HUMANITARIAN MEDICAL ALLOCATION FOR PUBLIC HEALTH EMERGENCIES

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CERTIFICATE OF ORIGINALITY

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it reproduces no material previously published or written, nor material that has been accepted for the award of any other degree or diploma, except where due acknowledgement has been made in the text.

________________________(Signed)

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Abstract

Emerging and re-emerging epidemic diseases pose an on-going threat to global public health security. Recent examples of epidemics, including the outbreak of Ebola virus disease, cholera and Middle East respiratory syndrome coronavirus (MERS-CoV), have caused physical and psychological pain of millions of people. Epidemic outbreaks are also very common in the aftermath of natural disasters and bioterrorist actions. According to the World Health Organization (WHO)'s Twelfth General Programme of Work, the improvement of prevention, preparedness, response and recovery activities is set as one of the WHO's five strategic imperatives.

In response to a large-scale epidemic, satisfying medical needs is crucial to the success of humanitarian-oriented operations and management. However, it differs from the general business logistics problems in many ways. This research underscores the importance of humanitarian medical allocation, and aims to explore a novel approach for improving humanitarian allocation of medical reliefs for response to unconventional large-scale epidemics, and draw managerial insights for health care practice. Several methods are adopted, including stochastic dynamic programming, linear programming, epidemic diffusion models and game theory. This inter-disciplinary research would contribute to the decision analysis of humanitarian medical allocation.

Specifically, three sub-topics are conducted focusing on humanitarian medical allocation in one area, allocation in multiple areas and allocation with cross-sector cooperation, respectively.

(1) In the sub-research of humanitarian medical allocation in one area, this research proposes a model of time-varying allocation of emergency medical relief for response to large-scale epidemics. Based on the trend of epidemic disease spreading, a stochastic

dynamic programming model is developed to optimize the temporal allocation policy of emergency medical relief in each time period. This new formulation is closer to the real logistics practice during epidemic outbreaks than the traditional ones. Additionally, this research obtains a general form of the optimal medical allocation decision in each time period and further develops a case study based on real data to demonstrate the applicability of the proposed model. According to the above analytical and numerical studies, some properties are provided and their implementations for policy makers are discussed.

(2) In the sub-research of humanitarian medical allocation in multiple areas, we present a novel model of emergency medical logistics for quick response to public health emergencies. The proposed methodology consists of two recursive mechanisms: (i) the time-varying forecasting of medical relief demand and (ii) relief distribution. The medical demand associated with each epidemic area is forecast according to a modified susceptible-exposed-infected-recovered (SEIR) model. A linear programming approach is then applied to facilitate distribution decision-making. Both the physical and psychological situations of those affected are considered. The modified SEIR model contributes to forecasting by considering not only physical factors, such as the differences in the infection conditions of survivors and the spatial interaction relationships among epidemic areas, but also the psychological demand of exposed and undiagnosed individuals. In the distribution model, psychological fragility is formulated and discussed in detail, unlike previous studies. The relationship between emergency medical logistics and the psychological effects on affected people is highlighted as well. Numerical studies are conducted. Results show that the consideration of survivor psychology significantly reduces the psychological fragility of affected people, but it barely influences physical fragility.

(3) Typically, humanitarian logistics engages a large number and variety of sectors, including central governments, local governments, the military, international organizations and private companies. Modern information technologies provide potential opportunities to share information among different sectors, and to work together to pursue effective and efficient relief operations. However, each of these sectors may have different missions and capacities. In this sub-research of cross-sector cooperation in allocation, a series of cross-sector decision models are developed to discuss different types

of cooperation and information sharing between public and private sectors. The basic model, which consists of a public sector (usually the government) and a private sector, is formulated to obtain the optimal decisions of the two sectors. Then this research presents three more cooperation mechanisms: semi-cooperation with a private leader, semi-cooperation with a government leader, and full cooperation. The optimal solutions of these four models are provided and compared. By solving and comparing their optimal solutions, this sub-research makes the first step to understand the differences among these four mechanisms. The results illustrate that full cooperation is not always the best choice, while semi-cooperation with information sharing would also achieve potential advantages, even if two sectors made their own decisions separately.

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Chapter 1

Introduction

This chapter is a brief introduction of this research. On the basis of introducing the research background and defining several key concepts, this chapter firstly presents the background and objectives in Section 1.1. Then Section 1.2 clarifies the content of this research and explains the methodology. Finally, the contributions are highlighted in Section 1.3.

1.1 Background and Objectives

1.1.1 Background

A public health emergency is defined by the U.S. National Disaster Medical System as the emergency need for healthcare or medical services in response to a disaster, the significant outbreak of an infectious disease, bioterrorist attack, and other significant or catastrophic events. Among all kinds of public health emergencies, unconventional emerging and re-emerging epidemic diseases pose an on-going threat to global public health security.

Recent examples of unconventional epidemics, including the outbreak of Ebola virus disease, cholera and Middle East respiratory syndrome coronavirus (MERS-CoV), have caused harm to millions of people. Globally, there were over 48 million cases of malaria and they caused an estimated 584 thousand deaths in 2013. The recent outbreak of Ebola disease has had a total of 28,141 confirmed, probable, and suspected cases reported, with 11,291 deaths. Epidemic outbreaks are also very common in the aftermath of natural disasters and bioterrorist actions. In addition to health threats and economic losses, public health emergencies also result in psychological suffering, such as feelings of helplessness, sorrow, and panic. Studies conducted on the worldwide effects of the SARS outbreak in 2003 suggest that the fear of SARS is a more severe pandemic than the disease itself (Cheng and Tang 2004). According to the World Health Organization (WHO)'s Twelfth General Programme of Work, the improvement of prevention, preparedness, response and recovery activities is set as one of the WHO's five strategic imperatives.

In response to a large-scale epidemic, satisfying medical needs is crucial to the success of humanitarian-oriented emergency operations and management. Although most public health emergencies cannot be avoided, their influence can be significantly reduced by efficient humanitarian medical logistics.

Humanitarian logistics is a branch of logistics which specializes in organizing the delivery and warehousing of supplies during complex emergencies to the affected areas and people. Types and quantities of the resources, ways of procurement and storage of the supplies, tools of tracking and means of transportation to the stricken area, specialization of teams participating in the operations and plans of cooperation between these teams, are some important issues that are connected directly to humanitarian logistics. Emergency logistics is one of the most significant contents of humanitarian logistics, so some literatures use this term instead of humanitarian logistics.

The needs in medical relief include medicines and disposable medical products. Compared with related business logistics problems and general humanitarian logistics of other kinds of reliefs (food as an example), the challenge of prompt and effective allocation of medical supplies arises from its unique characteristics of demand:

(1) Limited demand-related information. The lack of information such as the severity of injuries and the number of casualties, challenges distribution-related decision making. In particular, the incubation period results in a time delay in demand (Li et al. 1999, Zhang and Ma 2003). Additionally, uncertain probability distribution of medical demand leads to more difficulties. In business and general humanitarian logistics, although demand is uncertain, the probability distribution of demand is rela-

tively stable. However, the probability distribution of medical demand also vary with time because of the disease spreading and the growth of doctors experience and public knowledge.

(2) Unpredictable epidemic outbreaks in other areas. Demand in business logistics is limited in several known areas. In general humanitarian logistics, potential demand in new areas are predictable based on the forecasting of secondary disasters. However, epidemic may occur or break out in any areas unexpectedly. A disease can spread quickly from one area to another and can even become a large-scale epidemic. Infection, recovery, and mortality rates typically vary across areas because of different physical conditions of individuals, as well as habits, customs and medical services provided by the hospitals in each area (Brauer and van den Driessche 2001, Capaldi et al. 2012).

(3) Timeliness and imperfect substitutability. Deferred delivery of medical relief is not allowed. And unlike other forms of relief, the substitutability of medical relief is imperfect. A specific medicine usually cannot be substituted by another medicine (Mete and Zabinsky 2010).

Besides the unique characteristics of medical demand, challenges also arise from special requirements of storage and transport of medicines, which lead to the fact that medical logistics capacity cannot be expanded in short term.

1.1.2 Objectives

Humanitarian medical allocation that directly responses to public health emergencies are vital. However, this field faces many challenges that have not been addressed. This research is conducted to explore a novel analytical approach for improving humanitarian allocation of medical reliefs for response to unconventional large-scale epidemics, and draw managerial insights for health care practice. Specifically, this research aims

(1) to develop optimization models to formulate the objectives and constrains of humanitarian medical allocation, as well as the humanitarian allocation problem considering cross-sector cooperation and information sharing;

(2) to provide concise and detailed analyses of the proposed optimization models and obtain optimal strategies and properties of humanitarian medical allocation;

(3) to conduct several numerical studies with both real-case data and experimental data and show the availability and potential implications;

(4) to compare different allocation models and optimal strategies, and additionally discuss their advantages in different situations for practice.

1.2 Contents and Methodology

This research proposes several models and then conducts analytical and numerical analyses to pursue humanitarian medical allocation with the balance between effectiveness, efficiency and fairness.

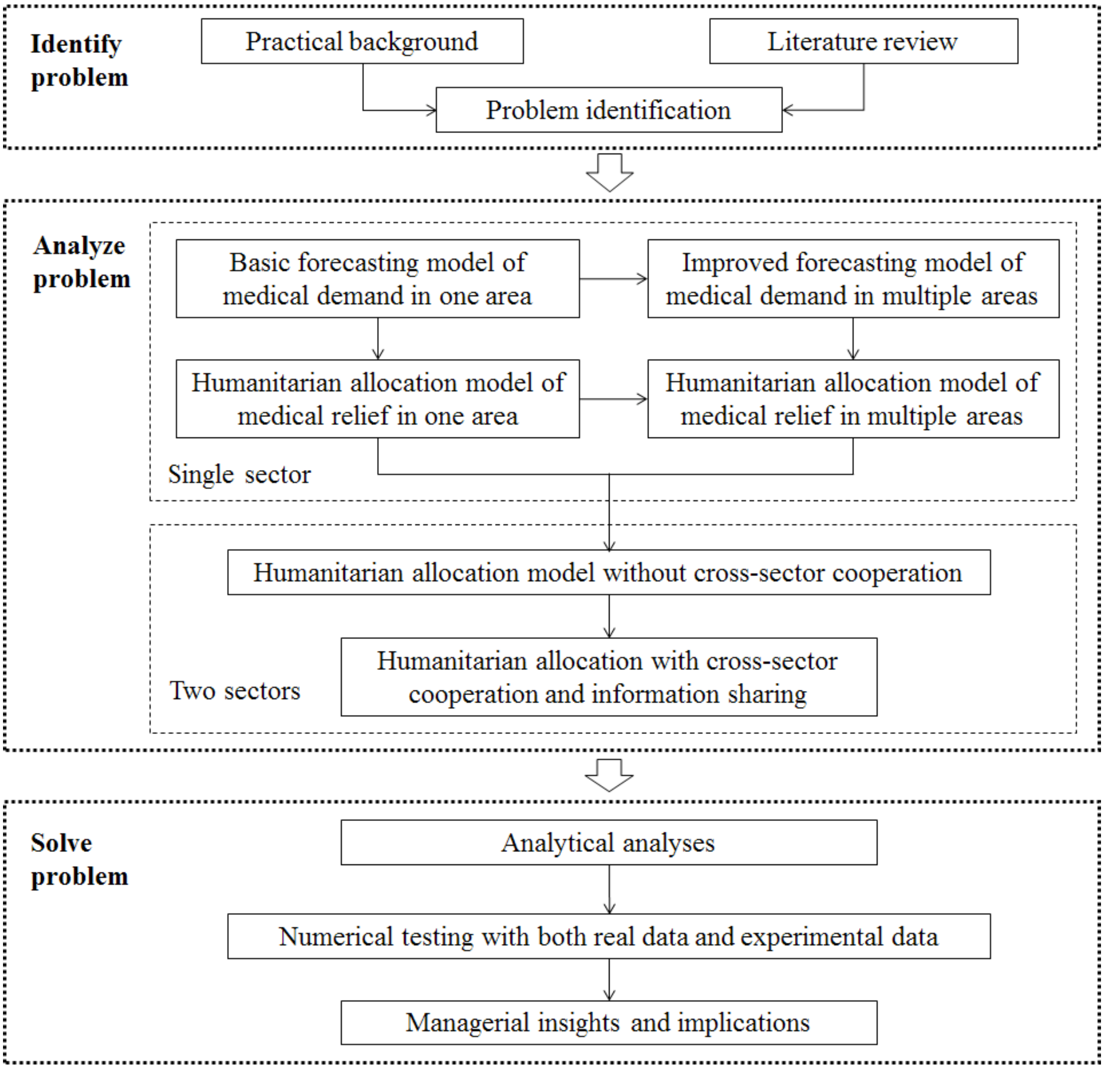


Figure 1.1: Technology roadmap

The technology roadmap of this research is shown as Fig. 1.1.

This research is deployed by the following three issues: the temporal allocation problem in a single epidemic area, the allocation problem in multiple areas, and the allocation problem with cross-sector cooperation.

(1) Humanitarian medical allocation in one area. Chapter 2 presents a temporal allocation model for response to large-scale epidemic outbreaks in one area. The proposed stochastic dynamic programming approach is developed based on unique characteristics of epidemic diseases. This chapter provides a general closed-form of the optimal allocation policy in each time period. Several properties of the problem and its optimal policy are derived. A case study based on a real epidemic outbreak is conducted and the relations between the optimal policy and each parameter are discussed. The results highlight some managerial implications for better response to epidemic outbreaks.

(2) Humanitarian medical allocation in multiple areas. Chapter 3 presents a novel model of this problem. The proposed methodology consists of two recursive mechanisms: the time-varying forecasting of medical relief demand and relief distribution. The medical demand associated with each epidemic area is forecast according to a modified susceptible-exposed-infected-recovered (SEIR) model. A linear programming approach is then applied to facilitate distribution decision-making. The physical and psychological fragility of affected people are discussed. Numerical studies are conducted. Results show that the consideration of survivor psychology significantly reduces the psychological fragility of affected people, but it barely influences physical fragility.

(3) Humanitarian medical allocation with cross-sector cooperation and information sharing. Chapter 4 presents a cross-sector decision methodology to achieve efficient and effective humanitarian medical allocation where multiple parties are involved in. Based on the theories and methods of public-private partnerships, the basic model, which contains a public sector (e.g. the government) and a private sector, formulates the optimal decision of the two sectors, respectively. Then this chapter provides and compares three cooperation mechanisms: semi-cooperation with a private leader, semi-cooperation with a government leader and full cooperation. In this chapter, we provide analytical solution to discuss the difference among four proposed models and conduct

some further numerical studies. The results highlight some managerial implications for better understanding of cross-sector cooperation in humanitarian allocation.

To sum up, the related fields and main methods are listed in Table 1.1.

Table 1.1: Related fields and methods

Model	Related Fields	Main Method
Humanitarian medical allocation in one area	Humanitarian logistics (Emergency logistics) Healthcare	Dynamic stochastic programming Epidemic diffusion model
Humanitarian medical allocation in multiple areas	Humanitarian logistics (Emergency logistics) Healthcare Survivor psychology	Linear programming Epidemic diffusion model
Humanitarian medical allocation with cross-sector cooperation	Humanitarian logistics (Emergency logistics) Public-private partnership	Stochastic programming Stackelberg game

1.3 Impacts and Contributions

This interdisciplinary study contributes to the decision analysis of humanitarian medical allocation problem in response to unconventional epidemic outbreaks. Humanitarian medical allocation differs from general humanitarian allocation problems and related business logistics problems in that the former problem involve many challenges that increase the complexity and difficulty of solving the logistical problems. Specifically,

(1) When discussing the humanitarian medical allocation problem in one epidemic area, Chapter 2 develops stochastic dynamic programming model and further obtains a general form of the optimal medical allocation decision in each time period.

(2) The models in Chapters 2 and 3 are developed based on the trend of epidemic disease spreading. This research applies epidemic diffusion models to forecast the demand of urgent medical reliefs. These new formulations are closer to the real logistics practice during epidemic outbreaks than the traditional ones.

(3) When forecasting medical demand, Chapter 3 also considers the differences in the infection conditions of survivors, the spatial interaction relationships among epidemic areas and the demand of exposed and undiagnosed individuals.

(4) To specify the objective functions, in Chapter 3, psychological fragility is formulated and discussed in detail, unlike in previous studies. The relationship between humanitarian medical logistics and the psychological effects on affected people is highlighted as well.

(5) In Chapter 5, this research proposes and compares four optimization models, including one allocation model without cross-sector cooperation and three cooperation models with different cooperation mechanisms. It illustrates the value of cross-sector cooperation between public and private sectors when making humanitarian allocation decisions.

Chapter 2

Humanitarian Medical Allocation in One Area

This chapter presents a temporal allocation model of medical relief in one area for response to large-scale epidemic outbreaks. The proposed stochastic dynamic programming approach is developed based on unique characteristics of epidemic diseases. This chapter provides a general closed-form of the optimal allocation policy in each time period, and several properties of the problem and its optimal policy are derived. In addition, a case study based on a real epidemic outbreak is conducted and the relations between the optimal policy and each parameter are discussed. These results highlight some managerial implications for better response to epidemic outbreaks.

2.1 Introduction

As discussed in Chapter 1, humanitarian medical logistics faces more challenges compared with related business logistics and general humanitarian logistics of other kinds of reliefs. Considering the gap mentioned in Section 2.2, this chapter proposes a model of time-varying allocation of emergency medical supply for response to large-scale epidemics. Based on the trend of epidemic disease spreading, a stochastic dynamic programming model is developed to optimize the allocation policy of emergency medical supply in each time period.

In this chapter, we consider the following situation: a local government collects local information of demand, supply and the trend of disease spreading, and makes allocated decision. If there is some redundant relief, central government would allocate them to other areas then. But the local government have no information about the supply and demand of other areas.

Specifically, this chapter contributes to the fields of logistics and healthcare in the following two ways:

(1) The stochastic dynamic programming model proposed in this chapter characterizes the temporal allocation problem of medical supply based on the trend of epidemic disease spreading. This new formulation is closer to the real logistics practice during epidemic outbreaks.

(2) This chapter obtains a general form of the optimal medical allocation decision in each time period and further develops a case study based on real data to demonstrate the applicability of the proposed model. According to the above analytical and numerical studies, some properties are provided and their implementations for policy makers are discussed.

The remainder of this chapter is organized as follows. Related literatures are reviewed in Section 2.2. Then decision models are developed in Section 2.3. Section 2.4 presents the analytical and numerical studies of the local decision problem, which is a stochastic dynamic programming model. Finally, Section 2.5 concludes this chapter and provides insights for policy makers. All the proofs are given in Section 3.6.

2.2 Literature Review

Although some studies try to combine medical rescue with emergency logistics (Sheu and Pan, 2014), only a few studies exist on emergency medical logistics for public health emergencies despite its importance and particularity. Only bioterror response logistics, a special case in emergency medical logistics, has been discussed (Kaplan et al. 2003, Craft et al. 2005, Miller et al. 2006, Zaric et al. 2008, Hu and Zhao 2011, Liu and Zhao 2011). These studies have aided in understanding the problems of evaluating existing proposals for logistics, distributing antibiotics, and providing hospital care after a bioterror attack. Methods such as atmospheric release models, dose-response models, disease progression models and epidemic diffusion models have been used. However, a terrorist attack has only two most feared biological agents, namely, smallpox and anthrax (Craft et al. 2005), but other public health emergencies may be aroused by other diseases that are less understood and the results of these studies are usually difficult to apply. Besides, this stream of studies does not discuss time-varying allocation strategies of medical supplies in detail.

Regarding emergency response to epidemic, most researches on medical resource allocation study a static problem while only a few take into consideration the time evolution and dynamic nature of medical demand. Zaric and Brandeau (2001, 2002) present dynamic models for epidemic resources allocation, develop approximation methods and present heuristics for solving the models. These studies additionally suggest that allowing for reallocation of funds may generate more health benefits. In Ekici et al. (2014), food distribution during epidemic is examined. Wang et al. (2009) build a multi-objective stochastic programming model to discuss the selection of logistics hubs and the distribution of medical supplies, and the model is solved by genetic algorithm based on Monte Carlo simulation. Rachaniotis et al. (2012) propose a deterministic resource scheduling model in epidemic control. The model is appropriate for large populations, where random interactions can be averaged out. He and Liu (2015) develop and compare three emergency medical logistics models. Their models consist of two recursive mechanisms: the time-varying forecasting of medical demand and the distribution of medical supplies. Liu and Xiao (2015) present a discrete time-space network model for a dynamic resource allocation problem following an epidemic outbreak in a region and a custom genetic algorithm is adopted to solve the proposed model. Several similar models are also built to discuss this problem (Liu and Liang 2013, Liu et al. 2015).

While these dynamic studies provide insights towards medical allocation and epidemic control, they often overlook some of the following critical aspects: (1) Although several periods are discussed, the models in most previous studies are essentially repetitious one-period problem, and decisions are optimized for one time period but not for the whole time horizon. (2) Demand of medical supplies is of stochastic nature while some previous works have been directed towards the deterministic case. (3) Analytical solutions are important references to understand medical allocation problem. However, analytical analyses of most stochastic dynamic researches focus on only two time periods, or the solutions are obtained by heuristic methods. To the best of our knowledge, in related areas, there has not been a research work providing a general analytical form of the optimal policy in each time period.

2.3 Model Development

2.3.1 Assumptions

The medical system considered in this chapter is a specific two-layer supply chain that involves one central warehouse and one epidemic area. One kind of medical supply needs to be allocated to the epidemic area several times. Only one kind of medical relief is considered because in practice the prescriptions for different patients with a specific disease are similar. Those essential medicines and disposable medical products can be regarded as one package.

The central warehouse gathers all medical relief from producers, charity organization and other logistics hubs, and distributes supply appropriately to the epidemic area. The epidemic area can only get medical supply from this specific central warehouse. At the beginning of each time period, the policy maker collects real-time inventory information, updates knowledge of the disease, and decides the amounts of medical supply sent to the area in the current and the optimal policies in the following time periods.

Based on the above description, four basic assumptions are made to facilitate the model formulation.

(1) In a specific area, the maximal available amounts of medical supply in each time period are the same.

(2) Allocation periods have been set in advance.

(3) The demand is strongly correlated with the number of quarantined patients. In practice, infectious patients can always be divided into two groups: quarantined and un-quarantined patients. Only a part of patients can be diagnosed, hospitalized and get treatments, and its proportion nearly remain unchanged in the short term.

Quarantined patients are separated from other residents so they do not infect others. Un-quarantined patients do not realize they have been infected and are not in the need of medical relief. (Chowell et al. 2006).

(4) Lead time of supply distribution is less than the length of one time period.

2.3.2 Notations

Notations in this chapter are as follows.

Parameters of the model:

S**:** Available amount of medical relief in each time period

Q_t**:** Inventory of medical relief in the epidemic area at the beginning of time period t

D_t**:** A stochastic variable, which refers to demand for medical relief in the epidemic area in time period t

$g_t(D_t)$**:** The probability distribution function of D_t

$G_t(D_t)$**:** The cumulative distribution function of D_t

α**:** Penalty per unit of oversupplied medical relief

β**:** Penalty per unit of unfulfilled demand for medical relief

P_t**:** Number of infectious people in the epidemic area in time period t

$v_t(P_t)$**:** The probability distribution function of P_t

$V_t(P_t)$**:** The cumulative distribution function of P_t

u**:** Un-quarantined rate of infectious people in the epidemic area

$m(uP_t)$**:** A function to forecast the number of infectious people in time period $t+1$, according to the number of infectious people in time period t

$M(D_t)$**:** A function to forecast future demands according to the demand in time period t

δ_t: The random variation of the number of infectious people in time period t

σ_t: The random variation of the demand in time period t

$w(\sigma_t)$: The probability distribution function of σ_t

$W(\sigma_t)$: The cumulative distribution function of σ_t

Decision variables of the model:

X_t: Amount of medical supply allocated to the epidemic area in time period t

2.3.3 Problem Modeling

Based on the description of the logistics system given above, the problem can be divided into n time periods, with a policy decision required at each time period. Inventory at the beginning of each time period varies among time periods. The state transition function is:

$$Q_{t+1} = (Q_t + X_t - D_t)^+, \quad 0 < t \leq n-1 \tag{2.1}$$

The objective is to find an optimal policy for the overall problem to minimize the expected sum of total penalty.

$$\begin{cases} f_{n+1}(Q_{n+1}, X_{n+1}) = 0 \\ f_t(Q_t, X_t) = \min\limits_{X_t} E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X_{t+1})\} \end{cases} \tag{2.2}$$

$$(0 \leq X_t \leq S, \quad t = 1, 2, \ldots, n)$$

where

$$Y_t(X_t, D_t) = \alpha(X_t + Q_t - D_t)^+ + \beta(D_t - X_t - Q_t)^+ \tag{2.3}$$

$Y_t(Q_t, X_t^L)$ is the contribution of time period t to the objective function. $(X_t + Q_t - D_t)^+$ is the oversupplied amount of medical relief in the epidemic area at the end of time period t, while $(D_t - X_t - Q_t)^+$ is the amount of unfulfilled demand in time period t. α and β are penalty coefficients. Local governments always tend to leave relief for themselves as much as possible. Thus the central government gives a penalty of oversupplied distribution to local governments. However, the penalty coefficient of the oversupply is usually smaller than that of the unfulfilled demand, i.e., $\alpha < \beta$.

2.3.4 Probability Distribution Function of Demand

In the proposed optimization model above, D_t is a random variable affected by the characteristics of epidemic disease. This subsection will discuss the probability distribution of D_t, which is strongly related to the number of quarantined patients.

Note that the number of quarantined patients also vary among time periods:

$$P_{t+1} = m(uP_t) + \delta_{t+1}, \quad 0 < t \leq n-1 \tag{2.4}$$

where $m(uP_t)$ is a forecasting model of the number of infectious people. u refers to un-quarantined rate. $m(uP_t) \geq 0$ and is increasing in P_t. $\delta_t (t = 1, 2, \ldots, n)$ are i.i.d. random variables following an exponential distribution.

Proposition 2.1.

(1) With any given P_t,

$$V_{t+1}(y) = \begin{cases} 0 & , \quad y < 0 \\ W\left(y - m(uP_t)\right) - W\left(y - m(uP_t)\right) W\left(-m(uP_t)\right) + W\left(-m(uP_t)\right) & , \quad y \geq 0 \end{cases}$$

and

$$v_{t+1}(y) = \begin{cases} 0 & , \quad y < 0 \\ w\left(y - m(uP_t)\right)\left[1 - W\left(-m(uP_t)\right)\right] & , \quad y > 0 \end{cases}$$

where $W(\delta_t)$ and $w(\delta_t)$ are the cumulative distribution and probability density functions of δ_t, respectively.

(2) With any given y, $V_{t+1}(y)$ is decreasing in P_t.

The demand in each time period is strongly related to the corresponding number of quarantined infectious people because only quarantined patients can be diagnosed and get treatments. Define θ as the demand of the medical relief per patient in each time period. Thus,

$$
\begin{aligned}
D_{t+1} &= \theta(1-u)P_{t+1} \\
&= \theta(1-u)\left(m(uP_t)+\delta_{t+1}\right) \\
&= \theta(1-u)\left(m\left(\frac{\theta}{\theta(1-u)}D_t\right)+\delta_{t+1}\right)
\end{aligned}
$$

The above equation can be simplified as

$$D_{t+1} = M(D_t) + \sigma_{t+1}, \quad 0 < t \le n-1 \tag{2.5}$$

where $M(D_t) = \theta(1-u)m\left(\frac{\theta}{\theta(1-u)}D_t\right)$ is a forecasting model of demand. This equation presents the relationship between D_{t+1} and D_t. $M(D_t) \ge 0$ and is increasing in D_t. $\sigma_{t+1} = \theta(1-u)\delta_{t+1}$ also follows an exponential distribution with mean λ. That is,

$$w'(\sigma_t) = \begin{cases} 0 & , \quad \sigma_t < 0 \\ \lambda e^{-\lambda\sigma_t} & , \quad \sigma_t > 0 \end{cases} \tag{2.6}$$

and

$$W'(\sigma_t) = \begin{cases} 0 & , \quad \sigma_t < 0 \\ 1-e^{-\lambda\sigma_t} & , \quad \sigma_t \ge 0 \end{cases} \tag{2.7}$$

Define $g_t(D_t)$ and $G_t(D_t)$ as the probability density function and cumulative distribution function of D_t, respectively. These two functions describe the natural characteristics of epidemic disease.

Based on Eqs.(2.5), (2.6) and (2.7), the following equations are obtained:

$$G_{t+1}(y) = \int_0^\infty W'\left(y - M(D_t)\right)|_{D_t} g_t(D_t)\, dD_t \tag{2.8}$$

$$g_{t+1}(y) = \int_0^\infty w'\left(y - M(D_t)\right)|_{D_t} g_t(D_t)\, dD_t \tag{2.9}$$

where

$$W'\left(y-M(D_t)\right)|_{D_t} = \begin{cases} 0 & , \quad y - M(D_t) < 0 \\ 1 - e^{-\lambda(y-M(D_t))} & , \quad y - M(D_t) \ge 0 \end{cases}$$

$$w'\left(y-M(D_t)\right)|_{D_t} = \begin{cases} 0 & , \quad y - M(D_t) < 0 \\ \lambda e^{-\lambda(y-M(D_t))} & , \quad y - M(D_t) > 0 \end{cases}$$

To further specify the function $M(D_t)$, previous studies have put efforts on mathematical models of disease spreading rules (Larson, 2007; Anderson, 2013; He and Liu, 2015), among them exponential model and susceptible-exposed-infected-recovered (SEIR) model draw considerable attention. The former is the most simplified to describe the essential characteristics of epidemic spreading in early periods while SEIR model is more complex and realistic. This chapter adopts exponential model and in the next chapter we will introduce and improve the SEIR model. One can also adopt other epidemic models proposed in the health-care literature.

The exponential model is:

$$P_{t+1} = kuP_t + \delta_{t+1}(0 < t \leq n-1) \tag{2.10}$$

where, parameter k is decided by the specific characteristics of disease.

Thus,

$$\begin{aligned} D_{t+1} &= \theta(1-u)P_{t+1} \\ &= \theta(1-u)(kuP_t + \delta_{t+1}) \\ &= \theta(1-u)\left(\frac{ku}{\theta(1-u)}D_t + \delta_{t+1}\right) \end{aligned}$$

Set $A = ku$ and $\sigma = \theta(1-u)\delta$. We have

$$M(D_t) = AD_t \tag{2.11}$$

and

$$D_{t+1} = AD_t + \sigma_{t+1} \tag{2.12}$$

Proposition 2.2 gives the probability density and cumulative distribution functions of D_t based on the above exponential model.

Proposition 2.2.

$$G_t(y) = \begin{cases} 0 & , \quad y < A^t D_0 \\ 1 - \frac{1}{\prod_{l=1}^{t-1}(1-A^l)} \sum_{i=0}^{t-1} B^i(t-1,A) e^{\frac{-\lambda(y-A^t D_0)}{A^i}} & , \quad y \geq A^t D_0 \end{cases}$$

$$g_t(y) = \begin{cases} 0 & , \quad y < A^t D_0 \\ \frac{\lambda}{\prod_{l=1}^{t-1}(1-A^l)} \sum_{i=0}^{t-1} B^i(t-1,A) e^{\frac{-\lambda(y-A^t D_0)}{A^i}} \frac{1}{A^i} & , \quad y \geq A^t D_0 \end{cases}$$

where, $B^i(t-1, A)$ is the coefficient of x^i in the expression of $(1-Ax)(1-A^2x)\ldots(1-A^{t-1}x)$.

2.4 Analytical Analyses

The decision problem is a stochastic dynamic programming model. This section will do analytical analyses on it and the next section will do numerical analyses.

2.4.1 General Form of the Optimal Solution

In this part, we solve the above model by first solving the two-period sub-problem consisting of time period *n-1* and time period n, and then find a general form of the optimal policy in each time period in the n-period stochastic dynamic programming model.

To facilitate model solving, a function $L_t(x)$ is defined as:

$$L_t(x) = \int_x^{\infty} (D_t - x) g_t(D_t)\, dD_t, \quad x \geq 0 \tag{2.13}$$

$L_t(x)$ can be proved to be strictly decreasing and convex in x: $\frac{dL_t(x)}{dx} = G_t(x) - 1 \leq 0$. In Eq. (2.2), $E_{D_t} Y_t(X_t, D_t)$ can be written as

$$\begin{aligned} E_{D_t} Y_t(X_t, D_t) =& E_{D_t}\{\alpha(X_t + Q_t - D_t)^+ \beta(D_t - X_t - Q_t)^+\} \\ =& \alpha \int_0^{X_t+Q_t} (X_t + Q_t - D_t) g_t(D_t)\, dD_t \\ &+ \beta \int_{X_t+Q_t}^{\infty} (D_t - X_t - Q_t) g_t(D_t)\, dD_t \\ =& \alpha(X_t + Q_t - \mu_t) + (\alpha + \beta) L_t(X_t + Q_t) \end{aligned} \tag{2.14}$$

where μ_t refers to the mean of D_t.

Proposition 2.3. *The optimal solution of time period n is*

$$X_n^* = \min\{S, \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - Q_n\right)^+\}$$

that is

$$X_n^* = \begin{cases} S & , \quad Q_n \leq G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S \\ G_n^{-1}(\frac{\beta}{\alpha+\beta}) - Q_n & , \quad G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S \leq Q_n \leq G_n^{-1}(\frac{\beta}{\alpha+\beta}) \\ 0 & , \quad Q_n \geq G_n^{-1}(\frac{\beta}{\alpha+\beta}) \end{cases}$$

Since $Q_n = (X_{n-1} + Q_{n-1} - D_{n-1})^+$ (from Eq. (2.1)),

$$X_n^* = \begin{cases} 0 & , \quad D_{n-1} \leq X_{n-1} + Q_{n-1} - G_n^{-1}(\frac{\beta}{\alpha+\beta}) \\ G_n^{-1}(\frac{\beta}{\alpha+\beta}) - Q_n & , \quad X_{n-1} + Q_{n-1} - G_n^{-1}(\frac{\beta}{\alpha+\beta}) \leq D_{n-1} \\ & \quad\quad \leq X_{n-1} + Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+ \\ S - \left(S - G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)\right)^+ & , \quad D_{n-1} \geq X_{n-1} + Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+ \end{cases}$$

Since $Q_n = (X_{n-1} + Q_{n-1} - D_{n-1})^+$ (Eq. (2.1)),

$$X_n^* = \begin{cases} 0 & , \quad D_{n-1} \leq X_{n-1} + Q_{n-1} - G_n^{-1}(\frac{\beta}{\alpha+\beta}) \\ G_n^{-1}(\frac{\beta}{\alpha+\beta}) - Q_n & , \quad X_{n-1} + Q_{n-1} - G_n^{-1}(\frac{\beta}{\alpha+\beta}) \leq D_{n-1} \\ & \quad\quad \leq X_{n-1} + Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+ \\ S - \left(S - G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)\right)^+ & , \quad D_{n-1} \geq X_{n-1} + Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+ \end{cases}$$

Thus,

$$
\begin{aligned}
& f_{n-1}(Q_{n-1}, X_{n-1}) \\
&= \min_{X_{n-1}} E_{D_{n-1}}\{Y_{n-1}(X_{n-1}, Q_{n-1}) + f_n(Q_n, X_n)\} \\
&= \min_{X_{n-1}} \{\alpha(Q_{n-1} + X_{n-1} - \mu_{n-1}) + (\alpha+\beta)L_{n-1}(Q_{n-1} + X_{n-1}) \\
& \quad + \int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} [\alpha(X_{n-1} + Q_{n-1} - D_{n-1} - \mu_{n-1}) \\
& \quad +(\alpha+\beta)L_{n-1}(Q_{n-1} + X_{n-1} - D_{n-1})]\, g_{n-1}(D_{n-1})\, dD_{n-1} \\
& \quad + \int_{X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})}^{X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+} \left[\alpha\left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - \mu_{n-1}\right)\right. \\
& \quad \left. +(\alpha+\beta)L_{n-1}\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})\right)\right] g_{n-1}(D_{n-1})\, dD_{n-1} \\
& \quad + \int_{X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+}^{X_{n-1}+Q_{n-1}} \left[\alpha\left(S - \left(S - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right)^+\right.\right. \\
& \quad \left. +Q_{n-1} + X_{n-1} - D_{n-1} - \mu_{n-1}\right) \\
& \quad \left.+(\alpha+\beta)L_{n-1}\left(S - \left(S - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right)^+ + Q_{n-1} + X_{n-1} - D_{n-1}\right)\right] \\
& g_{n-1}(D_{n-1})\, dD_{n-1} \\
& \quad + \int_{X_{n-1}+Q_{n-1}}^{\infty} \left[\alpha\left(S - \left(S - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right)^+ - \mu_{n-1}\right)\right. \\
& \quad \left. +(\alpha+\beta)L_{n-1}\left(S - \left(S - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right)^+\right)\right] g_{n-1}(D_{n-1})\, dD_{n-1}\}
\end{aligned}
$$

Proposition 2.4. *The objective function in the (n-1)th time period, which represents the expectation of the total penalty in the (n-1)th and n-th time periods) is convex in X_{n-1} for any given S, Q_{n-1}, $G_{n-1}(D_{n-1})$ and $G_n(D_n)$.*

Set

$$I_t = \frac{dE_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X_{t+1})\}}{dX_t}$$

I_{n-1} has the properties given in Propositions 2.5 and 2.6.

Proposition 2.5. *I_{n-1} is an increasing function of (1) Q_{n-1} and (2) S.*

Proposition 2.6. *(1)* $\lim_{S\to\infty} I_{n-1} = I_{n-1}|_{S=G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)}$;

(2) $\lim_{Q_{n-1}\to\infty} I_{n-1} \geq \alpha \geq 0$.

Set $I_t^0 = I_t|_{X_t=0}$ and $I_t^S = I_t|_{X_t=S}$.

They have the following properties:

Proposition 2.7.

(1) Existence and uniqueness of function $h_{n-1}^0(S)$: For any given S, there exists a unique $h_{n-1}^0(S) > 0$, s.t. $I_{n-1}^0|_{Q_{n-1}<h_{n-1}^0(S)} < 0$, $I_{n-1}^0|_{Q_{n-1}=h_{n-1}^0(S)} = 0$ and $I_{n-1}^0|_{Q_{n-1}>h_{n-1}^0(S)} > 0$. Additionally, $h_{n-1}^0(S)$ is decreasing in S.

(2) Existence and uniqueness of function $h_{n-1}^S(S)$: For any given S, there exists a unique $h_{n-1}^S(S)$, s.t. $I_{n-1}^S|_{Q_{n-1}<h_{n-1}^S(S)} < 0$, $I_{n-1}^S|_{Q_{n-1}=h_{n-1}^S(S)} = 0$ and $I_{n-1}^S|_{Q_{n-1}>h_{n-1}^S(S)} > 0$. Additionally, $h_{n-1}^S(S)$ is decreasing in S.

(3) $h_{n-1}^0(S) = h_{n-1}^S(S) + S$.

Therefore, the optimal solution can be obtained.

Proposition 2.8. *For given S, Q_{n-1}, $G_{n-1}(D_{n-1})$ and $G_n(D_n)$, the optimal solution of the two-period sub-problem is*

$$X_{n-1}^* = \begin{cases} 0 & , \quad Q_{n-1} \geq h_{n-1}^0(S) \\ h_{n-1}^0(S) - Q_{n-1} & , \quad \left(h_{n-1}^0(S) - S\right)^+ \leq Q_{n-1} < h_{n-1}^0(S) \\ S & , \quad Q_{n-1} < \left(h_{n-1}^0(S) - S\right)^+ \end{cases}$$

and

$$X_n^* = \min\{S, \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - (Q_{n-1} + X_{n-1}^* - D_{n-1})\right)^+\}$$

where $h_{n-1}^0(S)$ is a function of S defined as $I_{n-1}^0|_{Q_{n-1}=h_{n-1}^0(S)} = 0$.

Proposition 2.8 gives the optimal solutions of time periods n and $n-1$. Similar to Proposition 2.3, X_{n-1}^* is also piecewise. When initial inventory (Q_{n-1}) is small, medical relief is allocated as much as possible. When initial inventory is enough, no relief will be allocated. When inventory is moderate, the optimal allocation amount is a linear function of and decreasing in the inventory Additionally, X_n^* is not only related with inventory, but also affected by X_{n-1}^*.

Then we will provide a general form of the optimal solution at each time period. Similar to the two-period problem, the solution procedures of the n-period problem also begin from the last time period.

At the t-th period,

$$f_t(Q_t, X_t) = \min_{X_t} E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X^*_{t+1})\}$$
$$= \min_{X_t}\{\alpha(Q_t + X_t - \mu_t) + (\alpha+\beta)L_t(X_t + Q_t) + E_{D_t} f_{t+1}(Q_{t+1}, X^*_{t+1})\}$$
$$s.t. \quad 0 \le X_t \le S$$

Proposition 2.9. *Given an optimal policy of the (t+1)th time period, the objective function in the t-th time period $E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X^*_{t+1})\}$, which represents the expectation of the total penalty from the t-th time period to the n-th time period, is convex in X_t for any given S, Q_t and $G_i(D_i)$ $(i = t+1, t+2, \ldots, n)$.*

Propositions 2.10 and 2.11 are provided for discussing the properties of I_t, whose value is affected by the decision variable X_t and parameters Q_t and S.

Proposition 2.10. *I_t is an increasing function of (1) Q_t and (2) S, where I_t is defined as $I_t = \frac{dE_{D_t}\{Y_t(X_t,D_t)+f_{t+1}(Q_{t+1},X^*_{t+1})\}}{dX_t}$ $(t = 1, 2, \ldots, n)$.*

Proposition 2.11.

(1) Existence and uniqueness of function $h^0_t(S)$: For any given S, there exists a unique $h^0_t(S) > 0$, s.t. $I^0_t|_{Q_t<h^0_t(S)} < 0$, $I^0_t|_{Q_t=h^0_t(S)} = 0$ and $I^0_t|_{Q_t>h^0_t(S)} > 0$. Additionally, $h^0_t(S)$ is decreasing in S.

(2) Existence and uniqueness of function $h^S_t(S)$: For any given S, there exists a unique $h^S_t(S)$, s.t. $I^S_t|_{Q_t<h^S_t(S)} < 0$, $I^S_t|_{Q_t=h^S_t(S)} = 0$ and $I^S_t|_{Q_t>h^S_t(S)} > 0$. Additionally, $h^S_t(S)$ is decreasing in S.

(3) $h^0_t(S) = h^S_t(S) + S$.

Based on the above propositions, we obtain the optimal solution at time period t given in Proposition 2.12 and Figure 2.1.

Proposition 2.12. *With any given S, Q_t and $G_i(D_i)$, $(i = t, t+1, \ldots, n)$, for any time period t, a general form of the optimal solution is*

$$x^*_t = \begin{cases} 0 & , \quad Q_t \ge h^0_t(S) \\ h^0_t(S) - Q_t & , \quad \left(h^0_t(S) - S\right)^+ \le Q_t < h^0_t(S) \\ S & , \quad Q_t < \left(h^0_t(S) - S\right)^+ \end{cases} \tag{2.15}$$

where $h^0_t(S)$ is a function of S defined as $I^0_t|_{Q_t=h^0_t(S)} = 0$ and $h^0_t(S) > 0$.

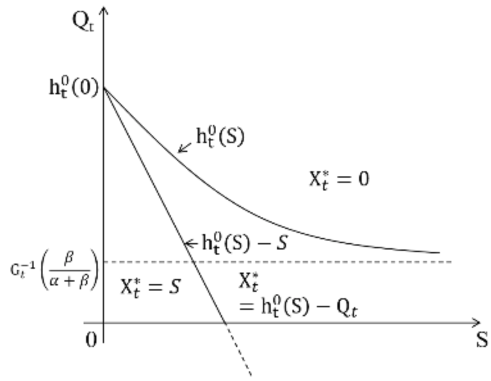


Figure 2.1: General form of the optimal solution

According to Proposition 2.12 and Figure 2.1, the optimal solution in each time period is a piecewise function of the allocated amount in the previous periods, maximum supply, demand in the previous periods and the distribution of demand in the following periods. Figure 2.1 shows the $Q_t - S$ plane. It is segmented into three parts by $h_t^0(S)$ and $h_t^0(S) - S$. The different combinations of S and Q_t determine the different forms of the optimal solution.

$$
\begin{aligned}
f_t(Q_t, X_t) =& \min_{X_t} E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X_{t+1}^*)\} \\
=& \alpha(Q_t + X_t^* - \mu_t) + (\alpha+\beta)L_t(X_t^* + Q_t) \\
& + E_{D_t} f_{t+1}\left((Q_t + X_t^* - D_t)^+, X_{t+1}^*\right) \\
=& \alpha(Q_t + X_t^* - \mu_t) + (\alpha+\beta)L_t(X_t^* + Q_t) \\
& + \int_0^{Q_t + X_t^* - h_{t+1}^0(S)} f_{t+1}(Q_t + X_t^* - D_t, 0) g_t(D_t)\, dD_t \\
& + \int_{Q_t + X_t^* - h_{t+1}^0(S)}^{Q_t + X_t^* - \left(h_{t+1}^0(S) - S\right)^+} f_{t+1}(Q_t + X_t^* - D_t, \\
& h_{t+1}^0(S) - Q_t - X_t^* + D_t) g_t(D_t)\, dD_t \\
& + \int_{Q_t + X_t^* - \left(h_{t+1}^0(S) - S\right)^+}^{Q_t + X_t^*} f_{t+1}(Q_t + X_t^* - D_t, S - (S - h_t^0(S))^+) g_t(D_t)\, dD_t \\
& + \int_{Q_t + X_t^*}^{\infty} f_{t+1}(0, S - (S - h_t^0(S))^+) g_t(D_t)\, dD_t
\end{aligned}
$$

2.4.2 Properties of the Optimal Solution

Based on the solution of the proposed stochastic dynamic programming model, several further properties are discussed.

First, the properties of $h_t^0(S)$, which is an important function in solving the proposed problem, are described in Proposition 2.13.

Proposition 2.13.

(1) $\frac{\partial h_t^0(S)}{\partial S} \leq 0$

(2) $h_t^0(S) \geq G_t^{-1}\left(\frac{\beta}{\alpha+\beta}\right)$

(3) $\frac{\partial h_{t+1}^0(S)}{\partial D_t} \geq 0 (D_t \neq X_t + Q_t)$

Proposition 2.13(1) illustrates that $h_t^0(S)$ is decreasing in S. The reason is when supply increases, the probability of demand un-fulfillment in following periods would decline. Thus, the allocated amount can be reduced. Proposition 2.13(2) shows in any time periods, the value of $h_t^0(S)$ is greater than or equal to the $G_t^{-1}(\frac{\beta}{\alpha+\beta})$, which is the optimal solution of the static problem (one-period problem). Proposition 2.13(3) states that if the demand in a specific time period increases, then in the next period, the value of $h_t^0(S)$ also increases, because increased demand indicates patients in following periods.

According to Proposition 2.12 and Eq.(2.1), Eq.(2.15) is equivalent to

$$x_{t+1}^* = \begin{cases} 0 & , \quad D_t \leq Q_t + X_t - h_{t+1}^0(S) \\ h_{t+1}^0(S) - (Q_t + X_t - D_t)^+ & , \quad Q_t + X_t - h_{t+1}^0(S) < D_t \\ & \qquad \leq Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \\ S - (S - h_t^0(S))^+ & , \quad D_t \geq Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \end{cases}$$

Based on the above equations and Proposition 2.12, $\frac{\partial X_t^*}{\partial Q_t}$, $\frac{\partial X_{t+1}^*}{\partial Q_t}$, $\frac{\partial X_{t+1}^*}{\partial X_t}$, $\frac{\partial Q_{t+1}}{\partial X_t}$, $\frac{\partial Q_{t+1}}{\partial Q_t}$, $\frac{\partial X_t^*}{\partial S}$ and $\frac{\partial X_{t+1}^*}{\partial D_t}$ can be calculated out.

Proposition 2.14.

(1) $\frac{\partial X_t^*}{\partial Q_t} = \begin{cases} 0 & , \quad Q_t < \left(h_t^0(S) - S\right)^+ \\ -1 & , \quad \left(h_t^0(S) - S\right)^+ < Q_t < h_t^0(S) \\ 0 & , \quad Q_t > h_t^0(S) \end{cases}$

$\frac{\partial^2 X_t^*}{\partial Q_t^2} = 0$ ($Q_t \neq h_t^0(S)$ *and* $Q_t \neq \left(h_t^0(S) - S\right)^+$)

(2) $\frac{\partial Q_{t+1}}{\partial X_t} = \frac{\partial Q_{t+1}}{\partial Q_t} = \begin{cases} 1 & , \quad D_t < Q_t + X_t \\ 0 & , \quad D_t > Q_t + X_t \end{cases}$

$\frac{\partial^2 Q_{t+1}}{\partial X_t^2} = \frac{\partial^2 Q_{t+1}}{\partial Q_t^2} = 0$ ($D_t \neq Q_t + X_t$)

(3) $\frac{\partial X_{t+1}^*}{\partial X_t} = \frac{\partial X_{t+1}^*}{\partial Q_t} = \begin{cases} 0 & , \quad D_t < Q_t + X_t - h_{t+1}^0(S) \\ -1 & , \quad Q_t + X_t - h_{t+1}^0(S) < D_t \\ & \quad\quad < Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \\ 0 & , \quad D_t < Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \end{cases}$

$\frac{\partial^2 X_{t+1}^*}{\partial X_t^2} = \frac{\partial^2 X_{t+1}^*}{\partial Q_t^2} = 0$ ($D_t \neq Q_t + X_t - h_{t+1}^0(S)$ *and* $D_t \neq Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+$)

(4) $\frac{\partial X_t^*}{\partial S} = \begin{cases} 1 & , \quad Q_t < \left(h_t^0(S) - S\right)^+ \\ \frac{dh_t^0(S)}{dS} \leq 0 & , \quad \left(h_t^0(S) - S\right)^+ < Q_t < h_t^0(S) \\ 0 & , \quad Q_t > h_t^0(S) \end{cases}$

(5) $\frac{\partial X_{t+1}^*}{\partial D_t} = \begin{cases} 1 & , \quad D_t < Q_t + X_t - h_{t+1}^0(S) \\ \frac{dh_{t+1}^0(S)}{dD_t} + 1 \geq 1 & , \quad Q_t + X_t - h_{t+1}^0(S) < D_t \\ & \quad\quad < Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \\ 0 & , \quad D_t < Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \end{cases}$

Proposition 2.14 shows the relationship between the optimal solution and the value of inventory, demand and supply. Specifically, Proposition 2.14(1) gives the relationship between the optimal solution and the initial inventory in a time period. When inventory is large or small enough, the solution is not affected by inventory; but when inventory is moderate, the solution is linearly decreasing in the inventory.

Proposition 2.14(2) shows how inventory and allocated amount in a specific time period affects the inventory in the next period. Note that $Q_{t+1} = (Q_t + X_t - D_t)^+$.

When demand is relatively small, if inventory and allocated amount increase one unit, then the inventory in the next period also increases one unit.

Proposition 2.14(3) gives the relationship between the optimal solution in a specific time period and the inventory and solution in the previous period. This relationship is similar to (1).

Proposition 2.14(4) gives the relationship between the optimal solution and supply. When inventory is relatively small, one unit increase in supply will lead to one unit increase in the solution. With more inventory, the solution will decline as supply increases but their relation is not linear.

Proposition 2.14(5) shows the relationship between the optimal solution in a specific time period and the demand in the previous period. Obviously, increased demand will lead to increase in the optimal solution. In addition, when demand is low ($D_t < Q_t + X_t - h^0_{t+1}(S)$), the optimal solution will increase one unit if demand in the previous period increases one unit. However, when demand exceeds a critical value (when $Q_t + X_t - h^0_{t+1}(S) < D_t < Q_t + X_t - \big(h^0_{t+1}(S) - S\big)^+$), one unit increase in demand will lead to more than one unit increase in the optimal solution in the next time period. When demand is greater then $Q_t + X_t - \big(h^0_{t+1}(S) - S\big)^+$, it will not affect the optimal solution in the next period.

The optimization problem in time period n can be regarded as a one-period problem. Consider the following static stochastic programming model:

$$\begin{aligned} &\min_{X_t'} E_{D_t}\{Y_t(X_t', D_t)\} \\ &s.t. \quad 0 \le X_t' \le S \end{aligned} \tag{2.16}$$

A general form of the optimal solution of this model is:

$$X_t'^{*} = \begin{cases} 0 & , \quad Q_t \ge G_t^{-1}(\frac{\beta}{\alpha+\beta}) \\ G_t^{-1}\left(\frac{\beta}{\alpha+\beta}\right) - Q_t & , \quad \left(G_t^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+ \le Q_t \le G_t^{-1}(\frac{\beta}{\alpha+\beta}) \\ S - \left(S - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right)^+ & , \quad Q_t \le \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+ \end{cases} \tag{2.17}$$

Comparing the above equation with Eq.(2.15), we obtain the following proposition:

Proposition 2.15. *Let $X_t'^{*}$ be the optimal solution of the static problem (Eq. (2.16)) in time period t, and X_t^* be the optimal solution of the dynamic problem developed in*

Section 2.3. In any time period t with any given Q_t and S, $X_t^{\prime *} \leq X_t^*$. And when $S \to \infty$, $X_t^{\prime *} \to X_t^*$.

2.5 Numerical Studies

This section develops numerical studies to demonstrate the applicability of the proposed methodology and to compare the optimal policies in different situations. All computational processes are conducted with MATLAB on a computer with a 2.69GHz CPU and 8G RAM. The probability and cumulative distribution functions are calculated according to Proposition 2.2.

2.5.1 Numerical Study 1

This study is designed as a simplification of the real case of Severe Acute Respiratory Syndromes (SARS) outbreak in China in the first quarter of 2003. Parameters are set according to the situation on 9 Feb 2003 in Guangzhou, a city in south China, and their values are as follows: $\lambda = 0.25$, $P_0 = 226$, $A = 1.08$, $u = 0.2$, $S = 200$, $n = 3$, $\alpha = 0.3$ and $\beta = 0.7$. Meanwhile, for generality, an experimental situation is tested: the number of patients is decreasing ($0 < A < 1$).

Table 2.1 reports the solutions of $h_t^0(S)$, $t = 1, \ldots, n$, which are compared to the corresponding $G_t^{-1}\left(\frac{\beta}{\alpha+\beta}\right)$ and the expectation of demands. Note that $G_t^{-1}\left(\frac{\beta}{\alpha+\beta}\right)$ refers to the optimal policy of the stochastic static programming model (Optimization Problem (2.16) and Eq. (2.17)).

Table 2.1: Policy comparison between dynamic and static models

Time Period	A=1.08			A=0.9		
	$h_t^0(S)$	$G_t^{-1}\left(\frac{\beta}{\alpha+\beta}\right)$	$E\{D_t\}$	$h_t^0(S)$	$G_t^{-1}\left(\frac{\beta}{\alpha+\beta}\right)$	$E\{D_t\}$
t=1	247.2	200.1	198.5	167.5	167.5	165.9
t=2	256.4	221.0	217.5	155.7	155.7	152.5
t=3	243.4	243.4	238.1	144.9	144.9	140.5

In Table 2.1, $h_t^0(S)$ is the optimal policy of the stochastic dynamic programming model. $G_t^{-1}\left(\frac{\beta}{\alpha+\beta}\right)$ is the optimal policy of the static model. $E\{D_t\}$ is the expectation of D_t.

Regardless of which model is adopted, the optimal available amounts of medical supply in each time period are larger than the expectation of the corresponding demands, because the penalty coefficient of the unfulfilled demand is greater than that of the oversupply. In addition, when the disease is still spreading ($A > 1$), the optimal available amounts obtained by the dynamic model are larger than that of the static model. The reason is that to meet increasing demand, more medical supplies are allocated to earlier time periods in advance. However, when the epidemic has been controlled and the number of patients is declining ($0 < A < 1$), the solutions of the two models are the same.

In the following numerical studies, only the first situation ($A = 1.08$) is considered.

2.5.2 Numerical Study 2

This study tests how the optimal policy changes when λ, S and β change. Other parameters are set the same as Study 1.

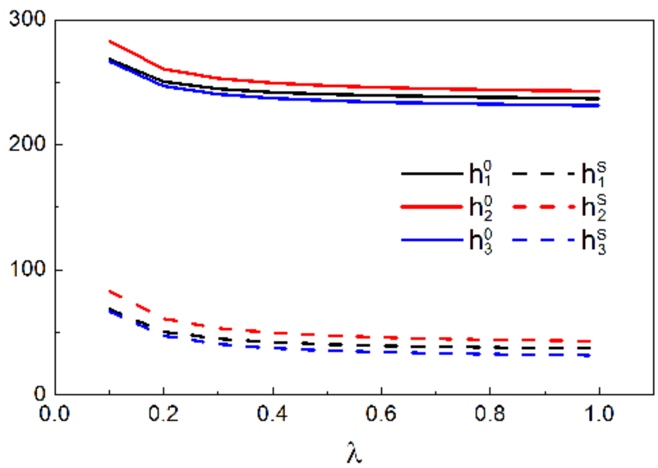


Figure 2.2: Sensitivity analysis of λ

Fig. 2.2 illustrates that with a given S, as λ increases, both $h_t^0(S)$ and $h_t^S(S)$ ($t = 1, , n$) decrease and the rates of decrease slow down. A smaller λ leads to larger expectation and variance of demands. Thus, more medical supply is allocated.

Fig. 2.3 shows how optimal policies are affected by the storage and transportation

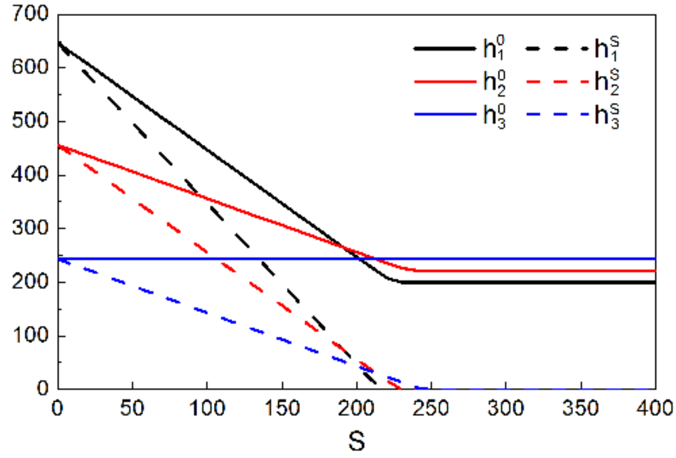


Figure 2.3: Sensitivity analysis of S

capacity of the central warehouse. $h_t^0(S)$ $(t = 1, 2)$ and $h_t^S(S)$ $(t = 1, 2, 3)$ decrease as S increases when S is less than the corresponding boundary points, while they remain unchanged if the capacity is enough. Additionally, $h_1^0(S)$ and $h_1^S(S)$ decrease the fastest. These results are consistent with the analytical analyses in Section 4 and they are also valuable references for the long-term decision of capacity investment.

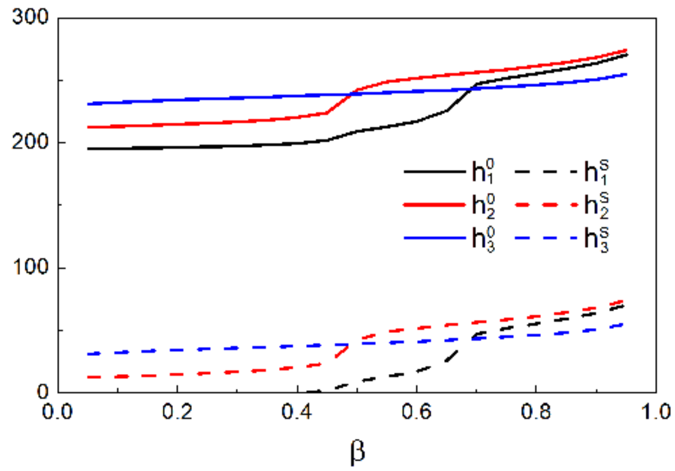


Figure 2.4: Sensitivity analysis of β

Fig. 2.4 presents the sensitivity analysis of penalty coefficients. In this analysis, $\alpha = 1 - \beta$. As β increases, $h_t^0(S)$ and $h_t^S(S)$ $(t = 1, 2, 3)$ increase. But these increases are very slow when the value of β is small. In addition, a stepped growth trend is found.

2.5.3 Numerical Study 3

This study aims to further discuss the relationship among the optimal policy, warehouse capacity and penalty coefficients. Fig. 2.5 reports how S and β affect $h_2^0(S)$. $h_1^0(S)$, $h_1^S(S)$ and $h_2^S(S)$ are also tested and have similar properties.

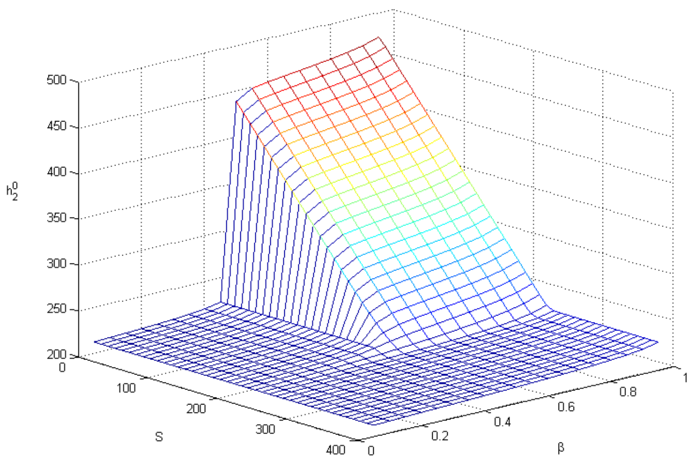


Figure 2.5: Optimal policies of different combinations of S and β

When β is small, the impacts of S on $h_2^0(S)$ is almost negligible. Only if β is larger than about 0.45, S and $h_2^0(S)$ show a similar relation as Fig. 2.3 in Study 2. On the other hand, when S is large (greater than about 260), $h_2^0(S)$ keeps a slight increase as β increases. However, $h_2^0(S)$ grows stepwise when S is less than about 260, and the smaller S, the greater the amount of $h_2^0(S)$.

2.6 Discussion

In this chapter, the relations between the optimal policy and each parameter are discussed, among which findings point to significant effects of local supply capacity. It is affected by local production, storage and transportation capacities. Since medicine usually requires strict conditions of storage and transport, storage and transportation capacities can hardly be expanded in short term, same as production capacity. When the capacity is not enough, the optimal amount of medical supply in each time period decreases as capacity increases, and the amounts in previous time periods decrease faster. But these changes are not significant if the penalty of the unfulfilled demand is relatively small.

Apart from the capacity, demand shows a leverage effect on the optimal policy, which means one percent change in demand will lead to more than one percent change in the optimal decision for the following time periods.

Furthermore, in the situation that an epidemic disease keeps spreading and number of patients is increasing, the optimal amounts solved by the proposed stochastic dynamic model are always not less than that of the stochastic static model. And their differences in previous time periods are larger than those in later time periods.

2.7 Summary

In this chapter, a stochastic dynamic programming model is developed to optimize the allocation of medical relief for response to epidemic outbreaks. The problem is divided into several finite time periods, with a policy decision required at each time period. The inventory of medical supply and the probability distribution of demand in the epidemic area change among time periods, and the demand in each time period is regarded as a stochastic parameter. To solve the model, this chapter provides a general analytical closed-form of the optimal allocation policy in each time period to minimize the expected sum of the overall penalty. Several properties of the optimal policy are derived and discussed. Additionally, numerical studies show the applicability of the proposed method. The results support and supplement the analytical analyses.

In addition, the analytical and numerical analyses highlight some managerial implications into improving decisions on medical allocation for better response to epidemic outbreaks.

Finally, researchers would extend this chapter to include more participants in emergency medical allocation. Chapter 4 will take local private sectors into consideration and discuss their cooperation with the government. Furthermore, this chapter only considers one epidemic area and focuses on the medical relief for treatment. Future studies would explore the allocation policies of preventative relief, such as vaccine. It would be studied in the next chapter.

2.8 Proofs of Propositions and Theorems in this Chapter

Proof of Proposition 2.1.

(1)$V_{t+1}(y) = Pr\{P_{t+1} \leq y\} = Pr\{(m(uP_t) + \delta_{t+1})^+ \leq y\}$.

If $y < 0$, then $Pr\{(m(uP_t) + \delta_{t+1})^+ \leq y\} = 0$;

If $y \geq 0$, then

$$\begin{aligned} & Pr\{(m(uP_t) + \delta_{t+1})^+ \leq y\} \\ =& Pr\{m(uP_t) + \delta_{t+1} \leq y\} Pr\{m(uP_t) + \delta_{t+1} \geq 0\} \\ & + Pr\{m(uP_t) + \delta_{t+1} < 0\} \\ =& W(y - m(uP_t)) [1 - W(-m(uP_t))] + W(-m(uP_t)) \\ =& W(y - m(uP_t)) - W(-m(uP_t)) W(-m(uP_t)) + W(-m(uP_t)) \end{aligned}$$

Thus,

$$V_{t+1}(y) = \begin{cases} 0 & , \quad y < 0 \\ W(y - m(uP_t)) + W(-m(uP_t)) & \\ -W(y - m(uP_t)) W(-m(uP_t)) & , \quad y \geq 0 \end{cases}$$

and

$$v_{t+1}(y) = \begin{cases} 0 & , \quad y < 0 \\ w(y - m(uP_t)) [1 - W(-m(uP_t))] & , \quad y > 0 \end{cases}$$

(2)

$$\begin{aligned} \frac{dV_{t+1}(y)}{dP_t} &= \begin{cases} 0 & , \quad y < 0 \\ -\frac{dm(uP_t)}{dP_t} [w(y - m(uP_t)) - W(y - m(uP_t)) w(-m(uP_t)) & \\ -w(y - m(uP_t)) W(-m(uP_t)) + w(-m(uP_t))] & , \quad y \geq 0 \end{cases} \\ &\leq \begin{cases} 0 & , \quad y < 0 \\ -\frac{dm(uP_t)}{dP_t} [w(y - m(uP_t)) - w(-m(uP_t))] & \\ [1 - W(y - m(uP_t))] & , \quad y \geq 0 \end{cases} \end{aligned}$$

Since $\frac{dm(uP_t)}{dP_t} \geq 0$, $w\left(y - m(uP_t)\right) \geq 0$, $w\left(-m(uP_t)\right) \geq 0$, $1 - W\left(y - m(uP_t)\right) \geq 0$, we can obtain $\frac{dV_{t+1}(y)}{dP_t} \leq 0$. So $V_{t+1}(y)$ is decreasing in P_t. □

Proof of Proposition 2.2.

According to the definition of $B_i(t-1, A)$, $B_i(t-1, A)$ has the following properties:

(1) $B_0(t-1, A) = 1$;

(2) $B_i(t-1, A) - B_{i-1}(t-1, A)A^t = B_i(t, A)$;

(3) $B_i(t-1, A) = (-1)^t A^{\frac{t(t+1)}{2}}$.

And we can calculate out

$$B_i(t-1, A) = (-1)^i \frac{A^{\frac{i(i+1)}{2}}(1 - A^{t-1})(1 - A^{t-2}) \dots (1 - A^{t-i})}{(1 - A^i)(1 - A^{i-1}) \dots (1 - A)}$$

Therefore,

$$\begin{aligned} G_1(y) =& Pr\{AD_0 + \sigma_1 \leq y\} \\ =& Pr\{\sigma_1 \leq y - AD_0\} \\ =& \begin{cases} 0 & , \quad y < AD_0 \\ 1 - e^{-\lambda(y - AD_0)} & , \quad y \geq AD_0 \end{cases} \end{aligned}$$

and

$$g_1(y) = \frac{dG_1(y)}{dy} = \begin{cases} 0 & , \quad y < AD_0 \\ \lambda e^{-\lambda(y - AD_0)} & , \quad y > AD_0 \end{cases}$$

Assume

$$G_t(y) = \begin{cases} 0 & , \quad y < A^t D_0 \\ 1 - \frac{1}{\prod_{j=1}^{t-1}(1 - A^j)} \sum_{i=0}^{t-1} B_i(t-1, A) e^{\frac{-\lambda(y - A^t D_0)}{A^i}} & , \quad y \geq A^t D_0 \end{cases}$$

Then

$$
\begin{aligned}
G_{t+1}(y) =& \int_0^\infty G_t(y)|_{D_t} g_t(D_t)\, dD_t \\
=& \int_{A^t D_0}^{\frac{y}{A}} 1 - e^{-\lambda(y-AD_t)} \frac{\lambda}{\prod_{j=1}^{t-1}(1-A^j)} \sum_{i=0}^{t-1} B_i(t-1,A) e^{\frac{-\lambda(D_t - A^t D_0)}{A^i}} \frac{1}{A^i}\, dD_t \\
=& \frac{\lambda}{\prod_{j=1}^{t-1}(1-A^j)} \int_{A^t D_0}^{\frac{y}{A}} \left[\sum_{i=0}^{t-1} B_i(t-1,A) e^{\frac{-\lambda(D_t - A^t D_0)}{A^i}} \frac{1}{A^i} \right. \\
& \left. - \sum_{i=0}^{t-1} B_i(t-1,A) e^{\lambda(\frac{D_t - A^t D_0}{A^i} + y - AD_t)} \frac{1}{A^i}\, dD_t \right] \\
=& -\frac{\lambda}{\prod_{j=1}^{t-1}(1-A^j)} \left[-\sum_{i=0}^{t-1} B_i(t-1,A) \frac{A^{k+1}}{1-A^{k+1}} e^{\frac{-\lambda(y - A^{t+1} D_0)}{A^{i+1}}} \right. \\
& \left. - \sum_{i=0}^{t-1} B_i(t-1,A) + \sum_{i=0}^{t-1} B_i(t-1,A) \frac{1}{1-A^{k+1}} e^{-\lambda(y - A^{t+1} D_0)} \right] \\
=& 1 - \frac{1}{\prod_{j=1}^{t-1}(1-A^j)} \left[\sum_{i=1}^{t} \frac{B_i(t,A)}{1-A^t} e^{\frac{-\lambda(y-A^{t+1}D_0)}{A^{i+1}}} + \frac{1}{1-A^t} e^{-\lambda(y-A^{t+1}D_0)} \right] \\
=& 1 - \frac{1}{\prod_{j=1}^{t}(1-A^j)} \sum_{i=0}^{t} B_i(t,A) e^{\frac{-\lambda(y-A^{t+1}D_0)}{A^i}} \, (y \geq A^{t+1} D_0)
\end{aligned}
$$

And easily get

$$
g_{t+1}(y) = \begin{cases} 0 & , \quad y < A^{t+1} D_0 \\ \frac{\lambda}{\prod_{j=1}^{t}(1-A^j)} \sum_{i=0}^{t} B_i(t-1,A) e^{\frac{-\lambda(y-A^{t+1}D_0)}{A^i}} \frac{1}{A^i} & , \quad y \geq A^{t+1} D_0 \end{cases}
$$

□

Proof of Theorem 2.3.

$$
f_n(Q_n, X_n) = \min_{X_n} \{\alpha(X_n + Q_n - \mu_n) + (\alpha+\beta) L_n(X_n + Q_n) + E_{D_n} f_{n+1}(Q_{n+1}, X_{n+1})\}
$$

$$
f_{n+1}(Q_{n+1}, X_{n+1}) = 0
$$

$$
\frac{d[\alpha(X_n + Q_n - \mu_n) + (\alpha+\beta) L_n(X_n + Q_n)]}{dX_n} = (\alpha+\beta) G_n(X_n + Q_n) - \beta
$$

$$
\frac{d^2[\alpha(X_n + Q_n - \mu_n) + (\alpha+\beta) L_n(X_n + Q_n)]}{dX_n^2} = (\alpha+\beta) g_n(X_n + Q_n) \geq 0
$$

The first order condition is $X_n = G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right) - Q_n$. Recall $0 \leq X_n \leq S$, the optimal solution is $X_n^* = \min\{S, \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - Q_n\right)^+\}$. □

Proof of Proposition 2.4.

$$
\begin{aligned}
&\frac{dE_{D_{n-1}}\{Y_{n-1}(X_{n-1}, D_{n-1}) + f_n(Q_n, X_n)}{dX_{n-1}} \\
=&\alpha G_{n-1}(X_{n-1} + Q_{n-1}) - \beta - \beta G_{n-1}\left(X_{n-1} + Q_{n-1} - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right) \\
&+ \beta G_{n-1}\left(X_{n-1} + Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+\right) \\
&+ (\alpha+\beta)\int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} G_n(X_{n-1} + Q_{n-1} - D_{n-1}) g_{n-1}(D_{n-1})\, dD_{n-1} \\
&+ (\alpha+\beta)\int_{X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+}^{X_{n-1}+Q_{n-1}} G_n\left(S - (S - G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+\right. \\
&\left.+X_{n-1} + Q_{n-1} - D_{n-1}\right) g_{n-1}(D_{n-1})\, dD_{n-1}
\end{aligned}
$$

If $S < G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$
\begin{aligned}
&\frac{d^2E_{D_{n-1}}\{Y_{n-1}(X_{n-1}, D_{n-1}) + f_n(Q_n, X_n)}{dX_{n-1}^2} \\
=&\alpha g_{n-1}(X_{n-1} + Q_{n-1}) + (\alpha+\beta) G_n(S) g_{n-1}(X_{n-1} + Q_{n-1}) \\
&+ (\alpha+\beta)\int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} g_n(X_{n-1} + Q_{n-1} - D_{n-1}) g_{n-1}(D_{n-1})\, dD_{n-1} \\
&+ (\alpha+\beta)\int_{X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)}^{X_{n-1}+Q_{n-1}} g_n(S + X_{n-1} + Q_{n-1} - D_{n-1}) g_{n-1}(D_{n-1})\, dD_{n-1} \\
\geq&0
\end{aligned}
$$

If $S \geq G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$
\begin{aligned}
&\frac{d^2E_{D_{n-1}}\{Y_{n-1}(X_{n-1}, D_{n-1}) + f_n(Q_n, X_n)}{dX_{n-1}^2} \\
=&\alpha g_{n-1}(X_{n-1} + Q_{n-1}) + \beta g_{n-1}(X_{n-1} + Q_{n-1}) \\
&+ (\alpha+\beta)\int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)} g_n(X_{n-1} + Q_{n-1} - D_{n-1}) g_{n-1}(D_{n-1})\, dD_{n-1} \\
\geq&0
\end{aligned}
$$

Thus $E_{D_{n-1}}\{Y_{n-1}(X_{n-1}, D_{n-1}) + f_n(Q_n, X_n)\}$ is convex in X_{n-1} for any given S, $G_{n-1}(D_{n-1})$ and $G_n(D_n)$. □

Proof of Proposition 2.5.

$$
\begin{aligned}
I_{n-1} =& \alpha G_{n-1}(X_{n-1}+Q_{n-1}) - \beta - \beta G_{n-1}\left(X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})\right) \\
&+ \beta G_{n-1}\left(X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+\right) \\
&+ (\alpha+\beta)\int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} G_n(X_{n-1}+Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1} \\
&+ (\alpha+\beta)\int_{X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+}^{X_{n-1}+Q_{n-1}} G_n\left(S-(S-G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+\right. \\
&\left.+X_{n-1}+Q_{n-1}-D_{n-1}\right) g_{n-1}(D_{n-1})\,dD_{n-1}
\end{aligned}
$$

(1) If $S < G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$
\begin{aligned}
\frac{\partial I_{n-1}}{\partial Q_{n-1}} &= \alpha g_{n-1}(X_{n-1}+Q_{n-1}) + (\alpha+\beta)G_n(S)g_{n-1}(X_{n-1}+Q_{n-1}) \\
&+ (\alpha+\beta)\int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)} g_n(X_{n-1}+Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1} \\
&+ (\alpha+\beta)\int_{X_{n-1}+Q_{n-1}-\left(G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)-S\right)}^{X_{n-1}+Q_{n-1}} g_n(S+X_{n-1}+Q_{n-1}-D_{n-1}) \\
& g_{n-1}(D_{n-1})\,dD_{n-1} \\
&\geq 0
\end{aligned}
$$

If $S \geq G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$
\begin{aligned}
\frac{\partial I_{n-1}}{\partial Q_{n-1}} &= \alpha g_{n-1}(X_{n-1}+Q_{n-1}) + \beta g_{n-1}(X_{n-1}+Q_{n-1}) \\
&+ (\alpha+\beta)\int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)} g_n(X_{n-1}+Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1} \\
&\geq 0
\end{aligned}
$$

(2) For S, If $S < G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$
\begin{aligned}
\frac{\partial I_{n-1}}{\partial S} =& (\alpha+\beta)\int_{X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)}^{X_{n-1}+Q_{n-1}} g_n(S+X_{n-1}+Q_{n-1}-D_{n-1}) \\
& g_{n-1}(D_{n-1})\,dD_{n-1} \geq 0
\end{aligned}
$$

If $S \geq G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$\frac{\partial I_{n-1}}{\partial S} = 0$$

□

Proof of Proposition 2.6.

(1) $S \geq G_n^{-1}(\frac{\beta}{\alpha+\beta})$, $\frac{\partial I_{n-1}}{\partial S} = 0$. Thus, according to the proof of Proposition 2.5, $\lim_{S\to\infty} I_{n-1} = I_{n-1}|_{S=G_n^{-1}(\frac{\beta}{\alpha+\beta})}$.

(2) We can calculate

$$\begin{aligned}
&\lim_{Q_{n-1}\to\infty} I_{n-1} \\
&= (\alpha+\beta) \lim_{Q_{n-1}\to\infty} \int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} G_n(X_{n-1}+Q_{n-1}-D_{n-1}) g_{n-1}(D_{n-1})\, dD_{n-1} \\
&\quad + \alpha - \beta \\
&= \alpha
\end{aligned}$$

□

Proof of Proposition 2.7.

(1) According to the definition of I_{n-1}^0,

$$\begin{aligned}
I_{n-1}^0 =& \alpha G_{n-1}(Q_{n-1}) - \beta - \beta G_{n-1}\left(Q_{n-1} - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right) \\
&+ \beta G_{n-1}\left(Q_{n-1} - (G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S)^+\right) \\
&+ (\alpha+\beta)\int_0^{Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} G_n(Q_{n-1}-D_{n-1}) g_{n-1}(D_{n-1})\, dD_{n-1} \\
&+ (\alpha+\beta)\int_{Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+}^{Q_{n-1}} G_n\left(S - (S - G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+ + Q_{n-1} - D_{n-1}\right) \\
&g_{n-1}(D_{n-1})\, dD_{n-1}
\end{aligned}$$

Obviously, $I_{n-1}^0|_{Q_{n-1}=0} = -\beta < 0$. According to Proposition 2.6 $\lim_{Q_{n-1}\to\infty} I_{n-1} \geq 0$. And since $\frac{\partial I_{n-1}^0}{\partial Q_{n-1}}$ (recall $\frac{\partial I_{n-1}}{\partial Q_{n-1}} \geq 0$ as proved in Proposition 2.5), we can obtain: for any given S, $\exists$ unique $h_{n-1}^0(S) > 0$, such that

$$\begin{cases}
I_{n-1}^0|_{Q_{n-1}<h_{n-1}^0(S)} \leq 0 \\
I_{n-1}^0|_{Q_{n-1}=h_{n-1}^0(S)} = 0 \\
I_{n-1}^0|_{Q_{n-1}>h_{n-1}^0(S)} \geq 0
\end{cases}$$

To prove $h_{n-1}^0(S)$ is unique for any given S, consider a small Δ, such that

$$I_{n-1}^0|_{Q_{n-1}=h_{n-1}^0(S)} = I_{n-1}^0|_{Q_{n-1}=h_{n-1}^0(S)+\Delta} = 0$$

. That is, $\frac{\partial I^0_{n-1}}{\partial Q_{n-1}}|_{Q_{n-1}=h^0_{n-1}(S)} = 0$.

If $S \geq G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$\begin{aligned}\frac{\partial I^0_{n-1}}{\partial Q_{n-1}} =&(\alpha+\beta)g_{n-1}(Q_{n-1})\\ &+(\alpha+\beta)\int_0^{Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} g_n(Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1}\\ \Leftrightarrow&(\alpha+\beta)g_{n-1}\left(h^0_{n-1}(S)\right)\\ &+(\alpha+\beta)\int_0^{h^0_{n-1}(S)-G_n^{-1}(\frac{\beta}{\alpha+\beta})} g_n\left(h^0_{n-1}(S)-D_{n-1})g_{n-1}(D_{n-1}\right)\,dD_{n-1}\\ &=0\end{aligned}$$

If $S < G_n^{-1}(\frac{\beta}{\alpha+\beta})$, then

$$\begin{aligned}\frac{\partial I^0_{n-1}}{\partial Q_{n-1}} =&\left(\alpha+G_n(S)\right)g_{n-1}(Q_{n-1})\\ &+(\alpha+\beta)\int_0^{Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} g_n(Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1}\\ &+(\alpha+\beta)\int_{Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})+S}^{Q_{n-1}} g_n(S+Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1}\\ \Leftrightarrow&\left(\alpha+G_n(S)\right)g_{n-1}(h^0_{n-1}(S))\\ &+(\alpha+\beta)\int_0^{h^0_{n-1}(S)-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)} g_n(h^0_{n-1}(S)-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1}\\ &+(\alpha+\beta)\int_{h^0_{n-1}(S)-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)+S}^{h^0_{n-1}(S)} g_n(S+h^0_{n-1}(S)-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1}\\ &=0\end{aligned}$$

Therefore, for any S,

$$\frac{\partial I^0_{n-1}}{\partial Q_{n-1}}|_{Q_{n-1}=h^0_{n-1}(S)} = 0 \Leftrightarrow \begin{cases} g_{n-1}\left(h^0_{n-1}(S)\right)=0\\ g_{n-1}\left(h^0_{n-1}(S)-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)\right)=0\end{cases}$$

Thus, $h^0_{n-1}(S) \leq M(D_{n-2})$, for any known D_{n-2}.

However, $h^0_{n-1}(S) \leq M(D_{n-2}) \Rightarrow I^0_{n-1}|_{Q_{n-1}=h^0_{n-1}(S)} = -\beta < 0$, which is contradictory to the definition of $h^0_{n-1}(S)$. Thus, there does not exist a Δ, s.t. $I^0_{n-1}|_{Q_{n-1}=h^0_{n-1}(S)} = I^0_{n-1}|_{Q_{n-1}=h^0_{n-1}(S)+\Delta} = 0$. So $h^0_{n-1}(S)$ is unique.

(2) According to the definition of I_{n-1}^S,

$$\begin{aligned}
I_{n-1}^S =& \alpha G_{n-1}(S+Q_{n-1}) - \beta - \beta G_{n-1}\left(S+Q_{n-1} - G_n^{-1}(\frac{\beta}{\alpha+\beta})\right) \\
&+ \beta G_{n-1}\left(S+Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+\right) \\
&+ (\alpha+\beta)\int_0^{S+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})} G_n(S+Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1} \\
&+ (\alpha+\beta)\int_{S+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+}^{S+Q_{n-1}} G_n\left(2S - (S-G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+ + Q_{n-1} - D_{n-1}\right) \\
&g_{n-1}(D_{n-1})\,dD_{n-1}
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial I_{n-1}^S}{\partial S} =& \alpha g_{n-1}(S+Q_{n-1}) \\
&+ \beta g_{n-1}\left(S+Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+\right) \\
&\frac{d\left(S+Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+\right)}{dS} \\
&+ (\alpha+\beta)\int_0^{S+Q_{n-1}-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)} g_n(S+Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1} \\
&+ (\alpha+\beta)\int_{S+Q_{n-1}-\left(G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)-S\right)^+}^{S+Q_{n-1}} g_n(2S - (S-G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+ \\
&+ Q_{n-1} - D_{n-1})\frac{d\left(2S - (S-G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+\right)}{dS} g_{n-1}(D_{n-1})\,dD_{n-1} \\
&+ G_n\left(S - (S-G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+\right) g_{n-1}(S+Q_{n-1}) \\
&- G_n\left(S - (S-G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+ - (G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S)^+\right) \\
&g_{n-1}\left(S+Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+\right) \\
&\frac{d\left(S+Q_{n-1} - \left(G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S\right)^+\right)}{dS} \\
\geq& 0
\end{aligned}$$

We can calculate out $I_{n-1}^S|_{Q_{n-1}=h_{n-1}^0(S)} \geq 0$, $I_{n-1}^S|_{Q_{n-1}=0,S=0} = -\beta < 0$ and

$$\begin{aligned}
&\lim_{S\to\infty} I_{n-1}^S|_{Q_{n-1}=0} \\
=&\alpha - \beta \\
&+ (\alpha+\beta) \lim_{S\to\infty} \int_0^{S+Q_{n-1}-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)} G_n(S+Q_{n-1}-D_{n-1}) g_{n-1}(D_{n-1})\, dD_{n-1} \\
\geq&\alpha - \beta \\
&+ (\alpha+\beta) \lim_{S\to\infty} G_n\left(G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)\right) G_{n-1}\left(S+Q_{n-1}-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)\right)|_{Q_{n-1}=0} \\
=&\alpha \\
\geq&0
\end{aligned}$$

Recall that $\frac{\partial I_{n-1}^S}{\partial S} \geq 0$, we can obtain that there exists $S_{n-1}^{'} > 0$, such that

$$\begin{cases} I_{n-1}^S|_{Q_{n-1}=0,S<S_{n-1}^{'}} \geq 0 \\ I_{n-1}^S|_{Q_{n-1}=0,S=S_{n-1}^{'}} = 0 \\ I_{n-1}^S|_{Q_{n-1}=0,S>S_{n-1}^{'}} \leq 0 \end{cases}$$

Thus, if $S < S_{n-1}^{'}$, then there exists $h_{n-1}^S(S)$, such that

$$\begin{cases} I_{n-1}^S|_{Q_{n-1}<h_{n-1}^S(S)} \leq 0 \\ I_{n-1}^S|_{Q_{n-1}=h_{n-1}^S(S)} = 0 \\ I_{n-1}^S|_{Q_{n-1}>h_{n-1}^S(S)} \geq 0 \end{cases}$$

If $S = S_{n-1}^{'}$, then $h_{n-1}^S(S_{n-1}^{'}) = 0$; if $S > S_{n-1}^{'}$, then $h_{n-1}^S(S) < 0$ and $I_{n-1}^S \geq 0$.

Similar to (1), one can prove that $h_{n-1}^S(S)$ is unique for given S. And according to Proposition 2.5, $h_{n-1}^S(S)$ is a decreasing function of S.

(3) According to the definition of I_{n-1}

$$I_{n-1} = \alpha G_{n-1}(X_{n-1}+Q_{n-1}) - \beta - \beta G_{n-1}\left(X_{n-1}+Q_{n-1}-G_n^{-1}(\frac{\beta}{\alpha+\beta})\right)$$
$$+\beta G_{n-1}\left(X_{n-1}+Q_{n-1}-\left(G_n^{-1}(\frac{\beta}{\alpha+\beta})-S\right)^+\right)$$
$$+(\alpha+\beta)\int_0^{X_{n-1}+Q_{n-1}-G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)} G_n(X_{n-1}+Q_{n-1}-D_{n-1})g_{n-1}(D_{n-1})\,dD_{n-1}$$
$$+(\alpha+\beta)\int_{X_{n-1}+Q_{n-1}-\left(G_n^{-1}\left(\frac{\beta}{\alpha+\beta}\right)-S\right)^+}^{X_{n-1}+Q_{n-1}} G_n\left(S-(S-G_n^{-1}(\frac{\beta}{\alpha+\beta}))^+\right.$$
$$\left.+X_{n-1}+Q_{n-1}-D_{n-1}\right)g_{n-1}(D_{n-1})\,dD_{n-1}$$

Thus, if $X'_{n-1}+Q'_{n-1}=X''_{n-1}+Q''_{n-1}$ (X'_{n-1}, X''_{n-1}, Q'_{n-1} and $Q''_{n-1}\geq 0$), then

$$I_{n-1}|_{X_{n-1}=X'_{n-1},Q_{n-1}=Q'_{n-1}} = I_{n-1}|_{X_{n-1}=X''_{n-1},Q_{n-1}=Q''_{n-1}}$$
$$I^S_{n-1}|_{Q_{n-1}=h^S_{n-1}(S)} = I_{n-1}|_{X_{n-1}=S,Q_{n-1}=h^S_{n-1}(S)} = I_{n-1}|_{X_{n-1}=0,Q_{n-1}=h^S_{n-1}(S)+S} = 0$$

Recall that

$$I^0_{n-1}|_{Q_{n-1}=h^0_{n-1}(S)} = I_{n-1}|_{X_{n-1}=0,Q_{n-1}=h^0_{n-1}(S)}$$

and note the uniqueness of $h^0_{n-1}(S)$, we can obtain $h^0_{n-1}(S)=h^S_{n-1}(S)+S$. □

Proof of Proposition 2.8.

Based on Proposition 2.7, for given S, Q_{n-1}, $G_{n-1}(y)$ and $G_n(y)$,

(1) If $Q_{n-1}\geq h^0_{n-1}(S)$, then $I^0_{n-1}|_{X_{n-1}=0}\geq 0$ and $I^0_{n-1}|_{X_{n-1}=S}\geq 0$;

(2) If $\left(h^S_{n-1}(S)\right)^+\leq Q_{n-1}<h^0_{n-1}(S)$, then $I^0_{n-1}|_{X_{n-1}=0}<0$ and $I^0_{n-1}|_{X_{n-1}=S}\geq 0$;

(3) If $Q_{n-1}<\left(h^S_{n-1}(S)\right)^+$, then $I^0_{n-1}|_{X_{n-1}=0}<0$ and $I^0_{n-1}|_{X_{n-1}=S}<0$.

And since $E_{D_{n-1}}\{Y_{n-1}(X_{n-1},D_{n-1}+f_n(Q_n,X_n)\}$ is convex in X_{n-1} (Proposition 2.4), the optimal solution is

$$X^*_{n-1}=\begin{cases}0 & , \quad Q_{n-1}\geq h^0_{n-1}(S)\\ \bar{X}_{n-1} & , \quad \left(h^S_{n-1}(S)\right)^+\leq Q_{n-1}<h^0_{n-1}(S)\\ S & , \quad Q_{n-1}<\left(h^S_{n-1}(S)\right)^+\end{cases}$$

where, $\bar{X}_{n-1}$ is defined as $I_{n-1}|_{X_{n-1}=\bar{X}_{n-1}}=0$. Recall that if $X'_{n-1}+Q'_{n-1}=X''_{n-1}+Q''_{n-1}$ (X'_{n-1}, X''_{n-1}, Q'_{n-1} and $Q''_{n-1}\geq 0$), then

$$I_{n-1}|_{X_{n-1}=X'_{n-1},Q_{n-1}=Q'_{n-1}} = I_{n-1}|_{X_{n-1}=X''_{n-1},Q_{n-1}=Q''_{n-1}}$$

We can get $\bar{X}_{n-1} = h^0_{n-1}(S) - Q_{n-1}$. Therefore,

$$X^*_{n-1} = \begin{cases} 0 & , \quad Q_{n-1} \geq h^0_{n-1}(S) \\ h^0_{n-1}(S) - Q_{n-1} & , \quad \left(h^0_{n-1}(S) - S\right)^+ \leq Q_{n-1} < h^0_{n-1}(S) \\ S & , \quad Q_{n-1} < \left(h^0_{n-1}(S) - S\right)^+ \end{cases}$$

□

Proof of Propositions 2.9, 2.10 and 2.11, and Theorem 2.12

We use induction to prove these propositions and the theorem.

First, assume these propositions and lemmas hold for time period $t+1 (t = 1, 2, \ldots, n-1)$, that is:

(1) Given an optimal policy of the (*i+2*)th time period, the objective function in the (*t+1*)th time period $E_{D_{t+1}}\{Y_{t+1}(X_{t+1}, D_{t+1}) + f_{t+2}(Q_{t+2}, X^*_{t+2})\}$, which represents the expectation of the total penalty from the (*t+1*)th to the n-th time period) is convex in X_{t+1} for any given S, Q_t and $G_i(D_i)$ $(i = t+2, t+3, \ldots, n)$;

(2) The optimal solution at time period *t+1* is

$$X^*_{t+1} = \begin{cases} 0 & , \quad Q_{t+1} \geq h^0_{t+1}(S) \\ h^0_{t+1}(S) - Q_{t+1} & , \quad \left(h^0_{t+1}(S) - S\right)^+ \leq Q_{t+1} < h^0_{t+1}(S) \\ S & , \quad Q_{t+1} < \left(h^0_{t+1}(S) - S\right)^+ \end{cases}$$

where $h^0_{t+1}(S)$ is a function of S and defined as $I^0_{t+1}|_{Q_{t+1}=h^0_{t+1}(S)} = 0$.

We have proved these propositions and the theorem hold for time periods *n-1* and n. Then we will prove they hold for time period t $(t = 1, 2, \ldots, n-2)$ as follows:

Proof of Proposition 2.9.

$$X_{t+1}^* = \begin{cases} 0 & , \quad Q_{t+1} \geq h_{t+1}^0(S) \\ h_{t+1}^0(S) - Q_{t+1} & , \quad \left(h_{t+1}^0(S) - S\right)^+ \leq Q_{t+1} < h_{t+1}^0(S) \\ S & , \quad Q_{t+1} < \left(h_{t+1}^0(S) - S\right)^+ \end{cases}$$

$$\because Q_{t+1} = (Q_t + X_t - D_t)^+$$

$$\therefore X_{t+1}^* = \begin{cases} 0 & , \quad D_t \leq Q_t + X_t - h_{t+1}^0(S) \\ h_{t+1}^0(S) - Q_{t+1} & , \quad Q_t + X_t - h_{t+1}^0(S) < D_t \\ & \quad\quad < Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \\ S - (S - h_{t+1}^0(S))^+ & , \quad D_t > Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \end{cases}$$

and since $\frac{\partial Q_{t+1}}{\partial X_t} = \begin{cases} 1 & , \quad D_t < Q_t + X_t \\ 0 & , \quad D_t > Q_t + X_t \end{cases}$, we get

$$\frac{\partial X_{t+1}^*}{\partial X_t} = \begin{cases} 0 & , \quad D_t < Q_t + X_t - h_{t+1}^0(S) \\ -1 & , \quad Q_t + X_t - h_{t+1}^0(S) < D_t < Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \\ 0 & , \quad D_t > Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+ \end{cases}$$

Thus, when $D_t \neq X_t + Q_t$ and $D_t \neq X_t + Q_t - h_{t+1}^0(S)$ and $D_t \neq X_t + Q_t - \left(h_{t+1}^0(S) - S\right)^+$, $\frac{\partial^2 Q_{t+1}}{\partial X_t^2} = \frac{\partial^2 X_{t+1}^*}{\partial X_t^2} = 0$. We can calculate out:

$$\frac{d^2 f_{t+1}(Q_{t+1}, X_{t+1}^*)}{dX_t^2}|_{D_t = Q_t + X_t} = \frac{d^2 f_{t+1}(0, S)}{dX_t^2} = 0$$

$$\frac{d^2 f_{t+1}(Q_{t+1}, X_{t+1}^*)}{dX_t^2}|_{D_t = Q_t + X_t - h_{t+1}^0(S)} = \frac{d^2 f_{t+1}\left(h_{t+1}^0(S), 0\right)}{dX_t^2} = 0$$

$$\frac{d^2 f_{t+1}(Q_{t+1}, X_{t+1}^*)}{dX_t^2}|_{D_t = Q_t + X_t - \left(h_{t+1}^0(S) - S\right)^+} = \frac{d^2 f_{t+1}\left(\left(h_{t+1}^0(S) - S\right)^+, S\right)}{dX_t^2} = 0$$

If $D_t \neq X_t + Q_t$ and $D_t \neq X_t + Q_t - h_{t+1}^0(S)$ and $D_t \neq X_t + Q_t - \left(h_{t+1}^0(S) - S\right)^+$,

$$\frac{d^2 f_{t+1}(Q_{t+1}, X_{t+1}^*)}{dX_t^2}$$

$$= \frac{d^2}{d\left(Q_{t+1} + X_{t+1}^*\right)^2} E_{D_{t+1}}\{Y_{t+1}(X_{t+1}, D_{t+1}) + f_{t+2}(Q_{t+2}, X_{t+2}^*)\} \left[\frac{d(Q_{t+1} + X_{t+1}^*)}{dX_t}\right]^2$$

$$= \frac{\partial^2}{\partial X_{t+1}^{*2}} E_{D_{t+1}}\{Y_{t+1}(X_{t+1}, D_{t+1}) + f_{t+2}(Q_{t+2}, X_{t+2}^*)\} \left[\frac{d(Q_{t+1} + X_{t+1}^*)}{dX_t}\right]^2$$

Since $E_{D_{t+1}}\{Y_{t+1}(X_{t+1}, D_{t+1}) + f_{t+2}(Q_{t+2}, X^*_{t+2})\}$ is convex in $(X_{t+1}$ (assumption of induction), $\frac{d^2 E_{D_{t+1}}\{Y_{t+1}(X_{t+1},D_{t+1})+f_{t+2}(Q_{t+2},X^*_{t+2})\}}{dX^2_{t+1}} \geq 0$. Thus, $\frac{d^2 f_{t+1}(Q_{t+1},X^*_{t+1})}{dX^2_t} \geq 0$.

Therefore, $\forall D_t \geq 0$, $\frac{d^2 f_{t+1}(Q_{t+1},X^*_{t+1})}{dX^2_t} \geq 0$.

$$
\begin{aligned}
&\frac{d^2 E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X^*_{t+1})\}}{dX_t^2} \\
=&\frac{d^2}{dX_t^2}\left[\alpha(Q_t + X_t - \mu_t) + (\alpha+\beta)L_t(X_t + Q_t) + E_{D_t} f_{t+1}(Q_{t+1}, X_{t+1})\right] \\
=&(\alpha+\beta)g_t(X_t + Q_t) + \int_0^\infty \frac{d^2 f_{t+1}(Q_{t+1}, X^*_{t+1})}{dX_t^2} g_t(D_t)\, dD_t \\
\geq& 0
\end{aligned}
$$

And since $\frac{d^2 E_{D_{n-1}}\{Y_{n-1}(X_{n-1},D_{n-1})+f_n(Q_n,X^*_n)\}}{dX^2_{n-1}} \geq 0$ (Proposition 2.4), we can obtain that for any $i = 1, 2, \ldots, n-1$, $\frac{d^2 E_{D_t}\{Y_t(X_t,D_t)+f_{t+1}(Q_{t+1},X^*_{t+1})\}}{dX^2_t} \geq 0$.

That is $E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X^*_{t+1})\}$ is convex in X_t. □

Proof of Proposition 2.10.

$$
\begin{aligned}
f_t(Q_t, X_t) =& \min_{X_t} E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X^*_{t+1})\} \\
=& \min_{X_t}\{\alpha(Q_t + X_t - \mu_t) + (\alpha+\beta)L_t(X_t + Q_t) \\
&+ \int_0^{Q_t+X_t-h^0_{t+1}(S)} f_{t+1}(Q_t + X_t - D_t, 0) g_t(D_t)\, dD_t \\
&+ \int_{Q_t+X_t-h^0_{t+1}(S)}^{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+} f_{t+1}(Q_t + X_t - D_t, h^0_{t+1}(S) - Q_t - X_t + D_t) g_t(D_t)\, dD_t \\
&+ \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} f_{t+1}(Q_t + X_t - D_t, S - (S - h^0_{t+1}(S))^+) g_t(D_t)\, dD_t \\
&+ \int_{Q_t+X_t}^\infty f_{t+1}(0, S - (S - h^0_{t+1}(S))^+) g_t(D_t)\, dD_t\}
\end{aligned}
$$

According to the definition of I_t,

$$
\begin{aligned}
I_t =& \frac{dE_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X^*_{t+1})\}}{dX_t} \\
=&(\alpha+\beta)G_t(X_t + Q_t) - \beta + \int_0^{Q_t+X_t-h^0_{t+1}(S)} \frac{df_{t+1}(Q_{t+1}, 0)}{dX_t} g_t(D_t)\, dD_t \\
&+ \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} \frac{df_{t+1}(Q_{t+1}, S - (S - h^0_{t+1}(S))^+)}{dX_t} g_t(D_t)\, dD_t
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\partial I_t}{\partial Q_t} =& \frac{\partial I_t}{\partial X_t} \\
=& \frac{d^2 E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X^*_{t+1})\}}{dX_t^2} \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial I_t}{\partial S} =& \frac{\partial}{\partial S} \int_0^{Q_t+X_t-h^0_{t+1}(S)} \frac{\partial f_{t+1}(Q_{t+1}, 0)}{\partial X_t} g_t(D_t)\, dD_t \\
&+ \frac{\partial}{\partial S} \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} \frac{\partial f_{t+1}(Q_{t+1}, S-(S-h^0_{t+1}(S))^+)}{\partial X_t} g_t(D_t)\, dD_t \\
=& \frac{\partial}{\partial S} \int_0^{Q_t+X_t-h^0_{t+1}(S)} \frac{\partial f_{t+1}(Q_{t+1}, 0)}{\partial Q_{t+1}} \frac{\partial Q_{t+1}}{\partial X_t} g_t(D_t)\, dD_t \\
&+ \frac{\partial}{\partial S} \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} \frac{\partial f_{t+1}(Q_{t+1}, S-(S-h^0_{t+1}(S))^+)}{\partial Q_{t+1}} \frac{\partial Q_{t+1}}{\partial X_t} g_t(D_t)\, dD_t \\
=& \frac{\partial}{\partial S} \int_0^{Q_t+X_t-h^0_{t+1}(S)} \frac{\partial f_{t+1}(Q_{t+1}, 0)}{\partial Q_{t+1}} g_t(D_t)\, dD_t \\
&+ \frac{\partial}{\partial S} \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} \frac{\partial f_{t+1}(Q_{t+1}, S-(S-h^0_{t+1}(S))^+)}{\partial Q_{t+1}} g_t(D_t)\, dD_t \\
=& \int_0^{Q_t+X_t-h^0_{t+1}(S)} \frac{\partial^2 f_{t+1}(Q_{t+1}, 0)}{\partial S \partial Q_{t+1}} g_t(D_t)\, dD_t \\
&+ \frac{\partial f_{t+1}(Q_{t+1}, 0)}{\partial Q_{t+1}} |_{D_t=Q_t+X_t-h^0_{t+1}(S)} g_t\left(Q_t+X_t-h^0_{t+1}(S)\right) \frac{d\left(Q_t+X_t-h^0_{t+1}(S)\right)}{dS} \\
&+ \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} \frac{\partial^2 f_{t+1}(Q_{t+1}, S-(S-h^0_{t+1}(S))^+)}{\partial S \partial Q_{t+1}} g_t(D_t)\, dD_t \\
&- \frac{\partial f_{t+1}(Q_{t+1}, S-(S-h^0_{t+1}(S))^+)}{\partial Q_{t+1}} |_{D_t=Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+} \\
&\quad g_t\left(Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+\right) \frac{d\left(Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+\right)}{dS}
\end{aligned}
$$

And

$$
\begin{aligned}
\because\quad & \frac{\partial^2 f_{t+1}(Q_{t+1},0)}{\partial S\partial Q_{t+1}} = 0 \\
& \frac{\partial f_{t+1}(Q_{t+1},0)}{\partial Q_{t+1}}|_{D_t=Q_t+X_t-h^0_{t+1}(S)} = \frac{\partial f_{t+1}(Q_{t+1},0)}{\partial Q_{t+1}}|_{Q_{t+1}=h^0_{t+1}(S)} = 0 \\
& \frac{\partial f_{t+1}(Q_{t+1},S-(S-h^0_{t+1}(S))^+)}{\partial Q_{t+1}}|_{D_t=Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+} \\
& = \begin{cases} \frac{\partial f_{t+1}(Q_{t+1},S)}{\partial Q_{t+1}}|_{Q_{t+1}=h^S_{t+1}(S)} = 0 \quad , & S \le h^0_{t+1}(S) \\ \frac{\partial f_{t+1}(Q_{t+1},h^0_{t+1}(S))}{\partial Q_{t+1}}|_{Q_{t+1}=0} = 0 \quad , & S > h^0_{t+1}(S) \end{cases} \\
\therefore\quad & \frac{\partial I_t}{\partial S} = \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} \frac{\partial^2 f_{t+1}(Q_{t+1},S-(S-h^0_{t+1}(S))^+)}{\partial S\partial Q_{t+1}} g_t(D_t)\,dD_t
\end{aligned}
$$

Assume $\frac{\partial^2 f_{t+2}(Q_{t+2},X^*_{t+2})}{\partial S\partial Q_{t+2}} \ge 0$, except finite number of points,

$$
\begin{aligned}
\because\quad & f_{t+1}(Q_{t+1},X^*_{t+1}) \\
& = \alpha(Q_{t+1}+X^*_{t+1}-\mu_{t+1}) + (\alpha+\beta)L_{t+1}(Q_{t+1}+X^*_{t+1}) \\
& + \int_0^\infty f_{t+2}(Q_{t+2},X^*_{t+2}) g_{t+1}(D_{t+1})\,dD_{t+1} \\
\therefore\quad & \frac{\partial f_{t+1}(Q_{t+1},X^*_{t+1})}{\partial Q_{t+1}} \\
& = (\alpha+\beta)G_{t+1}(Q_{t+1}+X^*_{t+1}) - \beta + \int_0^\infty \frac{\partial f_{t+2}(Q_{t+2},X^*_{t+2})}{\partial Q_{t+2}} \frac{\partial Q_{t+2}}{\partial Q_{t+1}} g_{t+1}(D_{t+1})\,dD_{t+1} \\
& = (\alpha+\beta)G_{t+1}(Q_{t+1}+X^*_{t+1}) - \beta + \int_0^{Q_{t+1}+X^*_{t+1}} \frac{\partial f_{t+2}(Q_{t+2},X^*_{t+2})}{\partial Q_{t+2}} g_{t+1}(D_{t+1})\,dD_{t+1} \\
\therefore\quad & \frac{\partial^2 f_{t+1}(Q_{t+1},X^*_{t+1})}{\partial S\partial Q_{t+1}} \\
& = \int_0^{Q_{t+1}+X^*_{t+1}} \frac{\partial^2 f_{t+2}(Q_{t+2},X^*_{t+2})}{\partial S\partial Q_{t+2}} g_{t+1}(D_{t+1})\,dD_{t+1} \ge 0
\end{aligned}
$$

And since $\frac{\partial^2 f_n(Q_n,X^*_n)}{\partial S\partial Q_n} = \begin{cases} 0 & , \quad Q_n > G_n^{-1}(\frac{\beta}{\alpha+\beta}) \\ (\alpha+\beta)g_n(S+Q_n) & , \quad Q_n < G_n^{-1}(\frac{\beta}{\alpha+\beta}) \end{cases} \ge 0$, $\forall Q_{t+1}$ (t=1, 2, …, $n-1$), $\frac{\partial^2 f_{t+1}(Q_{t+1},X^*_{t+1})}{\partial S\partial Q_{t+1}} \ge 0$ (except finite number of points).

Therefore, $\frac{\partial I_t}{\partial S} = \int_{Q_t+X_t-\left(h^0_{t+1}(S)-S\right)^+}^{Q_t+X_t} \frac{\partial^2 f_{t+1}(Q_{t+1},S-(S-h^0_{t+1}(S))^+)}{\partial S\partial Q_{t+1}} g_t(D_t)\,dD_t \ge 0$. I_t is an increasing function of Q_t and S. □

Proof of Proposition 2.11.

(1) According to the definition of I_t^0

$$\begin{aligned}I_t^0 =&(\alpha+\beta)G_t(Q_t)-\beta+\int_0^{Q_t-h_{t+1}^0(S)}\frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t}|_{X_t=0}g_t(D_t)\,dD_t\\&+\int_{Q_t-\left(h_{t+1}^0(S)-S\right)^+}^{Q_t}\frac{df_{t+1}(Q_t+X_t-D_t,S-(S-h_{t+1}^0(S))^+)}{dX_t}\Big|_{X_t=0}g_t(D_t)\,dD_t\end{aligned}$$

Thus,

$$I_t^0|_{Q_t=0}=-\beta<0$$

and

$$\begin{aligned}\lim_{Q_t\to\infty}I_t^0&=(\alpha+\beta)-\beta+\lim_{Q_t\to\infty}\int_0^{Q_t-h_{t+1}^0(S)}\frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t}|_{X_t=0}g_t(D_t)\,dD_t\\&\geq\alpha+\lim_{Q_t\to\infty}\frac{df_{t+1}(h_{t+1}^0(S)+X_t,0)}{dX_t}G_t\left(Q_t+X_t-h_{t+1}^0(S)\right)|_{X_t=0}\\&=\alpha\geq0\end{aligned}$$

Recall that $\frac{\partial I_t}{\partial Q_t}\geq0$, we can obtain: For any given S, there exists $h_t^0(S)>0$, such that

$$\begin{cases}I_t^0|_{Q_t<h_t^0(S)}\leq0\\I_t^0|_{Q_t=h_t^0(S)}=0\\I_t^0|_{Q_t>h_t^0(S)}\geq0\end{cases}$$

To prove $h_t^0(S)$ is unique for any given S, consider a small Δ, such that $I_t^0|_{Q_t=h_t^0(S)}=I_t^0|_{Q_t=h_t^0(S)+\Delta}=0$, that is, $\frac{\partial I_t^0}{\partial Q_t}|_{Q_t=h_t^0(S)}=0$.

If $S\geq h_{t+1}^{0-1}(S)$, then

$$\begin{aligned}\frac{\partial I_t^0}{\partial Q_t}=&(\alpha+\beta)g_t(Q_t)+\int_0^{Q_t-h_{t+1}^0(S)}\frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t}|_{X_t=0}g_t(D_t)\,dD_t\\&+\frac{df_{t+1}(h_{t+1}^0(S)+X_t,0)}{dX_t}|_{X_t=0}G_t\left(Q_t-h_{t+1}^0(S)\right)\\=&(\alpha+\beta)g_t(Q_t)+\int_0^{Q_t-h_{t+1}^0(S)}\frac{df_{t+1}(Q_t-D_t,0)}{dX_t}|_{X_t=0}g_t(D_t)\,dD_t\\\Leftrightarrow&(\alpha+\beta)g_t\left(h_t^0(S)\right)\\&+\int_0^{h_t^0(S)-h_{t+1}^0(S)}\frac{df_{t+1}(h_t^0(S)-D_t,0)}{dX_t}|_{X_t=0}g_t(D_t)\,dD_t\end{aligned}$$

If $S < h_{t+1}^{0-1}(S)$, then

$$\begin{aligned}
\frac{\partial I_t^0}{\partial Q_t} =&(\alpha+\beta)g_t(Q_t) + \int_0^{Q_t-h_{t+1}^0(S)} \frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \\
&+ \frac{df_{t+1}(h_{t+1}^0(S)+X_t,0)}{dX_t}|_{X_t=0} G_t\left(Q_t-h_{t+1}^0(S)\right) \\
&+ \int_{Q_t-h_{t+1}^0(S)+S}^{Q_t} \frac{df_{t+1}(Q_t+X_t-D_t,S)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \\
&+ \frac{df_{t+1}(X_t,S)}{dX_t}|_{X_t=0} G_t(Q_t) \\
&- \frac{df_{t+1}(h_{t+1}^0(S)-S+X_t,S)}{dX_t}|_{X_t=0} G_t(Q_t-h_{t+1}^0(S)+S) \\
=&(\alpha+\beta)g_t(Q_t) + \int_0^{Q_t-h_{t+1}^0(S)} \frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \\
&+ \int_{Q_t-h_{t+1}^0(S)+S}^{Q_t} \frac{df_{t+1}(Q_t+X_t-D_t,S)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \\
&+ \frac{df_{t+1}(X_t,S)}{dX_t}|_{X_t=0} G_t(Q_t) \\
\Leftrightarrow&(\alpha+\beta)g_t\left(h_t^0(S)\right) + \int_0^{h_t^0(S)-h_{t+1}^0(S)} \frac{df_{t+1}(h_t^0(S)+X_t-D_t,0)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \\
&+ \int_{h_t^0(S)-h_{t+1}^0(S)+S}^{h_t^0(S)} \frac{df_{t+1}(h_t^0(S)+X_t-D_t,S)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \\
&+ \frac{df_{t+1}(X_t,S)}{dX_t}|_{X_t=0} G_t\left(h_t^0(S)\right) = 0
\end{aligned}$$

Since

$$(\alpha+\beta)g_t\left(h_t^0(S)\right) \geq 0 \quad ,$$

$$\int_0^{h_t^0(S)-h_{t+1}^0(S)} \frac{df_{t+1}(h_t^0(S)+X_t-D_t,0)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \geq 0$$

and

$$\begin{aligned}
&\int_{h_t^0(S)-h_{t+1}^0(S)+S}^{h_t^0(S)} \frac{df_{t+1}(h_t^0(S)+X_t-D_t,S-(S-h_{t+1}^0(S))^+)}{dX_t}|_{X_t=0} g_t(D_t)\,dD_t \\
+&\frac{df_{t+1}(X_t,S)}{dX_t}|_{X_t=0} G_t\left(h_t^0(S)\right) \\
\geq&0
\end{aligned}$$

for any S,

$$\frac{\partial I_t^0}{\partial Q_t}|_{Q_t=h_t^0(S)} = 0 \Leftrightarrow \begin{cases} g_t\left(h_t^0(S)\right) = 0 \\ g_t\left(h_t^0(S)-h_{t+1}^0(S)\right) = 0 \end{cases}$$

Thus, $h_t^0(S) \leq M(D_{t-1})$, for any known D_{t-1}.

However, $h_t^0(S) \leq M(D_{t-1}) \Rightarrow I_t^0|_{Q_t=h_t^0(S)} = -\beta < 0$, which is contradictory to the definition of $h_t^0(S)$. So there does not exist a Δ, such that $I_t^0|_{Q_t=h_t^0(S)} = I_t^0|_{Q_t=h_t^0(S)+\Delta} = 0$. So $h_{n-1}^0(S)$ is unique.

Therefore, for any given S, there exists a unique $h_t^0(S) > 0$, such that $I_t^0|_{Q_t<h_t^0(S)} < 0$, $I_t^0|_{Q_t=h_t^0(S)} = 0$ and $I_t^0|_{Q_t>h_t^0(S)} > 0$.

And according to Proposition 2.10, $h_t^0(S)$ is decreasing in S.

(2) According to the definition of I_t^S,

$$\begin{aligned} I_t^S =&(\alpha+\beta)G_t(S+Q_t)-\beta \\ &+\int_0^{Q_t+S-h_{t+1}^0(S)} \frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t}|_{X_t=S} g_t(D_t)\,dD_t \\ &+\int_{Q_t+S-\left(h_{t+1}^0(S)-S\right)^+}^{Q_t+S} \frac{df_{t+1}(Q_t+X_t-D_t,S-(S-h_{t+1}^0(S))^+)}{dX_t}\bigg|_{X_t=S} g_t(D_t)\,dD_t \end{aligned}$$

We can calculate

$$\begin{aligned} &I_t^S|_{Q_t=h_t^0(S)} \geq 0 \\ &I_t^S|_{Q_t=0,S=0} = -\beta < 0 \\ &\lim_{S\to\infty} I_t^S|_{Q_t=0} = (\alpha+\beta)-\beta+\int_0^{S-h_{t+1}^0(S)} |_{X_t=S} g_t(D_t)\,dD_t \geq \alpha \geq 0 \end{aligned}$$

Recall that $\frac{\partial I_t}{\partial S} \geq 0$. There exists $S_t^{'} > 0$, such that

$$\begin{cases} I_t^S|_{Q_t=0,S<S_t'} \geq 0 \\ I_t^S|_{Q_t=0,S=S_t'} = 0 \\ I_t^S|_{Q_t=0,S>S_t'} \leq 0 \end{cases}$$

Thus if $S < S_t^{'}$, then there exists $h_t^S(S)$, such that

$$\begin{cases} I_t^S|_{Q_t<h_t^S(S)} \leq 0 \\ I_t^S|_{Q_t=h_t^S(S)} = 0 \\ I_t^S|_{Q_t>h_t^S(S)} \geq 0 \end{cases}$$

If $S = S_t^{'}$, then $h_t^S(S_t^{'}) = 0$; if $S > S_t^{'}$, then $h_t^S < 0$ and $I_t^S \geq 0$.

Similar to (1), one can prove $h_t^S(S)$ is unique for given S.

And according to Proposition 2.10, $h_t^S(S)$ is decreasing in S.

(3) We can get that if $X_t' + Q_t' = X_t'' + Q_t''$ $(t = 1, 2, \ldots, n)$, $I_t|_{X_t=X_t', Q_t=Q_t'} = I_t|_{X_t=X_t'', Q_t=Q_t''}$. So

$$I_t^S|_{Q_t=h_t^S(S)} = I_t|_{X_t=S, Q_t=h_t^S(S)} = I_t|_{X_t=0, Q_t=h_t^S(S)+S} = 0$$

Recall that $I_t^0|_{Q_t=h_t^0(S)} = I_t|_{X_t=0, Q_t=h_t^0(S)} = 0$ and the uniqueness of $h_t^0(S)$. We can obtain $h_t^0(S) = h_t^S(S) + S$. □

Proof of Theorem 2.12.

Based on Proposition 2.11, for any given S, Q_t and $G_i(x)$,$i = t, t+1, \ldots, n$,

(1) If $Q_t \geq h_t^0(S)$, then $I_t|_{X_t=0} \geq 0$ and $I_t|_{X_t=S} > 0$;

(2) If $h_t^S(S)^+ \leq Q_t < h_t^0(S)$, then $I_t|_{X_t=0} < 0$ and $I_t|_{X_t=S} \geq 0$;

(3) If $Q_t < h_t^S(S)^+$, then $I_t|_{X_t=0} < 0$ and $I_t|_{X_t=s} < 0$.

And since $E_{D_t}\{Y_t(X_t, D_t) + f_{t+1}(Q_{t+1}, X_{t+1}^*)\}$ is convex in X_t (Proposition 2.9), the optimal solution is

$$x_t^* = \begin{cases} 0 & , \quad Q_t \geq h_t^0(S) \\ \bar{X}_t & , \quad h_t^S(S)^+ \leq Q_t < h_t^0(S) \\ S & , \quad Q_t < h_t^S(S)^+ \end{cases}$$

where $\bar{X}_t$ is defined as $I_t|_{X_t=\bar{X}_t} = 0$ and $\bar{X}_t = h_t^0(S) - Q_t$.

When $t = n$, $h_n^0(S) = G_n^{-1}(\frac{\beta}{\alpha+\beta})$ and $h_n^S(S) = G_n^{-1}(\frac{\beta}{\alpha+\beta}) - S$.

Therefore, for any t

$$x_t^* = \begin{cases} 0 & , \quad Q_t \geq h_t^0(S) \\ h_t^0(S) - Q_t & , \quad \left(h_t^0(S) - S\right)^+ \leq Q_t < h_t^0(S) \\ S & , \quad Q_t < \left(h_t^0(S) - S\right)^+ \end{cases}$$

□

Proof of Proposition 2.13.

(1) According to the proof of Proposition 2.11, $\frac{\partial h_t^0(S)}{\partial S} \leq 0$.

(2) Since $\frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_{t+1}^*} \geq 0$, $\frac{df_{t+1}(Q_t+X_t-D_t,S-(S-h_{t+1}^0(S))^+)}{dX_{t+1}^*} \leq 0$ and Theorem 2.12, we can get $\frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t} \geq 0$ and $\frac{df_{t+1}(Q_t+X_t-D_t,S-(S-h_{t+1}^0(S))^+)}{dX_t} \leq 0$.

Thus,

$$\begin{aligned}
I_t^0 =&(\alpha+\beta)G_t(Q_t)-\beta+\int_0^{Q_t-h_{t+1}^0(S)} \frac{df_{t+1}(Q_t+X_t-D_t,0)}{dX_t}|_{X_t=0}g_t(D_t)dD_t \\
&+\int_{Q_t-(h_{t+1}^0(S)-S)^+}^{Q_t} \frac{df_{t+1}(Q_t+X_t-D_t,S-(S-h_{t+1}^0(S))^+)}{dX_t}|_{X_t=0}g_t(D_t)dD_t \\
\geq&(\alpha+\beta)G_t(Q_t)-\beta+\frac{df_{t+1}(X_t+h_{t+1}^0(S),0)}{dX_t}|_{X_t=0}G_t\left(Q_t-h_{t+1}^0(S)\right) \\
&+\frac{df_{t+1}\left(X_t-(h_{t+1}^0(S)-S)^+,S-(S-h_{t+1}^0(S))^+\right)}{dX_t}|_{X_t=0}G_t\left(Q_t-(h_{t+1}^0(S)-S)^+\right) \\
=&(\alpha+\beta)G_t(Q_t)-\beta
\end{aligned}$$

$\therefore I_t^0|_{Q_t=G_t^{-1}(\frac{\beta}{\alpha+\beta})} \geq (\alpha+\beta)G_t^{-1}(\frac{\beta}{\alpha+\beta})=0$

$\therefore h_t^0(S) \geq G_t^{-1}(\frac{\beta}{\alpha+\beta})$

(3) $\frac{\partial Q_{t+1}}{\partial D_t} = \begin{cases} -1 & , \quad D_t < X_t+Q_t \\ 0 & , \quad D_t > X_t+Q_t \end{cases}$ and $\frac{\partial I_{t+1}^0}{\partial Q_{t+1}^0} \geq 0$

$\therefore \frac{\partial I_{t+1}^0}{\partial D_t} \geq 0$ $(D_t \neq X_t+Q_t)$

Note that $h_{t+1}^0(S)$ is defined as $I_{t+1}^0|_{Q_{t+1}=h_{t+1}^0(S)}=0$, we can obtain $\frac{\partial h_{t+1}^0(S)}{\partial D_t} \geq 0$ $(D_t \neq X_t+Q_t)$. □

Chapter 3

Humanitarian Medical Allocation in Mutiple Areas

This chapter presents a novel model of emergency medical allocation for quick response to public health emergencies. The proposed methodology consists of two recursive mechanisms: (1) the time-varying forecasting of medical relief demand and (2) relief distribution. The medical demand associated with each epidemic area is forecast according to a modified susceptible-exposed-infected-recovered model. A linear programming approach is then applied to facilitate distribution decision-making. The physical and psychological fragility of affected people are discussed. Numerical studies are conducted. Results show that the consideration of survivor psychology significantly reduces the psychological fragility of affected people, but it barely influences physical fragility.

3.1 Introduction

In addition to health threats and economic losses, public health emergencies also result in psychological suffering, such as feelings of helplessness, sorrow, and panic. Studies conducted on the worldwide effects of the SARS outbreak in 2003 suggest that the fear of SARS is a more severe pandemic than the disease itself (Cheng and Tang 2004).

Most emergencies cannot be avoided, but their influence can be significantly reduced by an efficient framework of emergency medical logistics. Medical logistics that direct responses to public health emergencies are vital. However, this field of emergency logistics faces many challenges that have not been addressed effectively, and the available literature inadequately addresses emergency medical logistics.

The current chapter proposes a model of emergency logistics for rapid response to public health emergencies. In particular, a modified epidemic susceptible-exposed-infected-recovered (SEIR) model is developed to forecast time-varying demand as well as a linear programming model that optimizes decisions regarding the distribution of emergency medical reliefs.

Specifically, this chapter contributes to the decision analysis of logistical responses to public health emergencies in the following ways:

(1) This interdisciplinary study contributes to the fields of public health and emergency logistics. Emergency medical logistics differs from general emergency logistics in that the former involves many challenges that increase the complexity and difficulty of solving logistical problems.

(2) This chapter applies a novel methodology to forecast the demand of multiple urgent medi-cal reliefs and to distribute these reliefs to multiple epidemic areas. The physical and psycho-logical situations of those affected are considered. The modified SEIR model contributes to forecasting by considering not only physical factors, such as the differences in the infection conditions of survivors and the spatial interaction relationships among epidemic areas, but also the psychological demand of exposed and undiagnosed individuals. In the distribution model, psychological fragility is formulated and discussed in detail, unlike in previous studies. The relationship between emergency medical logistics and the psychological effects on affected people is highlighted as well.

(3) This chapter conducts a case study using real data and a continuation study with experi-mental data to demonstrate the applicability of the three proposed models. These models are then compared. Observations are provided and their implements are discussed on this basis.

The remainder of this chapter is organized as follows. Section 2.2 reviews related studies. Section 2.3 presents the proposed basic methodology, including time-varying demand forecasting according to the epidemic diffusion rule and the distribution of medical reliefs. Then this section introduces two extended models. Section 2.4 presents a numerical study and discusses the analytical results. Section 2.5 provides managerial insights. Section 2.6 concludes and discusses the directions for future work. Finally, Section 2.7 gives the proof and supplement information for this chapter.

3.2 Literature Review

Only bioterror response logistics, a special case in humanitarian medical logistics, has been discussed. A terrorist attack usually focuses on only one or several cities, whereas other public health emergencies may occur in large areas at the same time. Except for the distinctiveness of bioterrorism, these studies ignored the differences in the infection conditions and survivor psychology. In practice, vulnerable groups, such as children and the elderly, face different infection, recovery, and mortality rates. Moreover, the psychological suffering of affected people in a bioterror attack is usually more serious than physical pain.

As reviewed in Chapter 2, although some studies try to combine medical service with emergency logistics, most of them focus on emergency logistics after large-scale natural disasters and regard medical supplies as one common type of relief items like food or tents. These studies make remarkable advances in decision optimization of the locations of medical facilities and the distribution of medical supplies, but ignore the unique characteristics of epidemic diseases (Jia et al. 2007, Berman and Gavious 2007, Mete and Zabinsky 2010, Sheu and Pan 2014).

In addition, this work reviews the related literature by first focusing on demand forecasting and then discussing the approaches to logistics distribution for an emergency and their objective functions.

Typically, demand forecasting is studied based on general supply chain management in business logistics but limited to emergency logistics. The approaches adopted in business logistics forecast are based on historical values, which can be collected easily during business pro-cesses. By contrast, emergency medical logistics lacks historical data. Gaur et al. (2007) dis-cussed demand uncertainty in business logistics, but demand history was unavailable. Based on the characteristics of the predictions of emergency resource demand, Sheu (2010) presented a dynamic model of relief-demand management for emergency logistics operations under imperfect information conditions in large natural disasters. Mete and Zabinsky (2010) proposed fore-casting and optimization approaches to problems on medical storage and distribution for a wide variety of disaster types and magnitudes. Hasan and Ukkusuri (2011) developed a novel model to understand the cascade of the warning information flow in social networks during the hurricane evacuations. Fajardo and Gardner (2013) used a bilinear integer program to model diseases spreading through direct human interaction on a social-contact network. Ekici et al. (2014) created an interesting approach to demand forecasting based on the characteristics of disease epidemics. They also developed a SEIR model with a spatial component among communities, age-based structure, heterogeneous mixing, and night/day differentiation to plan food distribution. Few studies have forecast demand in this way (Wang and Wang 2008, Liu and Zhao 2011), even though scholars have conducted much research on preventing and controlling epidemics, as well as identifying their characteristics and models.

The mathematical models of epidemic diffusion rules can be used to facilitate demand forecasting for public health emergencies (such as Hamer 1906, Gani 1978, Hethcote and Tudor1980, Hethcote 1999, Rahman and Smith 2000, Wu and Feng 2000, Brauer and van den Driessche 2001, Eames and Keeling 2002, Gomes et al. 2002, Lu et al. 2002, Wang et al. 2003, Zhou et al. 2004, Eubank et al. 2004, Keeling and Eames 2005, Zhang et al. 2005, Zhen et al. 2006, Xiao and Ruan 2007, Tripathi 2007, Mukhopadhyay and Bhattacharyya 2008, El-Gohary and Alwasel 2009, Elmojtaba et al. 2010, Yuan and Wang 2010, Yang et al. 2010, Rawls and Turnquist 2010, Sun 2010, Capaldi et al 2012).

Most models, including the susceptible-infectious, susceptible-infectious-removed, susceptible-infectious-susceptible, susceptible-infectious-removed-susceptible, susceptible-exposed-infectious, and SEIR model, are suitable for studying the general laws of all epidemics. In particular, SEIR model, which considers incubation period, has drawn considerable attention. In the real world, there is some duration between the time that a person is infected and the time that he/she starts infecting others. SEIR model divided the people in epidemic areas into four classes: susceptible, exposed, infectious and recovered. The model describes the transition among these classes with differential equations. Basic reproduction number is an important concept in epidemic model. It determines the global dynamics and the outcome of disease. If it is less than or equal to 1, the disease-free equilibrium is globally stable and the disease always dies out; otherwise, there exists a unique positive endemic equilibrium and the disease persists at an endemic equilibrium state if it initially exists. (Hethcote and Tudor 1980, Greenhalgh 1992, Li and Muldowney 1995, Li et al. 1999, Zhang and Ma 2003).

Based on these standard models, some studies try to build multi-class models (Lajmanovich and Yorke 1976, Hethcote 1978, Aronsson and Mellander 1980, Diekmann et al. 1990, Guo and Li 2006, Guo et al. 2008). Since these problems are usually complex to solve, most of them are discussed with strict assumptions. Two main types of exploration are noteworthy, but neither is perfectly appropriate for general medical emergency logistics in this work. The first type of exploration covers studies that consider an ecosystem where two disease-affected populations thrive and epidemics can spread among these populations (Han et al. 2001, Jang 2007, Gonzlez-Parra et al2009, Chaudhuri et al. 2012). In these studies, the two populations compete for survival resource or have a predator-prey relation. The second type involves age-structured models (Cha et al. 1998, Zhang and Peng 2007, Li and Song 2011) that are unable to describe the vulnerable people because they can be distinguished not only by age but also by other characteristics, such as gender. Therefore, demand forecasting based on these models, including standard models and previous multi-class models, may forecast demand that differs markedly from the real demand of emergency logistics.

Additionally, considerable effort has been made to optimize logistics under emergencies (Carter 1992, Thomas 2002, Viswanath and Peeta 2003, Jotshi et al. 2009, Wang and He 2009, Zhang et al. 2012). Özdamar et al. (2004) constructed a distribution optimization model for a situation in which the supply is limited, current demand is known, future demand can be predicted, and commands on vehicle allocation are composed of a series of breakpoints. Advar and Mert (2010) proposed an international relief-planning model that can handle uncertain information while maximizing the credibility of international agencies in the most cost-efficient way. Ben-Tal et al. (2011) applied robust optimization for dynamic evacuation traffic-flow problems with time-dependent uncer-tainty on demand. Liu and Ye (2014) proposed a sequential approach for humanitarian logistics in natural disasters based on the Bayesian group information updates. Allahviranloo et al. (2014) proposed three new formulations to account for different optimization strategies under uncertain demand or utility level: reliable, robust, and fuzzy selective vehicle routing problems. And they developed three parallel genetic algorithms and a classic genetic algorithm. Stochastic programming is also an appropriate tool for making emergency logistics decisions and has been applied to different cases of emergency management (Barbarosolu and Arda 2004, Beraldi et al. 2004, Chang et al. 2007, Zhan et al. 2014). Han et al. (2013) present a novel approach to consider a vehicle routing problem with uncertain travel times in which a penalty is incurred for each vehicle that exceeds a given time limit.

Existing models can be divided based on their objective characterization. They can be cate-gorized into models that (1) minimize the distribution time or shipping distance (Zografos et al. 2002, Yan and Shih 2009, Zhan et al. 2014), (2) minimize cost of logistics (Haghani and Oh 1996, Ben-Tal et al. 2011), (3) minimize the number of wounded and dead people (Fiedrichet al. 2000, Yi and Kumar 2007), and (4) maximize level of satisfaction of the relief demand (Özdamar et al. 2004, Sheu 2007). Some studies developed multi-objective models. Tzeng et al. (2007) constructed a relief-distribution model by multi-objective programming to design relief delivery systems for a real case. As a part of the entire disaster salvaging system, sufficiently accurate information is needed before model application. Vitoriano et al. (2009) presented a Humanitari-an Aid Distribution System focusing on the transportation problem to distribute humanitarian aid to affected people after a disaster in a developing country. Vitoriano et al. (2011) proposed a new approach to solve humanitarian aid distribution problems, by constructing a goal program-ming model based upon cost, time, equity, priority, reliability and security. Ortuo et al. (2011) presented a novel lexicographical goal programming model with a first level of priority with the goal of delivering the planned goods in the operation verifying all the hard constraints or to deliver the largest possible quantity of commodity. Then the model concerned other targets in a second level of priority.

Despite remarkable advances in emergency logistics modeling, the relationship between relief distribution and psychological utility or psychological cost for those affected has barely been assessed or formulated, although characterizing this relationship may improve the rationality of models. Disasters have negative psychological effects on affected people and can even manifest as a major depressive episode, acute stress disorder or post-traumatic stress disorder. Hu and Sheu (2013) developed a multi-objective linear programming model to minimize total reverse logistical costs, corresponding environmental and operational risks, and psychological trauma experienced by local residents while they waited for medical treatment and removal of debris. In Hu et al. (2014), a mixed-integer linear program is constructed for multi-step evacuation and temporary resettlement under minimization of panic-induced psychological penalty cost, psychological intervention cost, and costs associated with transportation and building shelters. Negative psychological effects of widespread disease epidemics are usually more significant. For the public, the psychological effect of an epidemic may be greater than the danger to physical health (Cheng and Tang 2004). Maunder et al. (2003) examined the psychological and the occupational effects of SARS in a large hospital in Canada in 2003. Their study showed that patients with SARS reported fear, loneliness, boredom and anger and worried about the adverse effects of quarantine and their contagiousness on family members and friends. The patients experienced anxiety about fever and the effects of insomnia. Healthy people were also adversely affected by the fear of contagion and infecting their family, friends, and colleagues. Other studies on SARS have reported similar findings (Wang and Luo 2003, Cheng and Cheung 2005, Leung et al. 2005). These studies have identified approaches that can alleviate psychological suffering, such as symmetric information, on-time treatment, effective prophylactic methods, and social support. Most approaches have been considered part of medical logistics.

3.3 Model Development

This section presents the proposed methodology. A modified SEIR model is developed based on a specification of an emergency medical relief system, including the sequence of operational procedures and basic assumptions, to forecast time-varying demand. The final part of this section presents the model for medical relief distribution.

3.3.1 Assumptions

The hypothetical network of emergency medical logistics is a specific two-layer supply chain that involves (1) local Emergency Medical Reserve Centers (EMRCs) that serve as emergency medical logistics hubs, and (2) medical demand areas, namely epidemic areas. EMRCs gather medical reliefs and distribute them appropriately to epidemic areas.

The government decides whether or not to issue an emergency medical response shortly after an infectious disease has established itself. EMRCs then initiate a mechanism to forecast time-varying medical relief demand by collecting and estimating epidemic characteristics. The mechanism of medical relief distribution is then triggered based on updated information. Fig. 3.1 lists the sequence of the operation.

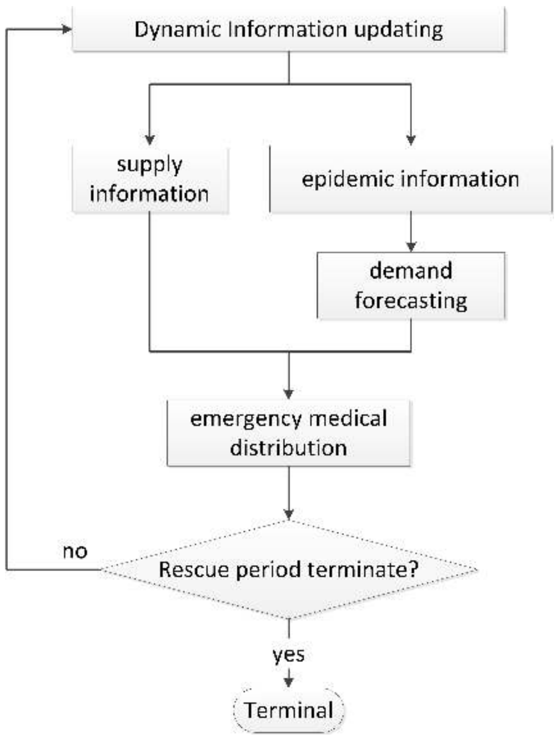


Figure 3.1: Sequence of operational procedures in emergency medical allocation

This figure presents the recurrent calculation time step in emergency medical logistics. Supply and epidemic information (e.g., mortality, recovery, exposure, and infection rates) is updated at the beginning of each step.

Five basic assumptions are made to facilitate model formulation:

(1) The number of and geographic information on epidemic areas are available, and the location of EMRCs is known because they have been established before emergencies.

(2) The corresponding socioeconomic statistics (e.g., population size, population composition, and natural mortality and birth rates) are determined for each epidemic area. Such data can generally be obtained from government databases.

(3) Emergency medical suppliers are known. The number and the type of available medical suppliers are identified at the beginning of each time step.

(4) Different medical reliefs can be loaded on a vehicle that serves affected areas. Correspondingly, a vehicle is allowed to load multiple medical reliefs for any distribution mission.

(5) Recovered individuals acquire permanent immunity.

3.3.2 Notations

On the basis of these assumptions, the next sections will present a demand forecasting model and a spacial allocation model of emergency medical relief. Notations used in this chapter are as follows:

Parameters:

A**:** A set of epidemic areas, $A = \{a_1, a_2, \ldots, a_J\}$

B**:** A set of local EMRCs, $B = \{b_1, b_2, \ldots, b_I\}$

M**:** A set of medical reliefs, $M = Mp + Mt$. Mp and Mt are sets of prophylactic and treatment reliefs, respectively. $Mp = \{m_1, m_2, \ldots, m_l\}$, $Mt = \{m_{l+1}, m_{l+2}, \ldots, m_{l+k}\}$

Q^p_{it}**:** Available amount of relief p in EMRC i in time period t

W_{it}**:** Available transportation capacity in EMRC i in time period t

C_t**:** The budget limit for humanitarian medical allocation in time period t

c^p_{ijt}**:** Unit distribution cost of relief p from EMRC i to area j in time period t

θ^p**:** Demand of medical relief p needed per unit time per demander

α^p: Mortality increase caused by unit dissatisfaction of medical relief p, $p \in Mt$

β^p: Infection rate increase caused by unit dissatisfaction of medical relief p, $p \in Mt$

α_M: The highest mortality rate that can be withstood

β_M: The highest infection rate that can be withstood

$S_j^c(t)$: Numbers of common susceptible people in area j at moment t

$S_j^v(t)$: Numbers of vulnerable susceptible people in area j at moment t

$E_j^c(t)$: Numbers of common exposed people in area j at moment t

$E_j^v(t)$: Numbers of vulnerable exposed people in area j at moment t

$I_j^c(t)$: Numbers of common infectious people in area j at moment t

$I_j^v(t)$: Numbers of vulnerable infectious people in area j at moment t

$R_j^c(t)$: Numbers of common recovered people in area j at moment t

$R_j^v(t)$: Numbers of vulnerable recovered people in area j at moment t

S_{jt}^c: The approximate average numbers of common susceptible people in area j in time period t

S_{jt}^v: The approximate average numbers of vulnerable susceptible people in area j in time period t

E_{jt}^c: The approximate average numbers of common exposed people in area j in time period t

E_{jt}^v: The approximate average numbers of vulnerable exposed people in area j in time period t

I_{jt}^c: The approximate average numbers of common infectious people in area j in time period t

I_{jt}^v: The approximate average numbers of vulnerable infectious people in area j in time period t

A_j^c: Constant net input of common people to area j

A_j^v: Constant net input of vulnerable people to area j

α_j^c: Disease mortality rate of common people in area j

α_j^v: Disease mortality rate of vulnerable people in area j

β_j^c: Common infection rate in area j

β_j^v: Vulnerable infection rate in area j

δ_j: Contact coefficient in area j

ε: Exposure rate (the rate at which exposed individuals become infectious)

γ_j^c: Recovered rate of common patient in area j

γ_j^v: Recovered rate of vulnerable patient in area j

μ_j: Diagnosis rate in area j

d_j^c: Natural mortality rate of common people in area j

d_j^v: Natural mortality rate of vulnerable people in area j

D_{jt}^p: Demand for medical relief p in area j in time period t

Inv_{jt}^p: The inventory of relief p in area j at the beginning of time period t

Decision variable:

x_{ijt}^p: The amount of medical relief p sent from EMRC i to area j in time period t

3.3.3 Medical Demand Forecasting Model

This mechanism forecasts the time-varying emergency medical demand of each affected area based on epidemic diffusion rules. On the basis of previous models, this study constructs a modified SEIR epidemic diffusion model that accommodates the differences between vulnerable and non-vulnerable or common groups to enhance forecast accuracy given that vulnerable groups usually report different infection, recovery, and mortality rates. The SEIR model is considered because it reflects the practical phenomena observed in incubation periods.

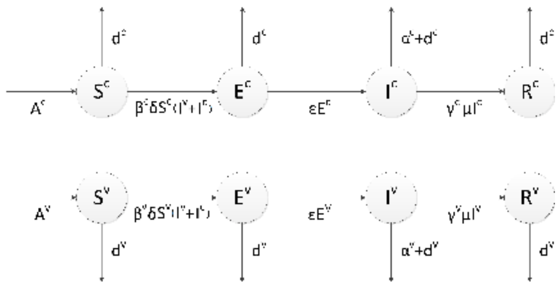


Figure 3.2: Modified SEIR model

Fig. 3.2 shows the relationship and transition among different groups in a specific area. S, E, I, and R denote the number of susceptible, exposed, infectious, and recovered people in an area, respectively. Superscripts c and v are the common and the vulnerable groups, respectively. S^c and S^v represent those who are susceptible to infection. E^c and E^v correspond to those who are subject to incubation periods (or latent periods). The incubation period is the time interval between exposure to a disease and the manifestation of initial signs or symptoms. An individual in this period has been infected but is not yet infectious himself/herself. I^c and I^v represent those who are infectious. R^c and R^v denote those who are cured and have permanent immunity. Generally, $\mu > 0$, $\delta > 0$, $\varepsilon > 0$, $A^c \geq 0$, $A^v \geq 0$ and $d^v > d^c > 0$, $\beta^v > \beta^c > 0$, $\alpha^v > \alpha^c > 0$, $\gamma^c > \gamma^v > 0$. These coefficients may vary with the increase in knowledge regarding an epidemic disease. For instance, diagnosis rate may increase and mortality rate may decrease as doctors amass knowledge.

The modified SEIR model in this study is formulated as follows:

$$\begin{cases} \frac{dS_j^c(t)}{dt} = A_j^c - d_j^c S_j^c(t) - \beta_j^c \delta_j S_j^c(t) \left(I_j^c(t) + I_j^v(t)\right) \\ \frac{dS_j^v(t)}{dt} = A_j^v - d_j^v S_j^v(t) - \beta_j^v \delta_j S_j^v(t) \left(I_j^c(t) + I_j^v(t)\right) \\ \frac{dE_j^c(t)}{dt} = \beta_j^c \delta_j S_j^c(t) \left(I_j^c(t) + I_j^v(t)\right) - d_j^v E_j^c(t) - \varepsilon E_j^c(t) \\ \frac{dE_j^v(t)}{dt} = \beta_j^v \delta_j S_j^v(t) \left(I_j^c(t) + I_j^v(t)\right) - d_j^v E_j^v(t) - \varepsilon E_j^v(t) \\ \frac{dI_j^c(t)}{dt} = \varepsilon E_j^c(t) - \left(\alpha_j^c + d_j^c\right) I_j^c(t) - \gamma_j^c \mu_j I_j^c(t) \\ \frac{dI_j^v(t)}{dt} = \varepsilon E_j^v(t) - \left(\alpha_j^v + d_j^v\right) I_j^v(t) - \gamma_j^v \mu_j I_j^v(t) \\ \frac{dR_j^c(t)}{dt} = \gamma_j^c \mu_j I_j^c(t) - d_j^c R_j^c(t) \\ \frac{dR_j^v(t)}{dt} = \gamma_j^v \mu_j I_j^v(t) - d_j^v R_j^v(t) \end{cases} \tag{3.1}$$

The first two equations in Eq. (3.1) describe how the number of susceptible people varies in epidemic area j. This number increases because of net input (such as susceptible input from non-epidemic areas and newborns) and decreases as susceptible people die naturally or are exposed. The third and fourth equations indicate the transition from susceptible to exposed groups and from exposed to infectious groups. The fifth and sixth equations describe the transition from exposed to infectious groups and from infectious to recovered groups. The last two equations indicate how the number of recovered people varies.

In this study, two types of emergency medical reliefs are considered: prophylactic reliefs for susceptible people to reduce the infection rate (denoted as Mp) and treatment reliefs for infectious people to lower the mortality rate (denoted as Mt).

The modified SEIR model operates under continuous time forecasting, and distribution decisions are made in discrete time periods (as explained in Section 3.1 and depicted in Fig. 3.1). Therefore, this study uses aggregation to approximate medical demand further. t represents the t-th time period, and the length of each time period depends on specific emergencies. Time periods usually range from 4h to 6h for an anthrax attack and one to several days for an influenza outbreak.

Notably, this work forecasts demand based on both the physical and psychological effects of reliefs. Demand for prophylactic reliefs is strongly correlated with the numbers of susceptible, exposed, and undiagnosed people. Prophylactic medicines do not affect exposed and undiagnosed people physically, but demand remains because susceptible, exposed, and infectious but undiagnosed individuals are impossible to distinguish. Demand for treatment reliefs is strongly correlated with the number of infectious and diagnosed individuals. Medical reliefs are consumed continuously; therefore, demand for both types is related to the length of each time period. Thus, the following demand forecasting model is established:

$$D_{jt}^{p} = \begin{cases} \theta^{p} \int_{t_0}^{t_0+L} \left[S_j(t) + E_j(t) + (1-\mu_j) I_j(t)\right] dt - Inv_{jt}^{p} & , \quad p \in Mp \\ \theta^{p} \int_{t_0}^{t_0+L} \mu_j I_j(t)\, dt - Inv_{jt}^{p} & , \quad p \in Mt \end{cases} \tag{3.2}$$

where $S_j(t) = S_j^c(t) + S_j^v(t)$; $E_j(t) = E_j^c(t) + E_j^v(t)$; $I_j(t) = I_j^c(t) + I_j^v(t)$; t_0 is the beginning moment of period t; L is the length of a time period.

3.3.4 Distribution Decision Model

In this section, a linear programming model is formulated and applied to distribute urgent medical reliefs from multiple EMRCs to multiple epidemic areas.

The objective function of the distribution model considers the physical fragility of affected individuals. This function is represented by mortality and infection rates. The objective function is:

$$\min \quad F_{phy} = \sum_{j=1}^{J} \left(\sigma_1 \bar{\beta}_{jt}^c S_{jt}^c + \sigma_2 \bar{\beta}_{jt}^v S_{jt}^v + \sigma_3 \bar{\alpha}_{jt}^c I_{jt}^c + \sigma_4 \bar{\alpha}_{jt}^v I_{jt}^v \right) \tag{3.3}$$

where,

$$\bar{\alpha}_{jt}^v = \alpha_j^v + \sum_{p=l+1}^{l+k} \alpha^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p} \quad , \quad \forall j, t \tag{3.4}$$

$$\bar{\alpha}_{jt}^c = \alpha_j^c + \sum_{p=l+1}^{l+k} \alpha^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p} \quad , \quad \forall j, t \tag{3.5}$$

$$\bar{\beta}_{jt}^v = \beta_j^v + \sum_{p=1}^{l} \beta^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p} \quad , \quad \forall j, t \tag{3.6}$$

$$\bar{\beta}_{jt}^c = \beta_j^c + \sum_{p=1}^{l} \beta^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p} \quad , \quad \forall j, t \tag{3.7}$$

The proposed objective function minimizes the aforementioned physical fragility (F_{phy}), which is given by a function of variable x_{ijt}^p.

The weights (σ_1,σ_2, σ_3 and σ_4) in Eq.(3.3) reflect the priorities of different groups. These weights are set by the decision maker (usually the government or experts) based on the trend and the characteristic of the spreading disease, as drawn from the modified SEIR model (Eq.(3.1)). For example, the priority of prophylactic medicines should be not lower than that of treatment medicines ($\sigma_1 \geq \sigma_3$ and $\sigma_2 \geq \sigma_4$) at the beginning because in this situation, the major object of emergency medical logistics is to avoid a large-scale epidemic. Furthermore, the priority of vulnerable groups should be not lower than that of common groups ($\sigma_2 \geq \sigma_1$ and $\sigma_4 \geq \sigma_3$) for humanitarian reasons. Recovered people are not included in Eq. (3.3) because they are healthy. Exposed people are not included because they are no longer at risk of infection and do not suffer illness until they become infectious.

In Eqs. (3.4) to (3.7), the demand for medical reliefs and the number of people in each group are forecasted using Eqs. (3.1) and (3.2). $\bar{\alpha}^v_{jt}$ and $\bar{\alpha}^c_{jt}$ are adopted to measure physical fragility. These variables refer to the expected epidemic mortality rates of vulnerable and common infectious individuals, respectively, after distribution in area j during time period t. Both variables are related to the amount of treatment reliefs sent to epidemic area j in a specific time period. Similarly, $\bar{\beta}^v_{jt}$ and $\bar{\beta}^c_{jt}$ indicate the expected epidemic infection rates of vulnerable and common susceptible people after distribution, respectively. These variables are related to the amount of prophylactic medical reliefs sent to epidemic area j in time period t.

The following constraints are proposed:

$$\sum_{j=1}^{J} x^p_{ijt} \leq Q^p_{it} \quad , \quad \forall i, p, t \tag{3.8}$$

$$\sum_{i=1}^{I} x^p_{ijt} \leq D^p_{it} \quad , \quad \forall j, p, t \tag{3.9}$$

$$\sum_{i=1}^{I} \sum_{p=1}^{l+k} x^p_{ijt} \leq W_{it} \quad , \quad \forall i, t \tag{3.10}$$

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{p=1}^{l+k} c^p_{ijt} x^p_{ijt} \leq C_t \quad , \quad \forall t \tag{3.11}$$

$$\bar{\alpha}^v_{jt}, \bar{\alpha}^c_{jt} \leq \alpha_M \quad , \quad \forall j, t \tag{3.12}$$

$$\bar{\beta}^v_{jt}, \bar{\beta}^c_{jt} \leq \beta_M \quad , \quad \forall j, t \tag{3.13}$$

$$x^p_{ijt} \geq 0 \quad , \quad \forall i, j, p, t \tag{3.14}$$

Among the constraints discussed above, Eq. (3.8) ensures that the aggregate amount of medical relief p sent from EMRC i does not exceed the corresponding amount available in this EMRC in the given time period t. Eq. (3.9) guarantees that the aggregate amount of medical relief p sent to a given area j does not exceed the corresponding demand in the given time period t. Eq. (3.10) ensures that the aggregate amount of medical reliefs distributed from any given EMRC i does not exceed the corresponding transportation capacity. Eq. (3.11) guarantees that the aggregate distribution cost does not exceed the budget in time period t. Eqs. (3.12) and (3.13) consider fairness and ensure that the mortality and infection rates in all epidemic areas do not exceed the limit. These two equations balance these rates among different areas to avoid the situation in which some areas receive enough reliefs, whereas other areas report high rates because of a lack of medicines. Eq. (3.14) characterizes a feasible numerical domain associated with x_{ijt}^p.

3.3.5 Model Extension

In this section, the model is extended to improve its representation of real problems. The mechanism for demand forecasting is improved in the first part, and the distribution mechanism is enhanced in the subsequent part.

Extended Model 1 The basic model assumes that epidemic areas are independent and that all of the people moving into these areas are susceptible. In reality, an epidemic area is affected by another in a public health emergency given the infectiousness of an epidemic. Therefore, an extension that considers the spatial interaction relationships among epidemic areas is developed in this section.

A migration matrix H is adopted:

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1j} & \cdots & h_{1J} \\ \vdots & \ddots & & & \vdots \\ h_{j1} & & h_{jj} & & h_{jJ} \\ \vdots & & & \ddots & \\ h_{J1} & \cdots & h_{Jj} & \cdots & h_{JJ} \end{bmatrix} \tag{3.15}$$

where $h_{j_1 j_2} \geq 0$ is the rate at which the population in j_1 moves to j_2. Moreover, $h_{11} = h_{22} = \cdots = h_{JJ} = 0$ and $\sum_{m=1}^{J} h_{jm} \leq 1, \forall j$. The corresponding epidemic model is formulated as follows:

$$\begin{cases} \frac{dS_j^c(t)}{dt} = \sum_{m=1}^{J} h_{mj} S_m^c(t) - \sum_{m=1}^{J} h_{jm} S_j^c(t) - d_j^c S_j^c(t) - \beta_j^c \delta_j S_j^c(t) I_j(t) \\ \frac{dS_j^v(t)}{dt} = \sum_{m=1}^{J} h_{mj} S_m^v(t) - \sum_{m=1}^{J} h_{jm} S_j^v(t) - d_j^v S_j^v(t) - \beta_j^v \delta_j S_j^v(t) I_j^(t) \\ \frac{dE_j^c(t)}{dt} = \sum_{m=1}^{J} h_{mj} E_m^c(t) - \sum_{m=1}^{J} h_{jm} E_j^c(t) + \beta_j^c \delta_j S_j^c(t) I_j(t) - d_j^v E_j^c(t) - \varepsilon E_j^c(t) \\ \frac{dE_j^v(t)}{dt} = \sum_{m=1}^{J} h_{mj} E_m^v(t) - \sum_{m=1}^{J} h_{jm} E_j^v(t) + \beta_j^v \delta_j S_j^v(t) I_j(t) - d_j^v E_j^v(t) - \varepsilon E_j^v(t) \\ \frac{dI_j^c(t)}{dt} = \sum_{m=1}^{J} h_{mj} I_m^c(t) - \sum_{m=1}^{J} h_{jm} I_j^c(t) + \varepsilon E_j^c(t) - (\alpha_j^c + d_j^c) I_j^c(t) - \gamma_j^c \mu_j I_j^c(t) \\ \frac{dI_j^v(t)}{dt} = \sum_{m=1}^{J} h_{mj} I_m^v(t) - \sum_{m=1}^{J} h_{jm} I_j^v(t) + \varepsilon E_j^v(t) - (\alpha_j^v + d_j^v) I_j^v(t) - \gamma_j^v \mu_j I_j^v(t) \\ \frac{dR_j^c(t)}{dt} = \sum_{m=1}^{J} h_{mj} R_m^c(t) - \sum_{m=1}^{J} h_{jm} R_j^c(t) + \gamma_j^c \mu_j I_j^c(t) - d_j^c R_j^c(t) \\ \frac{dR_j^v(t)}{dt} = \sum_{m=1}^{J} h_{mj} R_m^v(t) - \sum_{m=1}^{J} h_{jm} R_j^v(t) + \gamma_j^v \mu_j I_j^v(t) - d_j^v R_j^v(t) \end{cases} \tag{3.16}$$

As with Eq. (3.1), Eq. (3.16) describe how the numbers of susceptible individuals vary in epidemic area j. $\sum_{m=1}^{J} h_{mj} S_m^c(t)$ and $\sum_{m=1}^{J} h_{mj} S_m^v(t)$ indicate the numbers of common and vulnerable susceptible people moving to area j from other epidemic areas, respectively, whereas $\sum_{m=1}^{J} h_{jm} S_j^c(t)$ and $\sum_{m=1}^{J} h_{jm} S_j^v(t)$ are the numbers of common and vulnerable susceptible people that move from area j to other areas, respectively. Similar migration is also formulated the other equations in Eq. (3.16).

The demand forecasting model is:

$$D_{jt}^p = \begin{cases} \theta^p \int_{t_0}^{t_0+L} [S_j(t) + E_j(t) + (1-\mu_j) I_j(t)]\, dt - Inv_{jt}^p & , \quad p \in Mp \\ \theta^p \int_{t_0}^{t_0+L} \mu_j I_j(t)\, dt - Inv_{jt}^p & , \quad p \in Mt \end{cases} \tag{3.17}$$

The definitions of other notations are similar to those provided for the basic model.

Extended Model 2 The follows extends the model by integrating survivor psychology with medical relief logistics.

As mentioned in Chapters 1 and 2, previous studies presented strong evidence for the adverse psychological effects of public health emergencies on affected people. These studies also suggest that providing on-time treatment and applying an effective prophylactic method help alleviate psychological suffering. These approaches are closely related to on-time allocation and a sufficient amount of medical relief.

A distribution time threshold is set on the basis of the results of previous studies; that is, emergency medical distribution is completely effective only when it is delivered within the time limit. Affected people suffer extra psychological distress, such as anxiety, anger, panic, and fear if distribution time exceeds its limit. Therefore, a suffering coefficient is adopted to represent additional psychological suffering as a result of distribution delay. Similarly, an insufficient amount of medical reliefs can also increase psychological suffering. The mental states of different groups vary. Three additional suffering coefficients are then adopted to denote suffering due to the insufficient amount of relief. The new notations are as follows:

φ_1: Suffering coefficient for diagnosed people caused by unsatisfactory amounts of treatment reliefs

φ_2: Suffering coefficient for susceptible, exposed, and infectious but undiagnosed people caused by unfulfilled amounts of treatment reliefs

φ_3: Suffering coefficient for susceptible, exposed, and infectious but undiagnosed people caused by unfulfilled amounts of prophylactic reliefs

φ_4: Suffering coefficient caused by delay

T_M: Transportation time threshold

T_{ijt}: Transportation time from EMRC i to epidemic area j in time period t

The objective of the extended model is formulated as follows:

$$\min \quad F = F_{phy} + (1 - \omega) F_{psy} \tag{3.18}$$

where F_{phy} is given as in Eq. (3.3) and F_{psy} is the psychological fragility of affected people.

$$
\begin{aligned}
F_{psy} =& \varphi_1 \sum_{j=1}^{J} \sum_{p=l+1}^{l+k} \left(\alpha^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p} \right) \mu_j I_{jt} \\
&+ \varphi_2 \sum_{j=1}^{J} \sum_{p=l+1}^{l+k} \left(\alpha^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p} \right) (S_{jt} + E_{jt} + (1-\mu_j) I_{jt}) \\
&+ \varphi_3 \sum_{j=1}^{J} \sum_{p=1}^{l} \left(\beta^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p} \right) (S_{jt} + E_{jt} + (1-\mu_j) I_{jt}) \\
&+ \varphi_4 \sum_{j=1}^{J} \sum_{i=1}^{I} \left((T_{ijt} - T_M)^+ \sum_{p=1}^{l+k} x_{ijt}^p (S_{jt} + E_{jt} + I_{jt}) \right)
\end{aligned}
\tag{3.19}
$$

The first two terms in Eq. (3.19) reflect the additional psychological suffering caused by unsatisfactory amounts of treatment reliefs. Disease-caused mortality increases as unsatisfied demand increases. Thus, the fear of disease-caused death increases. That is, psychological fragility increases as unsatisfied demand for treatment relief increases. The third term in Eq. (3.19) reflects the additional psychological fragility of individuals who are susceptible, exposed, and infectious but undiagnosed as a result of the unsatisfactory amounts of prophylactic reliefs. Exposed people have already been infected, and prophylactic reliefs do not have physical effects on them. However, psychological effects remain because these individuals are impossible to distinguish from the susceptible group. The psychological suffering of infectious people who have not been diagnosed is also affected by prophylactic reliefs because they do not realize that they have been infected. The fourth term corresponds to the additional pain caused by delayed distribution. A long transportation time increases the psychological fragility of all affected individuals.

The other denotations and constraints are similar to those presented in the basic model.

3.4 Analytical Analyses

In this section, two theorems are proposed for further discussion. Theorem 3.1 helps analyze the trend of epidemic diffusion after relief distribution. Theorem 3.2 shows the condition in which the effectiveness and fairness of emergency medical logistics can be balanced.

Disease-free and endemic equilibrium are discussed, and the first theorem is proposed to explore the effects on the epidemic diffusion of emergency logistics.

An equilibrium is defined as a state where ($\frac{dX_i}{dt} = 0$ for all compartments $X_i \subseteq \{S_j^c(t), E_j^c(t), I_j^c(t), R_j^c(t), S_j^v(t), E_j^v(t), I_j^v(t), R_j^v(t)\}$ in a specific epidemic area j). In the system of Eq. (3.1), set

$$N_j^c(t) = S_j^c(t) + E_j^c(t) + I_j^c(t) + R_j^c(t)$$
$$N_j^v(t) = S_j^v(t) + E_j^v(t) + I_j^v(t) + R_j^v(t)$$
$$\Gamma = \{\big(E_j^c(t), I_j^c(t), R_j^c(t), N_j^c(t), E_j^v(t), I_j^v(t), R_j^v(t), N_j^v(t)\big) \subseteq R_+^8\}$$

In each area j, a disease free equilibrium can be found at $E_1 = (0,0,0,0,0,0,0,0)$ and $E_2 = (0,0,0,N_{je}^c,0,0,0,N_{je}^v)$ and they exist for all nonnegative values of their parameters.

Theorem 3.1. *In the system of Eq. (3.1),*

1. if $\Lambda_j^c - \delta_j\beta_j^c N_{je}^c > 0$ and $\Lambda_j^v - \delta_j\beta_j^v N_{je}^v > 0$,

(a) and if $R_j^0 > 1$, there is one and only one positive equilibrium $E_3 = (E_{je}^c,\ I_{je}^c,\ R_{je}^c,\ N_{je}^c,\ E_{je}^v,\ I_{je}^v,\ R_{je}^v,\ N_{je}^v)$.

(b) and if $R_j^0 \le 1$, there is no positive equilibrium.

2. if $\Lambda_j^c - \delta_j\beta_j^c N_{je}^c < 0$ and $\Lambda_j^v - \delta_j\beta_j^v N_{je}^v < 0$, there is one and only one positive equilibrium E_3.

3. if $(\Lambda_j^c - \delta_j\beta_j^c N_{je}^c)(\Lambda_j^v - \delta_j\beta_j^v N_{je}^v) \le 0$, there is one and only one positive equilibrium E_3.
where $\Lambda_j^k = \frac{1}{\varepsilon}(\gamma_j^k\mu_j + \alpha_j^k + d_j^k)(d_j^k + \varepsilon), k \in \{c, v\}$ and $R_j^0 = \frac{\delta_j^2\beta_j^c\beta_j^v N_{je}^c N_{je}^v}{(\Lambda_j^c - \delta_j\beta_j^c N_{je}^c)(\Lambda_j^v - \delta_j\beta_j^v N_{je}^v)}$.

Proof of Theorem 3.1. (1) For an equilibrium in system of Eqs. (3.1), we can obtain

$$
\begin{aligned}
I_j^v(t) &= \frac{i}{\delta_j \beta_j^c}\left(\frac{(d_j^c+\varepsilon)E_j^c(t)}{N_{je}^c - E_j^c(t) - I_j^c(t) - R_j^c(t)} - \delta_j \beta_j^c I_j^c(t)\right) \\
E_j^v(t) &= \frac{1}{\varepsilon}\left(\gamma_j^v \mu_j + \alpha_j^v + d_j^v\right) I_j^v(t) \\
R_j^v(t) &= \frac{\gamma_j^v \mu_j}{d_j^v} I_j^v(t) \\
I_j^c(t) &= \frac{i}{\delta_j \beta_j^v}\left(\frac{(d_j^v+\varepsilon)E_j^v(t)}{N_{je}^v - E_j^v(t) - I_j^v(t) - R_j^v(t)} - \delta_j \beta_j^v I_j^v(t)\right) \\
E_j^c(t) &= \frac{1}{\varepsilon}\left(\gamma_j^c \mu_j + \alpha_j^c + d_j^c\right) I_j^c(t) \\
R_j^c(t) &= \frac{\gamma_j^c \mu_j}{d_j^c} I_j^c(t)
\end{aligned}
$$

Let $\Upsilon_j^c = 1 + \frac{1}{\varepsilon}\left(\gamma_j^c \mu_j + \alpha_j^c + d_j^c\right) + \frac{\gamma_j^c \mu_j}{d_j^c}$ and $\Upsilon_j^v = 1 + \frac{1}{\varepsilon}\left(\gamma_j^v \mu_j + \alpha_j^v + d_j^v\right) + \frac{\gamma_j^v \mu_j}{d_j^v}$. We can deduce

$$
\begin{cases}
I_j^c(t) = \frac{\left(\Lambda_j^v - \delta_j \beta_j^v N_{je}^v\right) I_j^v(t) + \Upsilon_j^v \delta_j \beta_j^v I_j^v(t)^2}{\delta_j \beta_j^v \left(N_{je}^v - \Upsilon_j^v I_j^v(t)\right)} &, \quad I_j^v(t) \in [0, \frac{N_{je}^v}{\Upsilon_j^v}) \\
I_j^v(t) = \frac{\left(\Lambda_j^c - \delta_j \beta_j^c N_{je}^c\right) I_j^c(t) + \Upsilon_j^c \delta_j \beta_j^c I_j^c(t)^2}{\delta_j \beta_j^c \left(N_{je}^c - \Upsilon_j^c I_j^c(t)\right)} &, \quad I_j^c(t) \in [0, \frac{N_{je}^c}{\Upsilon_j^c})
\end{cases}
$$

An equilibrium exists ⇔ curve $I_j^c(t)$ intersect with curve $I_j^v(t)$ in the first quadrant of the plane whose dimensions are $I_j^c(t)$ and $I_j^v(t)$.

With Eq. (3.1) we can obtain

$$
\begin{aligned}
\frac{dI_j^c(t)}{dI_j^v(t)} &= \frac{\left(\Lambda_j^v - \delta_j \beta_j^v N_{je}^v\right) N_{je}^v + \Upsilon_j^v \delta_j \beta_j^v I_j^v(t)\left(2N_{je}^v - \Upsilon_j^v I_j^v(t)\right)}{\delta_j \beta_j^v \left(N_{je}^v - \Upsilon_j^v I_j^v(t)\right)^2} \\
\frac{d^2 I_j^c(t)}{dI_j^v(t)^2} &= \frac{2\Upsilon_j^v \Lambda_j^v N_{je}^v}{\delta_j \beta_j^v \left(N_{je}^v - \Upsilon_j^v I_j^c(t)\right)^3} \\
\frac{dI_j^v(t)}{dI_j^c(t)} &= \frac{\left(\Lambda_j^c - \delta_j \beta_j^c N_{je}^c\right) N_{je}^c + \Upsilon_j^c \delta_j \beta_j^c I_j^c(t)\left(2N_{je}^c - \Upsilon_j^c I_j^c(t)\right)}{\delta_j \beta_j^c \left(N_{je}^c - \Upsilon_j^c I_j^c(t)\right)^2} \\
\frac{d^2 I_j^v(t)}{dI_j^c(t)^2} &= \frac{2\Upsilon_j^c \Lambda_j^c N_{je}^c}{\delta_j \beta_j^c \left(N_{je}^c - \Upsilon_j^c I_j^c(t)\right)^3}
\end{aligned}
$$

Since $\frac{dI_j^c(t)}{dI_j^v(t)} > 0$ and $\frac{d^2 I_j^c(t)}{dI_j^v(t)^2} > 0$, $I_j^c(t)$ is a monotonically increasing convex function when (1)$I_j^v(t) \in [0, \frac{N_j e^v}{\Upsilon_j^v})$, and (2) when $I_j^v(t) \to \frac{N_j e^v}{\Upsilon_j^v}$, $I_j^v(t) \to \infty$. Similarly, $I_j^v(t)$ is a monotonically increasing convex function if (1)$I_j^c(t) \in [0, \frac{N_j e^c}{\Upsilon_j^c})$, and (2) when $I_j^c(t) \to \frac{N_j e^c}{\Upsilon_j^c}$, $I_j^c(t) \to \infty$.

And since if $I_j^v(t)=0$, $\frac{dI_j^c(t)}{dI_j^v(t)} = \frac{\Lambda_j^v}{\delta_j \beta_j^v N_{je}^v} - 1$; if $I_j^c(t)=0$, $\frac{dI_j^v(t)}{dI_j^c(t)} = \frac{\Lambda_j^c}{\delta_j \beta_j^c N_{je}^c} - 1$, $R_j^0 > 1$, curve $I_j^c(t)$ intersects with curve $I_j^v(t)$ only once in the first quadrant. That is, there exists only one positive equilibrium E_3 in the system of Eq. (3.1). With $R_j^0 \le 1$, no equilibrium exists.

(2) From (1) we get E_3 exists $\Leftrightarrow$ curve $I_j^c(t)$ intersect with curve $I_j^v(t)$ in the first quadrant when $I_j^c(t) \in (0, \frac{\delta_j \beta_j^c N_{je}^c - \Lambda_j^c}{\Upsilon_j^c \delta_j \beta_j^c})$,$I_j^v(t) < 0$.

$I_j^v(t)$ is a monotonically increasing convex function if $I_j^c(t) \in [\frac{\delta_j \beta_j^c N_{je}^c - \Lambda_j^c}{\Upsilon_j^c \delta_j \beta_j^c}, \frac{N_{je}^c}{\Upsilon_j^c})$. And when $I_j^c(t) \to \frac{N_{je}^c}{\Upsilon_j^c}$, $I_j^v(t) \to \infty$.

Similarly, when $I_j^v(t) \in (0, \frac{\delta_j \beta_j^v N_{je}^v - \Lambda_j^v}{\Upsilon_j^v \delta_j \beta_j^v})$,$I_j^c(t) < 0$.

$I_j^c(t)$ is a monotonically increasing convex function if $I_j^v(t) \in [\frac{\delta_j \beta_j^v N_{je}^v - \Lambda_j^v}{\Upsilon_j^v \delta_j \beta_j^v}, \frac{N_{je}^v}{\Upsilon_j^v})$ and when $I_j^v(t) \to \frac{N_{je}^v}{\Upsilon_j^v}$, $I_j^c(t) \to \infty$.

Thus, curve $I_j^c(t)$ and $I_j^v(t)$ have one and only one intersection point $(I_j e^c, I_j e^v)$ in the first quadrant. That is, there is one and only one positive equilibrium E_3.

(3) In the similar way of (2), we can prove there is one and only one positive equilibrium E_3 if $(\Lambda_j^c - \delta_j \beta_j^c N_{je}^c)(\Lambda_j^v - \delta_j \beta_j^v N_{je}^v) \le 0$. □

The three cases described in Theorem 3.1 are mutually exclusive and collectively exhaustive. R_j^0 is the basic reproduction number of the epidemic in epidemic area j and represents the number of cases that one case generates on average over the course of its infectious period. This metric is useful because it helps determine whether an epidemic can spread through a population.

Instead α_j^c, α_j^v, β_j^c and β_j^v in Theorem 3.1 by $\bar{\alpha}_j^c$, $\bar{\alpha}_j^v$, $\bar{\beta}_j^c$ and $\bar{\beta}_j^v$ respectively, we can deduce there is no positive equilibrium under the Conditions 1(b), that is,

$$\begin{aligned} &\bar{\Lambda}_j^c - \delta_j \bar{\beta}_j^c N_{je}^c > 0 \\ &\bar{\Lambda}_j^v - \delta_j \bar{\beta}_j^v N_{je}^v > 0 \\ &\bar{R}_j^0 = \frac{\delta_j^2 \bar{\beta}_j^c \bar{\beta}_j^v N_{je}^c N_{je}^v}{(\bar{\Lambda}_j^c - \delta_j \bar{\beta}_j^c N_{je}^c)(\bar{\Lambda}_j^v - \delta_j \bar{\beta}_j^v N_{je}^v)} < 1 \end{aligned}$$

where $\bar{\Lambda} = \frac{1}{\varepsilon}(\gamma_j^k \mu_j + \bar{\alpha}_j^k + d_j^k)(d_j^k + \varepsilon), k \in \{c, v\}$.

Epidemic diseases die out in the long run if no positive equilibrium exists, and diseases can continue to spread in a population under other situations. Theorem 3.1 can facilitate emergency medical allocation by (1) setting no positive equilibrium as another constraint in the distribution model or by (2) checking Condition 1(b) after distribution decisions have been made by the models regardless of equilibrium. This work applies the theorem in the second manner. Equilibrium is disregarded in the distribution model (Eqs. (3.3) to (3.14)). The existence of a positive equilibrium is then determined. If equilibrium exists in some areas, then the government should adjust relevant policies (not logistics policies) to prevent diseases from spreading in the long run. An example of such adjustments involves investments in exogenous parameters, such as diagnosis rate and contact coefficient. Equilibrium is not set as a constraint because this behavior is less important than the transient behavior for a particular application to a sudden emergency related to a new strain of epidemic disease. Therefore, this work regards equilibrium condition as a consequence and not a constraint of emergency allocation.

Theorem 3.2. *Under the constraint of the given model, if $\alpha_M < \alpha_U$ or $\beta_M < \beta_U$ then the problem has no feasible solution, in which*

$$\alpha_U = \alpha^v_{jt} + \sum_{p=l+1}^{l+k} \left(\alpha^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q^p_{it}}{D^p_{jt}}\} \right), \forall j, t$$

$$\beta_U = \beta^v_{jt} + \sum_{p=1}^{l} \left(\beta^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q^p_{it}}{D^p_{jt}}\} \right), \forall j, t$$

Proof of Theorem 3.2.

Since $0 \leq x^p_{ijt} \leq \sum_{j=1}^{J} x^p_{ijt} \leq Q^p_{it}, \forall i, j, p, t$ and $0 \leq \sum_{i=1}^{I} x^p_{ijt} \leq \sum_{i=1}^{I} Q^p_{it}, \forall j, p, t$, we can obtain

$$\alpha^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q^p_{it}}{D^p_{jt}}\} \leq \alpha^p \frac{D^p_{jt} - \sum_{i=1}^{I} x^p_{ijt}}{D^p_{jt}}, \forall j, p, t$$

and

$$\alpha^v_{jt} + \sum_{p=l+1}^{l+k} \left(\alpha^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q^p_{it}}{D^p_{jt}}\} \right) \leq \alpha^v_{jt} + \sum_{p=l+1}^{l+k} \left(\alpha^p \frac{D^p_{jt} - \sum_{i=1}^{I} x^p_{ijt}}{D^p_{jt}} \right) = \bar{\alpha}^v_{jt}, \forall j, t$$

Therefore, if $\alpha_M < \alpha^v_{jt} + \sum_{p=l+1}^{l+k} \left(\alpha^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q^p_{it}}{D^p_{jt}}\} \right)$, then the problem has no feasible solution as constrain $\bar{\alpha}^v_{jt} \leq \alpha_M$ is insatiable.

In a similar manner, one can prove that if

$$\alpha_M < \alpha^c_{jt} + \sum_{p=l+1}^{l+k} \left(\alpha^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q^p_{it}}{D^p_{jt}}\} \right) \quad , \forall j, t$$

or

$$\beta_M < \beta_{jt}^v + \sum_{p=1}^{l} \left(\beta^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q_{it}^p}{D_{jt}^p}\} \right) \quad , \forall j, t$$

or

$$\beta_M < \beta_{jt}^c + \sum_{p=1}^{l} \left(\beta^p \max\{0, 1 - \frac{\sum_{i=1}^{I} Q_{it}^p}{D_{jt}^p}\} \right) \quad , \forall j, t$$

then the problem has no feasible solution. And since $\alpha_{jt}^v > \alpha_{jt}^c$ and $\beta_{jt}^v > \beta_{jt}^c$, Theorem 3.2 can be obtained. □

α_j^v refers to the disease mortality rate of vulnerable people in area j and α^p refers to the mortality increase caused by unit dissatisfaction of medical relief p ($p \in Mt$). So $\bar{\alpha}_{jt}^v = \alpha_j^v + \sum_{p=l+1}^{l+k} \alpha^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p}$ is the morality rate of vulnerable people in area j after medical delivery in time period t. Similarly, $\bar{\beta}_{jt}^v = \beta_j^v + \sum_{p=1}^{l} \beta^p \frac{D_{jt}^p - \sum_{i=1}^{I} x_{ijt}^p}{D_{jt}^p}$ is the infection rate of vulnerable people in area j after medical delivery in time period t.

Note that the models have constraints $\bar{\alpha}_{jt}^v, \bar{\alpha}_{jt}^c \leq \alpha_M$ and $\bar{\beta}_{jt}^v, \bar{\beta}_{jt}^c \leq \beta_M$ ($\forall j, t$). Theorem 3.2 gives the upper-bounds of α_M and β_M, that is, α_U and β_U. If $\alpha_M < \alpha_U$ or $\beta_M < \beta_U$, then the model has no feasible solution.

The objective function of the model involves effectiveness. Eqs. (3.12) and (3.13) are concerned with fairness, which is achieved by ensuring that the mortality and infection rates do not exceed α_M and β_M in all areas. A feasible solution cannot be obtained when extreme fairness ($\alpha_M < \alpha_U$ or $\beta_M < \beta_U$) is required. Therefore, we can balance effectiveness and fairness only to some extent.

3.5 Numerical Study

This section conducts a case study using real data and a continuation study with experimental data to demonstrate the applicability of the proposed methodology. The case study involves the SARS outbreak in China during the first quarter of 2003. All computational processes are conducted with MATLAB on a personal computer with a 2.53Hz CPU and 2G RAM. Following an introduction of the case background, the main procedures executed in this numerical study are the validation of forecasting model, testing with real data, and experimental testing.

3.5.1 Case Background

The SARS outbreak originated in Guangdong Province, China, in November 2002. The epidemic spread across almost the entire country. When the severity of this public health emergency was determined, the Chinese central government responded. The government reported 305 cases and 5 deaths to the World Health Organization on 10th February 2003.

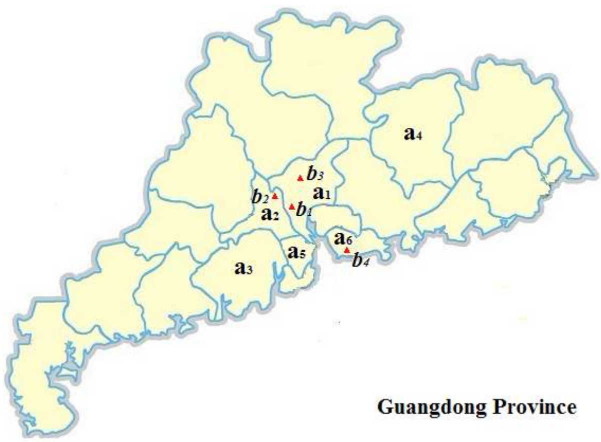
The numerical study focuses on Guangdong. It is also known as Canton or Kwangtung Province and is located on the South China Sea coast. Guangdong is among the most populous provinces in China and registers 79.1 million permanent residents, as well as 31 million migrants who lived in the province for at least six months of the year. These numbers account for 7.79% of Mainland China's population. Given the epidemic-related information and the basic state of the distribution network, the proposed methodology is used to forecast the trend of the spread of SARS and to make medical logistics decisions for Guangdong. The unit interval is one day.

Cases of SARS infection were reported in the following prefectures in Guangdong: Guangzhou, Foshan, Jiangmen, Heyuan, Zhongshan, and Shenzhen. The province has four EMRCs. Fig. 3.3 shows the simplified geographical relationships among these affected areas and among the EMRCs. Three types of treatment and two types of prophylactic reliefs were provided as urgently needed. The four EMRCs concentrated their supplies of reliefs and distributed them to the six epidemic areas. The case background indicates that a simplified 546 medical logistics network is formed (5 types of medical reliefs, 4 EMRCs, and 6 epidemic areas). People older than 45 are regarded as the vulnerable ones in this case.

To elucidate the methodology, this work explains the decisions made on 10th February 2003 in detail as an example.

3.5.2 Testing of the Forecasting Model

Demand is forecasted using Eqs. (3.2) and (3.17). These two forecasting models are compared with moving average method and standard SEIR model, which are presented in Appendix A.



$a_1 : Guangzhou \quad a_2 : Foshan \quad a_3 : Jiangmen \quad a_4 : Heyuan$

$a_5 : Zhongshan \quad a_6 : Shenzhen \quad b_1, b_2, b_3, b_4 : EMRCs$

Figure 3.3: Study areas

This study uses Eqs. (3.1) and (3.16) to compare the numbers of corresponding demanders instead of comparing demands because demands for both types of prophylactic reliefs are positively correlated with the numbers of susceptible, exposed, and undiagnosed people. Moreover, demands for all three types of treatment reliefs are correlated with the number of infectious and diagnosed individuals.

The parameters are presented in Section2.7, and they are set as follows: (1) the parameters for population are set according to the National Bureau of Statistics of P. R. China (2004). (2) The parameters for disease (recovery rate, diagnosed rate, mortality, and incubation period) are set. Doctors and medical experts can generally estimate the expected values of these parameters shortly after a new disease is recognized in a region. In this study, they are obtained from several medical reports and statistics (National Health and Family Planning Commission of P. R. China 2004; Chinese Center for Disease Control and Prevention; Kamps and Hoffmann 2003). (3) Infection rates are set with two methods. First, these rates can be obtained by fitting the forecasting curve with observations given epidemic areas with adequate infectious cases. In this study, the infection rates in Guangzhou and Zhongshan are set in this manner. The objective of fitting is to minimize $\sqrt{\sum_{k=1}^{N}(obs_k - est_k)^2} / \left(\sqrt{\sum_{k=1}^{N}(obs_k)^2} + \sqrt{\sum_{k=1}^{N}(est_k)^2} \right)$, where obs_k refers to the kth observed value and est_k refers to the kth estimated value.

Given epidemic areas with a few infectious cases, these rates can also be set by comparing specific areas with the areas listed above in terms of previous cases of similar epidemic outbreak. In this study, infection rates in the other four areas (j=2, 3, 4, 6) are set in this manner. For example, the infection rates of influenza and pneumonia in Guangzhou (j=1) is 0.05 lower than those in Foshan (j=2). Thus $\beta_2^c = \beta_1^c + 0.05$ and $\beta_2^v = \beta_1^v + 0.05$.

The following table compares demand forecasting in Guangzhou (j=1) as an example. Although more accurate parameters can be obtained at present, this study attempts to recreate a specific situation in which a new, little-know epidemic outbreak is reported. The original point of forecasting is 10th February.

Table 3.1: Comparison of forecasting methods for prophylactic demanders

Forecasting Methods	Average Forecasting Error		
	in one week	in two weeks	in one month
Basic Model	0.01%	0.03%	0.06%
Extended Model 1	0.01%	0.03%	0.06%
Standard SEIR model	0.02%	0.03%	0.05%
Moving average method	0.02%	0.03%	0.05%

Table 3.2: Comparison of forecasting methods for treatment demanders

Forecasting Methods	Average Forecasting Error		
	in one week	in two weeks	in one month
Basic Model	3.55%	3.99%	45.84%
Extended Model 1	3.54%	3.96%	44.80%
Standard SEIR model	12.02%	13.75%	43.07%
Moving average method	11.61%	8.17%	13.10%

Tables 3.1 shows that all four models are valid in terms of forecasting prophylactic demanders. In combination with Fig. 3.4,Table 3.2 illustrates that the performances of these models differ with respect to forecasting the numbers of treatment demanders. The methods proposed in this chapter perform better than standard SEIR model and moving average method in the initial two weeks, but then the error rate increases rapidly. The main reason for this trend is the development of preventive and therapeutic methods as doctors amass additional knowledge about SARS. Nonetheless,

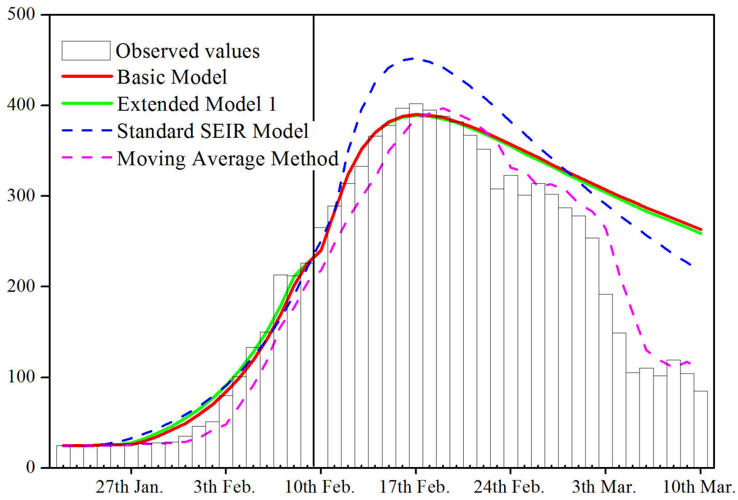


Figure 3.4: Forecasting of treatment demanders

accurate forecasting for two weeks is enough to facilitate distribution decision-making. In addition, the models proposed in this chapter can forecast every group in the population, unlike the moving average method. This advantage is useful in distributional decision-making.

3.5.3 Testing with Real Data

In this section, the three models proposed in Sections 3.2 are solved and their results compared. This study summarizes and compares the numerical analysis results for these three models. The parameters and statistics are reported in Section 2.7.

Table 3.3 shows the main indices of the three models. Extended forecasting method (proposed in Section 3.2.3) is adopted in Extended Model 2. The psychological fragility determined with the basic model and with Extended Model 1 are also calculated for comparison. The computation times for solving these three models are 1.94, 2.48 and 2.53 seconds, respectively.

The horizontal axis in Fig. 3.5 shows four EMRCs, whereas the vertical axis indicate epidemic areas. Circles with different colors correspond to the five types of reliefs, and the colored areas refer to the corresponding distributed amounts (specific numbers are reported in the supplement).

The following observations are made based on the results presented above.

Table 3.3: Numerical results of testing with real data

	Basic Model	Extended Model 1	Extended Model 2
average mortality rate*	0.497	0.497	0.091
average infection rate*	0.573	0.572	0.571
total amount of prophylactic reliefs	436994	436980	436405
total amount of treatment reliefs	181	195	770
cost	443342	520878	461523
physical fragility	3869667	3862785	3881985
psychological fragility	2269299	2291386	1172696

**Average mortality rate is the average of mortality rates of all areas after distribution, that is, the average of* $\bar{\alpha}_j^k$ *(j=1, 2, ..., 6; k=c, v); and average infection rate is the average of infection rates of all areas after distribution, that is, the average of* $\bar{\beta}_j^k$ *(j=1, 2, ..., 6; k=c, v).*

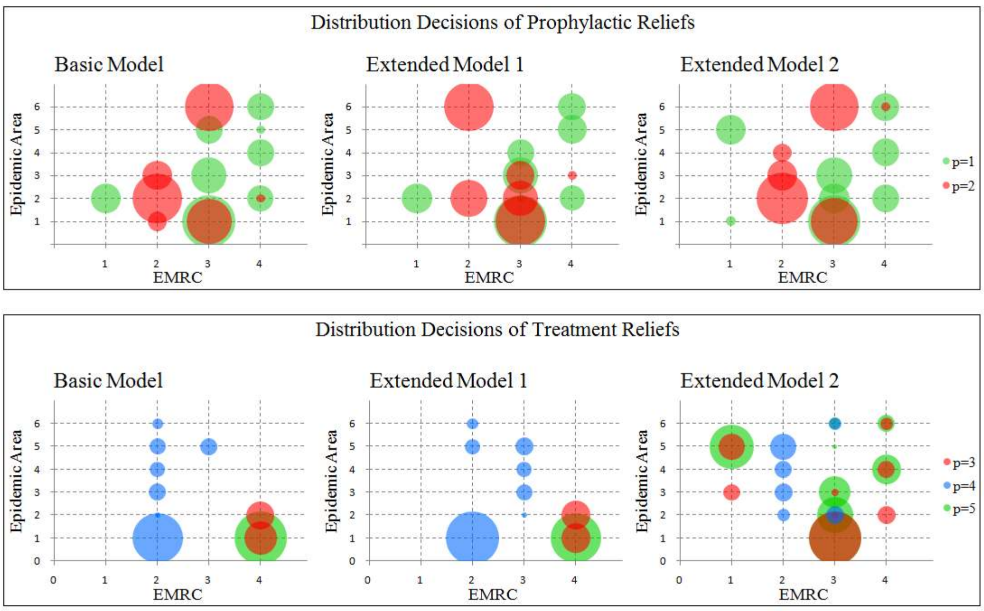


Figure 3.5: Comparison of three models

(1) The amounts of prophylactic reliefs distributed to specific epidemic areas differ when the spatial interaction relationships among epidemic areas are considered. As a result, the average infection rate decrease slightly although the total distributed amount does not increase.

(2) The total amount of treatment reliefs distributed increases when survivor psychology is considered. In the process, psychological fragility declines significantly.

(3) The amount of reliefs distributed increase for epidemic areas with a large input population that originates from severely infected areas when the spatial interaction relationships among epidemic areas are considered. In this case, additional prophylactic reliefs are sent to Shenzhen (j=6) to avoid the spread of SARS. The reason for this outcome is that this area has a large input population coming from Guangzhou (j=1) and Zhongshan (j=5). These areas are the most severely. Moreover, the number of residents in Shenzhen (j=6) is small. Thus Guangzhou (j=1) is less affected although the input from Zhongshan (j=5) to this area (j=1) is larger given that Guangzhou (j=1) has much more residents.

(4) The amount of prophylactic reliefs distributed increases for epidemic areas with low diagnosis rates when survivor psychology is considered. In this case, the diagnosis rate of Heyuan (j=4) is the lowest and many infectious people have not realized that they are infected. Infec-tious but undiagnosed individuals share a physical state with those who have been diagnosed, but their psychological characteristics differ. Therefore, more prophylactic relief is sent to Heyuan (j=4) in Extended Model 2 than in Extended Model 1.

(5) The number of distribution paths that report long transportation times decreases when survivor psychology is considered. A long distribution time increases psychological fragility. In this case, the distribution paths from EMRC 4 to Guangzhou (j=1) and to Jiangmen (j=3) are avoided in Extended Model 2, unlike in the other two models. In a case of emergency medical logistics following a large-scale disaster, the effect of survivor psychology on path selection is strengthened because destroyed infrastructure may increase transportation time.

3.5.4 Experimental Testing

Five observations are made based on the analysis presented with actual SARS data. An analysis using experimental data is conducted to test these observations in different epidemic situations.

We test four typical types of situations as follows:

Situation 1: not easily spread and not especially lethal (small α and small β);

Situation 2: easily spread but not especially lethal (small α and large β);

Situation 3: not easily spread but especially lethal (large α and small β);

Situation 4: easily spread and especially lethal (large α and large β).

The effects of α and β alone are reported and discussed in this section to induce brief observations given the similar effects of δ and β, as well as the inverse effects of α and γ.

The other parameters are similar to those in the last section. We make distribution decisions in the four situations as in the analysis with real data. The main indices and specific decisions are summarized in Appendix A.

The following four observations are made based on the results of experimental testing:

(1) All four situations exhibit results similar to those in the first and second observations made by analyzing real data. That is, the amounts of prophylactic reliefs distributed to specific epidemic areas differ when the spatial interaction relationships among epidemic areas are considered, although the total amount does not increase. Moreover, the total amount of treatment reliefs distributed increases when survivor psychology is considered, thus reducing psychological fragility considerably.

(2) All four situations show the results similar to those in the third observation made by analyzing real data. That is, more prophylactic reliefs are distributed to epidemic areas with large input populations that originate from severely infected areas when the spatial interaction relationships among such areas are considered.

(3) Situations 2 and 4 display results similar to those in the fourth observation made by analyzing real data. More prophylactic reliefs are sent to areas with low diagnosis rates when survivor psychology is considered. However, Situations 1 and 3 fail to

provide strong evidence for these changes. The effect of the variation in prophylactic relief on physical and psychological fragility is insignificant in these situations given the low infection rate.

(4) In all four experimental situations, distribution paths that report long transportation times are avoided when survivor psychology is considered, as in the analysis with real data.

3.6 Discussion

Academic literature and popular press both indicate that the decision-makers for public health emergencies have long been advised to control physical factors, such as mortality and infection rates, and economic factors, such as transportation cost. This study underscores the importance of survivor psychology in such emergencies and provides suggestions on efficient, effective, and fair medical rescue.

This work highlights the necessity and feasibility of reducing the psychological fragility of affected people during emergency medical logistics. Aside from inducing health threats and economic losses, public health emergencies also have negative psychological effects on both patients and healthy people. Affected people may feel helplessness, sorrow, panic, anxiety, and fear. Such individuals may even develop mental illnesses. Widespread serious psychological problems may also threaten economic order and public security. Numerical results show that the consideration of survivor psychology significantly reduces the psychological fragility of affected people and that it hardly affects the physical fragility. Aside from highlighting survivor psychology, this work also indicates the benefits of considering the spatial interaction relationships among epidemic areas, especially in response to public health emergencies with high infection rates in areas with high population density.

In addition, the specific effects of incubation period and diagnosis rate are emphasized in the discussion of the relationship between logistics and survivor psychology. Additional treatment reliefs generally help improve psychological states. Conversely, prophylactic reliefs are significant to epidemics with long latent periods. Such reliefs do not exert physical effects on exposed people, and such individuals cannot infect others. However, psychological effects remain. Similarly, the psychological suffering of infectious but undiagnosed people is affected by prophylactic rather than treatment reliefs. Therefore, additional prophylactic but not treatment reliefs are sent to areas with low diagnosis rates. The increase in diagnosis rate lowers the demand for prophylactic reliefs and the corresponding logistics capacity.

Overall, this work provides managerial insights to improve decisions made on medical distribution as per demand forecasting for quick response to public health emergencies. These insights also enhance the physical and psychological status of affected individuals.

3.7 Summary

This chapter presents a novel model of humanitarian medical allocation for rapid response to public health emergencies. The proposed model consists of two mechanisms: (1) medical demand forecasting and (2) relief distribution. The medical demand associated with each epidemic area is forecast using a modified SEIR model. This process is followed by a linear programming approach to making distribution decisions. On the basis of a basic model that applies the proposed method, two extended models are generated by assessing (1) the spatial interaction relationships among epidemic areas and (2) survivor psychology.

A numerical study conducted on a real SARS outbreak in China demonstrates the applicability of the proposed method. The numerical results of the three models are compared to identify the advantages of each model. The psychological status of affected people improves significantly when survivor psychology is considered. Four experimental situations are tested to support and supplement the analysis with real data. Managerial insights are also provided.

The performance of emergency medical logistics may be improved significantly. The first mechanism of the proposed model is used in demand forecasting and supports the distribution mechanism in which the government is the only real decision-maker. Nevertheless, the forecast demand information can be shared with non-governmental organizations and local charities for medical logistics coordination. This issue is worthy of further research. Chapter 4 provides a preliminary discussion of cooperation and information sharing between the government decision-maker and private sectors.

3.8 Supplement for this Chapter

3.8.1 Parameters in the Numerical Study

See Tables 3.4 to 3.10.

Table 3.4: Parameters for epidemic and population

	j=1	j=2	j=3	j=4	j=5	j=6
A_j^c	504	229	114	80	92	556
A_j^v	80	16	-8	-1	6	84
β_j^c	0.4	0.45	0.46	0.5	0.48	0.4
β_j^v	0.5	0.55	0.56	0.6	0.58	0.55
δ_j	1.02×10^{-8}	1.14×10^{-8}	0.33×10^{-8}	0.13×10^{-8}	1.05×10^{-8}	3.11×10^{-8}
ε	0.23	0.23	0.23	0.23	0.23	0.23
γ_j^c	0.1	0.13	0.1	0.08	0.1	0.1
γ_j^v	0.05	0.08	0.05	0.03	0.05	0.05
α_j^c	0.003	0.003	0.006	0.01	0.008	0.005
α_j^v	0.005	0.005	0.008	0.015	0.01	0.007
μ_j^c	0.8	0.7	0.6	0.5	0.7	0.6
d_j^c	1.00×10^{-5}	0.996×10^{-5}	1.03×10^{-5}	1.01×10^{-5}	0.991×10^{-5}	1.00×10^{-5}
d_j^v	3.00×10^{-5}	2.99×10^{-5}	3.00×10^{-5}	2.98×10^{-5}	2.99×10^{-5}	3.01×10^{-5}
$S_j^c(0)$	6875080	4174889	2738309	1657264	1739502	5898274
$S_j^v(0)$	2971785	1391632	1288616	823680	643378	1567896
$E_j^c(0)$	316	41	22	29	50	10
$E_j^v(0)$	136	20	10	15	15	3
$I_j^c(0)$	198	12	17	15	29	8
$I_j^v(0)$	85	7	8	7	11	2

Table 3.5: Parameters for medical reliefs

	j=1	j=2	j=3	j=4	j=5	j=6
c^1_{1jt}	0.04	0.24	0.94	2.07	0.91	1.63
c^1_{2jt}	0.23	0.11	0.78	2.23	0.74	1.91
c^1_{1jt}	0.10	0.28	1.03	2.05	1.04	1.73
c^1_{1jt}	1.63	1.91	2.63	1.70	2.54	0.06
c^2_{1jt}	0.08	0.48	1.88	4.14	1.82	3.26
c^2_{2jt}	0.46	0.22	1.56	4.46	1.48	3.82
c^2_{3jt}	0.20	0.56	2.06	4.10	2.08	3.46
c^2_{4jt}	3.26	3.82	5.26	3.40	5.08	0.12
c^3_{1jt}	0.04	0.24	0.94	2.07	0.91	1.63
c^3_{2jt}	0.23	0.11	0.78	2.23	0.74	1.91
c^3_{3jt}	0.10	0.28	1.03	2.05	1.04	1.73
c^3_{4jt}	1.63	1.91	2.63	1.70	2.54	0.06
c^4_{1jt}	0.12	0.72	2.82	6.21	2.73	4.89
c^4_{2jt}	0.69	0.33	2.34	6.69	2.22	5.73
c^4_{3jt}	0.30	0.84	3.09	6.15	3.12	5.19
c^4_{4jt}	4.89	5.73	7.89	5.10	7.62	0.18
c^5_{1jt}	0.04	0.24	0.94	2.07	0.91	1.63
c^5_{2jt}	0.23	0.11	0.78	2.23	0.74	1.91
c^5_{3jt}	0.10	0.28	1.03	2.05	1.04	1.73
c^5_{4jt}	1.63	1.91	2.63	1.70	2.54	0.06

Table 3.6: Parameters for medical inventory

	j=1	j=2	j=3	j=4	j=5	j=6
Inv^1_t	24000	12000	6000	5000	1500	3000
Inv^2_t	131000	41000	57000	10000	200	85000
Inv^3_t	21	5	0	0	0	2
Inv^4_t	20	0	0	0	0	0
Inv^5_t	300	8	0	0	6	2

Table 3.7: Parameters for medical reliefs

	p=1	p=2	p=3	p=4	p=5
θ^p	0.01	0.02	1	1	5
α^p			0.1	0.3	0.2
β^p	0.3	0.1			

Table 3.8: Parameters for EMRCs

	Q^1_{it}	Q^2_{it}	Q^3_{it}	Q^4_{it}	Q^5_{it}	W_{it}
i=1	25000	0	50	0	100	25000
i=2	0	100000	0	75	0	110000
i=3	130000	175000	150	25	270	300000
i=4	60000	2000	40	0	60	100000

Table 3.9: Migration matrix for Extended Model 1

	$j_2 = 1$	$j_2 = 2$	$j_2 = 3$	$j_2 = 4$	$j_2 = 5$	$j_2 = 6$
$j_1 = 1$	0	4.57	9.04	5.27	2.88	13.75
$j_1 = 2$	6.00	0	2.94	1.71	0.93	4.46
$j_1 = 3$	44.82	11.14	0	12.84	7.01	33.51
$j_1 = 4$	49.57	12.32	2.44	0	7.76	37.06
$j_1 = 5$	14.92	3.71	7.34	4.28	0	11.16
$j_1 = 6$	0.93	0.23	0.46	0.27	0.15	0

**The unit is* 1×10^{-7}.

Table 3.10: Additional parameters for Extended Model 2

T_{ijt}	j=1	j=2	j=3	j=4	j=5	j=6
i=1	0.25	0.40	1.57	2.78	1.52	2.05
i=2	0.88	0.52	1.30	3.38	1.23	3.18
i=3	0.67	0.47	1.72	3.42	4.13	2.08
i=4	2.72	2.35	4.38	2.83	4.23	0.43
$T_M = 2$						

3.8.2 Two Previous Forecasting Methods

Standard SEIR model

$$\begin{cases} \frac{dS_j(t)}{dt} = A_j - d_j S_j(t) - \beta_j \delta_j S_j(t) I_j(t) \\ \frac{dE_j(t)}{dt} = \beta_j \delta_j S_j(t) I_j(t) - d_j E_j(t) - \varepsilon E_j(t) \\ \frac{dI_j(t)}{dt} = \varepsilon E_j(t) - d_j I_j(t) - \alpha_j I_j(t) - \gamma_j I_j(t) \\ \frac{dR_j(t)}{dt} = \gamma_j I_j(t) - d_j R_j(t) \end{cases}$$

The number of prophylactic demanders in time period t is $\int_{t_0}^{t_0+L} [S_j(t) + E_j(t)]\, dt$ and the number of treatment demanders in time period t is $\int_{t_0}^{t_0+L} I_j(t)\, dt$, where t_0 refers to the beginning moment of period t and L is the length of a time period.

Parameters are set as follows (j=1): $A_1 = 584$, $d_1 = 0.00002$, $\alpha_1 = 0.0045$, $\beta_1 = 0.4$, $\delta_1 = 1.02 \times 10^{-8}$, $\varepsilon = 0.23$, $\gamma_1 = 0.08$, $S_1(0) = 9847978$, $E_1(0) = 452$, $I_1(0) = 283$, $R_1(0) = 0$.

Moving Average Model

$$N_{jt} = 0.7N_{jt-1} + 0.2N_{jt-2} + 0.1N_{jt-3}$$

where N_{jt} refers to the number of demanders in area j in time period t.

3.8.3 Results of the Numerical Study

See the following tables. The unit of the solutions is 1×10^{-7}.

Table 3.11: Optimal solutions of the basic model with real data

	j=1	j=2	j=3	j=4	j=5	j=6
i=1			m_1:25000			
i=2	m_2:10285 m_4:54.72	m_2:66175 m_4:0.5867	m_2:23540 m_4:6.403	m_4:4.983	m_4:5.867	m_4:2.567
i=3	m_1:75475 m_2:55665		m_1:34270		m_1:20225 m_4:6.259	m_2:64329
i=4	m_3:23.00 m_5:60.00	m_1:18666 m_2:2000 m_3:17.00		m_1:19810	m_1:2075	m_1:19449

Table 3.12: Optimal solutions of Extended Model 1 with real data

	j=1	j=2	j=3	j=4	j=5	j=6
i=1		m_1:25000				
i=2	m_4:66.20	m_2:35586			m_4:5.871	m_2:64414 m_4:2.933
i=3	m_1:74480 m_2:65960	m_1:1486 m_2:32533 m_4:0.6667	m_1:34244 m_2:21487 m_4:6.403	m_1:19790 m_4:5.750	m_4:7.569	
i=4	m_3:20.00 m_5:60.00	m_1:17185 m_3:20.00	m_2:2000		m_1:22328	m_1:20487

Table 3.13: Optimal solutions of Extended Model 2 with real data

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:2522		m_3:14.00		m_1:22328	
					m_3:36.00	
					m_5:100	
i=2		m_2:67694	m_2:23487	m_2:8819	m_4:35.00	
		m_4:8.000	m_4:17.00	m_4:15.00		
i=3	m_1:71958	m_1:23798	m_1:34244		m_5:1.000	m_2:62414
	m_2:57141	m_3:3.000	m_3:3.000			m_4:8.000
	m_3:144.0	m_4:17.00	m_5:52.00			m_5:7.000
	m_5:144.0	m_5:66.00				
i=4		m_1:19873		m_1:19790		m_1:20337
		m_3:17.00		m_3:15.00		m_2:2000
				m_5:45.00		m_3:8.000
						m_5:15.00

Table 3.14: Optimal solutions of the basic model in Situation 1

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:25000					
i=2	m_2:31821	m_2:68179	m_4:4.524	m_4:4.767	m_4:11.41	m_4:2.450
	m_4:51.63					
i=3	m_1:32254	m_1:43666	m_1:34270	m_1:19810		m_2:62329
	m_2:34129		m_2:23540			
			m_4:1.596			
i=4	m_1:18221	m_3:17.00			m_1:22330	m_1:19449
	m_3:23.00					m_2:2000
	m_5:60.00					

Table 3.15: Optimal solutions of Extended Model 1 in Situation 1

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:25000					
i=2	m_2:35586 m_4:62.69				m_4:9.511	m_2:64414 m_4:2.800
i=3	m_1:33154 m_2:30381	m_2:68114	m_1:34245 m_2:21489 m_4:6.480	m_1:19790 m_4:5.500	m_1:22328 m_4:3.329	m_1:20483
i=4	m_1:16329 m_3:20.00 m_5:60.00	m_1:43671 m_3:20.00	m_2:2000			

Table 3.16: Optimal solutions of Extended Model 2 in Situation 1

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:2522 m_3:11.00	m_3:3.000			m_1:22328 m_3:36.00 m_5:100.0	
i=2	m_2:65967	m_2:34033 m_4:8.000	m_4:18.00	m_4:15.00	m_4:34.00	
i=3	m_1:71961 m_3:132.0 m_5:142.0	m_1:23794 m_2:33652 m_4:17.00 m_5:66.00	m_1:34245 m_2:23489 m_3:18.00 m_5:54.00		m_5:1.000	m_2:62414 m_4:8.000 m_5:7.000
i=4		m_1:19877 m_3:17.00		m_1:19790 m_3:15.00 m_5:45.00		m_1:20333 m_2:2000 m_3:8.000 m_5:15.00

Table 3.17: Optimal solutions of the basic model in Situation 2

	j=1	j=2	j=3	j=4	j=5	j=6
i=1		m_1:5190		m_1:19810		
i=2	m_2:12131 m_4:52.92		m_2:23540 m_4:4.229	m_4:4.767	m_4:11.41	m_2:64329 m_4:2.450
i=3	m_1:74475 m_2:51820	m_1:33195 m_2:68178	m_4:1.891		m_1:22330	
i=4	m_2:2000 m_3:23.00 m_5:60.00	m_1:5282 m_3:17.00	m_1:34270			m_1:20448

Table 3.18: Optimal solutions of Extended Model 1 in Situation 2

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:25000					
i=2	m_2:35586 m_4:47.13		m_4:6.480	m_4:5.500	m_4:12.84	m_2:64414 m_4:2.800
i=3	m_1:32294 m_2:28374 m_4:15.93	m_1:43671 m_2:68121	m_1:34245 m_2:23489	m_1:19790		
i=4	m_1:17186 m_2:2000 m_3:20.00 m_5:60.00	m_3:20.00			m_1:22328	m_1:20486

Table 3.19: Optimal solutions of Extended Model 2 in Situation 2

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:2522 m_3:11.00	m_3:3.000			m_1:22328 m_3:36.00 m_5:100.0	
i=2	m_2:30417	m_2:67692 m_4:8.000	m_4:18.00	m_2:1891 m_4:15.00	m_4:34.00	
i=3	m_1:71958 m_2:33652 m_3:132.0 m_5:142.0	m_1:23797 m_4:17.00 m_5:66.00	m_1:34245 m_2:23489 m_3:18.00 m_5:54.00		m_5:1.000	m_2:62414 m_4:8.000 m_5:7.000
i=4		m_1:19874 m_3:17.00		m_1:19790 m_3:15.00 m_5:45.00		m_1:20336 m_2:2000 m_3:8.000 m_5:15.00

Table 3.20: Optimal solutions of the basic model in Situation 3

	j=1	j=2	j=3	j=4	j=5	j=6
i=1		m_1:25000				
i=2	m_2:35671 m_4:31.55	m_4:11.20			m_4:27.20	m_2:64329 m_4:5.050
i=3	m_1:74475 m_2:30280 m_3:103.6 m_5:270.0	m_1:15266 m_2:67781	m_2:21540 m_4:13.65	m_1:19810 m_4:11.00		m_1:20449
i=4	m_3:25.00 m_5:60.00	m_1:3400 m_3:15.00	m_1:34270 m_2:2000		m_1:22330	

Table 3.21: Optimal solutions of Extended Model 1 in Situation 3

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:25000					
i=2	m_2:12097 m_4:29.16	m_4:1.320	m_2:23489	m_4:11.92	m_4:27.20	m_2:64414 m_4:5.400
i=3	m_1:49480 m_2:53863 m_3:177.0 m_5:270.0	m_1:3461 m_3:16.00 m_4:10.44	m_1:34245 m_4:14.56		m_1:22328	m_1:20486
i=4	m_3:40.00 m_5:60.00	m_1:40210 m_2:2000		m_1:19790		

Table 3.22: Optimal solutions of Extended Model 2 in Situation 3

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:2522 m_3:50.00 m_5:75.72		m_5:24.28		m_1:22328	
i=2	m_2:32308 m_4:16.80	m_2:67692 m_4:18.00		m_4:13.00	m_4:27.20	
i=3	m_1:71958 m_2:33652 m_3:134.0 m_5:259.0	m_1:23797 m_4:3.000 m_5:11.00	m_1:34245 m_2:23489 m_3:16.00 m_4:16.00			m_2:62414 m_4:6.000
i=4	m_3:5.000	m_1:19874 m_3:16.00 m_5:44.00		m_1:19790 m_3:13.00		m_1:20336 m_2:2000 m_3:6.000 m_5:16.00

Table 3.23: Optimal solutions of the basic model in Situation 4

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:25000					
i=2	m_2:65951	m_2:10509	m_2:23540		m_4:27.20	m_4:5.05
	m_4:31.55	m_4:11.20				
i=3	m_1:29734	m_1:43667	m_1:34269	m_4:11.00	m_1:22330	m_2:64329
	m_3:89.59	m_2:55272	m_4:13.65			
	m_5:270.0	m_3:15.00				
i=4	m_1:19741	m_2:2000		m_1:19810		m_1:20449
	m_3:40.00					
	m_5:60.00					

Table 3.24: Optimal solutions of Extended Model 1 in Situation 4

	j=1	j=2	j=3	j=4	j=5	j=6
i=1		m_1:25000				
i=2	m_2:65960	m_4:1.320		m_4:11.92	m_4:27.20	m_2:34040
	m_4:29.16					m_4:5.400
i=3	m_1:74480	m_1:18671	m_1:34245		m_1:2604	m_2:28374
	m_3:136.7	m_2:67706	m_2:23489			
	m_5:270.0	m_4:10.44	m_4:14.56			
i=4	m_3:24.00	m_3:16.00		m_1:19790	m_1:19724	m_1:20486
	m_5:60.00					m_2:2000

Table 3.25: Optimal solutions of Extended Model 2 in Situation 4

	j=1	j=2	j=3	j=4	j=5	j=6
i=1	m_1:2522 m_3:34.00 m_5:67.33	m_5:11.00	m_3:16.00 m_5:21.67		m_1:23328	
i=2	m_2:65153 m_4:16.80	m_2:10551 m_4:18.00	m_2:23489	m_2:1807 m_4:13.00	m_4:27.20	
i=3	m_1:71958 m_3:150.0 m_5:270.0	m_1:23797 m_2:57141 m_4:3.000	m_1:34245 m_4:16.00			m_2:62414 m_4:6.000
i=4	m_3:5.000	m_1:19874 m_3:16.00 m_5:44.00		m_1:19790 m_3:13.00		m_1:20336 m_2:2000 m_3:6.000 m_5:16.00

Table 3.26: Numerical results of experimental testing in Situation 1

	Basic Model	Extended Model 1	Extended Model 2
average mortality rate	0.498	0.498	0.089
average infection rate	0.373	0.372	0.372
total amount of prophylactic reliefs	436998	436984	436455
total amount of treatment reliefs	177	191	770
cost	481090	610594	464306
physical fragility	2425069	2418231	2420013
psychological fragility	2297467	2323959	1149958

Table 3.27: Numerical results of experimental testing in Situation 2

	Basic Model	Extended Model 1	Extended Model 2
average mortality rate	0.498	0.498	0.089
average infection rate	0.772	0.772	0.772
total amount of prophylactic reliefs	436998	436984	436405
total amount of treatment reliefs	177	191	770
cost	527057	539718	451672
physical fragility	5307370	5307397	5312915
psychological fragility	2171765	2314800	1156000

Table 3.28: Numerical results of experimental testing in Situation 3

	Basic Model	Extended Model 1	Extended Model 2
average mortality rate	0.472	0.472	0.303
average infection rate	0.372	0.372	0.372
total amount of prophylactic reliefs	436601	436572	436405
total amount of treatment reliefs	574	603	770
cost	767627	555226	443918
physical fragility	2418860	2418920	2419981
psychological fragility	1942086	1776862	1151773

Table 3.29: Numerical results of experimental testing in Situation 4

	Basic Model	Extended Model 1	Extended Model 2
average mortality rate	0.472	0.472	0.304
average infection rate	0.772	0.772	0.772
total amount of prophylactic reliefs	436601	436569	436405
total amount of treatment reliefs	574	606	770
cost	472774	487172	467576
physical fragility	5308059	5308123	5312750
psychological fragility	1759643	1742787	1161607

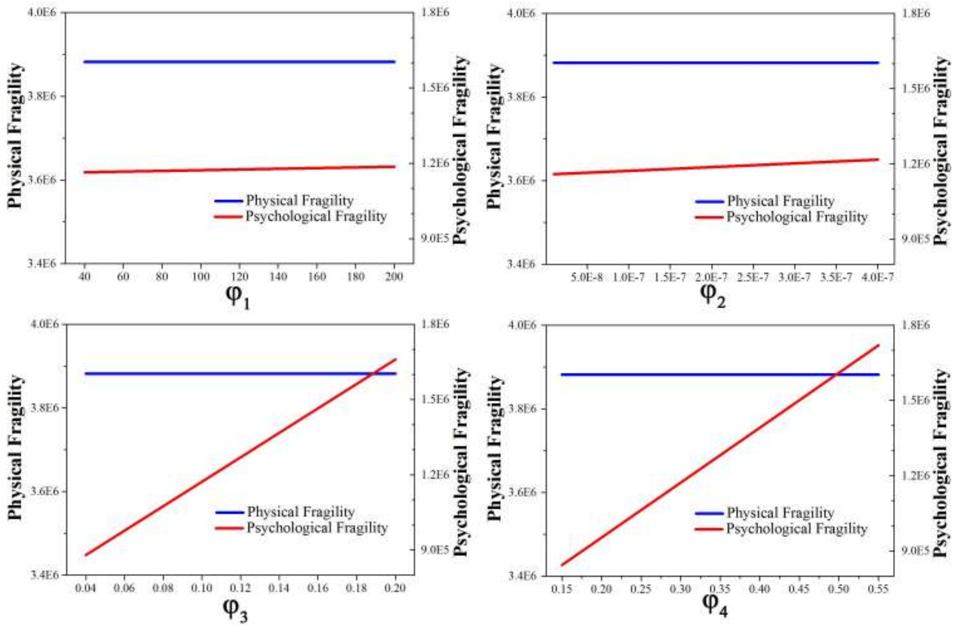


Figure 3.6: Sensitive analysis of suffering coefficients with real data

Chapter 4

Cross-sector Cooperation and Information Sharing in Humanitarian Medical Allocation

This chapter presents a cross-sector decision methodology to achieve efficient and effective humanitarian logistics of emergency reliefs, where a public sector (e.g., the government) and a private sector are involved. Optimization models of four mechanisms are developed: non-cooperation, semi-cooperation with a government leader, semi-cooperation with a private leader and full cooperation. Optimal solutions are provided to discuss the differences among these four models and numerical studies are conducted. The results illustrate that full cooperation is not always the best choice, while semi-cooperation with information sharing would also achieve potential advantages, even if two sectors made their own decisions separately.

4.1 Introduction

Humanitarian medical logistics is a branch of logistics problems which specializes in organizing the allocation and delivery of relief during natural disasters or complex emergencies to affected areas and people. Typically, humanitarian medical logistics engages a large number and variety of sectors, including central governments, local governments, the military, international organizations and private companies. In Chapter 3, the decision system with both central and local governments has been discussed. This chapter will focus on the relation between local governments and private sectors.

Modern information technologies provide potential opportunities to share information among different sectors, and to work together to pursue effective and efficient relief operations. However, each of these sectors may have different missions and capacity, which contributes to cooperation difficulties. Some of the sectors "often fail to make the effort, or simply find it too difficult to collaborate" (Fenton, 2003). As such, there have been few success stories of cooperation and information sharing between public and private sectors in humanitarian logistics.

Despite the importance and uniqueness of cross-sector cooperation and information sharing in relief operations, the literature inadequately addresses this problem. Considering the gap mentioned in Section 2, a series of cross-sector decision models are developed in this chapter to discuss different types of cooperation and information sharing between public and private sectors. Since logistics accounts for 80% of relief operations (Van Wassenhove 2006), this chapter focuses on the decision model of humanitarian logistics. The basic model, which contains a public sector (usually the government) and a private sector, formulates the optimal decisions of the two sectors, respectively. Then this chapter presents three more cooperation mechanisms: semi-cooperation with a private leader, semi-cooperation with a government leader and full cooperation. The optimal solutions of these four models are provided and compared. By solving and comparing their optimal solutions, this chapter makes the first step to understand the differences among these four mechanisms.

The remainder of this chapter is organized as follows. Section 4.2 reviews more related previous studies based on Chapters 2 and 3. Section 4.3 identifies the problem to be solved and then develops four optimization models. Optimal solutions and analytical analyses are provided in Section 4.4. Finally, Section 4.5 concludes the results and discusses their insights for humanitarian medical logistics.

4.2 Literature Review

Considerable effort has been made to optimize humanitarian logistics in the aftermath of large-scale disasters (Tofighi et al. 2016, Özdamar and Ertem 2015, Sheu 2014, Liu and Ye 2014, Galindo and Batta 2013, Zhan et al. 2014, Ben-Tal et al. 2011, Ortuo et al. 2011, Advar and Mert 2010, Sheu 2007, Altay and Green 2006, Özdamar et al. 2004, Beraldi et al. 2004). These studies used different methods to minimize distribution time or shipping distance, cost, the number of wounded and dead people, or to maximize level of satisfaction of the relief demand. Despite remarkable advances made by them, government was regarded as the only real decision maker. Nevertheless, the information and relief resource could be shared with non-governmental organizations and local private sectors for coordination.

In practice, many other participants, including international and local charity organizations, private companies and affected people, take part in humanitarian logistics of emergency relief. The complex situation of humanitarian logistics requires all parties to share information and work together (Balcik et al. 2010). Stewart et al. (2009) illustrated that the cooperation level between a government and a private sector after a disaster affected the resilience of local social and economic system. Maon et al. (2009) stated that cooperation between government and business could be conducted at three aspects: financial resources, capacity and entanglement. These studies analyzed the feasibility and potential advantages of the cooperation of government and private sectors to improve relief operations from the qualitative point of view, but there was a lack of further appropriate quantitative research.

Focusing on the relationship between public and private sectors, some research on public private partnership (referred as PPP) facilitates to understanding cross-sector cooperation in humanitarian logistics. PPP generally refers to various cooperative or contractual relationship established by public and private sectors so as to provide public products or services. The skills, information and resources of each sector are shared in delivering a service or facility for the use of the general public (Bovaird 2004). Since 80s in the 20th century, many countries have actively tried to apply PPP to the infrastructure and researchers discussed more use of this approach. Bhatak and Besley (2001) studied ownership structure between a public sector and a private sector with an incomplete contract when providing public goods. They studied the application of public-private cooperation in providing public goods or services, and further demonstrated that ownership structure and control configuration influenced cooperation efficiency. By investigating China's medical and health fields, Zhang et al. (2009) made a theoretical and empirical study of the cooperation efficiency in PPP. Kivleniece and Quelin (2012) identified the value creation through a theoretical framework of two conceptual public-private structural alternatives. Iossa and Martimort (2015) analyzed several main incentive issues in PPP and the shape of optimal contracts in each context by a basic model of procurement in a multi-task environment, in which a risk-averse firm chose non-contractible efforts in cost reduction and quality improvement. Some other research also studied PPP in the view of contract, risk management and management mechanism (Savas 2000, Hart 2003, Martimort and Pouyet 2008, Marin 2009, Grimsey and Lewis 2009, Garvin 2009, Shugart and Alexander 2009, Zhang 2011, Cruz and Marques 2013, Gurgun and Touran 2013, Hwang et al. 2013). However, as far as we know, there have not been any studies that combine government and private sectors together in humanitarian logistics in response to large-scale emergencies.

4.3 Model Development

4.3.1 Assumptions

This chapter considers a two-layer supply chain that involves (1) a logistics hub, and (2) an affected area in the emergency demand of a specific kind of relief. The emergency relief need to be delivered from the hub to the affected area. A public sector and a private sector take part in humanitarian logistics at the same time. The government usually plays the role of public sector and the private sector could be a private firm or a non-government organization. These two decision makers gather relief in logistics hub, and then distributes appropriately to the affected area, respectively.

Based on the above description, three basic assumptions are made to facilitate the model formulation.

(1) The affected area can only get relief from these two specific sectors from the logistics hub.

(2) The maximal available amounts of supply and the inventory in the affected area are known at the beginning of a decision period.

(3) The geographic information of both affected area and logistics hub is known because they have been established before emergencies.

On the basis of these assumptions, the following subsection presents four optimization models for the decision-making in humanitarian logistics. The differences among these models are summarized in Table 4.1.

Table 4.1: Description of the four proposed models

	Cross-sector Relationship	Decision-making Mode
Model 1	Non-cooperation	Independent decision
Model 2	Semi-cooperation with a private leader	Independent decision with information sharing
Model 3	Semi-cooperation with a government leader	Independent decision with information sharing
Model 4	Cooperation	The government makes decision

4.3.2 Notations

Notations used in this chapter are as follows:

Parameters:

D**:** A random variable referring to demand for relief in the affected area, $D \geq 0$

$g(D)$**:** The probability distribution function of D

$G(D)$**:** The cumulative distribution function of D

I**:** Inventory of relief in the affected area, $I \geq 0$

Q^G**:** Available amount of relief in the logistics hub owned by the government, $Q^G \geq 0$

Q^P**:** Available amount of relief in the logistics hub owned by the private sector, $Q^P \geq 0$

c**:** Unit supply cost from the logistics hub to the affected area, $c > 0$

R_0**:** Fixed rewards to the private sector for participating in humanitarian logistics, $R_0 > 0$

r**:** Rewards coefficient to the private sector for delivering per unit relief, $r > 0$

s**:** Extra subsidy given to the private sector for per unit relief delivered to the affected area, $s > 0$

α**:** Penalty per unit of oversupply, $\alpha > 0$

β**:** Penalty per unit of unfulfilled demand , $\beta > 0$

Decision variables:

x^G**:** Amount of relief, which are owned by the government and sent to the affected area

x^P**:** Amount of relief, which are owned by the private sector and sent to the affected area

4.3.3 Model 1: Non-cooperation

The government aims to maximize social welfare when making the humanitarian logistics decision. The optimization model for the government is

$$\min \quad U_1^G\left(x^G\right) = E\{\alpha\left(x^G + I - D\right)^+ + \beta\left(D - x^G - I\right)^+ + cx^G\} \tag{4.1}$$

$s.t.$

$$0 \leq x^G \leq Q^G \tag{4.2}$$

where E refers to the expected value. $\alpha\left(x^G + I - D\right)^+$ and $\beta\left(D - x^G - I\right)^+$ refer to the penalties of oversupply and unfulfilled demand, respectively. $(x^G + I - D)^+$ is the amount of oversupplied relief in the affected area, while $(D - x^G - I)^+$ is the amount of unfulfilled demand. α and β are the penalty coefficients. cx^G is the supply cost, including purchasing cost and delivery cost. The objective is to minimize the expectation of penalty and cost. Eq.(4.2) ensures that the amount of emergency relief sent by the government does not exceed the available amount.

Meanwhile for the private sector, the primary objective is to gain its reputation, which is related to the amount of helpful relief delivered by the private sector. The optimization model for the private sector is

$$\max \quad U_1^P\left(x^P\right) = E\{R_0 + r\min\{x^P, (D-I)^+\} - cx^P\} \tag{4.3}$$

$s.t.$

$$0 \leq x^P \leq Q^P \tag{4.4}$$

In Eq.(4.3), $\min\{x^P, (D-I)^+\}$ refers to the amount of emergency relief that would be used in the affected area. R_0 is fixed rewards to the private sector for participating in humanitarian logistics and r is the rewards coefficient for delivering per unit useful relief. Similar to the government, cx^P is supply cost and Eq.(4.4) ensures that the amount of allocated relief does not exceed the available amount of private sector.

On the basis of Model 1, we formulate three more decision models with cross-sector cooperation and information sharing.

4.3.4 Model 2: Semi-cooperation with a private leader

With the semi-cooperation strategy, the two sectors share information about the objective and constrains but make their own logistics decisions, respectively. In Model 2, the private sector is regarded as the leader and government follows its decision, while Model 3 in the next sub-section regards the government as the leader.

First stage (private sector)

$$\max \quad U_2^P\left(x^P\right) = E\{R_0 + r\min\{x^P, (D-I)^+\} - cx^P\} \tag{4.5}$$

$s.t.$

$$0 \leq x^P \leq Q^P \tag{4.6}$$

Second stage (government)

$$\min \quad U_2^G\left(x^G\right) = E\{\alpha\left(x^G + x^{P*} + I - D\right)^+ + \beta\left(D - x^G - x^{P*} - I\right)^+ + cx^G\} \tag{4.7}$$

$s.t.$

$$0 \leq x^G \leq Q^G \tag{4.8}$$

where x^{P*} is the optimal solution of the first stage.

4.3.5 Model 3: Semi-cooperation with a government leader

Similar to Model 2, The optimization mode in this situation is formulated as follows.

First stage (government):

$$\min \quad U_3^G\left(x^G\right) = E\{\alpha\left(x^G + x^P + I - D\right)^+ + \beta\left(D - x^G - x^P - I\right)^+ + cx^G\} \tag{4.9}$$

$s.t.$

$$0 \le x^G \le Q^G \tag{4.10}$$

Second stage (private sector)

$$\max \quad U_3^P\left(x^P\right) = E\{R_0 + r\min\{x^P, \left(D - I - x^{G*}\right)^+\} - cx^P\} \tag{4.11}$$

$s.t.$

$$0 \le x^P \le Q^P \tag{4.12}$$

where x^{G*} is the optimal solution of the first stage.

4.3.6 Model 4: Full cooperation

With full cooperation strategy, the two sectors share related information about each other and the government makes the logistics decision of both sectors to pursue better social welfare. The private sector gets subsidy from government for its contribution. The optimization model is

$$\min \quad U_4^G\left(x^G,\ x^P\right) = E\{\alpha\left(x^G + x^P + I - D\right)^+ + \beta\left(D - x^G - x^P - I\right)^+ + cx^G + sx^P\} \tag{4.13}$$

$s.t.$

$$E\{R_0 + r\min\{x^P, (D - I)^+\} + sx^P - cx^P\} \ge U_1^{P*} \tag{4.14}$$

$$0 \le x^G \le Q^G \tag{4.15}$$

$$0 \le x^P \le Q^P \tag{4.16}$$

where U_1^{P*} is the optimal value of U_1^P in Model 1.

In Eq.(4.13), sx^P refers to subsidy paid to the private sector and the other terms are similar to other models. Eq.(4.14) ensures that the private sector gains expected benefits not lower than in non-cooperation situation; otherwise, the private sector would refuse to cooperate with the government. Eqs.(4.15) and (4.16) ensure that the aggregate amounts of relief delivered by the government and the private sector do not exceed corresponding available amounts, respectively.

4.4 Analytical Analyses

This section firstly finds the optimal solutions of the four models respectively and then discusses their differences.

4.4.1 Analytical Solution of Model 1

For Model 1, Eq.(4.1) (the objective function of government) can be written as

$$U_1^G\left(x^G\right) = \alpha\int_0^{x^G+I}\left(x^G+I-D\right)g(D)\,dD+\beta\int_{x^G+I}^{\infty}\left(D-x^G-I\right)g(D)\,dD+cx^G$$

In Eq.(4.3) (the objective function of the private sector),

$$\min\{x^P,(D-I)^+\}=\begin{cases}0 & , \quad D\le I\\ x^P & , \quad D>I \quad \textit{and} \quad x^P<D-I\\ D-I & , \quad D>I \quad \textit{and} \quad x^P\ge D-I\end{cases}$$

That is,

$$\min\{x^P,(D-I)^+\}=\begin{cases}0 & , \quad D\le I\\ D-I & , \quad I<D\le x^P+I\\ x^P & , \quad D>x^P+I\end{cases}$$

Thus Eq.(4.3) can be written as

$$U_1^P\left(x^P\right)=R_0+r\int_I^{x^P+I}(D-I)\,g\left(D\right)\,dD+r\int_{x^P+I}^{\infty}x^Pg\left(D\right)\,dD-cx^P$$

Proposition 4.1 can be obtained.

Proposition 4.1.

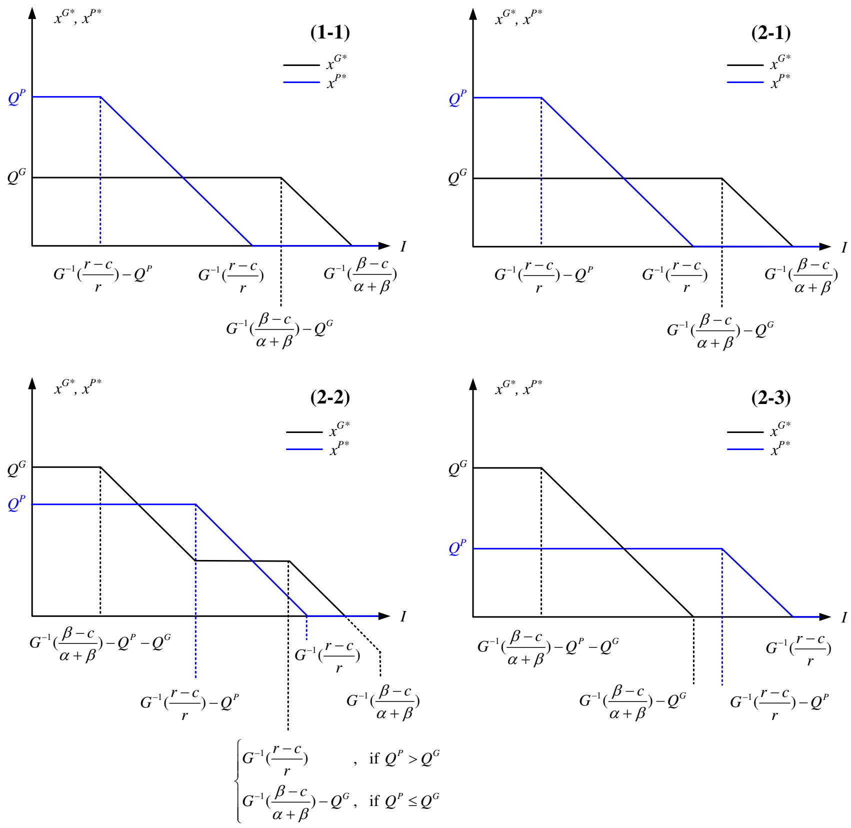
(1) U_1^G *is convex in* x^G *and* $x^{G*}=min\{Q^G,\left(G^{-1}(\frac{\beta-c}{\alpha+\beta})-I\right)^+\}$*;*

(2) U_1^P *is concave in* x^P *and* $x^{P*}=min\{Q^P,\left(G^{-1}(\frac{r-c}{r})-I\right)^+\}$*;*

where x^{G*} *and* x^{P*} *refer to the optimal solutions of decision variables* x^G *and* x^P*, respectively.*

Proposition 4.1 presents the optimal solutions of government and the private sector, respectively. This proposition and Fig. 4.1(1-1) show that both x^G and x^P are decreasing in inventory I. However, due to their non-cooperation, relief may be oversupplied and it would lead to a threat to rescue operations, such as traffic jams, unordered storage yards and secondary disasters.

It is worth noting that in some special situations, government may take no account of its cost for humanitarian reasons. That is, in the objection function $U_1^G(x^G)$, c is set as 0. On this condition, $x^{G*} = min\{Q^G, \left(G^{-1}(\frac{\beta}{\alpha+\beta}) - I\right)^+\}$ to reach maximum social welfare.



(1-1) Optimal solution of Model 1

(2-1) Optimal solution of Model 2 when $G^{-1}(\frac{r-c}{r}) \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G$

(2-2) Optimal solution of Model 2 when $G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G < G^{-1}(\frac{r-c}{r}) \le G^{-1}(\frac{\beta-c}{\alpha+\beta})$

(2-3) Optimal solution of Model 2 when $G^{-1}(\frac{r-c}{r}) > G^{-1}(\frac{\beta-c}{\alpha+\beta})$

Figure 4.1: Optimal solutions of Model 1 and Model 2

4.4.2 Analytical Solution of Model 2

For Model 2, similar to Model 1, the objective functions can be written as:

$$U_2^P\left(x^P\right) = R_0 + r\int_I^{x^P+I} (D-I)\,g(D)\,dD + r\int_{x^P+I}^{\infty} x^P g(D)\,dD - cx^P$$

and

$$\begin{aligned} U_2^G(x^G) =& \alpha\int_0^{x^G+x^{P*}+I} (x^G+x^{P*}+I-D)g(D)\,dD \\ &+\beta\int_{x^G+x^{P*}+I}^{\infty} (D-x^G-x^{P*}-I)g(D)\,dD + cx^G \end{aligned}$$

where x^{P*} refers to the optimal solution of the first stage.

Proposition 4.2 can be obtained.

Proposition 4.2.
(1) U_2^G is convex in x^G and U_2^P is concave in x^P.
(2) The optimal solution of Model 2 is

$$\begin{aligned} x^{P*} &= min\{Q^P, \left(G^{-1}(\frac{r-c}{r}) - I\right)^+\} \\ x^{G*} &= min\{Q^G, \left(G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - x^{P*}\right)^+\} \\ &= min\{Q^G, \left(G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - min\{Q^P, \left(G^{-1}(\frac{r-c}{r}) - I\right)^+\}\right)^+\} \end{aligned}$$

Proposition 4.2 and Fig. 4.1(2-1)(2-2)(2-3) present the optimal solution of Model 2. Since x^{P*} is decided in the first stage and the objective of the private sector is to gain its own reputation, the decision of the private sector is not affected by the government. Thus, the optimal solution of the private sector using Model 2 is the same as Model 1. In the second stage, the government knows the objective function and solution of the private sector. So its optimal solution is related with the private decision and is decreasing in both I and x^{P*}.

4.4.3 Analytical Solution of Model 3

Model 3 can be solved in a similar way. We defined vectors $\boldsymbol{z_1}$, $\boldsymbol{z_2}$, $\boldsymbol{z_3}$, $\boldsymbol{z_4}$ and $\boldsymbol{z_5}$ as:

$$\boldsymbol{z_1} = (\min\{Q^G, \left(G^{-1}(\frac{\beta - c}{\alpha + \beta}) - Q^P - I\right)^+\}, Q^P)$$
$$\boldsymbol{z_2} = (\min\{G^{-1}(\frac{r - c}{r}) - Q^P - I, \left(G^{-1}(\frac{\beta - c}{\alpha + \beta}) - Q^P - I\right)^+\}, Q^P)$$
$$\boldsymbol{z_3} = (0, G^{-1}(\frac{r - c}{r}) - I)$$
$$\boldsymbol{z_4} = (median\{Q^G, G^{-1}(\frac{r - c}{r}) - I, G^{-1}(\frac{\beta - c}{\alpha + \beta}) - I\}, 0)$$
$$\boldsymbol{z_5} = (\min\{Q^G, \left(G^{-1}(\frac{\beta - c}{\alpha + \beta}) - I\right)^+\}, 0)$$

Proposition 4.3 gives the optimal solution of Model 3.

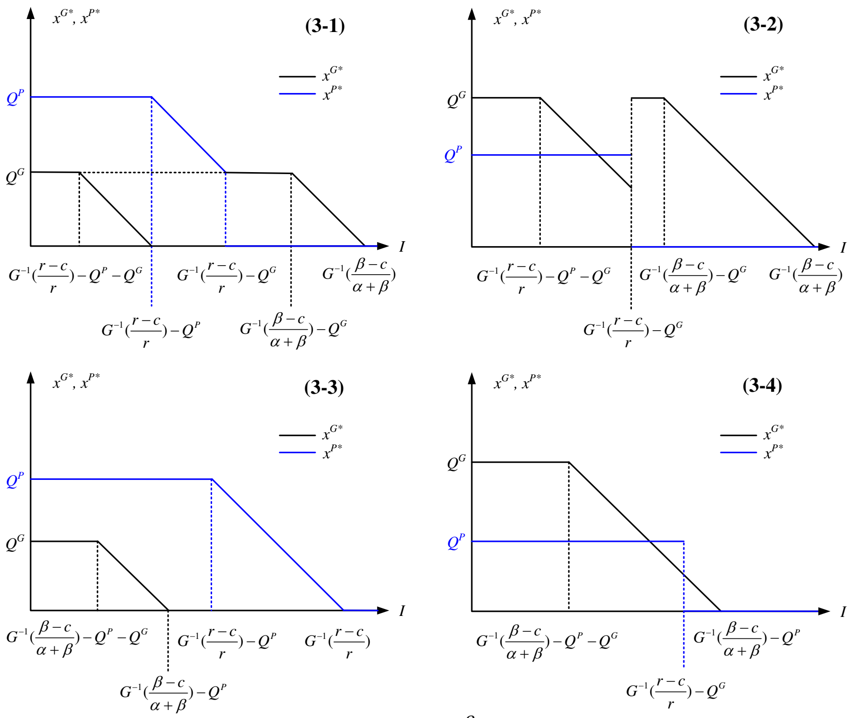
Proposition 4.3.

(1) U_3^G is convex in x^G and U_3^P is concave in x^P.

(2) The optimal solution of Model 3 is

$$(x^{G*}, x^{P*}) = \begin{cases} \boldsymbol{z_1} & , \quad I \leq G^{-1}(\frac{r-c}{r}) - Q^P - Q^G \\ \boldsymbol{z_2} & , \quad G^{-1}(\frac{r-c}{r}) - Q^P - Q^G < I \leq G^{-1}(\frac{r-c}{r}) - \max\{Q^P, Q^G\} \\ \boldsymbol{z_3} & , \quad G^{-1}(\frac{r-c}{r}) - \max\{Q^P, Q^G\} < I \leq G^{-1}(\frac{r-c}{r}) - \min\{Q^P, Q^G\} \text{ and } Q^P > Q^G \\ \boldsymbol{z_4} & , \quad G^{-1}(\frac{r-c}{r}) - \max\{Q^P, Q^G\} < I \leq G^{-1}(\frac{r-c}{r}) - \min\{Q^P, Q^G\} \text{ and } Q^P \leq Q^G \\ & \quad\ \text{or} \quad G^{-1}(\frac{r-c}{r}) - \min\{Q^P, Q^G\} < I \leq G^{-1}(\frac{r-c}{r}) \\ \boldsymbol{z_5} & , \quad I > G^{-1}(\frac{r-c}{r}) \end{cases}$$

Proposition 4.3 presents the optimal solution of Model 3. In the first stage, the government has to take the private sector into consideration according to its knowledge about the private objective. Then in the second stage, the optimal solution of the private sector is related with the government's decision. Therefore, both the two sectors make their decisions different from Model 1 and Model 2. It can be seen from the proposition and Fig. 4.2 that the optimal solutions are not only affected by inventory I, but also by the relationship between Q^G and Q^P. x^{P*} is still decreasing in I, while x^{G*} is discontinues and piecewise decreasing in I. The discontinues point is related with Q^G and Q^P.



(3-1) Optimal solution of Model 3 when $G^{-1}(\frac{r-c}{r}) \le G^{-1}(\frac{\beta-c}{\alpha+\beta})$ and $Q^P > Q^G$

(3-2) Optimal solution of Model 3 when $G^{-1}(\frac{r-c}{r}) \le G^{-1}(\frac{\beta-c}{\alpha+\beta})$ and $Q^P \le Q^G$

(3-3) Optimal solution of Model 3 when $G^{-1}(\frac{r-c}{r}) > G^{-1}(\frac{\beta-c}{\alpha+\beta})$ and $Q^P > Q^G$

(3-4) Optimal solution of Model 3 when $G^{-1}(\frac{r-c}{r}) > G^{-1}(\frac{\beta-c}{\alpha+\beta})$ and $Q^P \le Q^G$

Figure 4.2: Optimal solution of Model 3

4.4.4 Analytical Solution of Model 4

For Model 4,

$$\frac{\partial U_4^G\left(x^G,x^P\right)}{\partial x^G} = (\alpha+\beta)\,G\left(x^G+x^P+I\right)-\beta+c$$
$$\frac{\partial^2 U_4^G\left(x^G,x^P\right)}{\partial (x^G)^2} = (\alpha+\beta)\,g\left(x^G+x^P+I\right)\geq 0$$
$$\frac{\partial U_4^G\left(x^G,x^P\right)}{\partial x^P} = (\alpha+\beta)\,G\left(x^G+x^P+I\right)-\beta+s$$
$$\frac{\partial^2 U_4^G\left(x^G,x^P\right)}{\partial (x^P)^2} = (\alpha+\beta)\,g\left(x^G+x^P+I\right)\geq 0$$

Thus, U_4^G is convex in x^G and x^P, respectively.

Set $L(x^P) = E\{R_0 + r\min\{x^P,(D-I)^+\} + sx^P - cx^P\}$ and $U_1^{P*} = U_1^P(y^*)$ Obviously, $L(x^P)$ is concave in x^P and $\max L(y) \geq L(y^*) \geq U_1^P(y^*) = U_1^{P*}$ (note that $s>0$). We can obtain the following three properties of function $L(x^P)$:

$$L(0) = R_0$$

and

$$\lim_{x^P\to+\infty} L(x^P) = E\{R_0 + r\,(D-I)^+ + \lim_{x^P\to+\infty}(s-c)x^P\}$$
$$= \begin{cases} -\infty & , \quad s<c \\ R_0 + E\{r\,(D-I)^+\} & , \quad s=c \\ +\infty & , \quad s>c \end{cases}$$

and

$$\lim_{x^P\to-\infty} L(x^P) = R_0 + \lim_{x^P\to-\infty}(r+s-c)x^P = \begin{cases} +\infty & , \quad r+s<c \\ R_0 & , \quad r+s=c \\ -\infty & , \quad r+s>c \end{cases}$$

To facilitate solving of Model 4, the following paragraphs define three more notations: y_0, y_1 and y_2, and proof their existence. Then the first constraint of Model 4 can be simplified.

If $s>c$, then $\frac{dL(x^P)}{dx^P} = -rG(x^P+I) + r + s - c \geq s - c > 0$, so $L(x^P)$ is strictly increasing in x^P. And since $L(0) = R_0$ and $\lim\limits_{x^P\to+\infty} L(x^P) \geq U_1^{P*}$, there exists a unique

$y_0 \geq 0$, such that $L(y_0) = U_1^{P*}$. The first constraint of Model 4 is equivalent to $x^P \geq y_0$. Note $L(y^*) \geq U_1^P(y^*) = U_1^{P*} = L(y_0)$, we can obtain $y^* \geq y_0$. Since $0 \leq y^* \leq Q^P$, $y_0 \leq Q^P$. That is, $0 \leq y_0 \leq Q^P$.

If $s = c$, then $\frac{dL(x^P)}{dx^P} = -rG(x^P + I) + r + s - c \geq 0$, so $L(x^P)$ is increasing in x^P. Similar to $s > c$, there exists a y_0, such that $0 \leq y_0 \leq Q^P$ and $L(y_0) = U_1^{P*}$. The first constraint of Model 4 is equivalent to $x^P \geq y_0$.

If $c - r < s < c$, then $\lim_{x^P \to +\infty} L(x^P) = -\infty$, $\lim_{x^P \to -\infty} L(x^P) = -\infty$ and $\max L(y) \geq U_1^{P*}$. Thus, there exists a y_1 and a y_2, such that $y_1 < y_2$ and $L(y_1) = L(y_2) = U_1^{P*}$. The first constraint of Model 1 is equivalent to $y_1 \leq x^P \leq y_2$. Since $L(y^*) \geq U_1^P(y^*) = U_1^{P*} = L(y_1) = L(y_2)$ and $L(x^P)$ is concave in x^P, $y_1 \leq y^* \leq y_2$. Recall $0 \leq y^* \leq Q^P$, so $y_2 > 0$ and $y_1 < Q^P$.

If $s \leq c - r$, then $\frac{dL(x^P)}{dx^P} = -rG(x^P + I) + r + s - c \leq 0$, so $L(x^P)$ is decreasing in x^P. According to Model 1, when $c - r \geq s > 0$, $U_1^{P*} = U_1^P(0) = R_0$. So the first constraint of Model 4 is equivalent to $x^P = 0$.

Thus, Proposition 4.4 provides the optimal solution of Model 4.

Proposition 4.4.

(1) When $s > c$, the optimal solution of Model 4 is

$$(x^{G*}, x^{P*}) = \begin{cases} (Q^G, Q^P) & , \ I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P \\ (Q^G, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G) & , \ G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P < I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 \\ (Q^G, y_0) & , \ G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 < I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 \\ (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0, y_0) & , \ G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 < I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0 \\ (0, y_0) & , \ I > G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0 \end{cases}$$

(2) When $s = c$, define a set R^2 as $\{(x^G, x^P) : x^G \in [0, Q^G], x^P \in [y_0, Q^P], x^G + x^P = G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I\}$. The optimal solution of Model 4 is

$$(x^{G*}, x^{P*}) = \begin{cases} (Q^G, Q^P) & , \ I < G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P \\ \text{Any point in } R^2 & , \ G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P < I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_0 \\ (0, y_0) & , \ I > G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_0 \end{cases}$$

(3) When $c-r<s<c$, the optimal solution of Model 4 is

$$
(x^{G*},x^{P*}) = \begin{cases}
(Q^G,\min\{y_2,Q^P\}) & , \quad I \le G^{-1}(\frac{\beta-c}{\alpha+\beta})-\min\{y_2,Q^P\}-Q^G \\
(G^{-1}(\frac{\beta-c}{\alpha+\beta})-I-\min\{y_2,Q^P\},\min\{y_2,Q^P\}) & , \quad G^{-1}(\frac{\beta-c}{\alpha+\beta})-\min\{y_2,Q^P\}-Q^G \\
 & \quad < I \le G^{-1}(\frac{\beta-c}{\alpha+\beta})-\min\{y_2,Q^P\} \\
(0,\min\{y_2,Q^P\}) & , \quad G^{-1}(\frac{\beta-c}{\alpha+\beta})-\min\{y_2,Q^P\} \\
 & \quad < I \le G^{-1}(\frac{\beta-s}{\alpha+\beta})-\min\{y_2,Q^P\} \\
(0,G^{-1}(\frac{\beta-s}{\alpha+\beta})-I) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta})-\min\{y_2,Q^P\} \\
 & \quad < I \le G^{-1}(\frac{\beta-s}{\alpha+\beta})-y_1^+ \\
(0,y_1^+) & , \quad I > G^{-1}(\frac{\beta-s}{\alpha+\beta})-y_1^+
\end{cases}
$$

(4) When $s \le c-r$, the optimal solution of Model 4 is
$x^{G*}=\min\{\left(G^{-1}(\frac{\beta-c}{\alpha+\beta})-I\right)^+,Q^G\}$ *and* $x^{P*}=0$.

Proposition 4.4 gives the optimal solution of Model 4. Recall s refers to extra subsidy given to the private sector. The solution is affected by the value of subsidy and cost, because they affect benefits of the private sector and payments of the government. When $s \ne c$, x^{G*}, x^{P*} and $x^{G*}+x^{P*}$ are all decreasing in inventory I. When $s=c$ and $G^{-1}(\frac{\beta-s}{\alpha+\beta})-Q^G-Q^P<I\le G^{-1}(\frac{\beta-s}{\alpha+\beta})-y_0$, the model has infinite solutions but $x^{G*}+x^{P*}$ is also decreasing in inventory I.

4.4.5 Analytical Comparison of Four Models

Set x_i^{P*} and x_i^{G*} as the optimal solution of Model i. Theorem 4.5 compares the optimal solutions of the four models.

Theorem 4.5.

(1) If $I \le G^{-1}(\frac{\beta-c}{\alpha+\beta})-Q^G-Q^P$, then $x_i^{G*}=Q^G$ and $x_i^{P*}=Q^P$ $(i=1,2,3,4)$;

(2) If $I \ge \max\{G^{-1}(\frac{\beta-c}{\alpha+\beta}),G^{-1}(\frac{r-c}{r})\}$, then $x_i^{G*}=x_i^{P*}=0$ $(i=1,2,3,4)$;

(3) If $G^{-1}(\frac{\beta-c}{\alpha+\beta})-Q^G-Q^P<I<\max\{G^{-1}(\frac{\beta-c}{\alpha+\beta}),G^{-1}(\frac{r-c}{r})\}$, then $x_4^{P*}+x_4^{G*}\le x_i^{P*}+x_i^{G*}$ $(i=1,2,3)$.

The theorem indicates that when inventory is very low ($I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - Q^P$) or high ($I \geq \max\{G^{-1}(\frac{\beta-c}{\alpha+\beta}), G^{-1}(\frac{r-c}{r})\}$), the four models would get the same solutions. When inventory is moderate ($G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - Q^P < I < \max\{G^{-1}(\frac{\beta-c}{\alpha+\beta}), G^{-1}(\frac{r-c}{r})\}$), the results are different. It is generally believed that full cooperation is the best strategy to peruse better social welfare, because government is the only decision maker in this situation. However, this theorem shows different ideas, since the solution of full cooperation depends on the value of subsidy. Although the government tries to improve social welfare, he has to pay subsidy and ensures that the private sector would gain more or equal benefits to maintain their cooperation. As a result, the affected area may get less relief than other three models.

In addition, Theorem 4.6 makes a further comparison between Models 1, 2 and 3.

Theorem 4.6.

When $G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - Q^P < I < \max\{G^{-1}(\frac{\beta-c}{\alpha+\beta}), G^{-1}(\frac{r-c}{r})\}$, the optimal solution of Models 1, 2 and 3 have the following relation:

(1) If $G^{-1}(\frac{r-c}{r}) \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G$, then

$$x_1^{P*} = x_2^{P*} \begin{cases} \leq x_3^{P*} & , \quad Q^P > Q^G \\ \geq x_3^{P*} & , \quad Q^P \leq Q^G \end{cases}$$

$$x_1^{G*} = x_2^{G*} \geq x_3^{G*}$$

$$x_1^{P*} + x_1^{G*} = x_2^{P*} + x_2^{G*} \geq x_3^{P*} + x_3^{G*}$$

(2) If $G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G < G^{-1}(\frac{r-c}{r}) \leq G^{-1}(\frac{\beta-c}{\alpha+\beta})$, then

$$x_1^{P*} = x_2^{P*} \geq x_3^{P*}$$

$$x_1^{G*} \geq x_2^{G*} \geq \begin{cases} x_3^{G*} & , \quad Q^P > Q^G \\ 0 & , \quad Q^P \leq Q^G \end{cases}$$

$$x_1^{P*} + x_1^{G*} \geq x_2^{P*} + x_2^{G*} \geq x_3^{P*} + x_3^{G*}$$

(3) If $G^{-1}(\frac{r-c}{r}) > G^{-1}(\frac{\beta-c}{\alpha+\beta})$, then

$$x_1^{P*} = x_2^{P*} = x_3^{P*}$$

$$x_1^{G*} \geq x_2^{G*} = x_3^{G*}$$

$$x_1^{P*} + x_1^{G*} \geq x_2^{P*} + x_2^{G*} = x_3^{P*} + x_3^{G*}$$

Theorem 4.6 presents the differences of two semi-cooperation models and shows their potential advantages. The differences among Models 1, 2 and 3 depend on the relationship between $\frac{r-c}{r}$ and $\frac{\beta-c}{\alpha+\beta}$. With a smaller $\frac{r-c}{r}$, the private sector tends to deliver less relief, while it tends to deliver more relief with a greater $\frac{r-c}{r}$. This preference affects the government's decision. But regardless of the relationship between $\frac{r-c}{r}$ and $\frac{\beta-c}{\alpha+\beta}$, the total amounts delivered to the affected area by semi-cooperation models are less than or equal to Model 1 (non-cooperation model), and the amount decided by Model 3 (government leader) is always less than or equal to that by Model 2 (private leader).

4.5 Numerical Studies

This section develops numerical studies to further understand and compare the four proposed models. All computational processes are conducted with MATLAB on a computer with a 2.69GHz CPU and 8G RAM.

4.5.1 Study 1

This study is designed to analyze how optimal solution would change with different value of inventory (I) and unit subsidy (s). Parameters are set as follows: $\alpha = 0.3$, $\beta = 0.7$, $c = 0.4$, $r = 0.6$, $Q^G = 15$, $Q^P = 25$ and $R_0 = 10$. D follows a normal distribution, of which the mean is 55 and variance is 5. s is set as $s = 0.5$ when doing sensitivity analysis of I, while $I = 20$ when analyze s.

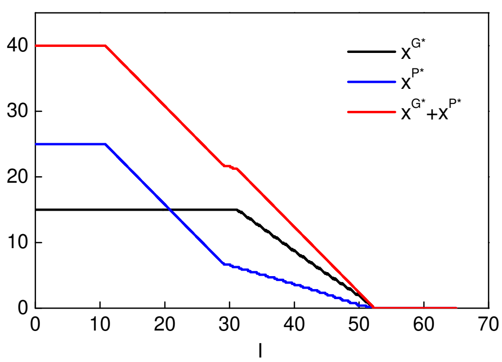


Figure 4.3: Optimal solution of Model 4 regarding I

Fig. 4.3 illustrates that as I increases, both x^{P*} and x^{G*} decrease and the total amount of delivered relief also decrease. But the rates of decrease are less than other three models if I is greater than 28 and less than 52. When inventory is enough ($I > 52$), $x^{P*} = x^{G*} = 0$.

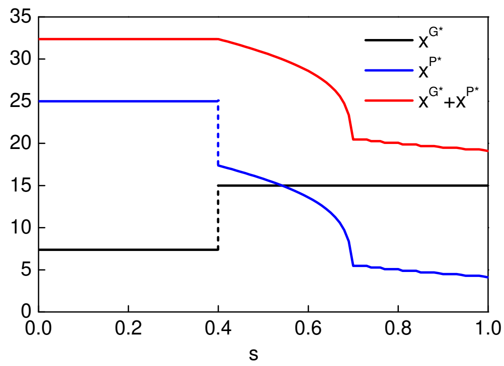


Figure 4.4: Optimal solution of Model 4 regarding s

Fig. 4.4 reports the effects of s. Recall $c = 0.4$. When unit subsidy is less than unit cost, the government tends to deliver more relief by the private sector and less relief by itself. When unit subsidy is greater than unit cost, the government delivers relief by itself as much as possible ($x^{G*} = Q^G$) and the private sector delivers less. However, the relation between s and x^{P*} is not linear. As a result, the total amount of relief ($x^{P*} + x^{Q*}$) also decreases as s increases when subsidy is greater than cost.

4.5.2 Study 2

This study makes a comparison of the four proposed models. Table 4.2 and Table 4.3 report the optimal solutions of the four models. I is set as 20 and 45, respective. Other parameters are the same as Study 1.

Table 4.2: Comparison of four models ($I = 20$)

	Model 1	Model 2	Model 3	Model 4
x^{G*}	15.00	7.38	7.38	15.00
x^{P*}	25.00	25.00	25.00	15.79
$x^{G*} + x^{P*}$	40.00	32.38	32.38	30.79
$G^{-1}(\frac{\beta}{\alpha+\beta}) - I - x^{G*} - x^{P*}$	-7.62	0.00	0.00	1.59

Table 4.3: Comparison of four models ($I = 45$)

	Model 1	Model 2	Model 3	Model 4
x^{G*}	7.38	0.00	7.85	5.23
x^{P*}	7.85	7.85	0.00	2.15
$x^{G*} + x^{P*}$	15.22	7.85	7.85	7.38
$G^{-1}(\frac{\beta}{\alpha+\beta}) - I - x^{G*} - x^{P*}$	-7.84	-0.46	-0.46	0.01

Note that $G^{-1}\left(\frac{\beta}{\alpha+\beta}\right)$ is the optimal value of social welfare. $G^{-1}(\frac{\beta}{\alpha+\beta}) - I - x^{G*} - x^{P*} < 0$ refers to oversupplied amount and $G^{-1}(\frac{\beta}{\alpha+\beta}) - I - x^{G*} - x^{P*} > 0$ refers to the amount of unfulfilled demand. The results are in accordance with analytical analysis. The total amounts delivered to the affected area by two semi-cooperation models are less than Model 1 and greater than Model 4. Full cooperation may result in insufficient delivery of relief.

4.5.3 Study 3

This study tests how the optimal solutions change when α and r change. Other parameters are set the same as Study 1.

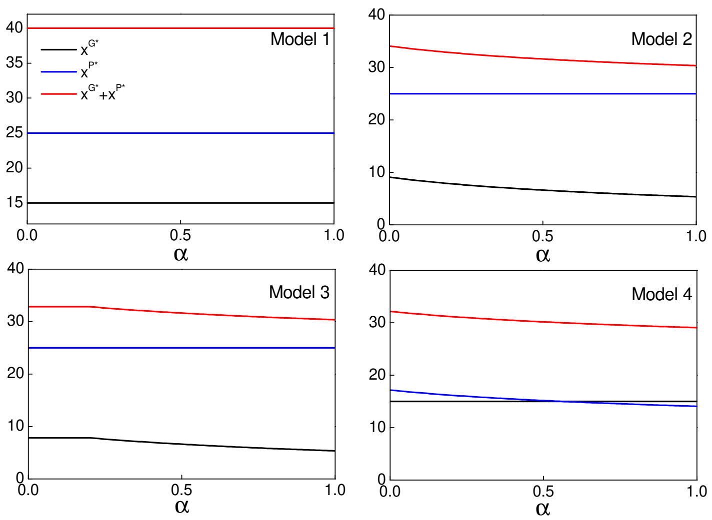


Figure 4.5: Sensitivity analysis of α

It can be seen in Fig. 4.5 that x^{G*} decreases as α increases and x^{P*} remains unchanged in Models 2 and 3, since α is not considered in the objective of the private

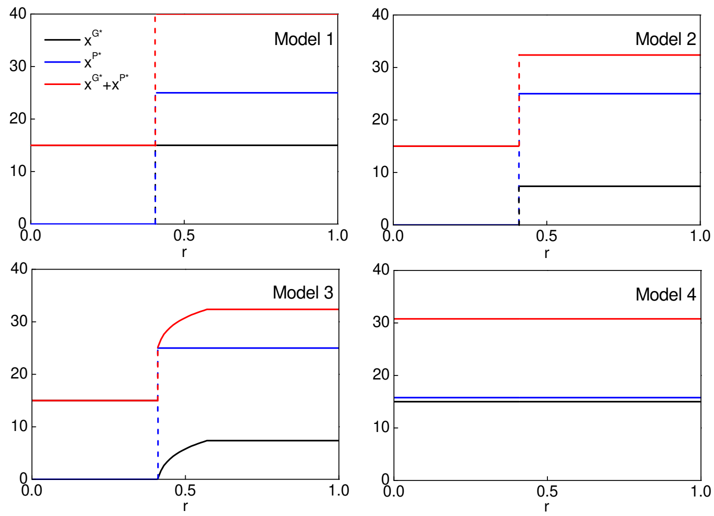


Figure 4.6: Sensitivity analysis of r

sector. But x^{G*} in Model 1 also remains unchanged because the maximum available amounts of a single sector are not enough. Additionally, in model 4, x^{P*} decreases as α increases, which is different from the other three models. The reason is in full cooperation, government is the decision maker.

Fig. 4.6 shows how the value of r affects optimal solutions. In Models 1, 2 and 3, $x^{P*} = 0$ when $r = c$ ($c = 0.4$), while x^{P*} is greater than 0 and remains unchanged when $r > c$. x^{G*} in Model 1 is not affected by r. In semi-cooperation (Models 2 and 3), the government's decision takes x^{P*} into account so x^{G*} is reduced when $r > c$. However, x^{G*} increases as r increases in the interval $0.4 < r < 0.6$. The result of Model 4 illustrates that both x^{G*} and x^{P*} are not affected by r. It would be a potential advantage of full cooperation model in practice if r is difficult to estimate.

4.6 Discussion

The results in this chapter demonstrate several managerial insights. First, the results illustrate significant impacts of subsidy in full cooperation mechanism. The government has to pay extra subsidy to the private sector to ensure their cooperation relationship. With low subsidy and corresponding benefits, the private sector may refuse to cooperate with the government. That is why in practice policy makers find it challenging to cooperate effectively with private sectors. However, in some situations, more subsidy results in insufficient delivery of relief.

Second, regarding the operations in semi-cooperation mechanism, the government and the private sector share related information with each other, but make their decisions independently. Semi-cooperation models show the same solutions as non-cooperation model when inventory is very low or high. When inventory is moderate, the optimal solutions of the private sector are still the same while solutions of the government are different. The total amount of relief delivered to the affected area by the two semi-cooperation models is always less than or equal to that by the non-cooperation model, and the allocated amount in semi-cooperation with a government leader is always less than or equal to that with a private leader. Our study also finds that the difference among these three models is affected by rewards coefficient, supply cost and penalty coefficients.

In addition, the results provide the private sector with important strategic advice on full cooperation with the government, because full cooperation always leads to benefits no less than non-cooperation mechanism. Meanwhile, the choice of the government is much more complex. In general, when inventory is adequately low or high, the non-cooperation model is suggested; while when inventory is moderate, semi-cooperation models based on information sharing become the best choice. Particularly, full cooperation would have advantages if the reward coefficient is difficult to estimate. However, in the long term, note the optimal solution of full cooperation model is affected by the subsidy paying to the private sector. Thus, if the two sectors could sign an agreement about corresponding subsidy before the emergency occurs, then the full cooperation model would have opportunities to achieve better social welfare.

4.7 Summary

This chapter focuses on cross-sector cooperation and information sharing in humanitarian logistics of emergency relief in the aftermath of emergencies. The goals of government and private sectors conflict in some situations since the former aims to maximize social welfare, while private sectors pursue their own reputation rewards. In terms of the relationship between government and a private sector, four optimization models are developed: non-cooperation, semi-cooperation with a private leader, semi-cooperation with a government leader and full cooperation models. This chapter provides the optimal solutions of these four models and compares them by both analytical and numerical studies.

4.8 Proofs of Propositions and Theorems in this Chapter

Proof of Proposition 4.1.

For the government:

$$\frac{dU_1^G\left(x^G\right)}{dx^G} = \alpha \int_0^{x^G+I} g(D)\,dD - \beta \int_{x^G+I}^{\infty} g(D)\,dD + c = (\alpha+\beta)\,G\left(x^G+I\right) - \beta + c$$

$$\frac{d^2U_1^G\left(x^G\right)}{d(x^G)^2} = (\alpha+\beta)\,g\left(x^G+I\right) \geq 0$$

Thus, U_1^G is convex in x^G,

$$\begin{aligned}
&(\alpha+\beta)\,G\left(x^G+I\right) - \beta + c = 0\\
&\Leftrightarrow x^G = G^{-1}\left(\frac{\beta-c}{\alpha+\beta}\right) - I\\
&\Rightarrow x^{G*} = min\{Q^G, \left(G^{-1}\left(\frac{\beta-c}{\alpha+\beta}\right) - I\right)^+\}
\end{aligned}$$

For the private sector:

$$\frac{dU_1^P\left(x^P\right)}{dx^P} = r\int_{x^P+I}^{\infty} g(D)\,dD - c = -rG(x^p+I) + r - c$$

$$\frac{d^2U_1^P\left(x^P\right)}{d(x^P)^2} = -rg(x^p+I) \leq 0$$

Thus, U_1^P is concave in x^P, and

$$\begin{aligned}
&-rG(x^p+I) + r - c = 0\\
&\Leftrightarrow x^P = G^{-1}(\frac{r-c}{r}) - I\\
&\Rightarrow x^{P*} = min\{Q_P, \left(G^{-1}(\frac{r-c}{r}) - I\right)^+\}
\end{aligned}$$

□

Proof of Proposition 4.2.

We firstly solve the second stage and then find optimal solution of the first stage.

The second stage (decision of government):

$$\frac{dU_2^G(x^G)}{dx^G} = (\alpha+\beta)G(x^G + x^{P*} + I) - \beta + c$$

$$\frac{d^2U_2^G(x^G)}{d(x^G)^2} = (\alpha+\beta)g(x^G + x^{P*} + I) \geq 0$$

Thus, U_2^G is convex in x^G, and the optimal solution of the second stage is

$$x^{G*} = min\{Q^G, \left(G^{-1}(\frac{\beta - c}{\alpha+\beta}) - I - x^{P*}\right)^+\}$$

First stage (decision of the private sector):

According to the formulation of $U_2^P(x^P)$, the decision of the private sector is not affected by the government. Thus, the solution of this stage is the same as that of Model 1. That is, $x^{P*} = min\{Q^P, \left(G^{-1}(\frac{r-c}{r}) - I\right)^+\}$

Therefore, the optimal solution of Model 2 is

$$x^{P*} = min\{Q^P, \left(G^{-1}(\frac{r-c}{r}) - I\right)^+\}$$

and

$$\begin{aligned} x^{G*} &= min\{Q^G, \left(G^{-1}(\frac{\beta - c}{\alpha+\beta}) - I - x^{P*}\right)^+\} \\ &= min\{Q^G, \left(G^{-1}(\frac{\beta - c}{\alpha+\beta}) - I - min\{Q^P, \left(G^{-1}(\frac{r-c}{r}) - I\right)^+\right)^+\} \end{aligned}$$

□

Proof of Proposition 4.3.

Second stage (decision of the private sector):

Similar to Model 1, we can obtain with any given x^G

$$x^{P*} = min\{Q^P, \left(G^{-1}(\frac{r-c}{r}) - I - x^G\right)^+\}$$

where x^{G*} refers to the optimal solution of the first stage.

First stage (decision of the government):

(a) If $x^G \geq G^{-1}(\frac{r-c}{r}) - I$, then $x^{P*} = 0$ and

$$x^G \in \begin{cases} \emptyset & , \quad I < G^{-1}(\frac{r-c}{r}) - Q^G \\ [G^{-1}(\frac{r-c}{r}) - I, Q^G] & , \quad G^{-1}(\frac{r-c}{r}) - Q^G \leq I \leq G^{-1}(\frac{r-c}{r}) \\ [0, Q^G] & , \quad I > G^{-1}(\frac{r-c}{r}) \end{cases}$$

$$\begin{aligned} U_3^G(x^G) &= E\{\alpha\left(x^G + I - D\right)^+ + \beta\left(D - x^G - I\right)^+ + cx^G\} \\ &= \alpha \int_0^{x^G+I} \left(x^G + I - D\right) g(D)\, dD + \beta \int_{x^G+I}^{\infty} \left(D - x^G - I\right) g(D)\, dD + cx^G \end{aligned}$$

Thus,

$$\begin{aligned} \frac{dU_3^G\left(x^G\right)}{dx^G} &= (\alpha+\beta)G(x^G + I) - \beta + c \\ \frac{d^2U_3^G\left(x^G\right)}{d(x^G)^2} &= (\alpha+\beta)g(x^G + I) \geq 0 \end{aligned}$$

$U_3^G(x^G)$ is convex in x^G and the first order condition is $x^G = G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I$. So in this situation, the optimal solution is

$$x^{G*} = \begin{cases} no\ feasible\ solution & , \quad I < G^{-1}(\frac{r-c}{r}) - Q^G \\ median\{Q^G, G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I, G^{-1}(\frac{r-c}{r}) - I\} & , \quad G^{-1}(\frac{r-c}{r}) - Q^G \leq I \\ & \quad\quad \leq G^{-1}(\frac{r-c}{r}) \\ min\{Q^G, \left(G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I\right)^+\} & , \quad I > G^{-1}(\frac{r-c}{r}) \end{cases}$$

(b) If $G^{-1}(\frac{r-c}{r}) - I - Q^P < x^G < G^{-1}(\frac{r-c}{r}) - I$, then $x^{P*} = G^{-1}(\frac{r-c}{r}) - I - x^G$.

$$\begin{aligned} U_3^G(x^G) =& E\{\alpha\left(x^G + G^{-1}(\frac{r-c}{r}) - I - x^G + I - D\right)^+ \\ &+ \beta\left(D - x^G - G^{-1}(\frac{r-c}{r}) + I - x^G - I\right)^+ + cx^G\} \\ =& E\{\alpha\left(G^{-1}(\frac{r-c}{r}) - D\right)^+ + \beta\left(D - G^{-1}(\frac{r-c}{r})\right)^+ + cx^G\} \end{aligned}$$

Since in this situation, $\frac{dU_3^G(x^G)}{dx^G} = c$ is a constant,

$$(x^{G*}, x^{P*}) = \begin{cases} no\ feasible\ solution & , \quad I \leq G^{-1}(\frac{r-c}{r}) - Q^P - Q^G \\ (G^{-1}(\frac{r-c}{r}) - Q^P - I, Q^P) & , \quad G^{-1}(\frac{r-c}{r}) - Q^P - Q^G < I \\ & \quad\quad \leq G^{-1}(\frac{r-c}{r}) - Q^P \\ (0, G^{-1}(\frac{r-c}{r}) - I) & , \quad G^{-1}(\frac{r-c}{r}) - Q^P < I \leq G^{-1}(\frac{r-c}{r}) \\ no\ feasible\ solution & , \quad I > G^{-1}(\frac{r-c}{r}) \end{cases}$$

.

(c) If $x^G \le G^{-1}(\frac{r-c}{r}) - I - Q^P$, then $x^{P*} = Q^P$ and

$$x^G \in \begin{cases} [0, Q^G] & , \quad I \le G^{-1}(\frac{r-c}{r}) - Q^P - Q^G \\ [0, G^{-1}(\frac{r-c}{r}) - I - Q^P] & , \quad G^{-1}(\frac{r-c}{r}) - Q^P - Q^G < I \le G^{-1}(\frac{r-c}{r}) - Q^P \\ \emptyset & , \quad I > G^{-1}(\frac{r-c}{r}) - Q^P \end{cases}$$

$$U_3^G(x^G) = E\{\alpha \left(x^G + Q^P + I - D\right)^+ + \beta \left(D - x^G - Q^P - I\right)^+ + cx^G\}$$
$$\frac{dU_3^G\left(x^G\right)}{dx^G} = (\alpha + \beta)\, G\left(x^G + Q^P + I\right) - \beta + c$$
$$\frac{d^2U_1^G\left(x^G\right)}{d(x^G)^2} = (\alpha + \beta)\, g\left(x^G + Q^P + I\right) \ge 0$$

Therefore, $U_3^G(x^G)$ is convex in x^G and the first order condition is $x^G = G^{-1}\left(\frac{\beta-c}{\alpha+\beta}\right) - Q^P - I$. In this situation, the optimal solution is

$$x^{G*} = \begin{cases} min\{Q^G, \left(G^{-1}\left(\frac{\beta-c}{\alpha+\beta}\right) - Q^P - I\right)^+\} & , \\ \qquad if \quad I \le G^{-1}(\frac{r-c}{r}) - Q^P - Q^G & \\ min\{G^{-1}(\frac{r-c}{r}) - Q^P - I, \left(G^{-1}\left(\frac{\beta-c}{\alpha+\beta}\right) - Q^P - I\right)^+\} & , \\ \qquad if \quad G^{-1}(\frac{r-c}{r}) - Q^P - Q^G < I \le G^{-1}(\frac{r-c}{r}) - Q^P & \\ no\ feasible\ solution & , \\ \qquad if \quad I > G^{-1}(\frac{r-c}{r}) - Q^P & \end{cases}$$

Therefore, when $Q^P > Q^G$ the optimal solution of Model 3 is

$$(x^{G*}, x^{P*}) = \begin{cases} \boldsymbol{z_1} & , \quad I \le G^{-1}(\frac{r-c}{r}) - Q^P - Q^G \\ \boldsymbol{z_2} & , \quad G^{-1}(\frac{r-c}{r}) - Q^P - Q^G < I \le G^{-1}(\frac{r-c}{r}) - Q^P \\ \boldsymbol{z_3} & , \quad G^{-1}(\frac{r-c}{r}) - Q^P < I \le G^{-1}(\frac{r-c}{r}) - Q^G \\ \boldsymbol{z_4} & , \quad G^{-1}(\frac{r-c}{r}) - Q^G < I \le G^{-1}(\frac{r-c}{r}) \\ \boldsymbol{z_5} & , \quad I > G^{-1}(\frac{r-c}{r}) \end{cases}$$

When $Q^P \leq Q^G$ the optimal solution of Model 3 is

$$(x^{G*}, x^{P*}) = \begin{cases} \boldsymbol{z_1} & , \quad I \leq G^{-1}(\frac{r-c}{r}) - Q^P - Q^G \\ \boldsymbol{z_2} & , \quad G^{-1}(\frac{r-c}{r}) - Q^P - Q^G < I \leq G^{-1}(\frac{r-c}{r}) - Q^G \\ \boldsymbol{z_4} & , \quad G^{-1}(\frac{r-c}{r}) - Q^G < I \leq G^{-1}(\frac{r-c}{r}) - Q^P \\ \boldsymbol{z_4} & , \quad G^{-1}(\frac{r-c}{r}) - Q^P < I \leq G^{-1}(\frac{r-c}{r}) \\ \boldsymbol{z_5} & , \quad I > G^{-1}(\frac{r-c}{r}) \end{cases}$$

Combine the situations of $Q^P > Q^G$ and $Q^P \leq Q^G$ together, we can get this proposition. □

Proof of Proposition 4.4(1).

In this situation, the optimal solution (x^{G*}, x^{P*}) is subjected to $x^{G*} = Q^G$ or $x^{P*} = y_0$. We can calculate that given $x^G = Q^G$, $x^{P*} = \min\{\max\{y_0, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G\}, Q^P\}$, and given $x^P = y_0$, $x^{G*} = \min\{\Big(G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0\Big)^+, Q^G\}$.

Thus, $(x^{G*}, x^{P*}) \in \{\boldsymbol{z_6}, \boldsymbol{z_7}\}$, where

$$\boldsymbol{z_6} = \begin{cases} (Q^G, y_0) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G < y_0 \\ (Q^G, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G) & , \quad y_0 \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G < Q^P \\ (Q^G, Q^P) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G \geq Q^P \end{cases}$$

and

$$\boldsymbol{z_7} = \begin{cases} (0, y_0) & , \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0 < 0 \\ (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0, y_0) & , \quad 0 \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0 < Q^G \\ (Q^G, y_0) & , \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0 \geq Q^G \end{cases}$$

That is

$$\boldsymbol{z_6} = \begin{cases} (Q^G, Q^P) & , \quad I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P \\ (Q^G, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P < I \\ & \qquad \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 \\ (Q^G, y_0) & , \quad I > G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 \end{cases}$$

and

$$\boldsymbol{z_7} = \begin{cases} (Q^G, y_0) & , \quad I \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 \\ (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0, y_0) & , \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 < I \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0 \\ (0, y_0) & , \quad I > G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0 \end{cases}$$

Since $s > c$ and $y_0 \le Q^P$, $G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 < G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G$.

(1) If $I \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P$, then $\boldsymbol{z_6} = (Q^G, Q^P)$ and $\boldsymbol{z_7} = (Q^G, y_0)$. Note that with given $x^G = Q^G$, $\forall x^P \in [y_0, Q^P]$, $U_4(\boldsymbol{z_6}) \le U_4(Q^G, x^P)$. So $U_4(\boldsymbol{z_6}) \le U_4(\boldsymbol{z_7})$. Thus the optimal solution in this situation is $(x^{G*}, x^{P*}) = (Q^G, Q^P)$.

(2) If $G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P < I \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0$, then $\boldsymbol{z_6} = (Q^G, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G)$ and $\boldsymbol{z_7} = (Q^G, y_0)$. Similar to (1), we can obtain the optimal solution in this situation is $(x^{G*}, x^{P*}) = (Q^G, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G)$.

(3) If $G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 < I \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0$, then $\boldsymbol{z_6} = \boldsymbol{z_7} = (Q^G, y_0)$. The optimal solution in this situation is $(x^{G*}, x^{P*}) = (Q^G, y_0)$.

(4) If $G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 < I \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0$, then $\boldsymbol{z_6} = (Q^G, y_0)$ and $\boldsymbol{z_7} = (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0, y_0)$. Similar to (1), we can obtain the optimal solution in this situation is $(x^{G*}, x^{P*}) = (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0, y_0)$.

(5) If $y > G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0$, then $\boldsymbol{z_6} = (Q^G, y_0)$ and $\boldsymbol{z_7} = (0, y_0)$. The optimal solution in this situation is $(x^{G*}, x^{P*}) = (0, y_0)$.

Therefore, when $s > c$ the optimal solution of Model 4 is

$$(x^{G*}, x^{P*})$$
$$= \begin{cases} (Q^G, Q^P) & , \quad I \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P \\ (Q^G, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I - Q^G) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - Q^P < I \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 \\ (Q^G, y_0) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - Q^G - y_0 < I \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 \\ (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - y_0, y_0) & , \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - Q^G - y_0 < I \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0 \\ (0, y_0) & , \quad I > G^{-1}(\frac{\beta-c}{\alpha+\beta}) - y_0 \end{cases}$$

□

Proof of Proposition 4.4(2).

$$\frac{\partial U_4^G\left(x^G, x^P\right)}{\partial x^G} = \frac{\partial U_4^G\left(x^G, x^P\right)}{\partial x^P}$$
$$\frac{\partial^2 U_4^G\left(x^G, x^P\right)}{\partial (x^G)^2} = \frac{\partial^2 U_4^G\left(x^G, x^P\right)}{\partial (x^P)^2}$$

The first order condition is $x^{G*} + x^{P*} = G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I = G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I$. Recall that $x^{G*} \in [0, Q^G]$ and $x^{P*} \in [y_0, Q^P]$, we can get the solution. □

Proof of Proposition 4.4(3).

In this situation, the optimal solution (x^{G*}, x^{P*}) is subjected to $x^{G*} = 0$ or $x^{P*} = \min\{y_2, Q^P\}$, and $x^P \in [y_1^+, \min\{y_2, Q^P\}]$. Thus, $(x^{G*}, x^{P*}) \in \{\boldsymbol{z_8}, \boldsymbol{z_9}\}$, where

$$\boldsymbol{z_8} = (0, median\{\left(G^{-1}(\frac{\beta - s}{\alpha+\beta}) - I\right)^+, y_1^+, \min\{y_2, Q^P\}\})$$
$$\boldsymbol{z_9} = (\min\{\left(G^{-1}(\frac{\beta - c}{\alpha+\beta}) - I - \min\{y_2, Q^P\}\right)^+, Q^G\}, \min\{y_2, Q^P\})$$

They can be written as

$$\boldsymbol{z_8} = \begin{cases} (0, y_1^+) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I < y_1^+ \\ (0, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I) & , \quad y_1^+ \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I < \min\{y_2, Q^P\} \\ (0, \min\{y_2, Q^P\}) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I \ge \min\{y_2, Q^P\} \end{cases}$$

and

$$\boldsymbol{z_9} = \begin{cases} (0, \min\{y_2, Q^P\}) \\ \qquad if \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\} < 0 \\ (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\}, \min\{y_2, Q^P\}) \\ \qquad if \quad 0 \le G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\} < Q^G \\ (Q^G, \min\{y_2, Q^P\}) \\ \qquad if \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\} \ge Q^G \end{cases}$$

That is

$$\boldsymbol{z_8} = \begin{cases} (0, \min\{y_2, Q^P\}) & , \quad I \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - \min\{y_2, Q^P\} \\ (0, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I) & , \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - \min\{y_2, Q^P\} < I \le G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_1^+ \\ (0, y_1^+) & , \quad I > G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_1^+ \end{cases}$$

and

$$\boldsymbol{z_9} = \begin{cases} (Q^G, \min\{y_2, Q^P\}) \\ \qquad if \quad I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} - Q^G \\ (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\}, \min\{y_2, Q^P\}) \\ \qquad if \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} - Q^G < I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} \\ (0, \min\{y_2, Q^P\}) \\ \qquad if \quad I > G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} \end{cases}$$

Since $c - r < s < c$, $y_1 \leq Q^P$, $y_1 < y_2$ and $y_2 \geq 0$, $G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} - Q^G \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} < G^{-1}(\frac{\beta-s}{\alpha+\beta}) - \min\{y_2, Q^P\} \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_1^+$.

(1) If $I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} - Q^G$, then $\boldsymbol{z_8} = (0, \min\{y_2, Q^P\})$ and $\boldsymbol{z_9} = (Q^G, \min\{y_2, Q^P\})$. Note that with given $x^P = \min\{y_2, Q^P\}$, $\forall x^G \in [0, Q^G]$, $U_4(\boldsymbol{z_9}) \leq U_4(x^G, \min\{y_2, Q^P\})$. So $U_4(\boldsymbol{z_9}) \leq U_4(\boldsymbol{z_8})$. Thus the optimal solution in this situation is $(x^{G*}, x^{P*}) = (Q^G, \min\{y_2, Q^P\})$.

(2) If $G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} - Q^G < I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\}$, then $\boldsymbol{z_8} = (0, \min\{y_2, Q^P\})$ and $\boldsymbol{z_9} = (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\}, \min\{y_2, Q^P\})$. Similar to (1), we can obtain the optimal solution in this situation is $(x^{G*}, x^{P*}) = (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\}, \min\{y_2, Q^P\})$.

(3) If $G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} < I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - \min\{y_2, Q^P\}$, then $\boldsymbol{z_8} = \boldsymbol{z_9} = (0, \min\{y_2, Q^P\})$. The optimal solution in this situation is $(x^{G*}, x^{P*}) = (0, \min\{y_2, Q^P\})$.

(4) If $G^{-1}(\frac{\beta-s}{\alpha+\beta}) - \min\{y_2, Q^P\} < I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_1^+$, then $\boldsymbol{z_8} = (0, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I)$ and $\boldsymbol{z_9} = (0, \min\{y_2, Q^P\})$. Similar to (1), we can obtain the optimal solution in this situation is $(x^{G*}, x^{P*}) = (0, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I)$.

(5) If $I > G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_1^+$, then $\boldsymbol{z_8} = (0, y_1^+)$ and $\boldsymbol{z_9} = (0, \min\{y_2, Q^P\})$. The optimal solution in this situation is $(x^{G*}, x^{P*}) = (0, y_1^+)$.

Therefore, when s¿c the optimal solution of Model 4 is

$$(x^{G*}, x^{P*}) = \begin{cases} (Q^G, \min\{y_2, Q^P\}) \\ \quad if \quad I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} - Q^G \\ (G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I - \min\{y_2, Q^P\}, \min\{y_2, Q^P\}) \\ \quad if G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} - Q^G < I \leq G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} \\ (0, \min\{y_2, Q^P\}) \\ \quad if \quad G^{-1}(\frac{\beta-c}{\alpha+\beta}) - \min\{y_2, Q^P\} < I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - \min\{y_2, Q^P\} \\ (0, G^{-1}(\frac{\beta-s}{\alpha+\beta}) - I) \\ \quad if \quad G^{-1}(\frac{\beta-s}{\alpha+\beta}) - \min\{y_2, Q^P\} < I \leq G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_1^+ \\ (0, y_1^+) \\ \quad if \quad I > G^{-1}(\frac{\beta-s}{\alpha+\beta}) - y_1^+ \end{cases}$$

□

Proof of Proposition 4.4(4).

Given $x^{P*} = 0$, The first order condition is $x^G = G^{-1}(\frac{\beta-c}{\alpha+\beta}) - I$. Recall that $x^G \in [0, Q^G]$, we can get the solution. □

Proof of Theorem 4.5.

Based on Propositions 4.1, 4.2, 4.3 and 4.4, one can get the optimal solutions of the proposed models in each situation and then compare them to obtain this proposition.

□

Proof of Theorem 4.6.

Based on Propositions 4.1, 4.2, 4.3 and 4.4, one can get the optimal solutions of Models 1, 2 and 3 in each situation and then compare them to obtain this proposition.

□

Chapter 5

Conclusions

5.1 Conclusions

This research underscores the importance of humanitarian medical allocation for public health emergencies, proposes analytical models and optimal strategies from three different perspectives, and provides suggestions on efficient, effective, and fair medical rescue.

In Chapter 2, a stochastic dynamic programming model is developed to optimize the temporal allocation problem in one epidemic area. This problem is divided into several finite time periods, with a policy decision required at each time period. The inventory of medical relief and the probability distribution of demand in the epidemic area change among time periods, and the demand in each time period is regarded as a stochastic parameter. To solve the model, this chapter provides a general analytical closed-form of the optimal allocation policy in each time period to minimize the expected sum of the overall penalty. In addition, a case study based on a real epidemic outbreak is conducted and the relations between the optimal policy and each parameter are discussed. These results highlight some managerial implications for better response to epidemic outbreaks.

Chapter 3 presents a novel allocation model of humanitarian medical allocation in multiple areas. The proposed model consists of two mechanisms: medical demand forecasting and relief distribution. The medical demand associated with each epidemic area is forecast using a modified SEIR model. This process is followed by a linear programming approach to make distribution decisions. On the basis of a basic model that applies the proposed method, two extended models are generated by assessing (1) the spatial interaction relationships among epidemic areas and (2) survivor psychology. A numerical study conducted on a real SARS outbreak in China demonstrates the applicability of the proposed method. The numerical results of the three models are compared to identify the advantages of each model. The psychological status of affected people improves significantly when survivor psychology is considered. Four experimental situations are tested to support and supplement the analysis with real data. This work highlights the necessity and feasibility of reducing the psychological fragility of affected people during emergency medical logistics, and indicates the benefits of considering the spatial interaction relationships among epidemic areas, especially in response to public health emergencies with high infection rates in areas with high population density. Additionally, the specific effects of incubation period and diagnosis rate are emphasized in the discussion of the relationship between logistics and survivor psychology.

These two chapters regard the government as the only decision-maker, while Chapter 4 focuses on the situation when both the government and a private sector take part in humanitarian allocation. In terms of the cross-sector relationship and decision-making mode, four optimization models are formulated: (1) non-cooperation, (2) semi-cooperation with a private leader, (3) semi-cooperation with a government leader and (4) full cooperation owned by the government. The objective of government is to maximize social welfare while the private sector aims to increase its reputation with lower cost. This chapter provides the optimal solutions of these four models and compares them by both analytical and numerical studies. The results provide the private sector with advice on full cooperation with the government and discuss the choice of the government in different situations.

Finally, the proposed model and analyses in this research on emergency medical allocation not only provide the optimal allocation policies, but also set a milestone in the fields of humanitarian logistics and public health to highlight the importance of studying and improving epidemic response operations.

5.2 Future Studies

Future studies may improve the current analyses in this chapter.

First of all, the stochastic models are developed with an assumption of stochastic demand, deterministic supply and deterministic transportation infrastructure. Nevertheless, in some public health emergencies, supply capacity and transportation infrastructure may also be uncertain. This issue should be further discussed.

Furthermore, survivor psychology is discussed on the basis of providing the public with transparent information. However, emergency logistics may influence affected people in different ways as a result of imperfect information. Therefore, additional research must be conducted to explain this issue.

In addition, only one private sector is taken into consideration in Chapter 4. Nevertheless, in practice more private sectors may take part in humanitarian logistics in public health emergencies and lead to a more complex situation. This issue is worthy of future work.

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