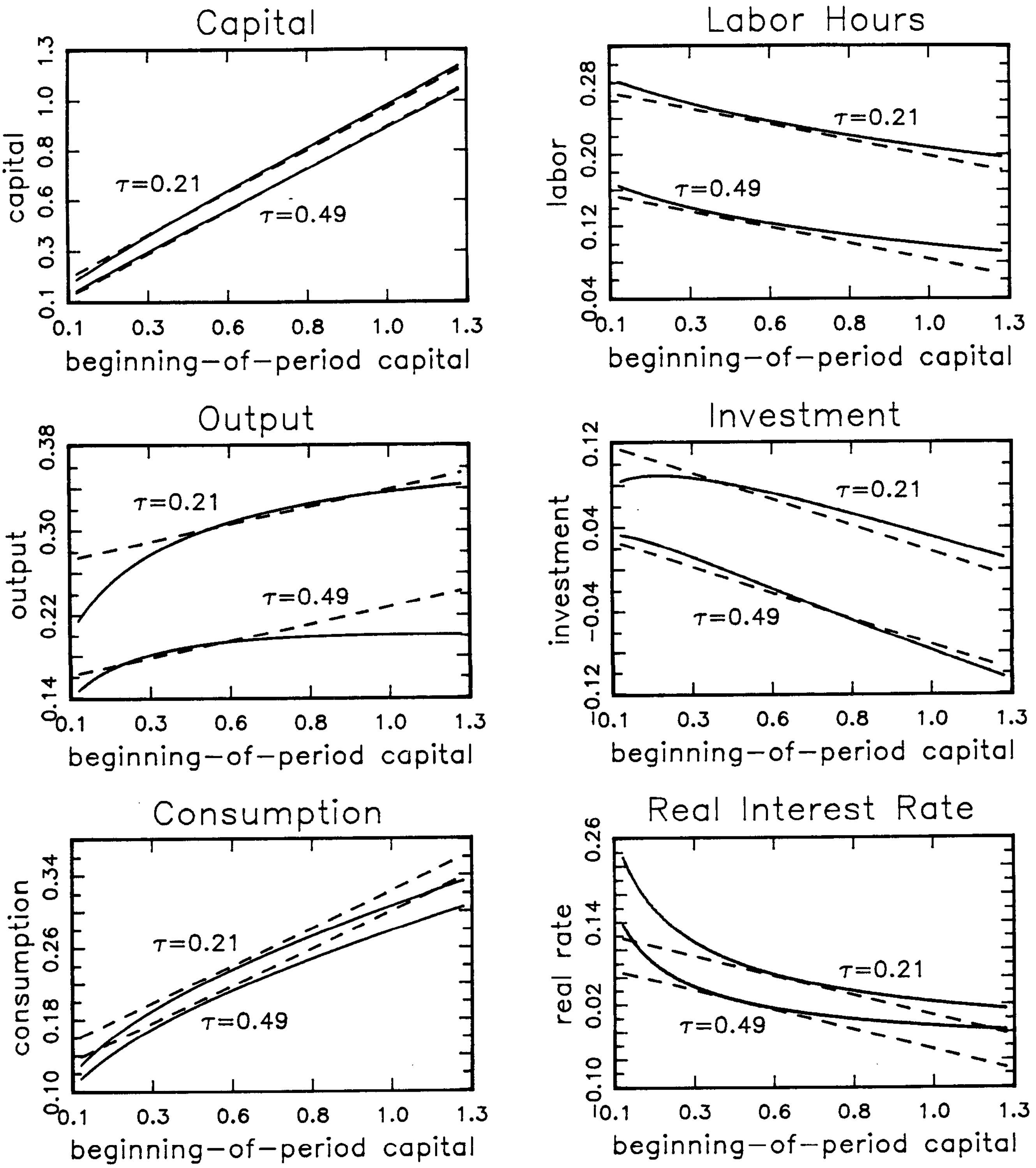
Figure 1c: Policy Functions (DSS vs. KPR1)
(parameters: $\sigma = 1$, variance of $\tau = 0.02$)



DSS: ———   KPR1: - - -

Figure 1d: Policy Functions (DSS vs. KPR2)

(parameters: $\sigma = 1$, variance of $\tau = 0.02$)



DSS: ———   KPR2: - - -

Figure 1e: Distribution of Capital

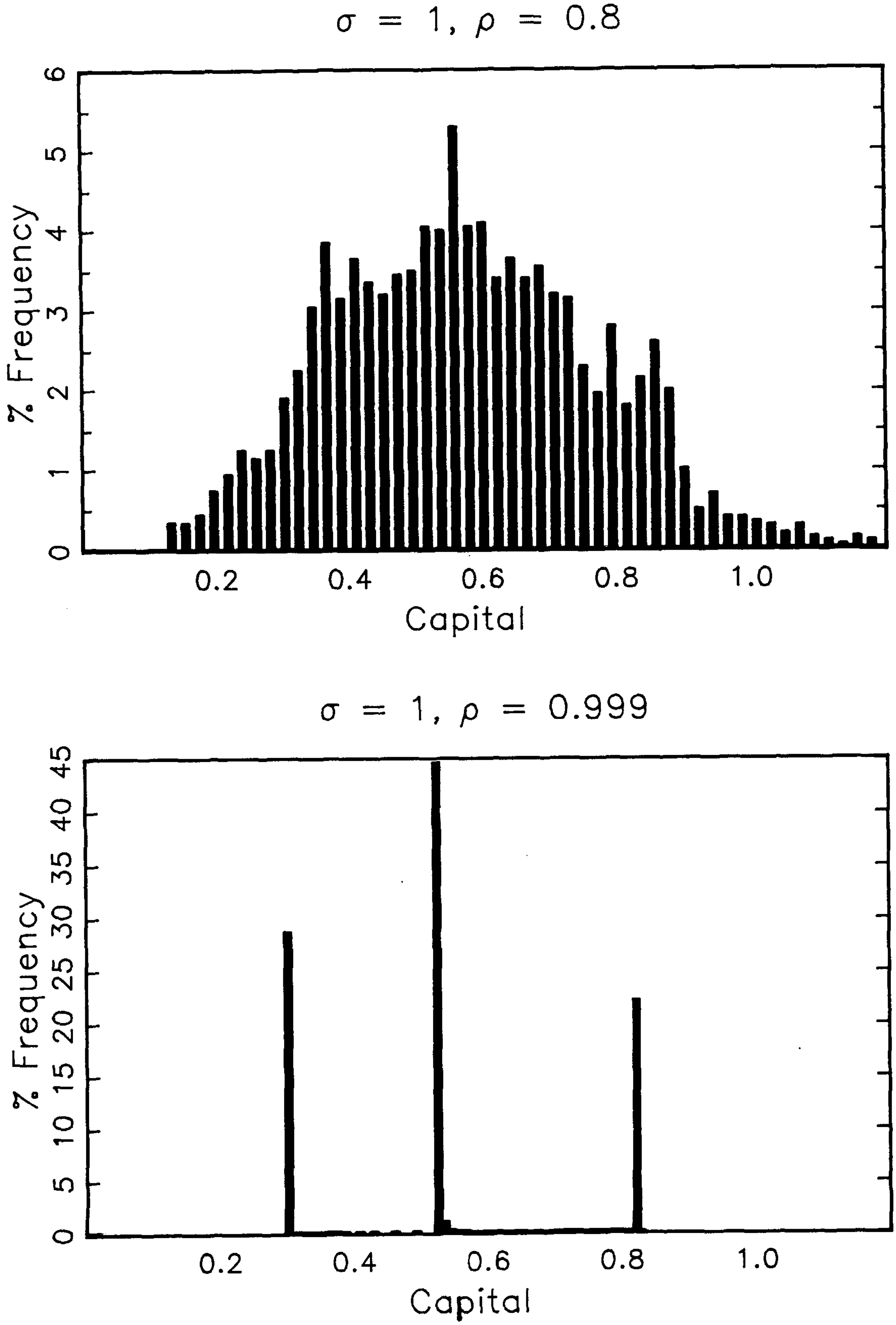


Figure 2: Coherence (DSS vs. LIN, LOG, KPR1 and KPR2)
(parameters: σ = 1, variance of τ = 0.02)