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Abstract

We propose and analyze a discontinuous Galerkin approximation for the Stokes problem. The finite element triangulation employed is not required to be conforming and we use discontinuous pressures and velocities. No additional unknown fields need to be introduced, but only suitable bilinear forms defined on the interfaces between the elements, involving the jumps of the velocity and the average of the pressure. We consider hp approximations using $\mathbb{Q}_{k'}$-$\mathbb{Q}_k$ velocity-pressure pairs with $k' = k+2, k+1, k$. Our methods show better stability properties than the corresponding conforming ones. We prove that our first two choices of velocity spaces ensure uniform divergence stability with the respect to the mesh size h. Numerical results show that they are uniformly stable with respect to the local polynomial degree k, a property that has no analog in the conforming case. An explicit bound in k which is not sharp is also proven. Numerical results show that if equal order approximation is chosen for the velocity and pressure, no spurious pressure modes are present but the method is not uniformly stable either with respect to h or k. We derive a priori error estimates generalizing the abstract theory of mixed methods. Optimal error estimates in h are proven. As for discontinuous Galerkin methods for scalar diffusive problems, half power of k is lost for p and hp approximations independently of the divergence stability.

Keywords: Mixed problems, hp approximations, spectral elements, discontinuous Galerkin approximations, non-conforming approximations

Subject Classification: 65N30, 65N35, 65N12, 65N15

1. Introduction. Discontinuous Galerkin (DG) methods have a long history and have recently become more and more popular. They have been heavily tested and studied, and they present considerable advantages for certain types of problems, especially those modeling phenomena where convection is strong; see the monograph [15].

Their main idea relies in the choice of approximation spaces consisting of piecewise polynomial functions with no kind of continuity constraints across the interface between the elements of a triangulation. Consistency and well-posedness are achieved by introducing suitable bilinear forms defined on the interface. In this respect they are closely related to finite volume methods as they relies on the definition of numerical fluxes. As for conforming finite element approximations, the corresponding discrete problem is given in terms of finite dimensional subspaces and bilinear forms.

One of the main advantages of DG methods is that they allow a much greater flexibility in the design of the mesh and in the choice of the approximation spaces. Indeed, if one abandons the idea of a *conforming* approximation, one may as well abandon the idea of a *conforming* triangulation. This was soon realized and exploited in some DG methods; see, e.g., [19]. A mixed domain decomposition approach is also natural where conforming approximations are considered on single subdomains or patches, and DG interface terms are introduced on the boundaries between the subdomains; see [27, 8, 30]. In order to illustrate this point, we mention the problem of singularity and boundary layer resolution as an example. Suitable meshes (radical for h, or geometric for p version finite elements) are required for the approximation of edge and vertex singularities. Due to the different types of these singularities, finding conforming global triangulations without excessive overrefinement is not always an easy task in practice; see, e.g., [3, 4] for h and [5, 22, 23, 31] for hp approximations. A DG approach allows local independent refinement strategies and independent polynomial degrees can be employed on different elements, thus allowing more flexible hp adaptivity strategies.

DG methods however require a considerable increase in the number of degrees of freedom. If for instance trilinear elements on a uniform mesh on a cube are employed, for one nodal value in a conforming approximation we have *eight* degrees of freedom in a DG approach. Such increase can be prohibitively expensive for h approximations of large, three dimensional, vector problems, unless there are other requirements, such as suitable refinement strategies or the treating of convective terms. The situation is somewhat different for p and hp approximations, where the additional degrees of freedom in a DG approximation do not have the same order of magnitude as the number of degrees of freedom of a conforming discretization. We also note that non-conforming meshes with hanging nodes can be employed for p and hp finite element discretizations. Even if multiply constrained nodes are possible in theory, they bring in considerable complications in the implementation of practical codes, and often only simply constrained nodes are treated. Such complications are removed in a DG approach.

We finally note that even if convection may be the dominant effect of a problem, diffusive terms still need to be accounted for and correctly discretized. In particular, the finite element approximation of the Oseen or the incompressible Navier-Stokes equations require the introduction of suitable velocity-pressure spaces that ensure stability and approximability. If convective terms are properly treated, such properties only depend on the diffusive part of the operator and can then be studied for the simpler Stokes problem; see, e.g., [29, 17, 13, 21, 9]. This is indeed the purpose of this paper.

We propose a DG approximation together with suitable finite element spaces consisting of discontinuous velocities and pressures for the Stokes problem. We aim to a method where no additional unknown fields are introduced, which involves the same local bilinear forms on each element as those employed for conforming approximations and only adds interface contributions on the interelement boundaries. We believe that this approach can also be more easily exploited in a domain decomposition framework.

One remarkable property of our method is that the corresponding modified divergence bilinear form and the velocity-pressure pairs exhibit greater stability than the corresponding conforming approximations. We believe that this is related to the stabilizing effect obtained when Dirichlet conditions are imposed weakly, as it can easily be seen in the case of one element $\Omega = (-1, 1)^n$. If we consider pressures in $\mathbb{Q}_k(\Omega)$, i.e., polynomials of degree k in each variable, with mean value zero, referring to [9, Th. 24.1], we see that the spurious pressure modes p, which satisfy

$$b(\mathbf{v}, p) = -\int_\Omega \nabla \cdot \mathbf{v}\, p\, dx = 0, \quad \mathbf{v} \in \mathbb{Q}_k(\Omega)^n \cap H_0^1(\Omega)^n,$$

are all related to the fact that the velocities vanish on $\partial\Omega$. If Dirichlet conditions are imposed weakly (see (4.4), with $\Gamma_{int} = \emptyset$), spurious pressure modes satisfy

$$b(\mathbf{v}, p) = -\int_\Omega \nabla \cdot \mathbf{v}\, p\, dx + \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, p\, dx = \int_\Omega \mathbf{v} \cdot \nabla p\, dx = 0, \quad \mathbf{v} \in \mathbb{Q}_k(\Omega)^n,$$

and thus vanish, as it can be seen by taking $\mathbf{v} = \nabla p$. This remark only gives an idea of the reason why greater stability is achieved with DG approximations and is no proof that the case of more then one element is also stable. In addition, *quantifying* the stability in terms of an explicit dependence on the mesh size and the polynomial degree is not a trivial matter and a complete theory is beyond the scope of this work. Here we show that the pairs $\mathbb{Q}_{k'}$-$\mathbb{Q}_k$, $k' = k+2, k+1$, are uniformly stable with respect to the mesh size h. Our tests show that the choice $\mathbb{Q}_k$-$\mathbb{Q}_k$ is also free from spurious pressure modes, but is not uniformly stable with respect to h. Our numerical results show that the pairs $\mathbb{Q}_{k'}$-$\mathbb{Q}_k$, $k' = k+2, k+1$, are also uniformly stable with respect to k in two dimensions, while $\mathbb{Q}_k$-$\mathbb{Q}_k$ is not. In this paper we only prove an algebraic bound for the inf-sup constant that decreases like k^{-4} for the case of $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ elements. For the pair $\mathbb{Q}_{k+2}$-$\mathbb{Q}_k$ a better bound, which is not sharp either, is obtained using a stability result for conforming approximations. As is the case of DG approximations for scalar second order problems, a loss of optimality for p approximations is also found in our error analysis. Such loss is independent of the divergence stability of the method and is related to the interface contributions involving the gradient of the velocity.

Ours is not the first work on DG approximations of the Stokes problem. We mention [7, 20], where an interior penalty approximation with discontinuous, piecewise divergence-free velocities and continuous pressures are employed for the Stokes and incompressible Navier-Stokes equations, respectively. In [14] a *local* DG approximation of the Stokes problem is proposed. There the introduction of the fluxes as additional unknowns appears to have a stabilizing effect, and equal order flux, velocity, and pressure spaces can be chosen. Optimal error estimates for h approximations are proved.

The work in [18] deserves particular mention. There an h approximation for incompressible and nearly incompressible elasticity based on a DG method is introduced and studied. Triangular and tetrahedral meshes are employed together with polynomial spaces of total degree $k+1$ and k for the velocity and pressure, respectively. Optimal error estimates in h are proven for velocity and pressure, which remain valid in the incompressible limit. We note that the interface and modified divergence bilinear forms that we employ are basically the same. Here we consider a positive-definite, non-symmetric velocity bilinear form instead of an indefinite, symmetric one, which requires restrictions on the penalization coefficient. Our focus here is on hp approximations on quadrilateral and hexahedral meshes with hanging nodes and on the stability properties of some DG approximations also in terms of the order of the approximation. We also note that the proof for the divergence stability in [18], which employs the so called BDM spaces, does not seem to extend to our types of meshes. Here we also present a proof for the a priori estimates since the standard theory of mixed methods cannot be directly applied.

The rest of the paper is organized as follows:
In section 2 we introduce our continuous problem. Finite element spaces are defined in section 3, while our DG methods are derived in section 4. In section 5 we present some numerical tests and estimate the stability constants for certain h and p approximations, while we prove explicit theoretical bounds in section 6. Section 7 is devoted to the well-posedness and consistency of the discrete problem, and a priori error estimates for the velocity and pressure are proven in section 8. We conclude with some remarks on other choices of velocity-pressure pairs in section 9.

2. Problem setting. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded polyhedral domain. Given two vectors $\mathbf{f} \in L^2(\Omega)^n$ and $\mathbf{g} \in H^{1/2}(\partial\Omega)^n$, with

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, ds = 0,$$

we consider the following system for a velocity $\mathbf{u}$ and a pressure p:

$$\begin{aligned} &\mathbf{u}_{|\partial\Omega} = \mathbf{g}, \\ &-\nu\Delta\mathbf{u} + \nabla p \;=\; \mathbf{f}, \quad \text{in } \Omega, \\ &\nabla \cdot \mathbf{u} \;=\; 0, \quad \text{in } \Omega. \end{aligned} \tag{2.1}$$

For a vector $\mathbf{u}$, the tensor $\nabla\mathbf{u}$ is defined by

$$(\nabla\mathbf{u})_{ij} = u_{i/j} = \frac{\partial u_i}{\partial x_j},$$

with u_i the i-th component of $\mathbf{u}$. If I is the identity matrix in $\mathbb{R}^n$, we can rewrite the second of (2.1) in terms of the divergence of a stress tensor τ:

$$-\nabla \cdot \tau = -\nabla \cdot (\nu\nabla\mathbf{u} - pI) = \mathbf{f}, \tag{2.2}$$

where

$$(\nabla \cdot \tau)_i = \sum_{j=1}^{n} \tau_{ij/j}.$$

For tensors τ and ϵ, and a vector $\mathbf{v}$, we define the products

$$\tau \cdot \epsilon = \sum_{i,j=1}^{n} \tau_{ij}\epsilon_{ij},$$
$$(\tau : \mathbf{v})_i = \sum_{j=1}^{n} \tau_{ij} v_j.$$

Using the following Green formula,

$$\int_{\mathcal{D}} ((\nabla\cdot\tau)\cdot\mathbf{v} + \tau\cdot\nabla\mathbf{v})\,dx = \int_{\partial\mathcal{D}} \mathbf{v}\cdot(\tau:\mathbf{n})\,ds, \quad \mathcal{D}\subset\Omega, \tag{2.3}$$

we find the following variational formulation of Problem (2.1): find $\mathbf{u}\in H^1(\Omega)^n$ and $p\in L^2_0(\Omega)$, such that

$$\begin{aligned} &\mathbf{u}_{|\partial\Omega} = \mathbf{g}, \\ &\nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega - (p,\nabla\cdot\mathbf{v})_\Omega = (\mathbf{f},\mathbf{v})_\Omega, \qquad \mathbf{v}\in V := H^1_0(\Omega)^n, \\ &(\nabla\cdot\mathbf{u},q)_\Omega = 0, \qquad q\in M := L^2_0(\Omega). \end{aligned} \tag{2.4}$$

Here $L^2_0(\Omega)$ denotes the subspace of $L^2(\Omega)$ of functions with vanishing mean value in Ω and, for $\mathcal{D}\subseteq\mathbb{R}^n$, $(u,v)_{\mathcal{D}}$, $(\mathbf{u},\mathbf{v})_{\mathcal{D}}$, and $(\tau,\epsilon)_{\mathcal{D}}$ denote the scalar products in $L^2(\mathcal{D})$, $L^2(\mathcal{D})^n$, and $L^2(\mathcal{D})^{n\times n}$, respectively, with $\|u\|_{\mathcal{D}}$, $\|\mathbf{u}\|_{\mathcal{D}}$, and $\|\tau\|_{\mathcal{D}}$ the corresponding norms. We denote the norm of $H^s(\mathcal{D})$ or $H^s(\mathcal{D})^n$, $s\in\mathbb{R}$, by $\|\cdot\|_{s,\mathcal{D}}$. Analogous notations are employed for the corresponding seminorms for $s>0$. In case $\mathcal{D}=\Omega$, we drop the subscript Ω and, in case $s=0$, we also drop the subscript 0. We recall that the seminorm $|\mathbf{u}|_{1,\Omega} = \|\nabla\mathbf{u}\|_{0,\Omega}$ is a norm in $H^1_0(\Omega)^n$, the subspace of $H^1(\Omega)^n$ of vectors that vanish on $\partial\Omega$.

We note that the second of (2.4) can also be written in terms of the stress tensor τ, since

$$\nu\nabla\mathbf{u}\cdot\nabla\mathbf{v} - p\nabla\cdot\mathbf{v} = (\nu\nabla\mathbf{u} - pI)\cdot\nabla\mathbf{v} = \tau\cdot\nabla\mathbf{v}.$$

The well-posedness of problem (2.4) is ensured by the two stability conditions

$$\nu(\mathbf{u},\mathbf{v}) \le \nu|\mathbf{u}|_1\,|\mathbf{v}|_1, \tag{2.5}$$

$$(\nabla\cdot\mathbf{u},p) \le |\mathbf{u}|_1\,\|p\|, \tag{2.6}$$

the coercivity condition

$$\nu(\nabla\mathbf{u},\nabla\mathbf{u}) \ge \nu|\mathbf{u}|_1^2, \quad \mathbf{u}\in H^1_0(\Omega)^n, \tag{2.7}$$

and the divergence stability condition

$$\sup_{0\neq\mathbf{v}\in H^1_0(\Omega)^n} \frac{(\nabla\cdot\mathbf{v},p)}{|\mathbf{v}|_1} \ge \gamma\|p\|, \quad p\in L^2_0(\Omega), \quad \gamma>0. \tag{2.8}$$

We refer to, e.g., [13, Ch. II] for a comprehensive analysis.

3. Finite element spaces. Given a shape-regular affine quadrilateral or hexahedral mesh $\mathcal{T}=\mathcal{T}_h$, of maximum diameter h, and polynomial degrees k' and k, we consider the following finite element spaces:

$$\begin{aligned} V_{k'} &= \left\{\mathbf{u}\in L^2(\Omega)^n \mid \quad \mathbf{u}_{|\kappa}\in\mathbb{Q}_{k'}(\kappa)^n \quad \kappa\in\mathcal{T}\right\}, \\ M_k &= \left\{p\in L^2_0(\Omega) \mid \quad p_{|\kappa}\in\mathbb{Q}_k(\kappa) \quad \kappa\in\mathcal{T}\right\}. \end{aligned} \tag{3.1}$$

Here $\mathbb{Q}_k(\kappa)$ is the space of polynomials of maximum degree k in each variable on κ. We note that we have considered discontinuous finite element spaces for the velocity that do not vanish on $\partial\Omega$, since we impose Dirichlet conditions weakly. We still require that pressures have mean value zero in Ω.

The mesh $\mathcal{T}$ is said to be *conforming* or *regular* if the intersection between two different elements is either empty, or a vertex, an edge, or face that is common to *both* elements. Meshes that are not conforming are sometimes called *irregular* and they contain hanging nodes; see, e.g., [23, 24]. We allow irregular meshes in general, but suppose that the intersection between neighboring elements is either a common vertex, or an entire edge, or an entire face of *one* of the two elements. We also allow non quasi-uniform meshes, but assume that the diameters of neighboring elements are not too different:

Assumption 3.1. *There exist constants independent of $\mathcal{T}$, such that*

$$ch_{\kappa'} \leq h_\kappa \leq Ch_{\kappa'}, \quad \kappa, \kappa' \in \mathcal{T},$$

if κ and κ' are distinct and they share an entire edge, if $n = 2$, or an entire face, if $n = 3$, of κ or κ'.

We will make particular choices for the velocity and pressure spaces in the next section.

We recall the following inverse estimates; see [26, Eqq. 4.6.4 and 4.6.5]. Let $q \in M_k$, then

$$|q|^2_{1,\kappa} \leq C\frac{k^4}{h^2_\kappa}\|q\|^2_{0,\kappa}, \tag{3.2}$$

$$\|q\|^2_{0,\gamma} \leq C\frac{k^2}{h_\kappa}\|q\|^2_{0,\kappa}, \tag{3.3}$$

where γ is either $\partial\kappa$ or one of its faces, and h_κ denotes the diameter of $\kappa \in \mathcal{T}$. Similar estimates hold for vector functions in V_k.

We will also need a multiplicative trace inequality:

$$\|q\|^2_{0,\partial\kappa} \leq C\,\left(\|q\|_{0,\kappa}\,\|\nabla q\|_{0,\kappa} + h_\kappa^{-1}\|q\|^2_{0,\kappa}\right), \quad q \in H^1(\kappa), \tag{3.4}$$

with $\kappa \in \mathcal{T}$ and C independent of h_κ; see [19]. An analogous estimate holds for vector functions.

The following approximation property can be found in [6, 24].

Lemma 3.1. *Let $q \in H^{n_\kappa}(\kappa)$, $\kappa \in \mathcal{T}$. Then, there exists $\Pi_\kappa q = \Pi_{\kappa,k}q \in \mathbb{Q}_k(\kappa)$ and C, only depending on the shape–regularity of κ , s, and n_κ, such that*

$$\|q - \Pi_\kappa q\|_{s,\kappa} \leq C\,\frac{h_\kappa^{m-s}}{k^{n_\kappa - s}}\,\|q\|_{n_\kappa,\kappa}\,, \quad 0 \leq s \leq m, \tag{3.5}$$

where $m := \min\{k+1, n_\kappa\}$.

It is possible to define a *global* operator $\Pi_{h,k}q$ on M_k by

$$\Pi_{h,k}q_{|_\kappa} := \Pi_{\kappa,k}q, \quad \kappa \in \mathcal{T}.$$

Similar operators and estimates hold for vector functions in $V_{k'}$.

4. Discrete problem. We now derive our DG formulation. The idea is to consider Problem (2.1) on each element $\kappa \in \mathcal{T}$ and impose Dirichlet conditions weakly on the boundary $\partial\kappa$ using the value of the velocity on the boundary of the neighboring elements. In addition, a suitable numerical flux needs to be chosen in order to approximate the flux $\tau : \mathbf{n}$ on $\partial\kappa$. Finally an interface term, penalizing the jumps of the velocity, will be added as for similar DG approximations of second order problems. This is a standard procedure in the derivation of DG formulations; see, e.g., [19, 14, 18].

Here, the choice of flux $\tau : \mathbf{n}$ instead of $\nu\nabla\mathbf{u} : \mathbf{n}$ seems most reasonable and physically meaningful. As in [18], this will bring a modification of *both* the bilinear form $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$; see (4.6).

We consider Problem (2.1) on $\kappa \in \mathcal{T}$, multiply the second equation (or, equivalently, (2.2)) by a test velocity $\mathbf{v}$, the third equation by a test pressure q, and sum. We obtain

$$\int_\kappa (-\nabla\cdot\tau)\cdot\mathbf{v} + \int_\kappa \nabla\cdot\mathbf{u}\, q\, dx = \int_\kappa \mathbf{f}\cdot\mathbf{v}\, dx.$$

By using the Green's formula (2.3), we find

$$\int_\kappa (\tau\cdot\nabla\mathbf{v} + \nabla\cdot\mathbf{u}q)\, dx - \int_{\partial\kappa} \mathbf{v}\cdot(\tau : \mathbf{n}_\kappa)\, ds = \int_\kappa \mathbf{f}\cdot\mathbf{v}\, dx,$$

where $\mathbf{n}_\kappa$ is the outward normal to $\partial\kappa$.

Let $\mathbf{u}^{out}$ be the value of a velocity $\mathbf{u}$ on $\partial\kappa$ from the neighboring elements. On $\partial\kappa \cap \partial\Omega \neq \emptyset$, we set $\mathbf{u}^{out} = \mathbf{g}$, the Dirichlet data. In addition we define ϵ as the flux relative to the pair $\{\mathbf{v}, q\}$:

$$\epsilon = \nu\nabla\mathbf{v} - qI.$$

Assuming that $\mathbf{u}$ is continuous and equal to $\mathbf{g}$ on $\partial\Omega$, we can write

$$\begin{aligned} \int_\kappa (\tau\cdot\nabla\mathbf{v} + \nabla\cdot\mathbf{u}q)\, dx - \int_{\partial\kappa} \mathbf{v}\cdot(\tau : \mathbf{n}_\kappa)\, ds \quad &+ \quad \rho\int_{\partial\kappa} \mathbf{u}\cdot(\epsilon : \mathbf{n}_\kappa)\, ds \\ = \int_\kappa \mathbf{f}\cdot\mathbf{v}\, dx \quad &+ \quad \rho\int_{\partial\kappa} \mathbf{u}^{out}\cdot(\epsilon : \mathbf{n}_\kappa)\, ds, \end{aligned} \tag{4.1}$$

where ρ is equal to one on $\partial\kappa \cap \partial\Omega$ and to one half elsewhere.

We now replace $\tau : \mathbf{n}_\kappa$ and $\epsilon : \mathbf{n}_\kappa$ with suitable numerical fluxes. In order to do so, we first need to define some geometrical objects related to the partition $\mathcal{T}$. We denote by $\mathcal{E}_{int}$ the set of all open $(n-1)$–dimensional intersections of neighboring elements

$$\mathcal{E}_{int} = \{e| \quad e = \partial\kappa \cap \partial\kappa',\ \kappa,\ \kappa' \in \mathcal{T},\ meas_{n-1}(e) > 0\}$$

and Γ_{int} their union, such that

$$\bar{\Gamma}_{int} = \bigcup_{e\in\mathcal{E}_{int}} \bar{e}.$$

Thanks to our assumptions on $\mathcal{T}$, these intersections are entire faces of elements in $\mathcal{T}$ for, e.g., $n = 3$. For the sake of brevity we will refer to such intersections as 'faces' in the following, even for $n = 2$. The boundary $\partial\Omega$ can also be partitioned into contributions from single elements. We define

$$\mathcal{E}_{out} = \{e| \quad e = \partial\kappa \cap \partial\Omega,\ \kappa \in \mathcal{T},\ meas_{n-1}(e) > 0\}$$

and

$$\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{E}_{int}.$$

In the following, we generically refer to elements in $\mathcal{E}_{out}$ as faces, even though they may actually consist of union of element faces.

Given an interior face $e \in \mathcal{E}_{int}$, there are two elements κ_i, κ_j, with, e.g., $i > j$, that share this face. We define the jump $[v]$ and the average $< v >$ by

$$[v]_e = v_{\kappa_i|_e} - v_{\kappa_j|_e}, \quad < v >_e = \frac{1}{2}(v_{\kappa_i|_e} + v_{\kappa_j|_e}),$$

and $\mathbf{n}$ as the unit normal which points from κ_i to κ_j, i.e., $\mathbf{n} = \mathbf{n}_{\kappa_i}$. We note that if $[v]_e = 0$, then $< v >_e = v_{\kappa_i|_e} = v_{\kappa_j|_e}$. For $e \in \mathcal{E}_{out}$, we define

$$[v]_e = v_{|_e}, \quad < v >_e = v_{|_e},$$

and $\mathbf{n}$ as the unit outward normal to $\partial\Omega$. Jumps and averages for vector functions are defined component by component.

Our choice for the numerical fluxes is

$$< \tau : \mathbf{n}_\kappa >, \quad < \epsilon : \mathbf{n}_\kappa >, \quad \text{on } \Gamma_{int} \cup \partial\Omega.$$

Replacing the fluxes in (4.1) with the numerical fluxes, we obtain

$$\begin{aligned}\int_\kappa (\tau \cdot \nabla \mathbf{v} + \nabla \cdot \mathbf{u} q)\, dx - \int_{\partial\kappa} \mathbf{v} \cdot < \tau : \mathbf{n}_\kappa > \, ds \quad + \quad \rho \int_{\partial\kappa} \mathbf{u} \cdot < \epsilon : \mathbf{n}_\kappa > \, ds \\ = \int_\kappa \mathbf{f} \cdot \mathbf{v}\, dx \quad + \quad \rho \int_{\partial\kappa} \mathbf{u}^{out} \cdot < \epsilon : \mathbf{n}_\kappa > \, ds,\end{aligned}$$

or, equivalently,

$$\left\{ \begin{aligned} \int_\kappa \tau \cdot \nabla \mathbf{v}\, dx - \int_{\partial\kappa} \mathbf{v} \cdot < \tau : \mathbf{n}_\kappa > \, ds \quad + \quad \rho \int_{\partial\kappa} \mathbf{u} \cdot < \nu \nabla \mathbf{v} : \mathbf{n}_\kappa > \, ds, \\ = \int_\kappa \mathbf{f} \cdot \mathbf{v}\, dx \quad + \quad \rho \int_{\partial\kappa} \mathbf{u}^{out} \cdot < \nu \nabla \mathbf{v} : \mathbf{n}_\kappa > \, ds \\ - \int_\kappa \nabla \cdot \mathbf{u} q\, dx + \rho \int_{\partial\kappa} < q > (\mathbf{u} \cdot \mathbf{n}_\kappa)\, ds = \rho \int_{\partial\kappa} < q > (\mathbf{u}^{out} \cdot \mathbf{n}_\kappa)\, ds. \end{aligned} \right. \tag{4.2}$$

Our derivation is concluded by summing over the elements. We start with the second equation in (4.2) and find

$$- \sum_{\kappa \in \mathcal{T}} \int_\kappa \nabla \cdot \mathbf{u} q\, dx + \int_{\Gamma_{int}} < q > [\mathbf{u} \cdot \mathbf{n}]\, ds + \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} q\, ds = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}\, q\, ds. \tag{4.3}$$

We note that, by summing over the elements, the integral on an interface $e = \partial\kappa \cap \partial\kappa'$ consists of two contributions, from κ and κ', and that ρ is equal to $1/2$ on e.

Equation (4.3) motivates our choice for a modified divergence bilinear form:

$$\begin{aligned} b(\mathbf{v}, p) &= - \sum_{\kappa \in \mathcal{T}} (\nabla \cdot \mathbf{v}, p)_\kappa + \int_{\Gamma_{int} \cup \partial\Omega} < p > [\mathbf{v} \cdot \mathbf{n}]\, ds \\ &= - \sum_{\kappa \in \mathcal{T}} (\nabla \cdot \mathbf{v}, p)_\kappa + \int_{\partial\Omega} p\, \mathbf{v} \cdot \mathbf{n}\, ds + \int_{\Gamma_{int}} < p > [\mathbf{v} \cdot \mathbf{n}]\, ds. \end{aligned} \tag{4.4}$$

Before considering the first equation in (4.2), we introduce a penalization coefficient for the velocity space. Let σ_0 be a positive constant and σ be the function defined on $\Gamma_{int} \cup \partial\Omega$ by

$$\sigma_{|_e} = \sigma_0 \nu \frac{k'^2}{h_e}, \quad e \in \mathcal{E},$$

with h_e the diameter of e and k' the polynomial order chosen for the approximation of the velocity; see below. Summing then over the elements and adding penalization terms, we obtain

$$\begin{aligned}
&\sum_{\kappa\in\mathcal{T}} \int_\kappa \nu \nabla\mathbf{u}\cdot\nabla\mathbf{v}\,dx + \int_{\partial\Omega} \sigma\mathbf{u}\cdot\mathbf{v}\,ds + \int_{\Gamma_{int}} \sigma[\mathbf{u}]\cdot[\mathbf{v}]\,ds \\
&+ \int_{\partial\Omega} (\mathbf{u}\cdot(\nu\,\nabla\mathbf{v}:\mathbf{n}) - (\nu\,\nabla\mathbf{u}:\mathbf{n})\cdot\mathbf{v})\;ds \\
&+ \int_{\Gamma_{int}} ([\mathbf{u}]\cdot <\nu\,\nabla\mathbf{v}:\mathbf{n}> - <\nu\,\nabla\mathbf{u}:\mathbf{n}>\cdot[\mathbf{v}])\;ds \\
&+ b(\mathbf{v},p) \\
&= \sum_{\kappa\in\mathcal{T}} \int_\kappa \mathbf{f}\cdot\mathbf{v}\,dx + \int_{\partial\Omega} (\sigma\,\mathbf{g}\cdot\mathbf{v} + \mathbf{g}\cdot(\nu\,\nabla\mathbf{v}:\mathbf{n}))\;ds,
\end{aligned} \tag{4.5}$$

We note that the penalization terms vanish if $\mathbf{u}$ is continuous across the elements and is equal to $\mathbf{g}$ on $\partial\Omega$.

We then define the following velocity bilinear form:

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= \sum_{\kappa\in\mathcal{T}} \nu(\nabla\mathbf{u},\nabla\mathbf{v})_\kappa + \int_{\Gamma_{int}\cup\partial\Omega} \sigma[\mathbf{u}]\cdot[\mathbf{v}]\,ds \\
&+ \int_{\Gamma_{int}\cup\partial\Omega} ([\mathbf{u}]\cdot <\nu\,\nabla\mathbf{v}:\mathbf{n}> - <\nu\,\nabla\mathbf{u}:\mathbf{n}>\cdot[\mathbf{v}])\;ds \\
&= \sum_{\kappa\in\mathcal{T}} \nu(\nabla\mathbf{u},\nabla\mathbf{v})_\kappa + \int_{\partial\Omega} \sigma\mathbf{u}\cdot\mathbf{v}\,ds + \int_{\Gamma_{int}} \sigma[\mathbf{u}]\cdot[\mathbf{v}]\,ds \\
&+ \int_{\partial\Omega} (\mathbf{u}\cdot(\nu\,\nabla\mathbf{v}:\mathbf{n}) - (\nu\,\nabla\mathbf{u}:\mathbf{n})\cdot\mathbf{v})\;ds \\
&+ \int_{\Gamma_{int}} ([\mathbf{u}]\cdot <\nu\,\nabla\mathbf{v}:\mathbf{n}> - <\nu\,\nabla\mathbf{u}:\mathbf{n}>\cdot[\mathbf{v}])\;ds.
\end{aligned}$$

Given the pressure space M_k, $k \geq 0$, and a velocity space $V_{k'}$, $k' \geq k$, (4.5) and (4.3) define our discrete problem:
Find $\mathbf{u} \in V_{k'}$ and $p \in M_k$, such that

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f},\mathbf{v})_\Omega + \int_{\partial\Omega} (\sigma\,\mathbf{g}\cdot\mathbf{v} + \mathbf{g}\cdot(\nu\,\nabla\mathbf{v}:\mathbf{n}))\;ds, && \mathbf{v}\in V_{k'}, \\
b(\mathbf{u},q) &= \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\,q\,ds, && q\in M_k.
\end{aligned} \tag{4.6}$$

We note that, by integrating by parts on each element, the bilinear form $b(\cdot,\cdot)$

can also be written as

$$b(\mathbf{v}, p) = \sum_{\kappa \in \mathcal{T}} (\mathbf{v}, \nabla p)_\kappa - \int_{\Gamma_{int}} [p] < \mathbf{v} \cdot \mathbf{n} > \, ds. \tag{4.7}$$

In particular we see that the $b(\mathbf{u}, 1)$ is identically zero for every $\mathbf{u}$. Consequently the second equation in (4.6) is satisfied for every $\mathbf{u} \in V_{k'}$ and q a constant function on Ω. As for conforming approximations, the pressure is determined up to an additive constant and uniqueness is achieved by requiring a vanishing mean value on Ω.

We consider three choices of approximation spaces.

1. **Method 1**. We choose V_{k+2} and M_k, $k \geq 0$. This is a generalization of the conforming $\mathbb{Q}_{k+2} - \mathbb{Q}_k$ spaces with discontinuous pressure; see, e.g., [28].
2. **Method 2**. We take V_{k+1} and M_k, $k \geq 0$. This is a generalization of the conforming $\mathbb{Q}_{k+1} - \mathbb{Q}_k$ spaces with continuous pressures, also known as Taylor-Hood elements; see, e.g., [12, 13].
3. **Method 3**. We consider equal polynomial degrees for the velocity and pressure: V_k and M_k, $k \geq 1$.

Methods 1, 2, and 3 correspond to the choices $k' = k+2, k+1, k$, respectively, for the velocity space $V_{k'}$ in problem (4.6). Our numerical results show however that an inf-sup condition holds for Method 3 with a constant that decreases as $Chk^{-1/2}$. Method 3 is stable but not uniformly in h and k, and thus unsuitable both for h and p-version approximations.

Given a velocity space $V_{k'}$, a pressure space M_k, $k', k \geq 0$, and $\mathbf{g} \in H^{1/2}(\partial\Omega)$, we define the space

$$Z(\mathbf{g}) = Z_{k',k}(\mathbf{g}) = \left\{ \mathbf{u} \in V_{k'} \;\middle|\; b(\mathbf{u}, q) = \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, q \, ds, \; q \in M_k \right\} \subset V_{k'},$$

and $Z = Z(0)$. For elements in $V_{k'}$ we employ a discrete norm defined by

$$|\mathbf{u}|_h^2 = \sum_{\kappa \in \mathcal{T}} |\mathbf{u}|_{1,\kappa}^2 + \int_{\Gamma_{int} \cup \partial\Omega} \sigma |[\mathbf{u}]|^2 \, ds = \sum_{\kappa \in \mathcal{T}} |\mathbf{u}|_{1,\kappa}^2 + \int_{\partial\Omega} \sigma |\mathbf{u}|^2 \, ds + \int_{\Gamma_{int}} \sigma |[\mathbf{u}]|^2 \, ds.$$

5. Numerical investigation of the divergence stability. The stability and accuracy of the discrete mixed problem depend on the a discrete inf-sup condition for the bilinear form $b(\cdot, \cdot)$ and the approximation spaces of velocities and pressures:

$$\sup_{0 \neq \mathbf{v} \in V_{k'}} \frac{b(\mathbf{v}, p)}{|\mathbf{v}|_h} \geq \gamma_i \|p\|, \quad p \in M_k, \quad \gamma_i > 0, \tag{5.1}$$

for $i = 1, 2, 3$, corresponding to the choices $k' = k+2, k+1, k$.

In this section we show some estimates of the inf-sup constant of Methods 1, 2, and 3. We only consider two dimensional problems on the unit square $\Omega = (0, 1)^2$ and uniform triangulations consisting of $n \times n$ square elements.

5.1. Numerical results for the h-version. For the results in this section, we fix the degree k and only consider the dependence on the mesh-size h. Figure 5.1 shows the estimated inf-sup conditions for Methods 1 and 2, as functions of the mesh size $h = 1/n$, for $k = 0, 1, 2, 3$. The results plotted on the left for $k' = k + 2$ are well-known since in this case our pressure space coincides with that of the standard $\mathbb{Q}_{k+2}$-$\mathbb{Q}_k$ elements but with a larger (discontinuous) velocity space. In this case the inf-sup constant can only improve. Our results are consistent with a stability constant which is independent of h, as stated in Lemma 6.1.

The results plotted on the right however cannot be deduced from the corresponding ones for the $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ Taylor-Hood elements with continuous pressure. Indeed $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ elements with discontinuous pressure and continuous velocity may show spurious pressure modes; see, e.g., [13]. On the other hand, our DG method employing $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ elements with discontinuous pressure and velocity does not have any spurious pressure modes and our results are consistent with a stability constant which is independent of h, as stated in Lemma 6.3. We also note that the $\mathbb{Q}_1$-$\mathbb{Q}_0$ elements are stable. This case is not covered by Lemma 6.3.

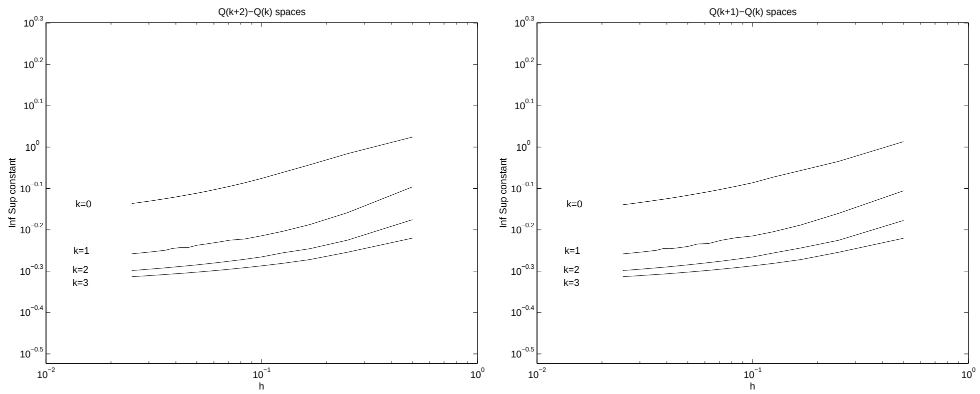

FIG. 5.1. *Estimated inf-sup conditions for Method 1 (left) and 2 (right), as functions of the mesh size $h = 1/n$ for different values of k*

Figure 5.2 shows the estimated inf-sup conditions for Method 3, as functions of the mesh size $h = 1/n$ for $k = 1, 2, 3, 4$. We note that $\mathbb{Q}_k$-$\mathbb{Q}_k$ elements with discontinuous pressure and continuous velocity can show spurious pressure modes. Our results show that our DG method with discontinuous pressure and velocity does not have spurious pressure modes. However the stability constant decreases with h, thus making Method 3 not suitable for h approximations. Our results are consistent with a linear dependence

$$\gamma_3 = ch,$$

with c depending on k. The error of the exact solution is then suboptimal of at least one power of h.

5.2. Numerical results for the p-version. The numerical tests presented in the previous section for the h version show that our DG methods exhibits better stability properties than the corresponding conforming approximations. It is natural to ask then if this is also the case for p-approximations and if, in particular, the stability constants of our methods exhibit a weaker dependence on the polynomial degree k.

Figure 5.3 shows the estimated inf-sup conditions for Methods 1 (left) and 2 (right), as functions of the polynomial degree k, for different uniform triangulations of $\Omega = (0, 1)^2$. We have also shown results for the corresponding conforming approximations on a 3×3 mesh: $\mathbb{Q}_{k+2}$-$\mathbb{Q}_k$ with discontinuous pressure on the left, and $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ Taylor-Hood elements with continuous pressure on the right.

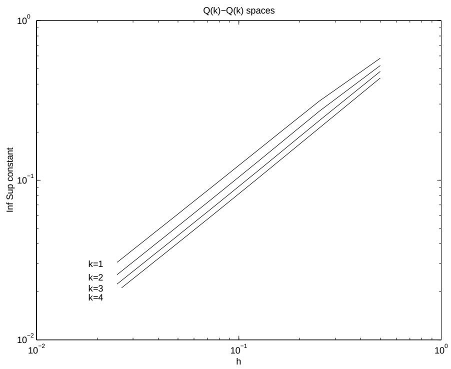

FIG. 5.2. *Estimated inf-sup conditions for Method 3, as functions of the mesh size $h = 1/n$ for different values of k*

For the plot on the left, we first remark the typical behavior of conforming approximations, where the decrease of the stability constant as $ck^{-1/2}$ is only observed for large k. Our results for the corresponding DG approximation show that such dependence is removed when the velocity space is made discontinuous and the divergence bilinear form suitably modified: the stability constants tend to a constant value when k becomes large.

What is remarkable is that exactly the same pattern is observed if we decrease the velocity space by one order (see Figure 5.3, right). We are unaware of any theoretical bound for p approximations using Taylor-Hood elements. Our results for conforming approximations show that the constants decrease like $ck^{-1/2}$ in two dimensions; see also Figure 3 in [2]. However such dependence is removed if velocities and pressures are made discontinuous. We also note that there is no appreciable difference when switching from a DG approximation based on $\mathbb{Q}_{k+2}$-$\mathbb{Q}_k$ elements to one based on $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ elements. Indeed the constants are only slightly smaller.

DG approximations using $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ appear particularly attractive since they exhibit a stability constant that does not depend on k with only a gap of one order between the velocity and the pressure. We note however that half a power of k is lost in our error estimates; see Lemmas 8.1 and 8.2. This loss is typical of p-version DG finite elements for second order problems; see, e.g., [19].

We now consider Method 3. Figure 5.4 shows the estimated inf-sup conditions as functions of the polynomial degree k, for different uniform triangulations. It is clear that choosing finite element spaces of equal order removes the uniform stability with respect to k as well as h. Our results are consistent with a dependence

$$\gamma_3 = c\,hk^{-1/2}$$

for the inf-sup condition. This dependence on h and k is likely to be removed if suitable stabilization procedures, as those in [16], are employed, but this generalization is beyond the scope of this paper. We will not consider Method 3 in our analysis.

We summarize the evidence found by our numerical results for two dimensional problems in the following remarks.

REMARK 1 (**Method 1**). *There exists a constant γ_1, independent of h and k,*

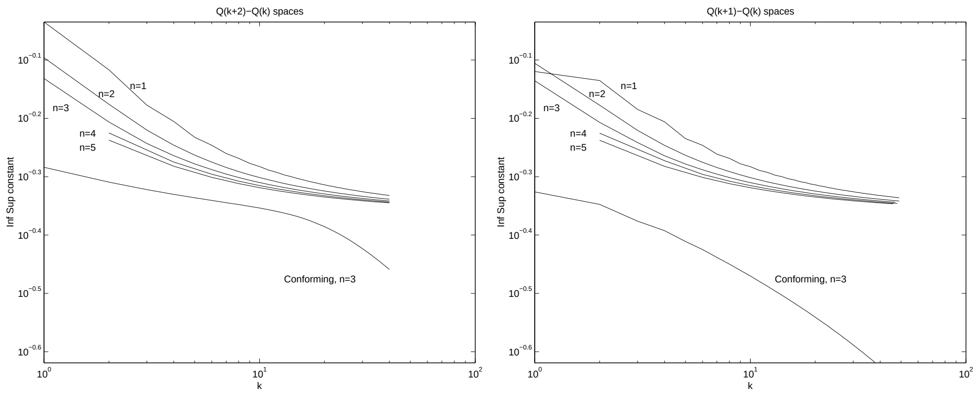

FIG. 5.3. *Estimated inf-sup conditions for Method 1 (left) and 2 (right), as functions of k for different values of the mesh-size.*

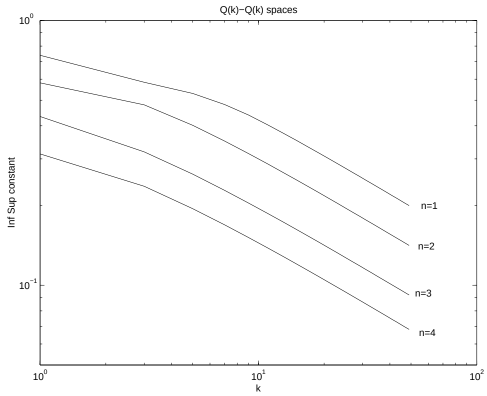

FIG. 5.4. *Estimated inf-sup conditions for Method 3, as functions of k for different values of the mesh-size.*

such that, for $k \geq 0$,

$$\sup_{0 \neq \mathbf{v} \in V_{k+2}} \frac{b(\mathbf{v}, p)}{|\mathbf{v}|_h} \geq \gamma_1 \|p\|, \quad p \in M_k, \quad \gamma_1 > 0. \tag{5.2}$$

REMARK 2 (**Method 2**). *There exists a constant γ_2, independent of h and k, such that, for $k \geq 0$,*

$$\sup_{0 \neq \mathbf{v} \in V_{k+1}} \frac{b(\mathbf{v}, p)}{|\mathbf{v}|_h} \geq \gamma_2 \|p\|, \quad p \in M_k, \quad \gamma_2 > 0. \tag{5.3}$$

REMARK 3 (**Method 3**). *There exists a constant c, independent of h and k, such that, for $k \geq 1$,*

$$\sup_{0 \neq \mathbf{v} \in V_k} \frac{b(\mathbf{v}, p)}{|\mathbf{v}|_h} \geq \gamma_3 \|p\| = chk^{-1/2} \|p\|, \quad p \in M_k. \tag{5.4}$$

6. Divergence stability for the *hp* version. This section is devoted to the divergence stability for the *hp* version of Methods 1 and 2. In view of the numerical results presented in the previous section, the dependence on h in our bounds is sharp, but that on k is not.

6.1. Method 1. The stability for the choice of spaces of Method 1 is a direct consequence of the corresponding property for conforming approximations using $\mathbb{Q}_{k+2} - \mathbb{Q}_k$ elements. We refer to [28]; in particular, see [23] and [31] for the case of non-conforming meshes with hanging nodes in two and three dimensions, respectively. Indeed, the pressure space is the same and the space of velocities that are continuous across the elements and vanish on $\partial\Omega$ is contained in V_{k+2}. Finally, the integral contributions on $\Gamma_{int} \cup \partial\Omega$ in the definitions of $b(\cdot,\cdot)$ and $|\cdot|_h$ vanish for continuous velocities. We thus have the following lemma

LEMMA 6.1 (**Method 1**). *There exists a constant $c > 0$, independent of h and k, such that, for $k \geq 0$,*

$$\sup_{0 \neq \mathbf{v} \in V_{k+2}} \frac{b(\mathbf{v}, p)}{|\mathbf{v}|_h} \geq \gamma_1 \|p\| \geq c k^{(1-n)/2} \|p\|, \quad p \in M_k. \tag{6.1}$$

We remark that, even though this bound is not sharp with respect to k, thanks to [23] and [31] the inf-sup constant γ_1 is independent of the aspect ratio of suitable boundary layer meshes.

6.2. Method 2. For the choice corresponding to Method 2, the proof proposed in [18] for simplicial meshes, employing the so called BDM spaces, does not seem to extend to our case. However stability can be proven in the same way as for the conforming Taylor-Hood elements $\mathbb{Q}_{k+1} - \mathbb{Q}_k$ consisting of continuous velocities and pressures, *despite* the fact that we consider here discontinuous pressures; see [12]. In addition, using some properties of the Legendre polynomials we are able to track down the dependence on k as well; for some crucial steps of our analysis we rely on [25]. We are unaware of a proof for the Taylor-Hood elements which gives an explicit dependence on the polynomial degree k: numerical evidence however shows that the inf-sup constant *does* depend on k; see, e.g., [2].

Even if our bound is not sharp, we have chosen to present the proof here for various reasons:
It is indeed a proof of an *algebraic* bound in k. For p- and hp-approximations of problems with piecewise analytic data, exponential convergence is ensured in case the solution of the continuous problem can be properly characterized; see, e.g., [24, Sect. 4.5] for more details. In addition our proof is the same as for Taylor-Hood elements and can be carried out exactly in the same way for conforming approximations with continuous pressure, thus giving an algebraic bound for them as well. Even for conforming approximations this bound does not appear to be sharp, but is sufficient to ensure exponential convergence. Our argument is dimension-independent and remains valid in the case of meshes with hanging nodes.

We finally note that a bound for Method 2 gives also a bound for Method 1. For Method 1 however a better bound, even if not sharp, is given in Lemma 6.1.

We first need some additional notations and results. Given an integer $k \geq 1$, we denote by $GGL(k)$ the set of Gauss-Lobatto points $\{a_i;\ 0 \leq i \leq k\}$ on $I = (-1,1)$ in increasing order and by $\{w_i > 0\}$ the corresponding weights; see, e.g., [21, Ch. 4]. For the square $(-1,1)^2$ we set $GGL(k)^2 = \{a_{ij} = (a_i, a_j);\ 0 \leq i,j \leq k\}$ and denote by $\{w_{ij} = w_i w_j > 0\}$ the corresponding weights. These definitions carry on in the three-dimensional case with the obvious modifications. We recall that the quadrature formula based on $GGL(k)$ has order $2k-1$ and that

$$\|p\|_{0,I}^2 \le \sum_{i=0}^{k} p(a_i)^2\, w_i \le \left(2+\frac{1}{k}\right) \|p\|_{0,I}^2, \quad p \in \mathbb{Q}_k(I); \tag{6.2}$$

see [21, Eq. 4.4.16]. In the following, we use the same notation for the mapped Gauss-Lobatto nodes and corresponding weights for an element $\kappa \in \mathcal{T}$. Similar estimates as (6.2) hold in two and three dimensions and for affinely mapped elements.

We also need the following lemma.

LEMMA 6.2. *Let $S = I^2$, $I = (-1,1)$, and $q \in \mathbb{Q}_k(S)$ such that*

$$\int_S q(x,y)\, dxdy = 0.$$

If $k \ge 1$, then

$$\|q\|_{0,S}^2 \le \int_S \left((1-x^2)\left|\frac{\partial q}{\partial x}\right|^2 + (1-y^2)\left|\frac{\partial q}{\partial y}\right|^2\right) dxdy \le 4k^2 \|q\|_{0,S}^2.$$

Analogous estimates hold in three dimensions for I^3.

Proof. We only give the proof for the two-dimensional case. Since $q \in \mathbb{Q}_k(S)$ has mean value zero, it can be written as

$$q(x,y) = \sum_{i=0}^{k}\sum_{j=0}^{k} q_{ij} L_i(x) L_j(y),$$

with $q_{00} = 0$. Here $\{L_i\}$ are the Legendre polynomials; see [9, Sect. 3].

We can write

$$(1-x^2)\left|\frac{\partial q}{\partial x}\right|^2 = \sum_{i,n=1}^{k}\sum_{j,m=0}^{k} q_{ij} q_{nm} \left((1-x^2) L_i'(x) L_n'(x)\right)\left(L_j(y) L_m(y)\right).$$

Using the conditions, see [9],

$$\int_{-1}^{1} L_i(x) L_n(x)\, dx = \|L_i\|_{0,I}^2\, \delta_{in},$$
$$\int_{-1}^{1} (1-x^2) L_i'(x) L_n'(x)\, dx = i(i+1)\int_{-1}^{1} L_i(x) L_n(x)\, dx,$$

we find

$$\int_S (1-x^2)\left|\frac{\partial q}{\partial x}\right|^2 dxdy = \sum_{i=1}^{k}\sum_{j=0}^{k} q_{ij}^2\, i(i+1) \|L_i\|_{0,I}^2 \|L_j\|_{0,I}^2.$$

Using similar arguments for the $\partial q/\partial y$, we find

$$\begin{aligned}
&\int_S \left((1-x^2)\left|\frac{\partial q}{\partial x}\right|^2 + (1-y^2)\left|\frac{\partial q}{\partial y}\right|^2\right) dxdy \\
&= \sum_{i=1}^{k}\sum_{j=0}^{k} q_{ij}^2\, i(i+1)\|L_i\|_{0,I}^2 \|L_j\|_{0,I}^2 + \sum_{i=0}^{k}\sum_{j=1}^{k} q_{ij}^2\, j(j+1)\|L_i\|_{0,I}^2 \|L_j\|_{0,I}^2 \\
&\ge \sum_{i=0}^{k}\sum_{j=0}^{k} q_{ij}^2 \|L_i\|_{0,I}^2 \|L_j\|_{0,I}^2 = \|q\|_{0,S}^2.
\end{aligned}$$

The upper bound can be found in a similar way. □

We are now ready to prove the following lemma.

LEMMA 6.3 (**Method 2**). *There exists a constant $c > 0$, independent of h and k, such that, for $k \geq 1$,*

$$\sup_{0 \neq \mathbf{v} \in V_{k+1}} \frac{b(\mathbf{v}, p)}{|\mathbf{v}|_h} \geq \gamma_2 \|p\| \geq c k^{-4} \|p\|, \quad p \in M_k. \tag{6.3}$$

Proof. The proof is similar to [12, Th. 3.1 and 3.2]. We also refer to [11] where similar ideas where first employed. Here, we only consider the two-dimensional case and find an explicit dependence on k of the constants. The extension to three dimensions is straightforward.

Given $p \in M_k$, we first decompose it as

$$p = p_0 + (p - p_0) = p_0 + \tilde{p}, \tag{6.4}$$

with $p_0 \in M_0$ the L^2 projection of p. If we set $p_\kappa = \tilde{p}_{|\kappa}$, $\kappa \in \mathcal{T}$, we have

$$\int_\kappa p_\kappa \, dx = 0.$$

Thanks to [28, Th. 5.1], there exists $\mathbf{v}_0 \in V_2 \cap H_0^1(\Omega)^2$, such that

$$b(\mathbf{v}_0, p_0) = \int_\Omega \nabla \cdot \mathbf{v}_0 \, p_0 \, dx = \|p_0\|^2, \quad |\mathbf{v}_0|_h = |\mathbf{v}_0|_{1,\Omega} \leq C_0 \|p_0\|, \tag{6.5}$$

with a constant C_0 that is independent of h and k. If hanging nodes are present (6.5) is a consequence of [23, Th. 4.9] (see [31, Lem. 7.3] for the three dimensional case).

We next consider $\kappa \in \mathcal{T}$, of diameter h_κ, and construct a velocity $\mathbf{v}_\kappa \in \mathbb{Q}_{k+1}^2$ in κ. We first note that it is enough to assign the values of each component at the $(k+2)^2$ nodes $GGL(k+1)^2$. We set

$$\mathbf{v}_\kappa(a_{ij}) = h_\kappa^2 \, \nabla p_\kappa(a_{ij})$$

at all the k^2 internal nodes. In addition, we define

$$\begin{aligned} &\mathbf{v}_\kappa(a_{ij}) \cdot \mathbf{n} = 0, \\ &\mathbf{v}_\kappa(a_{ij}) \times \mathbf{n} = h_\kappa^2 \, \nabla p_\kappa(a_{ij}) \times \mathbf{n}, \end{aligned}$$

on all the nodes on $\partial\kappa$ (i.e., a_{ij} with $i = 0$, $j = 0$, $i = k+1$, or $j = k+1$) and $\mathbf{v}_\kappa = 0$ at the four vertices of κ. These nodal values are then interpolated in order to obtain a polynomial in $\mathbb{Q}_{k+1}^2$. We note in particular that the normal component of $\mathbf{v}_\kappa$ vanishes on $\partial\kappa$.

A global function $\tilde{\mathbf{v}} \in V_{k+1}$ can be defined by

$$\tilde{\mathbf{v}}_{|\kappa} = \mathbf{v}_\kappa.$$

We remark that $\tilde{\mathbf{v}}$ has a vanishing, and thus continuous, normal component on Γ_{int}, but that its tangential component is in general discontinuous. In addition its normal component vanishes on $\partial\Omega$.

Since each component of $\mathbf{v}_\kappa$ belongs to $\mathbb{Q}_{k+1}$, using the inverse estimate (3.2) and (6.2), we find, for $\kappa \in \mathcal{T}$,

$$|\tilde{\mathbf{v}}|_{1,\kappa}^2 \le Ck^4 h_\kappa^{-2} \|\tilde{\mathbf{v}}\|_{0,\kappa}^2 \le Ck^4 h_\kappa^{-2} \sum_{0\le i,j\le k+1} |\tilde{\mathbf{v}}(a_{ij})|^2 w_{ij}.$$

Since the values of $\tilde{\mathbf{v}}$ at the nodes $GLL(k+1)^2$ are either equal to $h_\kappa^2 \nabla p_\kappa$ or vanish and the weights w_{ij} are positive, we can write

$$k^4 h_\kappa^{-2} \sum_{0\le i,j\le k+1} |\tilde{\mathbf{v}}(a_{ij})|^2 w_{ij} \le k^4 h_\kappa^2 \sum_{0\le i,j\le k+1} |\nabla p(a_{ij})|^2 w_{ij} = k^4 h_\kappa^2\, |\tilde{p}|_{1,\kappa}^2,$$

where for the last equality we have used the fact that $|\nabla \tilde{p}|^2$ belongs to $\mathbb{Q}_k^2$ and the quadrature formula on $GLL(k+1)^2$ is thus exact. Using these last two estimates and the inverse inequality (3.2) we find

$$\sum_{\kappa\in\mathcal{T}} |\tilde{\mathbf{v}}|_{1,\kappa}^2 \le C_1 \|\tilde{p}\|_0^2 = \tilde{C}_1 k^8 \|\tilde{p}\|_0^2, \tag{6.6}$$

with $\tilde{C}_1$ independent of h and k.

We next consider the interface contributions. Let $e \in \mathcal{E}_{int}$. Since each component of the jump $[\tilde{\mathbf{v}}]$ belongs to $\mathbb{Q}_{k+1}$ on e, using (6.2) and the definition of σ, we find

$$\int_e \sigma |[\tilde{\mathbf{v}}]|^2\, ds \le Ck^2 h_e^{-1} \sum_{0\le i\le k+1} |[\tilde{\mathbf{v}}(a_i)]|^2 w_i,$$

where the sum is taken over the nodes $GLL(k+1)$ on e. Proceeding as before and noting that the normal component of $\tilde{\mathbf{v}}$ vanishes on e, we can write

$$\begin{aligned} k^2 h_e^{-1} \sum_{0\le i\le k+1} |[\tilde{\mathbf{v}}(a_i)]|^2 w_i &\le k^2 h_e^3 \sum_{0\le i\le k+1} \left| \left[\frac{d\tilde{p}}{dt}(a_i) \right] \right|^2 w_i \\ = \quad k^2 h_e^3 \sum_{0\le i\le k+1} \left| \left(\frac{d\,[\tilde{p}]}{dt} \right)(a_i) \right|^2 w_i &= k^2 h_e^3\, |[\tilde{p}]|_{1,e}^2 \le Ck^6 h_e \int_e [\tilde{p}]^2\, ds, \end{aligned} \tag{6.7}$$

where t is the arc length along e and for the last inequality we have employed the inverse inequality (3.2). Combining these last two estimates and noting that a similar argument can be employed for $e \in \mathcal{E}_{out}$, we obtain

$$\sum_{e\in\mathcal{E}} \int_e \sigma |[\tilde{\mathbf{v}}]|^2\, ds \le C \sum_{e\in\mathcal{E}} k^6 h_e \int_e [\tilde{p}]^2\, ds. \tag{6.8}$$

Combining (6.6) and (6.8), and using the inverse inequality (3.3), we find

$$|\tilde{\mathbf{v}}|_h \le C_2\, \|\tilde{p}\|_0 = \tilde{C}_2 k^4 \|\tilde{p}\|_0. \tag{6.9}$$

We next consider $b(\tilde{\mathbf{v}}, \tilde{p})$. We first note that thanks to (4.7) and the fact that the normal component of $\tilde{\mathbf{v}}$ vanishes on Γ_{int}, we have

$$b(\tilde{\mathbf{v}}, \tilde{p}) = \sum_{\kappa\in\mathcal{T}} \int_\kappa \tilde{\mathbf{v}} \cdot \nabla \tilde{p}\, dx.$$

Since $\tilde{\mathbf{v}} \cdot \nabla \tilde{p}$ belongs to $\mathbb{Q}_{2k+1}$ in each element $\kappa \in \mathcal{T}$, we have

$$\begin{aligned}\int_\kappa \tilde{\mathbf{v}} \cdot \nabla \tilde{p}\, dx &= \sum_{0\le i,j\le k+1} \tilde{\mathbf{v}}(a_{ij}) \cdot \nabla \tilde{p}(a_{ij})\, w_{ij} \\ &= h_\kappa^2 \sum_{\substack{1\le i\le k\\ 0\le j\le k+1}} \left|\frac{\partial \tilde{p}}{\partial x}(a_{ij})\right|^2 w_{ij} + h_\kappa^2 \sum_{\substack{0\le i\le k+1\\ 1\le j\le k}} \left|\frac{\partial \tilde{p}}{\partial y}(a_{ij})\right|^2 w_{ij}.\end{aligned} \tag{6.10}$$

We now assume for simplicity that

$$\kappa = \left(\frac{-h_\kappa}{2}, \frac{h_\kappa}{2}\right)^2;$$

the more general case of an affinely mapped element can be dealt with in a similar way. Since

$$-h_\kappa/2 = a_0 < a_1 < \cdots < a_k < a_{k+1} = h_\kappa/2,$$

for the first term on the right hand side of (6.10) we find

$$\begin{aligned}& h_\kappa^2 \sum_{\substack{1\le i\le k\\ 0\le j\le k+1}} \left|\frac{\partial \tilde{p}}{\partial x}(a_{ij})\right|^2 w_{ij} \\ &= h_\kappa^2 \sum_{\substack{i=0,k+1\\ 0\le j\le k+1}} \left(1-\left(\frac{2a_i}{h_\kappa}\right)^2\right) \left|\frac{\partial \tilde{p}}{\partial x}(a_{ij})\right|^2 w_{ij} + h_\kappa^2 \sum_{\substack{1\le i\le k\\ 0\le j\le k+1}} \left|\frac{\partial \tilde{p}}{\partial x}(a_{ij})\right|^2 w_{ij} \\ &\ge h_\kappa^2 \sum_{\substack{0\le i\le k+1\\ 0\le j\le k+1}} \left(1-\left(\frac{2a_i}{h_\kappa}\right)^2\right) \left|\frac{\partial \tilde{p}}{\partial x}(a_{ij})\right|^2 w_{ij} \\ &= h_\kappa^2 \int_\kappa \left(1-\left(\frac{2x}{h_\kappa}\right)^2\right) \left|\frac{\partial \tilde{p}}{\partial x}(x,y)\right|^2 dxdy \\ &= h_\kappa^2 \int_{-1}^1 \int_{-1}^1 (1-\hat{x}^2) \left|\frac{\partial \hat{p}}{\partial \hat{x}}\right|^2 d\hat{x}d\hat{y},\end{aligned}$$

where $\hat{p}(\hat{x},\hat{y}) = \tilde{p}(x(\hat{x}), y(\hat{y}))$ and $[x(\hat{x}), y(\hat{y})]$ maps the reference square into κ. We note that, since $\tilde{p} \in \mathbb{Q}_k$, the function

$$\left(1-\left(\frac{2x}{h_\kappa}\right)^2\right) \left|\frac{\partial \tilde{p}}{\partial x}(x,y)\right|^2$$

belongs to $\mathbb{Q}_{2k}$ and the quadrature formula based on $GGL(k+1)^2$ is exact. Using similar arguments for the second term on the right hand side of (6.10) and Lemma 6.2, we find a constant C_3, independent of h and k, such that

$$\int_\kappa \tilde{\mathbf{v}} \cdot \nabla \tilde{p}\, dx \ge C_3\, \|\tilde{p}\|_0^2. \tag{6.11}$$

We next define

$$\mathbf{v} = \mathbf{v}_0 + \delta \tilde{\mathbf{v}},$$

with $\delta > 0$ to be specified later.

We first note that

$$b(\tilde{\mathbf{v}}, p_0) = -\sum_{\kappa\in\mathcal{T}} \int_\kappa \nabla\cdot\tilde{\mathbf{v}}\, p_0\, dx + \int_{\Gamma_{int}\cup\partial\Omega} [\tilde{\mathbf{v}}\cdot\mathbf{n}] < p_0 > ds = 0, \tag{6.12}$$

since p_0 is constant on each element and $\tilde{\mathbf{v}}\cdot\mathbf{n}$ vanishes on $\Gamma_{int}\cup\partial\Omega$. We also have, thanks to (6.5),

$$\left|\sum_{\kappa\in\mathcal{T}} \int_\kappa \nabla\cdot\mathbf{v}_0\, \tilde{p}\, dx\right| \le C\, |\mathbf{v}_0|_1\, \|p-p_0\|_0 \le C\, \|p_0\|_0\, \|p-p_0\|_0, \tag{6.13}$$

and, since $\mathbf{v}_0$ is continuous and vanishes on $\partial\Omega$,

$$\int_{\Gamma_{int}\cup\partial\Omega} [\mathbf{v}_0\cdot\mathbf{n}] < \tilde{p} > ds = 0. \tag{6.14}$$

Combining (6.13) and (6.14) yields

$$|b(\mathbf{v}_0, \tilde{p})| \le C_4 \|p_0\|_0\, \|p-p_0\|_0, \tag{6.15}$$

with C_4 independent of h and k. Using (6.5), (6.11), (6.12), and (6.15), we can write

$$b(\mathbf{v}, p) \ge \|p_0\|_0^2 + C_3\,\delta\, \|p-p_0\|_0^2 - C_4\|p_0\|_0\, \|p-p_0\|_0 \ge \frac{1}{2}\|p_0\|_0^2 + \left(\delta C_3 - \frac{C_4^2}{2}\right) \|p-p_0\|_0^2,$$

and thus

$$b(\mathbf{v}, p) \ge \frac{1}{2}\|p\|_0^2,$$

with the choice $\delta = (1 + C_4^2)/(2C_3)$. Finally (6.5) and (6.9) give

$$|\mathbf{v}|_h \le (C_0 + \delta C_2)\|p\|_0.$$

The last two estimates thus give

$$\gamma_2 = \frac{1}{2(C_0 + \delta C_2)} \ge ck^{-4}.$$

□

We remark that the proof of the previous lemma is valid for general meshes with hanging nodes. It also carries out in exactly the same way for the case of conforming Taylor-Hood elements with hanging nodes.

7. Stability and consistency of the discrete problem. Throughout this and the following section, we assume that discrete inf-sup conditions hold for Methods 1 and 2; see Lemmas 6.1 and 6.3, or, for p and hp approximations, Remarks 1 and 2. The following two corollaries are consequences of these discrete inf-sup conditions; see [13, Pr. 1.2, Pg. 39].

COROLLARY 7.1. *Let k' be equal to $k+2$ or $k+1$. If $p \in M_k$ satisfies*

$$b(\mathbf{v}, p) = 0, \quad \mathbf{v} \in V_{k'},$$

then $p = 0$.

COROLLARY 7.2. *For $i = 1, 2$, corresponding to the choices $k' = k+2, k+1$, we have*

$$\sup_{q \in M_k} \frac{b(\mathbf{v}, q)}{||q||} \geq \gamma_i \inf_{\mathbf{z} \in Z} |\mathbf{v} + \mathbf{z}|_h, \quad \mathbf{v} \in V_{k'}.$$

Before proceeding, we note that our discrete bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are not continuous on the original spaces $H^1(\Omega)^n$ and $L_0^2(\Omega)$, due to the interface contributions. This makes the analysis more complicated. However two weaker continuity properties hold.

We need to define two suitable stronger norms. For a velocity $\mathbf{V}$ we set

$$|||\mathbf{V}|||_v^2 \quad = \quad |\mathbf{V}|_h^2 + \sum_{e \in \mathcal{E}} \int_e \frac{\nu {k'}^2}{h_e} |[\mathbf{V}]|^2 \, ds + \sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa} \frac{\nu^2}{\sigma} |\nabla \mathbf{V}|^2 ds.$$

We note however that, in case $\mathbf{v} \in V_{k'}$, the inverse estimate (3.3) and the definition of σ ensure that

$$|\mathbf{v}|_h \leq |||\mathbf{v}|||_v \leq C|\mathbf{v}|_h, \tag{7.1}$$

with a constant C that only depends on σ_0.

We have the following continuity property.

LEMMA 7.3. *Let $\mathbf{V} \in L^2(\Omega)^n$, such that $\mathbf{V} \in H^2(\kappa)^n$, for $\kappa \in \mathcal{T}$, and $\mathbf{w} \in V_{k'}$. Then there exist constants independent of $\mathbf{V}$, $\mathbf{w}$, h, and k' such that*

$$|a(\mathbf{V}, \mathbf{w})| \leq \alpha |||\mathbf{V}|||_v \, |\mathbf{w}|_h,$$

and, in case $\mathbf{V} \in V_{k'}$,

$$|a(\mathbf{V}, \mathbf{w})| \leq \alpha' |\mathbf{V}|_h \, |\mathbf{w}|_h.$$

Proof. The proof is the same as that of [19, Lem. 4.3] which can be adapted to the vector case in a straightforward way. The second bound is a consequence of (7.1). □

Analogously, we define a stronger norm for the pressure:

$$|||Q|||_p^2 = \|Q\|_{0,\Omega}^2 + \sum_{\kappa \in \mathcal{T}} \int_{\partial\kappa} \frac{1}{\sigma} Q^2 \, ds.$$

In case $q \in M_k$, the inverse estimate (3.3) yields

$$\|q\|_{0,\Omega} \leq |||q|||_p \leq C\|q\|_{0,\Omega}, \tag{7.2}$$

with a constant that depends on σ_0 and ν.

LEMMA 7.4. *Let $Q \in L_0^2(\Omega)$ and $\mathbf{v} \in L^2(\Omega)^n$, such that $Q \in H^1(\kappa)$ and $\mathbf{v} \in H^1(\kappa)^n$, for $\kappa \in \mathcal{T}$. Then there exist constants independent of Q, $\mathbf{v}$, h, k, and k', such that*

$$|b(\mathbf{v}, Q)| \leq \beta |\mathbf{v}|_h \, |||Q|||_p,$$

and, in case $Q \in M_k$,

$$|b(\mathbf{v}, Q)| \le \beta |\mathbf{v}|_h \, \|Q\|_0.$$

Proof. We have

$$|b(\mathbf{v}, Q)| \le \left| \sum_{\kappa\in\mathcal{T}} (\nabla\cdot\mathbf{v}, Q)_\kappa \right| + \left| \int_{\Gamma_{int}} < Q > [\mathbf{v}\cdot\mathbf{n}]\, ds \right| + \left| \int_{\partial\Omega} Q\,\mathbf{v}\cdot\mathbf{n}\, ds \right| = I_1 + I_2 + I_3.$$

We consider each one of the three terms. We clearly have

$$I_1^2 \le \left(\sum_{\kappa\in\mathcal{T}} \|\nabla\cdot\mathbf{v}\|_{0,\kappa}^2 \right) \|Q\|_0^2 \le C|\mathbf{v}|_h^2 \, \|Q\|_0^2. \tag{7.3}$$

We next consider $e \in \mathcal{E}_{int}$, with $e = \partial\kappa \cap \partial\kappa'$. Using the definition of σ, we find

$$\left| \int_e < Q > [\mathbf{v}\cdot\mathbf{n}]\, ds \right|^2 \le \int_e \sigma^{-1} < Q >^2 \, ds \int_e \sigma[\mathbf{v}\cdot\mathbf{n}]^2\, ds.$$

In a similar way, for $e \in \mathcal{E}_{out}$, we find

$$\left| \int_e Q\mathbf{v}\cdot\mathbf{n}\, ds \right|^2 \le \int_e \sigma^{-1} Q^2\, ds \int_e \sigma|\mathbf{v}\cdot\mathbf{n}|^2\, ds.$$

The proof of the first bound is concluded by summing over $e \subset \Gamma_{int}\cup\partial\Omega$ and combining the result with (7.3). The second bound is a consequence of (7.2). □

We finally recall that the bilinear form $a(\cdot,\cdot)$ is coercive, i.e.,

$$a(\mathbf{u},\mathbf{u}) = |\mathbf{u}|_h^2, \quad \mathbf{u} \in V_{k'}. \tag{7.4}$$

Existence and uniqueness of the discrete problem (4.6) are ensured by (7.4), the continuity properties in Lemmas 7.3 and 7.4, and the discrete inf-sup conditions; see [13, Th. 1.1, Sect. II.1.1].

LEMMA 7.5. *Let $k \ge 0$. Then problem (4.6) has a unique solution $\{\mathbf{u}, p\} \in V_{k'} \times M_k$, for the two choices $k' = k+2, k+1$, corresponding to Methods 1 and 2.*

As is the case for DG approximations, consistency is ensured under some more stringent regularity assumptions on the exact solution. In order to prove this property, we need some preliminary results.

For $\mathcal{D} \subset \Omega$, we define $H(\mathrm{div}\,, \mathcal{D})$ as the space of tensors $\tau \in L^2(\Omega)^{n\times n}$, such that $\nabla\cdot\tau \in L^2(\Omega)^n$, equipped with the graph norm

$$\left(\|\tau\|_{\mathcal{D}}^2 + \|\nabla\cdot\tau\|_{\mathcal{D}}^2\right)^{1/2}.$$

If $e \subset \partial\mathcal{D}$ has non-vanishing $(n-1)$-dimensional measure, we define the space $H_{00}^{-1/2}(e)^n$ as the dual of $H_{00}^{1/2}(e)^n$, the space of vectors of $H^{1/2}(\partial\mathcal{D})^n$ that vanish on $\partial\mathcal{D} \setminus e$. In case $e = \partial\mathcal{D}$, we have $H_{00}^{-1/2}(e)^n = H^{-1/2}(e)^n$.

The following result can be proved using analogous techniques as those for spaces of vectors; see Sections III.1.1 and III.1.2 in [13].

LEMMA 7.6. *Let $\mathcal{D} \subset \Omega$ and $e \subset \partial\mathcal{D}$ with positive measure.*

1. *There exists a continuous trace operator from* $H(\mathrm{div}\,,\mathcal{D})$ *onto* $H_{00}^{-1/2}(e)^n$ *that coincides with*

$$\tau : \mathbf{n}_{|_e}$$

for $\tau \in C^\infty(\Omega)^{n\times n}$.
2. *The Green's formula (2.3) holds for* $\tau \in H(\mathrm{div}\,,\mathcal{D})$ *and* $\mathbf{v} \in H^1(\mathcal{D})^n$*, where the integral on the right hand side is to be understood as the duality pairing between* $H^{1/2}(\partial\mathcal{D})^n$ *and* $H^{-1/2}(\partial\mathcal{D})^n$.
3. *Let* $\Omega_i \subset \Omega$, $i = 1, 2$*, two open disjoint subsets with outward normals* $\mathbf{n}_i$*, such that the union of their closures coincides with* $\overline{\Omega}$*. Given* $\tau_i \in H(\mathrm{div}\,,\Omega_i)$*, the tensor in* Ω *defined by*

$$\tau_{|\Omega_i} = \tau_i$$

belongs to $H(\mathrm{div}\,,\Omega)$ *if and only if*

$$\tau_1 : \mathbf{n}_1 = -\tau_2 : \mathbf{n}_2 = \tau_2 : \mathbf{n}_1, \quad \text{in } H_{00}^{-1/2}(\partial\Omega_1 \cap \partial\Omega_2)^n.$$

We are now ready to prove the consistency of our methods.

LEMMA 7.7. *Let* $\{\mathbf{U}, P\} \in H^1(\Omega)^n \times L_0^2(\Omega)$ *be the solution of the continuous problem (2.4). If* $\mathbf{U} \in H^2(\kappa)^n$ *and* $P \in H^1(\kappa)$*, for* $\kappa \in \mathcal{T}$*, then* $\{\mathbf{U}, P\}$ *satisfies the discrete problem*

$$\begin{array}{lcll} a(\mathbf{U},\mathbf{v}) + b(\mathbf{v},P) & = & (\mathbf{f},\mathbf{v})_\Omega + \displaystyle\int_{\partial\Omega} (\sigma\,\mathbf{g}\cdot\mathbf{v} + \mathbf{g}\cdot(\nu\,\nabla\mathbf{v} : \mathbf{n}))\; ds, & \mathbf{v} \in V_{k'}, \\ b(\mathbf{U},q) & = & \displaystyle\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\, q\, ds, & q \in M_k, \end{array} \tag{7.5}$$

with $k' = k+2, k+1, k$ *and* $k \geq 0$.

Proof. We first note that, if T is the stress tensor of the exact solution, we have

$$\begin{array}{rcl} -\nu\Delta\mathbf{U} + \nabla P = -\nabla\cdot(\nu\nabla\mathbf{U} - PI) = -\nabla\cdot T & = & \mathbf{f} \in L^2(\Omega)^n, \\ \nabla\cdot\mathbf{U} & = & 0. \end{array}$$

Then $T \in H(\mathrm{div}\,,\Omega)$ and Lemma 7.6 holds. In particular, the normal component $T : \mathbf{n}$ is well defined and continuous across every $e \in \mathcal{E}_{int}$.

We will show that the residual

$$\begin{array}{rcl} R(\mathbf{v},q) & = & (\mathbf{f},\mathbf{v})_\Omega + \displaystyle\int_{\partial\Omega} (\sigma\,\mathbf{g}\cdot\mathbf{v} + \mathbf{g}\cdot(\nu\,\nabla\mathbf{v} : \mathbf{n}))\; ds - \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n}\, q\, ds \\ & - & a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},P) + b(\mathbf{U},q) \end{array}$$

vanishes for every $\mathbf{v} \in V_{k'}$ and $q \in M_k$.

Using the fact that $[\mathbf{U}] = 0$ on every $e \in \mathcal{E}_{int}$ and that $\mathbf{U} = \mathbf{g}$ on $\partial\Omega$, we can write

$$\begin{array}{rcl} R(\mathbf{v},q) & = & \displaystyle\sum_{\kappa\in\mathcal{T}} \int_\kappa (\mathbf{f}\cdot\mathbf{v} - \nu\nabla\mathbf{U}\cdot\nabla\mathbf{v} + \nabla\cdot\mathbf{v}P - \nabla\cdot\mathbf{U}q)\; dx \\ & + & \displaystyle\int_{\Gamma_{int}} < \nu\,\nabla\mathbf{U} : \mathbf{n} > \cdot[\mathbf{v}]\, ds - \int_{\Gamma_{int}} < P > \cdot[\mathbf{v}\cdot\mathbf{n}]\, ds \\ & + & \displaystyle\int_{\partial\Omega} (\nu\,\nabla\mathbf{U} : \mathbf{n})\cdot\mathbf{v} ds - \int_{\partial\Omega} P\mathbf{v}\cdot\mathbf{n} ds. \end{array}$$

Taking into account the identities

$$\nabla \cdot \mathbf{v}P = (PI) : \nabla \mathbf{v}, \quad < P > \cdot [\mathbf{v} \cdot \mathbf{n}] =< (PI) : \mathbf{n} > \cdot \mathbf{v},$$

and the definition of T, we obtain

$$\begin{aligned} R(\mathbf{v}, q) &= \sum_{\kappa \in \mathcal{T}} \int_\kappa (\mathbf{f} \cdot \mathbf{v} - T \cdot \nabla \mathbf{v} - \nabla \cdot \mathbf{U} q) \, dx \\ &+ \int\limits_{\Gamma_{int}} < T : \mathbf{n} > \cdot [\mathbf{v}] \, ds + \int\limits_{\partial\Omega} (T : \mathbf{n}) \cdot \mathbf{v} \, ds \\ &= \sum_{\kappa \in \mathcal{T}} \left(\int_\kappa (\mathbf{f} \cdot \mathbf{v} - T \cdot \nabla \mathbf{v} - \nabla \cdot \mathbf{U} q) \, dx + \int_{\partial\kappa} < T : \mathbf{n} > \cdot [\mathbf{v}] \, ds \right). \end{aligned}$$

Since $T : \mathbf{n}$ is continuous across the interelement boundaries, and thus equal to $< T : \mathbf{n} >$, and the Green's formula (2.3) can be applied, we find

$$R(\mathbf{v}, q) = \sum_{\kappa \in \mathcal{T}} \int_\kappa ((\mathbf{f} + \nabla \cdot T) \cdot \mathbf{v} - \nabla \cdot \mathbf{U} q) \, dx.$$

The proof is concluded by using the differential equation (2.1). □

8. A priori error estimates. This section is devoted to the proof of a priori error estimates. We proceed similarly as in [13]; see Section II.2.2, Propositions 2.4, 2.6, 2.7. However our proofs are more involved due to the lack of continuity of the bilinear forms in the continuous spaces; see Lemmas 7.3 and 7.4.

The next lemma gives a bound for the velocity.

LEMMA 8.1. *Let the exact solution* $\{\mathbf{U}, P\} \in H^1(\Omega)^n \times L^2_0(\Omega)$ *be in* $H^{m_\kappa}(\kappa)^n \times H^{n_\kappa}(\kappa)$, $\kappa \in \mathcal{T}$, *with* $m_\kappa \geq 2$ *and* $n_\kappa \geq 1$. *Then for* $i = 1, 2$, *corresponding to the choices* $k' = k+2, k+1$, *there exists a constant* C, *independent of* h *and* k, *but depending on* ν *and* σ_0, *such that*

$$|\mathbf{U} - \mathbf{u}|_h \leq C \sum_{\kappa \in \mathcal{T}} \left(\frac{1}{\gamma_i} \frac{h_\kappa^{s_\kappa - 1}}{k^{m_\kappa - 3/2}} |\mathbf{U}|_{m_\kappa, \kappa} + \frac{h_\kappa^{r_\kappa}}{k^{n_\kappa}} |P|_{n_\kappa, \kappa} \right), \tag{8.1}$$

with $1 \leq s_\kappa \leq \min\{k'+1, m_\kappa\}$, $1 \leq r_\kappa \leq \min\{k+1, n_\kappa\}$, *and* γ_i *the inf-sup constant of Method* i.

Proof. We consider a vector $\mathbf{w} \in Z(\mathbf{g})$. We have

$$|\mathbf{w} - \mathbf{u}|_h^2 = a(\mathbf{w} - \mathbf{u}, \mathbf{w} - \mathbf{u}) = a(\mathbf{w} - \mathbf{U}, \mathbf{w} - \mathbf{u}) + a(\mathbf{U} - \mathbf{u}, \mathbf{w} - \mathbf{u}).$$

Using Lemma 7.7 and (4.6), we can write

$$|\mathbf{w} - \mathbf{u}|_h^2 = a(\mathbf{w} - \mathbf{U}, \mathbf{w} - \mathbf{u}) - b(\mathbf{w} - \mathbf{u}, P - p).$$

We then note that, since $(\mathbf{w} - \mathbf{u}) \in Z$, the discrete pressure p can be replaced by any function $q \in M_k$. We have

$$|\mathbf{w} - \mathbf{u}|_h^2 = a(\mathbf{w} - \mathbf{U}, \mathbf{w} - \mathbf{u}) - b(\mathbf{w} - \mathbf{u}, P - q),$$

and using Lemmas 7.4 and 7.3

$$|\mathbf{w} - \mathbf{u}|_h \leq \alpha |||\mathbf{w} - \mathbf{U}|||_v + \beta |||P - q|||_p, \quad \mathbf{w} \in Z(\mathbf{g}), \quad q \in M_k. \tag{8.2}$$

A bound for the error is obtained using the triangle inequality

$$|\mathbf{U}-\mathbf{u}|_h \leq |\mathbf{U}-\mathbf{w}|_h + \alpha|||\mathbf{w}-\mathbf{U}|||_v + \beta|||P-q|||_p, \quad \mathbf{w} \in Z(\mathbf{g}),\ q \in M_k. \tag{8.3}$$

Our second step is to find bounds that involve an arbitrary function of $V_{k'}$ instead of $Z(\mathbf{g})$. In order to do so, given $\mathbf{v} \in V_{k'}$, we consider the problem of finding a $\mathbf{z}(\mathbf{v}) \in V_{k'}$, such that

$$b(\mathbf{z}(\mathbf{v}), q) = b(\mathbf{U}-\mathbf{v}, q), \quad q \in M_k.$$

Thanks to Corollary 7.2 and [13, Pr. 1.2, Pg. 39], we can find a solution $\mathbf{z} \in V_{k'}$, such that

$$\gamma_i\,|\mathbf{z}(\mathbf{v})|_h \leq \sup_{q\in M_k} \frac{b(\mathbf{z}, q)}{||q||} = \sup_{q\in M_k} \frac{b(\mathbf{U}-\mathbf{v}, q)}{||q||} \leq \beta|\mathbf{U}-\mathbf{v}|_h. \tag{8.4}$$

Since

$$b(\mathbf{z}(\mathbf{v})+\mathbf{v}, q) = b(\mathbf{U}, q) = \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n} q\, ds, \quad q \in M_k,$$

we have

$$\mathbf{w}(\mathbf{v}) = \mathbf{z}(\mathbf{v}) + \mathbf{v} \in Z(\mathbf{g}).$$

We next go back to (8.3), and take $\mathbf{w} = \mathbf{w}(\mathbf{v})$, with $\mathbf{v} \in V_{k'}$

$$|\mathbf{U}-\mathbf{u}|_h \leq |\mathbf{U}-\mathbf{v}|_h + |\mathbf{z}(\mathbf{v})|_h + \alpha|||\mathbf{v}-\mathbf{U}|||_v + \alpha|||\mathbf{z}(\mathbf{v})|||_v + \beta|||P-q|||_p, \tag{8.5}$$

where we have used a triangle inequality for $|||\cdot|||_v$.

Using (8.4), (8.5), and (7.2), we find

$$|\mathbf{U}-\mathbf{u}|_h \leq \frac{C}{\gamma_i}|\mathbf{U}-\mathbf{v}|_h + \beta|||P-q|||_p + \alpha|||\mathbf{U}-\mathbf{v}|||_v, \tag{8.6}$$

where we have assumed that $\gamma_i \leq 1$. Here C is independent of h, k, and ν, but depends on σ_0.

We finally make a particular choice for $\mathbf{v}$ and p. We choose

$$\mathbf{v} = \Pi_{h,k'}\mathbf{V}, \quad q = \Pi_{h,k}P.$$

We bound the single terms in the $|\cdot|_h$ and $|||\cdot|||$ norms. They consists of integrals over elements or part of the element boundaries. We start with the pressure terms: Thanks to the definition of $|||\cdot|||_p$ and σ, and the trace inequality (3.4), we can write

$$|||P-q|||_p^2 \leq C\|P-q\|_{0,\Omega}^2 + C\sum_{\kappa\in\mathcal{T}} \left(\frac{h_\kappa}{k^2}\|P-q\|_{0,\kappa}\,|P-q|_{1,\kappa} + \frac{1}{k^2}\|P-q\|_{0,\kappa}^2\right),$$

and, using Lemma 3.1 with $s = 0, 1$,

$$|||P-q|||_p^2 \leq C\sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2r_\kappa}}{k^{2n_\kappa}}|P|_{n_\kappa,\kappa}^2. \tag{8.7}$$

We next consider the velocity contributions. Using the trace estimate (3.4) and the shape-regularity of $\mathcal{T}$, we find

$$\sum_{e\in\mathcal{E}} \sigma_0\, \nu \frac{k'^2}{h_e} \int_e |[\mathbf{U}-\mathbf{v}]|^2\, ds \le C \sum_{\kappa\in\mathcal{T}} \left(\frac{k'^2}{h_\kappa} \|\mathbf{U}-\mathbf{v}\|_{0,\kappa}\, |\mathbf{U}-\mathbf{v}|_{1,\kappa} + \frac{k'^2}{h_\kappa^2} \|\mathbf{U}-\mathbf{v}\|_{0,\kappa}^2 \right),$$

and, using Lemma 3.1 with $s = 0, 1$,

$$\sum_{e\in\mathcal{E}} \sigma \int_e |[\mathbf{U}-\mathbf{v}]|^2\, ds \le C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{k^{2m_\kappa-3}} |\mathbf{U}|_{m_\kappa,\kappa}^2. \tag{8.8}$$

Similarly, for the gradient contributions Lemma 3.1 yields

$$\sum_{\kappa\in\mathcal{T}} \frac{\nu^2}{\sigma} \int_{\partial\kappa} |\nabla(\mathbf{U}-\mathbf{v})|^2\, ds \le C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{k^{2m_\kappa-1}} |\mathbf{U}|_{m_\kappa,\kappa}^2. \tag{8.9}$$

Combining (8.8) and (8.9) with the definitions of $|||\cdot|||_v$, $|\cdot|_h$, and σ, we obtain

$$|\mathbf{U}-\mathbf{v}|_h^2 \le |||\mathbf{U}-\mathbf{v}|||_v^2 \le C \sum_{\kappa\in\mathcal{T}} \frac{h_\kappa^{2s_\kappa-2}}{k^{2m_\kappa-3}} |\mathbf{U}|_{m_\kappa,\kappa}^2. \tag{8.10}$$

The proof is concluded by combining (8.6), (8.7), and (8.10). ∎

A bound for the pressure is given in the following lemma.

LEMMA 8.2. *Let the exact solution* $\{\mathbf{U}, P\} \in H^1(\Omega)^n \times L_0^2(\Omega)$ *be in* $H^{m_\kappa}(\kappa)^n \times H^{n_\kappa}(\kappa)$, $\kappa \in \mathcal{T}$, *with* $m_\kappa \ge 2$ *and* $n_\kappa \ge 1$. *Then for* $i = 1, 2$, *corresponding to the choices* $k' = k+2, k+1$, *there exists a constant* C, *independent of* h *and* k, *but depending on* ν *and* σ_0, *such that*

$$\|P-p\|_0 \le C \sum_{\kappa\in\mathcal{T}} \left(\frac{1}{\gamma_i^2} \frac{h_\kappa^{s_\kappa-1}}{k^{m_\kappa-3/2}} |\mathbf{U}|_{m_\kappa,\kappa} + \frac{1}{\gamma_i} \frac{h_\kappa^{r_\kappa}}{k^{n_\kappa}} |P|_{n_\kappa,\kappa} \right),$$

with $1 \le s_\kappa \le \min\{k'+1, m_\kappa\}$, $1 \le r_\kappa \le \min\{k+1, n_\kappa\}$, *and* γ_i *the inf-sup constant of Method* i.

Proof. Let $q = \Pi_{h,k} P \in M_k$. Using the discrete inf-sup conditions for $b(\cdot,\cdot)$, Lemmas 7.7, 7.3, and 7.4, we find find

$$\begin{aligned}
\|q-p\| &\le \frac{1}{\gamma_i} \sup_{0\neq\mathbf{v}\in V_{k'}} \frac{b(\mathbf{v}, q-p)}{|\mathbf{v}|_h} = \frac{1}{\gamma_i} \sup_{0\neq\mathbf{v}\in V_{k'}} \frac{b(\mathbf{v}, q-P) + b(\mathbf{v}, P-p)}{|\mathbf{v}|_h} \\
&= \frac{1}{\gamma_i} \sup_{0\neq\mathbf{v}\in V_{k'}} \frac{b(\mathbf{v}, q-P) - a(\mathbf{U}-\mathbf{u}, \mathbf{v})}{|\mathbf{v}|_h} \\
&\le \frac{\beta}{\gamma_i} |||P-q|||_p + \frac{\alpha}{\gamma_i} |||\mathbf{U}-\mathbf{u}|||_v.
\end{aligned} \tag{8.11}$$

A bound for the velocity contribution can be found using the triangle inequality and (7.1). Let $\mathbf{v} = \Pi_{h,k'} \mathbf{V}$. We have

$$\begin{aligned}
|||\mathbf{U}-\mathbf{u}|||_v &\le |||\mathbf{U}-\mathbf{v}|||_v + |||\mathbf{u}-\mathbf{v}|||_v \le |||\mathbf{U}-\mathbf{v}|||_v + C|\mathbf{u}-\mathbf{v}|_h \\
&\le |||\mathbf{U}-\mathbf{v}|||_v + C|\mathbf{U}-\mathbf{u}|_h + C|\mathbf{U}-\mathbf{v}|_h \\
&\le C(|||\mathbf{U}-\mathbf{v}|||_v + |\mathbf{U}-\mathbf{u}|_h).
\end{aligned} \tag{8.12}$$

Since

$$\|P - p\| \leq \|P - q\| + \|q - p\|,$$

the proof is concluded by combining (8.11) and (8.12), with the error estimates (8.7), (8.10), and (8.1). ∎

We note the loss of optimality of half a power of k in the estimates of Lemmas 8.1 and 8.2, typical of DG approximations of second order problems; see [19].

Remark 4. *We note that, if we assume that an inf-sup condition also holds for Method 3, as stated in Remark 3, then Lemma 7.5 and the error estimates in Lemmas 8.1 and 8.2 are also valid for Method 3 ($i = 3$). Lemma 7.7 is valid for an arbitrary $k' \geq 0$.*

We conclude with some comments on the optimality of the methods proposed. We assume that the exact solution satisfies

$$P \in H^{n_k}(\kappa), \quad \mathbf{U} \in H^{n_k+1}(\kappa)^n, \quad \kappa \in \mathcal{T}.$$

We then consider Lemmas 8.1 and 8.2 with

$$n_\kappa \geq k, \quad m_\kappa = n_\kappa + 1 \geq k + 1, \quad r_\kappa = k \quad s_\kappa = k + 1.$$

For the h-version, Lemmas 6.1 and 6.3 ensure that the given error estimates for Methods 1 and 2 are optimal:

$$\begin{aligned} |\mathbf{U} - \mathbf{u}|_h &\leq C \sum_{\kappa \in \mathcal{T}} h_\kappa^k \left(|\mathbf{U}|_{k+1,\kappa} + |P|_{k,\kappa}\right), \\ \|P - p\| &\leq C \sum_{\kappa \in \mathcal{T}} h_\kappa^k \left(|\mathbf{U}|_{k+1,\kappa} + |P|_{k,\kappa}\right). \end{aligned}$$

We note that the two methods have the same rate of convergence. Since the pressure spaces are the same, the increase in the velocity space of Method 1 does not present any advantage. Error estimates for Method 3 are suboptimal: half a power of h is lost for the velocity, and one full power for the pressure; see Remark 3.

We now consider p-approximations in two dimensions. Remarks 1 and 2 ensure

$$\begin{aligned} |\mathbf{U} - \mathbf{u}|_h &\leq Ck^{-(n_\kappa - 1/2)} \sum_{\kappa \in \mathcal{T}} \left(|\mathbf{U}|_{n_k+1,\kappa} + |P|_{n_k,\kappa}\right), \\ \|P - p\| &\leq Ck^{-(n_\kappa - 1/2)} \sum_{\kappa \in \mathcal{T}} \left(|\mathbf{U}|_{n_k+1,\kappa} + |P|_{n_k,\kappa}\right), \end{aligned}$$

where half a power of k is lost both for the velocity and the pressure. We note that for the case of conforming $\mathbb{Q}_{k+2}$-$\mathbb{Q}_k$ approximations we have

$$\begin{aligned} |\mathbf{U} - \mathbf{u}|_h &\leq Ck^{-(n_\kappa - 1/2)} \sum_{\kappa \in \mathcal{T}} \left(|\mathbf{U}|_{n_k+1,\kappa} + |P|_{n_k,\kappa}\right), \\ \|P - p\| &\leq Ck^{-(n_\kappa - 1)} \sum_{\kappa \in \mathcal{T}} \left(|\mathbf{U}|_{n_k+1,\kappa} + |P|_{n_k,\kappa}\right), \end{aligned}$$

since the inf-sup constant decreases as $k^{-1/2}$. Conforming Taylor-Hood $\mathbb{Q}_{k+1}$-$\mathbb{Q}_k$ elements in two dimensions appear to satisfy the same error estimate, since numerical results show the same behavior for the inf-sup constant; see Figure 5.3 and Figure 3 in [2].

9. Extensions to other velocity-pressure pairs. As already noted for Lemma 6.1, stability results for approximations using continuous velocities and discontinuous pressures give lower bounds for the inf-sup condition of the corresponding DG approximations employing discontinuous velocities. In particular, we can choose for velocities and pressures

$$V_{k+1}, \qquad \{q \in L_0^2(\Omega) | \quad q_{|\kappa} \in \mathbb{P}_k(\kappa),\ \kappa \in \mathcal{T}\},$$

with $\mathbb{P}_k(\kappa)$ the space of polynomials of total degree k in κ. Alternatively, we can employ

$$V_k, \qquad M_{[\lambda k]},$$

with $[\lambda k]$ the integer part of λk, with $0 < \lambda < 1$, and $k - \lambda k \geq 2$. Uniform divergence stability is ensured by the results in [10] for the corresponding conforming approximations. We also refer to [1] for additional choices of velocity-pressure pairs.

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