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Abstract.

This paper analyzes the volatility structure of commodity derivatives markets. The model encompasses stochastic volatility that may be unspanned by futures contracts. A generalized hump-shaped volatility specification is assumed that entails a finite-dimensional affine model for the commodity futures curve and quasi-analytical prices for options on commodity futures. An empirical study of the crude oil futures volatility structure is carried out using an extensive database of futures prices as well as futures option prices spanning 21 years. The study supports a hump-shaped, partially spanned stochastic volatility specification. Factor hedging, which takes into account shocks to both the volatility processes and the futures curve, depicts the presence of unspanned components in the volatility of commodity futures and the outperformance of the hump-shaped volatility in comparison to the more popular exponential decaying volatility. This hump shaped feature is more pronounced when the market is volatile.

Key words: Commodity derivatives, Crude oil derivatives, Unspanned stochastic volatility, Hump-shaped volatility, Pricing, Hedging

[1]*University of Technology Sydney, Finance Discipline Group, UTS Business School, carl.chiarella@uts.edu.au, boda.kang@uts.edu.au, christina.nikitopoulos@uts.edu.au
†University of New South Wales, Australian School of Business. td.to@unsw.edu.au

1 Introduction

Commodity derivatives serve the very important role of helping to manage the volatility of commodity prices. Apart from hedgers, the volatility of commodity prices are also of keen interest to speculators, who have become more dominant in these markets in recent years, see Barone-Adesi, Geman, and Theal (2010). However, these derivatives have their own volatility, of which the understanding and management is of paramount importance. In this paper, we will provide a tractable model for this volatility, and carry out empirical analysis for the most liquid commodity derivative market, namely the crude oil market.

The model used in this paper focuses directly on the volatility of derivatives. It is set up under the Heath, Jarrow, and Morton (1992) framework that treats the entire term structure of futures prices as the primary modelling element. Due to the standard feature that commodity futures prices are martingales under the risk-neutral measure, the model is completely identified by the volatility of futures prices and the initial forward curve. We model this volatility as a multifactor stochastic volatility, which may be partially unspanned by the futures contracts. Spot commodity prices are uniquely determined without the need to specify the dynamics of the convenience yield. Option prices can be obtained quasi-analytically and complex derivative prices can be determined via simulation.

Commodity derivatives have been previously studied under the Heath, Jarrow, and Morton (1992) framework. However, previous works such as those of Miltersen and Schwartz (1998), Clewlow and Strickland (2000) and Miltersen (2003) were restricted to deterministic volatility. Trolle and Schwartz (2009b) extended the literature significantly by considering unspanned stochastic volatility. However, there are two differences between this paper and the Trolle and Schwartz (2009b) paper. First, Trolle and Schwartz (2009b) start by modelling the spot commodity and convenience yield. Convenience yield is unobservable and therefore modelling it adds complexity to model assumptions and estimation. Moreover, sensitivity analysis has to rely on applying shocks to this unobserved convenience yield, which makes it less intuitive. Second, the volatility function in the Trolle and Schwartz (2009b) paper has an exponential decaying form, predicting that long term contracts will always be less volatile than short term contracts. Our model, on the other hand, uses a hump-shaped volatility (which can be reduced to an exponential decaying one), and therefore allows for increasing volatility at the short end of the curve.

The model in this paper falls under the generic framework provided by Andersen (2010) for the construction of Markovian models for commodity derivatives. As an extension to his work, we provide full results for models that allow for hump-shaped, unspanned stochastic

volatility. A hump is an important factor in other markets, such as interest rate markets, see for example Litterman, Scheinkman, and Weiss (1991), Dai and Singleton (2000) and Bekaert, Hodrick, and Marshall (2001). On the other hand, limited evidence exists in the crude oil market. As far as we are aware, this feature has only been studied in the working paper version of Trolle and Schwartz (2009b). It is reported that a hump shaped volatility function had been tried, but resulted in very similar estimates and almost indistinguishable price performance compared to the exponential volatility function. We will re-examine the volatility structure of the crude oil derivatives market. We use a larger panel dataset of crude oil futures and options traded on the NYMEX, spanning 21-years from 1 January 1990 to 31 December 2010. We find that a three-factor stochastic volatility model works well. Two of the volatility functions have a hump shape that cannot be captured by the exponential decaying specification. We also find that the hump shaped volatility matters a lot more when the market is volatile than when the market is relatively stable. The extent to which the volatility can be spanned by futures contracts varies over time, with the lowest spanning being in the recent period of 2006-2010.

The fact that volatility in the market cannot be spanned by futures contracts highlights the importance of options for hedging purposes. We analyze the hedging of straddle contracts, the pricing of which is highly sensitive to volatility. Given the multifactor nature of the model, factor hedging is employed. Factor hedging has been used successfully for deterministic and local volatility[2], such as in Clewlow and Strickland (2000) or Fan, Gupta, and Ritchken (2003). We expand the method to hedge the random shocks coming from stochastic volatility. We show that the hedging performance increases dramatically when options contracts are added to the hedging instrument set. The hedging performance is measured under various different factor hedging schemes, from delta-neutral to delta-vega and delta-gamma neutral.

An alternative approach to the HJM framework is modelling the spot commodity prices directly. A representative list of relevant literature would include Gibson and Schwartz (1990), Litzenberger and Rabinowitz (1995), Schwartz (1997), Hilliard and Reis (1998) and Casassus and Collin-Dufresne (2005). These models have been successful in depicting essential and critical features of distinct commodity market prices, for instance, the mean-reversion of the agricultural commodity market, the seasonality of the natural gas market, the spikes and regime switching of the electricity market and the inverse leverage in the oil market. The disadvantage of the spot commodity models is the requirement to specify and estimate the

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[2]Local volatility refers to model where there is a dependence between volatility and the level of the state variables.

unobservable convenience yield. The futures prices are then determined endogenously. In addition, HJM models can naturally embed unspanned stochastic volatility, a feature some spot commodity models cannot accommodate.[3]

The paper is organized as follows. Section 2 presents a generalised unspanned stochastic volatility model for pricing commodity derivatives within the HJM framework. Section 3 describes and analyzes the data for crude oil derivatives and explains the estimation algorithm. Section 4 presents the results. Section 5 examines the hedging performance. Section 6 concludes. Technical details are presented in the Appendix.

2 The HJM framework for commodity futures prices

We consider a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$, $T \in (0, \infty)$ with $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0,T]}$, satisfying the usual conditions.[4] We introduce $\mathbf{V} = \{\mathbf{V_t}, t \in [0,T],\}$ a generic stochastic volatility process modelling the uncertainty in the commodity market. We denote as $F(t, T, \mathbf{V_t})$, the futures price of the commodity at time $t \geq 0$, for delivery at time T, (for all maturities $T \geq t$). Consequently, the spot price at time t of the underlying commodity, denoted as $S(t, \mathbf{V_t})$ satisfies $S(t, \mathbf{V_t}) = F(t, t, \mathbf{V_t})$, $t \in [0,T]$. The futures price process is equal to the expected future commodity spot price under an equivalent risk-neutral probability measure Q, see Duffie (2001), namely

$$F(t, T, \mathbf{V_t}) = \mathbb{E}^Q[S(T, \mathbf{V_T}) | \mathcal{A}_t].$$

This leads to the well-known result that the futures price of a commodity is a martingale under the risk-neutral measure, thus the commodity futures price process follows a driftless stochastic differential equation. Let $W(t) = \{W_1(t), \ldots, W_n(t)\}$ be an $n-$dimensional Wiener process driving the commodity futures prices and $W^V(t) = \{W_1^V(t), \ldots, W_n^V(t)\}$ be the $n-$dimensional Wiener process driving the stochastic volatility process $\mathbf{V}_t$, for all $t \in [0,T]$.[5]

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[3]See the discussion in Collin-Dufresne and Goldstein (2002) for example.

[4]The usual conditions satisfied by a filtered complete probability space are: (a) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$ and (b) the filtration is right continuous. See Protter (2004) for technical details.

[5]We essentially assume that the filtration $\mathcal{A}_t$ includes $\mathcal{A}_t = \mathcal{A}_t^f \vee \mathcal{A}_t^V$, where

$$(\mathcal{A}_t^f)_{t \geq 0} = \{\sigma(W(s) : 0 \leq s \leq t)\}_{t \geq 0},$$
$$(\mathcal{A}_t^V)_{t \geq 0} = \{\sigma(W^V(s) : 0 \leq s \leq t)\}_{t \geq 0}.$$

Assumption 2.1 *The commodity futures price process follows a driftless stochastic differential equation under the risk-neutral measure of the form*

$$\frac{dF(t, T, \mathbf{V_t})}{F(t, T, \mathbf{V_t})} = \sum_{i=1}^{n} \sigma_i(t, T, \mathbf{V_t}) dW_i(t), \tag{2.1}$$

where $\sigma_i(t, T, \mathbf{V_t})$ are the $\underline{\mathcal{A}}$-adapted futures price volatility processes, for all $T > t$. The volatility process $\mathbf{V_t} = \{\mathbf{V_t^1}, \ldots, \mathbf{V_t^n}\}$ is an $n-$ dimensional well-behaved Markovian process evolving as

$$d\mathbf{V_t}^i = a_i^V(t, \mathbf{V_t}) dt + \sigma_i^V(t, \mathbf{V_t}) dW_i^V(t), \tag{2.2}$$

for $i = 1, \ldots, n$, where $a_i^V(t, \mathbf{V_t}), \sigma_i^V(t, \mathbf{V_t})$ are $\underline{\mathcal{A}}$-adapted stochastic processes and

$$\mathbb{E}^Q[dW_i(t) \cdot dW_j^V(t)] = \begin{cases} \rho_i dt, & i = j; \\ 0, & i \neq j. \end{cases} \tag{2.3}$$

Assume that all the above processes are $\underline{\mathcal{A}}$-adapted bounded processes with drifts and diffusions that are regular and predictable so that the proposed SDEs admit unique strong solutions. The proposed volatility specification expresses naturally the feature of unspanned stochastic volatility in the model. The correlation structure of the innovations determines the extent to which the stochastic volatility is unspanned. If the Wiener processes $W_i(t)$ are uncorrelated with $W_i^V(t)$ then the volatility risk is unhedgeable by futures contracts. When the Wiener processes $W_i(t)$ are correlated with $W_i^V(t)$, then the volatility risk can be partially spanned by the futures contracts. Thus the volatility risk (and consequently options on futures contracts) cannot be completely hedged by using only futures contracts.

Conveniently, the system (2.1) and (2.2) can be expressed in terms of independent Wiener processes. By considering the $n-$dimensional independent Wiener processes $W^1(t) = W(t)$ and $W^2(t)$, then one possible representation is

$$\frac{dF(t, T, \mathbf{V_t})}{F(t, T, \mathbf{V_t})} = \sum_{i=1}^{n} \sigma_i(t, T, \mathbf{V_t}) dW_i^1(t), \tag{2.4}$$

$$d\mathbf{V_t}^i = a_i^V(t, \mathbf{V_t}^i) dt + \sigma_i^V(t, \mathbf{V_t}^i) \left(\rho_i dW_i^1(t) + \sqrt{1 - \rho_i^2} dW_i^2(t) \right). \tag{2.5}$$

Clearly, the volatility risk of any volatility factors $\mathbf{V_t}^i$ with $\rho_i = 0$ cannot be spanned by futures contracts.

Let $X(t,T) = \ln F(t,T,\mathbf{V_t})$ be the logarithm of the futures prices process, then from (2.4) and an application of Ito's formula, it follows that

$$dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t,T,\mathbf{V_t})dt + \sum_{i=1}^{n}\sigma_i(t,T,\mathbf{V_t})dW_i(t). \tag{2.6}$$

Lemma 2.2 *Under the Assumption 2.1 for the commodity futures price dynamics, the commodity spot prices satisfy the SDE*

$$\frac{dS(t,\mathbf{V_t})}{S(t,\mathbf{V_t})} = \zeta(t)dt + \sum_{i=1}^{n}\sigma_i(t,t,\mathbf{V_t})dW_i(t), \tag{2.7}$$

with the instantaneous spot cost of carry $\zeta(t)$ satisfying the relationship

$$\zeta(t) = \frac{\partial}{\partial t}\ln F(0,t) - \frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t,t,\mathbf{V_t}) \tag{2.8}$$

$$- \sum_{i=1}^{n}\int_0^t \sigma_i(u,t,\mathbf{V_u})\frac{\partial}{\partial t}\sigma_i(u,t,\mathbf{V_u})du + \sum_{i=1}^{n}\int_0^t \frac{\partial}{\partial t}\sigma_i(u,t,\mathbf{V_u})dW_i(u).$$

Proof: See Appendix A. ∎

The commodity HJM model is Markovian in an infinite dimensional state space due to the fact that the futures price curve is an infinite dimensional object (one dimension for each maturity T). In addition, the path dependent nature of the integral terms in the drift (2.8) of the commodity spot prices also gives the process an infinite dimensional nature.

2.1 Finite Dimensional Realisations for a Commodity Forward Model

We specify functional forms for the futures price volatility functions $\sigma_i(t,T,\mathbf{V_t})$ that will allow the proposed commodity forward model to admit finite dimensional realisations (FDR).

Assumption 2.3 *The commodity futures price volatility functions $\sigma_i(t,T,\mathbf{V_t})$ are of the form*

$$\sigma_i(t,T,\mathbf{V_t}) = \alpha_i(t,\mathbf{V_t})\varphi_i(T-t), \tag{2.9}$$

where $\alpha_i : \mathbb{R}^+ \to \mathbb{R}$ are $\underline{A}$-adapted square-integrable stochastic processes and $\varphi_i : \mathbb{R} \to \mathbb{R}$

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are quasi exponential functions. A quasi-exponential function $\varphi : \mathbb{R} \to \mathbb{R}$ has the general form

$$\varphi(x) = \sum_i e^{m_i x} + \sum_j e^{n_j x}[p_j(x)\cos(k_j x) + q_j(x)\sin(k_j x)], \qquad (2.10)$$

where m_i, n_i and k_i are real numbers and p_j and q_j are real polynomials.

These very general volatility specifications have been proposed in Björk, Landén, and Svensson (2004) and can be adapted for commodity forward models. Björk, Landén, and Svensson (2004) have demonstrated, by employing methods of Lie algebra, that this functional form is a necessary condition for a forward interest rate model with stochastic volatility to admit FDR. In the spirit of Chiarella and Kwon (2001b) and Björk, Landén, and Svensson (2004), α_i may also depend on a finite set of commodity futures prices with fixed tenors. When level dependent (or constant direction) volatility is considered, it becomes very difficult to obtain tractable analytical solutions for futures option prices. For this reason, even though FDR can be obtained for a level dependent stochastic volatility model (clearly with a higher dimensional state space), we consider the dependence of α_i only on stochastic volatility.

These volatility specifications have the flexibility of generating a wide range of shapes for the futures price volatility surface. Some typical examples of interest rate volatility curves include, the exponentially declining stochastic volatility structures of the Ritchken and Sankarasubramanian (1995), and the hump-shaped volatility structures discussed in Chiarella and Kwon (2001a) and Trolle and Schwartz (2009a), which are special cases of these general specifications. Furthermore some special examples of commodity volatility curves include the exponentially declining stochastic volatility structures of Trolle and Schwartz (2009b) and the gas volatility structures following a regular pattern as discussed in Björk, Blix, and Landen (2006). Note that the latter authors do not consider a stochastic volatility model.

2.2 Hump-Shaped Unspanned Stochastic Volatility

Next we propose certain volatility specifications within the general functional form (2.9) which are not only multi-factor stochastic volatility of Heston (1993) type but also allow for humps.

Assumption 2.4 *The commodity futures price volatility processes $\sigma_i(t, T, \mathbf{V_t})$ are of the*

form

$$\sigma_i(t, T, \mathbf{V_t}) = (\kappa_{0i} + \kappa_i(T - t))e^{-\eta_i(T-t)}\sqrt{\mathbf{V_t}^i}, \tag{2.11}$$

where κ_{0i}, κ_i and η_i are constants.

When the commodity futures prices volatilities are expressed in this functional form then finite dimensional realisations of the state space are possible.

Proposition 2.5 *Under the volatility specifications of Assumption 2.4, the logarithm of the instantaneous futures prices at time t with maturity T, namely $\ln F(t, T, \mathbf{V_t})$, is expressed in terms of $6n$ state variables as*

$$\ln F(t, T, \mathbf{V_t}) = \ln F(0, T, V_0) - \sum_{i=1}^{n} \left(\frac{1}{2} \left(\gamma_{i1}(T - t)\mathbf{x}_i(t) + \gamma_{i2}(T - t)\mathbf{y}_i(t) + \gamma_{i3}(T - t)\mathbf{z}_i(t) \right) + \right.$$
$$\left. (\beta_{i1}(T - t)\phi_i(t) + \beta_{i2}(T - t)\psi_i(t)) \right), \tag{2.12}$$

where for $i = 1, 2, \ldots, n$

$$\beta_{i1}(T - t) = (\kappa_{0i} + \kappa_i(T - t))e^{-\eta_i(T-t)}, \tag{2.13}$$
$$\beta_{i2}(T - t) = \kappa_i e^{-\eta_i(T-t)}, \tag{2.14}$$
$$\gamma_{i1}(T - t) = \beta_{i1}(T - t)^2, \tag{2.15}$$
$$\gamma_{i2}(T - t) = 2\beta_{i1}(T - t)\beta_{i2}(T - t), \tag{2.16}$$
$$\gamma_{i3}(T - t) = \beta_{i2}(T - t)^2. \tag{2.17}$$

The state variables $\mathbf{x}_i(t), \mathbf{y}_i(t), \mathbf{z}_i(t), \phi_i(t)$ and $\psi_i(t)$, $i = 1, 2, \ldots, n$ evolve according to

$$d\mathbf{x}_i(t) = (-2\eta_i\mathbf{x}_i(t) + \mathbf{V_t}^i)dt,$$
$$d\mathbf{y}_i(t) = (-2\eta_i\mathbf{y}_i(t) + \mathbf{x}_i(t))dt,$$
$$d\mathbf{z}_i(t) = (-2\eta_i\mathbf{z}_i(t) + 2\mathbf{y}_i(t))dt, \tag{2.18}$$
$$d\phi_i(t) = -\eta_i\phi_i(t)dt + \sqrt{\mathbf{V_t}^i}dW_i(t),$$
$$d\psi_i(t) = (-\eta_i\psi_i(t) + \phi_i(t))dt,$$

subject to $\mathbf{x}_i(0) = \mathbf{y}_i(0) = \mathbf{z}_i(0) = \phi_i(0) = \psi_i(0) = 0$. The above-mentioned $5n$ state variables are associated with the stochastic volatility process $\mathbf{V_t} = \{\mathbf{V_t}^1, \ldots, \mathbf{V_t}^n\}$ which

is assumed to be an $n-$ dimensional of Heston (1993) type process such that

$$d\mathbf{V_t}^i = \mu_i^V(\nu_i^V - \mathbf{V_t}^i)dt + \varepsilon_i^V \sqrt{\mathbf{V_t}^i} dW_i^V(t), \tag{2.19}$$

where μ_i^V, ν_i^V, and ε_i^V are constants (they can also be deterministic functions).

Proof: See Appendix B for technical details. ∎

Note that the model admits FDR within the affine class of Duffie and Kan (1996). Additionally, the model is consistent, by construction, with the currently observed futures price curve, consequently it is a time-inhomogeneous model. However for estimation purposes, it is necessary to reduce the model to a time-homogeneous one as presented in Section 3.3 below. Note that the proposed volatility conditions in Andersen (2010) lead to time-inhomogeneous models, which cannot be directly applied for estimation purposes.

The price of options on futures can be obtained in closed form as a tractable expression for the characteristic function exists. By employing Fourier transforms, call and put options on futures contracts can be priced. These results are summarised in the following proposition which is a natural extensions of existing literature and are quoted here for completeness.

Proposition 2.6 *Under the stochastic volatility specifications (2.19) and for $t \leq T_o \leq T$, the transform $\phi(t; v, T_o, T) =: \mathbb{E}_t[\exp\{v \ln F(T_o, T, V_{T_o})\}]$ is expressed as*

$$\phi(t; v, T_o, T) = \exp\{M(t; v, T_o) + \sum_{i=1}^n N_i(t; v, T_o)\mathbf{V_t}^i + v \ln F(t, T, \mathbf{V_t})\}, \tag{2.20}$$

where $M(t) = M(t; v, T_o)$ and for $i = 1, \ldots, n$, $N_i(t) = N_i(t; v, T_o)$ satisfy the Ricatti ordinary differential equations

$$\frac{dM(t)}{dt} = -\sum_{i=1}^n \mu_i^V \nu_i^V N_i(t), \tag{2.21}$$

$$\frac{dN_i(t)}{dt} = -\frac{v^2 - v}{2}(\varphi_i)^2 - (\varepsilon_i^V v \rho_i \varphi_i - \mu_i^V) N_i(t) - \frac{1}{2}\varepsilon_i^{V^2} N_i^2(t), \tag{2.22}$$

subject to the terminal conditions $M(T_o) = N_i(T_o) = 0$, where $\varphi_i = (\kappa_{0i} + \kappa_i(T - t))e^{-\eta_i(T-t)}$.

The price at time t of a European put option maturing at T_o with strike K on a futures

contract maturing at time T, is given by

$$\mathcal{P}(t, T_o, T, K) = \mathbb{E}_t^Q[e^{-\int_t^{T_o} r_s ds} (K - F(T_o, T))^+]$$
$$= P(t, T_o)[KG_{0,1}(\log(K)) - G_{1,1}(\log(K))] \qquad (2.23)$$

where $P(t, T_o)$ is the price at time t of a zero-coupon bond maturing at T_o and $G_{a,b}(y)$ is given by

$$G_{a,b}(y) = \frac{\phi(t; a, T_o, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\phi(t; a + \mathbf{i}bu, T_o, T)e^{-\mathbf{i}uy}]}{u} du. \qquad (2.24)$$

Note that $\mathbf{i}^2 = -1$.

Proof: Follows along the lines of Duffie, Pan, and Singleton (2000) and Collin-Dufresne and Goldstein (2002). Technical details of the characteristic function are also presented in Appendix C. ∎

For the market price of volatility risk, a "complete" affine specification is assumed, see Doran and Ronn (2008) (where they have shown that the market price of volatility risk is negative) and in particular Dai and Singleton (2000). Accordingly, the market price of risk is specified as,

$$dW_i^{\mathbb{P}}(t) = dW_i(t) - \lambda_i \sqrt{\mathbf{V_t}^i} dt,$$
$$dW_i^{\mathbb{P}V}(t) = dW_i^V(t) - \lambda_i^V \sqrt{\mathbf{V_t}^i} dt, \qquad (2.25)$$

for $i = 1, \ldots, n$, where $W_i^{\mathbb{P}}(t)$ and $W_i^{\mathbb{P}V}(t)$ are Wiener processes under the physical measure $\mathbb{P}$. Note that under these specifications, the model parameters are $9n$, namely; λ_i, λ_i^V, κ_{0i}, κ_i, η_i, μ_i^V, ν_i^V, ε_i^V, ρ_i that we will estimate next by fitting the proposed model to crude oil derivative prices.

3   Data and the estimation method

3.1   Data

We estimate the model using an extended dataset of crude oil futures and options traded on the NYMEX[6]. The database spans the 21 years from 1 January 1990 to 31 December 2010. This is one of the richest databases available on commodity derivatives. In addition, over this period, noteworthy financial market events with extreme market movements, for instance the oil price crisis in 1990 and the financial crisis in 2008, have occurred.

Throughout the sample period, the number of available futures contracts with positive open interest per day has increased from 17 on 1st of January 1990 to 67 on 31st of December 2010. The maximum maturity of futures contracts with positive open interest has also increased from 499 (calendar) days to 3128 days. We can see that the price surfaces change significantly throughout the sample period. The maximum futures price was US40 per barrel in 1990 reaching US140 per barrel in 2008.

Given the large number of available futures contracts per day, we make a selection of contracts for estimation purposes based on their liquidity. Liquidity has increased across the sample. For instance, the open interest for the futures contract with 6 months to maturity has increased from 13,208 contracts in 1990 to 38,766 contracts in 2010. For contracts with less than 14 days to expiration, liquidity is very low, while for contracts with more than 14 days to expiration, liquidity increases significantly. We begin with the first seven monthly contracts, near to the trade date, namely m_1, m_2, m_3, m_4, m_5, m_6, and m_7. Note that the first contract should have more than 14 days to maturity. After that liquidity is mostly concentrated in the contracts expiring in March, June, September and December. Thus the first seven monthly contracts are followed by the three contracts which have either March, June, September or December expiration months. We name them q_1, q_2 and q_3. Beyond that, liquidity is concentrated in December contracts only, therefore the next five December contracts, namely y_1, y_2, y_3, y_4 and y_5, are included. As a result, the total number of futures contracts to be used in our analysis is 70,735, with the number of contracts to be used on a daily basis varying between a maximum of 15 and a minimum of 8. Figure 3.1 plots the selected futures prices on Wednesdays during the sample period.

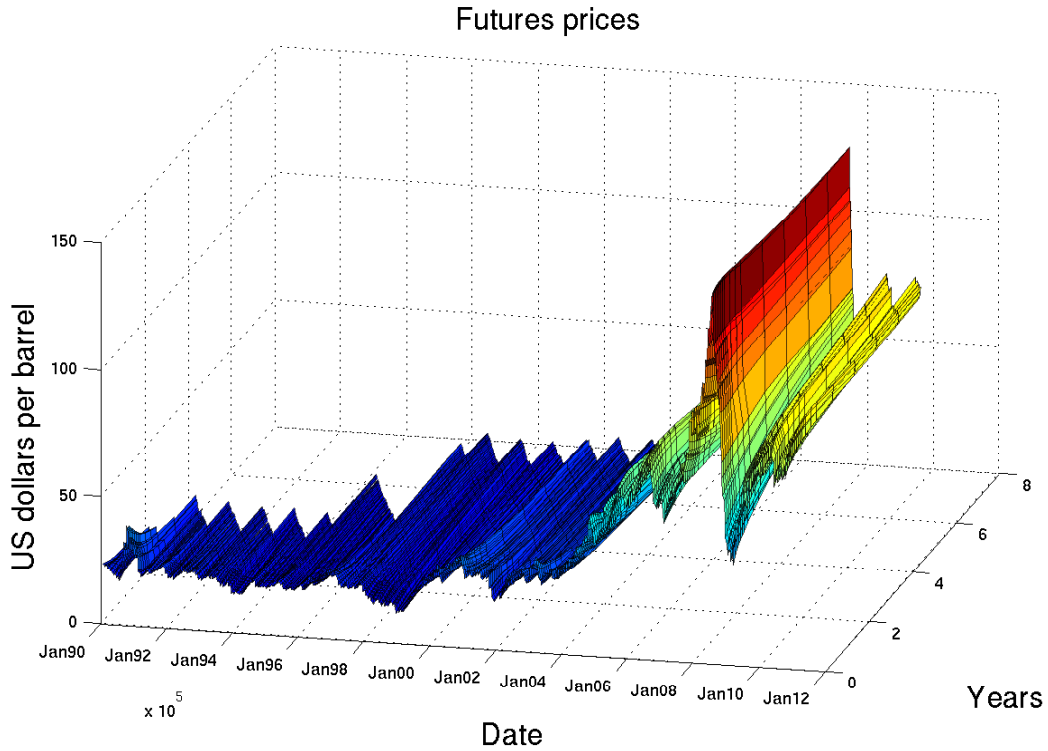[6]The database has been provided by CME.

Figure 3.1: Futures curve - 1 January 1990 to December 2010

With regard to option data, we consider the options on the first ten futures contracts only, namely the futures contracts m_1-m_7 and q_1-q_3. We avoid the use of longer maturities because in the proposed model we have not taken into account interest rates that vary considerably over the sample period and probably stochastically. Due to this model constraint, the option pricing equation (2.23) is not very accurate for longer maturities. Furthermore, the option pricing equation (2.23) provides the price for European options, not American options that are the options of our database. For the conversion of American prices to European prices, including the approximation of the early exercise premium, we follow the same approach proposed by Broadie, Chernov, and Johannes (2007) for equity options and applied by Trolle and Schwartz (2009b) for commodity options.

For each option maturity, we consider six moneyness intervals, $0.86-0.90, 0.91-0.95, 0.96-1.00, 1.01-1.05, 1.06-1.10, 1.11-1.15$. Note that moneyness is defined as option strike divided by the price of the underlying futures contract. In each moneyness interval, we use only the out-of-the-money (OTM) and at-the-money (ATM) options that are closest to the interval mean. OTM options are generally more liquid and we also benefit by a reduction in the errors that occurred in the early exercise approximation.

Based on this selection criteria, we consider 433,137 option contracts over the 21 years, with the daily range varying between 29 and 100 contracts (per trading day). Note that the total number of trading days where both futures and options data are available is 5272. ATM implied volatilities for options on the first ten oil futures contracts were computed by using the Barone-Adesi and Whaley (1987) option pricing formula and are displayed in Figure 3.2.
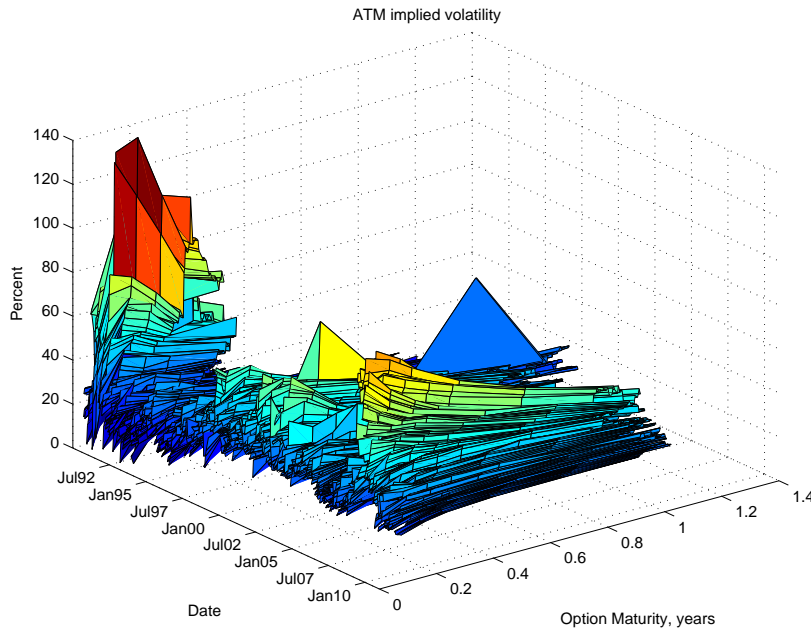
Figure 3.2: ATM implied volatilities of options on oil futures.

3.2   Sample selection

Figure 3.1 shows that the prices of futures contracts change significantly during the 21-year period from 1990 - 2010. In their study of crude oil futures from October 1991 to October 2007, Bekiros and Diks (2008) show that the two periods before and after 1999 differ considerably in their statistical features. They argue that there are economic reasons behind the change, namely the reduction in OPEC spare capacity and the increase in the US and China's oil consumption and imports.

Our data coverage is longer than that of Bekiros and Diks (2008), namely from January 1990 until December 2010. It covers two more important events related to crude oil prices, namely the Gulf War and the Global Financial Crisis. It can be seen in Figure 3.1 that not only the futures prices surged up during the two periods but there is also a lot of variation.

We therefore break the data further down further into smaller subsamples and analyze their statistical features in Table 3.1.

Period: 1990 - 1994				
	1M	4M	7M	13M
Mean	-0.00022	-0.00017	-0.00013	-0.00010
Standard Deviation	0.02631	0.01837	0.01616	0.01419
Sample Variance	0.00069	0.00034	0.00026	0.00020
Kurtosis	50.19	50.62	31.53	14.35
Skewness	-2.96698	-3.26779	-2.22760	-1.03636

Period: 1995 - 1999				
	1M	4M	7M	13M
Mean	0.00030	0.00023	0.00017	0.00009
Standard Deviation	0.02232	0.01580	0.01354	0.01167
Sample Variance	0.00049	0.00025	0.00018	0.00014
Kurtosis	6.84	7.12	6.60	5.58
Skewness	0.20058	0.14862	0.06334	0.03245

Period: 2000 - 2005				
	1M	4M	7M	13M
Mean	0.00062	0.00069	0.00075	0.00081
Standard Deviation	0.02426	0.01952	0.01736	0.01528
Sample Variance	0.00059	0.00038	0.00030	0.00023
Kurtosis	6.03	5.09	5.12	4.75
Skewness	-0.62528	-0.41719	-0.39037	-0.32096

Period: 2006 - 2010				
	1M	4M	7M	13M
Mean	0.00029	0.00029	0.00029	0.00028
Standard Deviation	0.02720	0.02272	0.02115	0.01922
Sample Variance	0.00074	0.00052	0.00045	0.00037
Kurtosis	7.71	5.77	5.69	5.59
Skewness	0.15898	-0.15396	-0.12377	-0.11221

Table 3.1: Daily log returns descriptive statistics 1990 - 2010.

Over the last 20 years, the futures returns have increased consistently until the last period, when the financial crisis hit. The variance of returns started high in the first period surrounding the Gulf War, reducing in the second period 1995-1999, bounced back in 2000-2005 and finally reached the highest level in 2006-2010. The variance were also mostly driven by extreme values (kurtosis) in the Gulf War period. Given these changes, we will later

empirically estimate the model for each period separately.

3.3 Estimation method

The estimation approach is quasi-maximum likelihood in combination with the extended Kalman filter. The model is cast into a state-space form, which consists of the system equations and the observation equations.

For estimation purposes, a time-homogeneous version of the model (2.12) is considered, by assuming for all T, $F(0, T) = f_o$, where f_o is a constant representing the long-term futures price (at infinite maturity). This constant is an additional parameter that is also to be estimated. In the estimation we normalized the long run mean of the volatility process, ν_i^V, to one to achieve identification. [7]

The system equations describe the evolution of the underlying state variables. In our case, the state vector is $X_t = \{X_t^i, i = 1, 2, \ldots, n\}$ where X_t^i consists of the six state variables $\mathsf{x}_i(t), \mathsf{y}_i(t), \mathsf{z}_i(t), \phi_i(t), \psi_i(t)$ and $\mathbf{V_t}^i$. The continuous time dynamics (under the physical probability measure) of these state variables are defined by (2.18), (2.19) and (2.25). The corresponding discrete evolution is

$$X_{t+1} = \Phi_0 + \Phi_X X_t + w_{t+1}, \ w_{t+1} \sim iid\, N(0, Q_t), \tag{3.26}$$

where Φ_0, Φ_X and Q_t can be computed in closed form. Details can be found in Appendix D.

The observation equations describe how observed options and futures prices are related to the state variables, namely

$$z_t = h(X_t) + u_t, \ u_t \sim iid\, N(0, \Omega). \tag{3.27}$$

In particular, log futures prices are linear functions of the state variables (as described in (2.12)) and the options prices are nonlinear functions of the state variables (as described in (2.23) and (2.24)) so the function h will have to vary accordingly.

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[7]For details see for example the discussion on invariant transformations in Dai and Singleton (2000).

3.4 Other considerations

3.4.1 Number of stochastic factors

The number of driving stochastic factors affecting the evolution of the futures curve can be determined by performing a principal component analysis (PCA) of futures price returns. Table 3.2 show that we do not need more than 3 factors to capture the variations in the futures returns. We therefore will estimate a 3-factor model for all sample periods in our empirical analysis and check their performance against corresponding 2-factor models.

Time Period	One factor	Two factor	Three factor
1990 - 1994	0.9056	0.9805	0.9961
1995 - 1999	0.8913	0.9667	0.9951
2000 - 2005	0.8229	0.9059	0.9549
2006 - 2010	0.9275	0.9715	0.9887

Table 3.2: Accumulated percentage of contribution towards return variation.

3.4.2 The discount function

The discount function $P(t, T)$ is obtained by fitting a Nelson and Siegel (1987) curve each trading day to LIBOR and swap data consisting of 1-, 3-, 6-, 9- and 12-month LIBOR rates and the 2-year swap rate, similar to Trolle and Schwartz (2009b).

Let $f(t, T)$ denote the time$-t$ instantaneous forward interest rate to time T. Nelson and Siegel (1987) parameterize the forward interest rate curve as

$$f(t, T) = \beta_0 + \beta_1 e^{-\theta(T-t)} + \beta_2 \theta(T-t) e^{-\theta(T-t)} \tag{3.28}$$

from which we can price LIBOR and swap rates. This also yields for zero-coupon bond prices the expression

$$P(t, T) = \exp\left\{\beta_0(T-t) + (\beta_1 + \beta_2)\frac{1}{\theta}\left(1 - e^{-\theta(T-t)}\right) + \beta_2(T-t)e^{-\theta(T-t)}\right\}. \tag{3.29}$$

On each observation date, we recalibrate the parameters $\beta_0, \beta_1, \beta_2$ and θ, by minimizing the mean squared percentage differences between the model implied forward rates (3.28) and the observed LIBOR and swap curve consisting of the 1-, 3-, 6-, 9- and 12-month LIBOR rates and the 2-year swap rate on that date.

3.4.3 Computational details

The loglikehood function is maximised by using the constrained optimization routine "e04jy" in the NAG library. We begin with several different initial hypothetical parameter values, firstly on monthly data, then on weekly data and finally on daily data, aimed at obtaining global optima.

The ODE's (2.21) and (2.22) are solved by a standard fourth-order Runge-Kutta algorithm using complex arithmetic. The integral in (2.24), is approximated by the Gauss-Legendre quadrature formula with 30 integration points and truncating the integral at 400.

4   Empirical Results

4.1   Parameter Estimation

Parameter estimates for the two-factor hump-shaped stochastic volatility model are presented in Table 4.3. Parameter estimates for the three-factor hump-shaped stochastic volatility model can be found in Table 4.4 and Table 4.5. Estimation is carried out for four different subsamples due to the marked difference in their price behaviour, as can be seen in Figure 3.1 and Table 3.1.

The combination of different hump-shaped curves can result in a rich pattern of volatility behaviour. From the parameter estimates, the significance of κ in all subsamples indicates the existence of a hump shape. Figure 4.3 and Figure 4.4 display the shape of each volatility component and the total volatility of the futures prices for the two-factor models and the three-factor models, respectively.[8] For all sample periods, only one of the three volatility factors can be described by an exponential decaying function. Table 4.6 shows the contribution of each volatility factor to the total variance. Note that "hump(e)" indicates that the volatility function has a humped shape, however, at the relevant maturity range (less than 5 years), the volatility appears to be increasing since the hump occurs at a later point in time. The total contribution of the two hump-shaped volatility factors is at least 78% of the total futures return variation.

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[8]The likelihood ratio tests strongly reject the 2-factor models in favour of the 3-factor ones.

	1990 - 1994		1995-1999		2000 - 2005		2006-2010	
	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$	$i = 1$	$i = 2$
κ_{0i}	0.1509	1.1013	0.1409	0.6226	0.0010	0.6945	0.3510	0.8906
	(0.008)	(0.0137)	(0.0016)	(0.0056)	(0.0005)	(0.0057)	(0.0026)	(0.0076)
κ_i	1.9551	0.1252	1.8965	0.1523	0.1159	0.5768	2	0.7115
	(0.0126)	(0.0013)	(0.0112)	(0.0013)	(0.0016)	(0.0043)	(0.0162)	(0.0063)
η_i	1.0338	0.2116	0.7939	0.0538	0.0010	0.4094	0.3697	0.2415
	(0.0114)	(0.0021)	(0.0072)	(0.042)	(0.0004)	(0.0032)	(0.0042)	(0.0022)
μ_i^V	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010
	(0.0003)	(0.0004)	(0.0002)	(0.0003)	(0.0004)	(0.0003)	(0.0003)	(0.0005)
ε_i^V	2	2	0.6041	1.2011	0.2687	1.5040	0.9475	1.7268
	(0.0187)	(0.0193)	(0.0054)	(0.0112)	(0.0017)	(0.0123)	(0.0094)	(0.0123)
ρ_i	0.03428	-0.14828	-0.0606	0.0678	-0.0105	-0.1593	-0.2130	-0.0614
	(0.0005)	(0.0013)	(0.0005)	(0.0004)	(0.0008)	(0.0013)	(0.0025)	(0.0004)
λ_i^V	-0.3492	1.5923	-0.9754	1.0451	1.1228	3.9996	-0.8979	2.3767
	(0.0025)	(0.0123)	(0.0092)	(0.0107)	(0.0125)	(0.0283)	(0.0087)	(0.0206)
λ_i	-1.6256	1.02266	0.7156	1.0227	-0.2281	0.3499	-0.1093	1.5340
	(0.0147)	(0.0118)	(0.0065)	(0.0102)	(0.0027)	(0.0034)	(0.0085)	(0.0111)
F	2.9737		1.9575		3.0696		3.8649	
	(0.0225)		(0.0121)		(0.0305)		(0.0218)	
σ_f	0.0010		0.0010		0.0010		0.0010	
	(0.0000)		(0.0001)		(0.0000)		(0.0001)	
σ_o	0.0100		0.0100		0.0904		0.5287	
	(0.0006)		(0.0005)		(0.0008)		(0.0045)	
$\log L$	-69284.78		-89356.46		-52708.30		-88174.21	

Table 4.3: The parameter estimates for the two-factor hump-shaped volatility model.

	1990 - 1994			1995-1999		
	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$
κ_{0i}	0.0954	0.5418	0.3698	0.4173	1.0789	0.2701
	(0.0011)	(0.0087)	(0.0056)	(0.0062)	(0.0098)	(0.0024)
κ_i	1.8975	0.4884	0.0010	1.2634	0.4559	0.0010
	(0.0101)	(0.0069)	(0.0000)	(0.0095)	(0.0057)	(0.0000)
η_i	0.3495	0.0010	0.5720	1.3273	0.2104	0.9548
	(0.0036)	(0.0001)	(0.0076)	(0.0105)	(0.0025)	(0.0099)
μ_i^V	0.0010	0.0010	1.5480	0.0010	0.0010	2.0000
	(0.0001)	(0.0001)	(0.0112)	(0.0001)	(0.0001)	(0.0129)
ε_i^V	0.5212	1.2462	0.4184	0.4483	1.5386	0.0011
	(0.0074)	(0.0113)	(0.0046)	(0.0041)	(0.0132)	(0.0001)
ρ_i	-0.0078	-0.0800	0.2363	0.1076	-0.5136	0.2154
	(0.0005)	(0.0010)	(0.0039)	(0.0021)	(0.0042)	(0.0018)
λ_i^V	-1.0068	1.0279	-1.0058	-0.8542	0.9683	-0.9559
	(0.0270)	(0.0162)	(0.0183)	(0.0187)	(0.0224)	(0.0198)
λ_i	0.7370	1.0312	-0.7494	0.9585	1.0204	-0.7919
	(0.0127)	(0.0218)	(0.0115)	(0.0227)	(0.0321)	(0.0175)
F		1.9250			1.9029	
		(0.0127)			(0.0109)	
σ_f		0.0010			0.0010	
		(0.0000)			(0.0001)	
σ_o		0.0180			0.0100	
		(0.0005)			(0.0004)	
$\log L$		-43294.92			-80252.78	

Table 4.4: The parameter estimates for the three-factor hump-shaped volatility model for the periods 1990 - 1995 and 1996 - 1999.

	2000 - 2005			2006 - 2010		
	$i=1$	$i=2$	$i=3$	$i=1$	$i=2$	$i=3$
κ_{0i}	0.4394	0.1447	0.0010	0.1047	0.4782	0.5435
	(0.0068)	(0.0011)	(0.0000)	(0.0009)	(0.0071)	(0.0059)
κ_i	0.0033	1.1032	0.0940	1.9057	0.4443	0.0010
	(0.0002)	(0.0111)	(0.0008)	(0.0186)	(0.0054)	(0.0001)
η_i	1.3300	0.9989	0.0010	0.2685	0.0053	0.5524
	(0.0122)	(0.0101)	(0.0001)	(0.0021)	(0.0004)	(0.0045)
μ_i^V	0.0010	0.0010	7.9991	0.0010	0.0010	1.5728
	(0.0001)	(0.0001)	(0.0358)	(0.0001)	(0.0001)	(0.0112)
ε_i^V	2.3831	3.0000	3.0000	0.5186	1.2218	0.3085
	(0.0173)	(0.0214)	(0.0225)	(0.0046)	(0.0099)	(0.0024)
ρ_i	-0.3803	-0.1123	0.7199	-0.0156	-0.0852	0.2346
	(0.0033)	(0.0008)	(0.0040)	(0.0007)	(0.0012)	(0.0022)
λ_i^V	-4.0000	-3.9995	0.3563	-1.0017	1.0357	-1.0003
	(0.0211)	(0.0225)	(0.0030)	(0.0102)	(0.0098)	(0.0101)
λ_i	2.2278	1.7151	4.0000	0.7387	1.0238	-0.7623
	(0.0210)	(0.0164)	(0.0267)	(0.0043)	(0.0099)	(0.0065)
F	3.1410			1.9324		
	(0.0087)			(0.0015)		
σ_f	0.0010			0.0010		
	(0.0000)			(0.0000)		
σ_O	0.0487			0.1321		
	(0.0002)			(0.0008)		
$\log L$	-48460.89			-36962.34		

Table 4.5: The parameter estimates for the three-factor hump-shaped volatility model for the periods 2000 - 2005 and 2006 - 2010.

Appendix E presents the model specifications that allow for exponential volatility structures only and the estimation results for the exponential volatility model.

All of the volatility factors are highly persistent (evidenced by the very low value of μ_i^V), suggesting that they are important for the pricing of futures and options of all maturities. For each of the subsamples, the innovation to at least one of the volatility factors has a very low correlation (absolute values from $0.7\% - 11\%$) with the innovations to the futures prices, implying the large extent to which the volatility is unspanned by the futures contracts.

We note that there were three major events that affected the volatility of the crude oil market, namely the Gulf War 1990-1991, the Iraq War 2003 and the Global Financial Crisis 2008. The implied volatility especially for short-dated options increased by more than 100% over the 1991 and 2003 crises while implied volatilities for both short-dated and long-dated op-

Subsample	σ_1	Shape	σ_2	Shape	σ_3	Shape
1990 - 1994	28.80%	Hump	57.27%	Hump(e)	13.93%	Exp decaying
1995 - 1999	1.18%	Hump	77.55%	Hump	21.27%	Exp decaying
2000 - 2005	0.75%	Exp decaying	76.04%	Hump	23.21%	Hump(e)
2006 - 2010	26.12%	Hump	52.64%	Hump(e)	21.24%	Exp decaying

Table 4.6: Shape and contribution of each volatility factor to the total variance for the three-factor models.

tions increased by 90% and 50%, respectively, over the 2008 crisis. Furthermore the effect of the shock to the implied volatility was more persistent over the 2008 crisis. We certainly expect the parameter estimates to be affected by these extreme market conditions. Nevertheless, Figure 4.4 shows that our estimates did pick up some, if not all, of these effects.

4.2 Pricing performance

Figure 4.5 graphs the RMSEs of the percentage differences between actual and fitted futures prices as well as of the difference between actual and fitted implied option volatilities, whereas Table 4.7 gives the average values. The overall goodness of fit is quite good, except during the special events of 1991, 2003 and 2008. Table 4.7 also compares the goodness of the fit of the hump-shaped volatility specification to the exponential decaying specification. The log likelihood ratio tests clearly favour the hump volatility specification. The improvement for the fit of futures prices averages at 2.4%. The improvement for the fit of option implied volatility is not much for the period 1995-2005, but very significant during the periods 1990-1994 and 2006-2010 at 4.15% and 10.86% respectively.

Sample	hump-shaped		improvement compared to exponential decaying		
	futures	option	futures	options	$\ln L_0 - \ln L_1$
1990 - 1994	0.0145	0.0339	1.69%	4.15%	20401.56
1995 - 1999	0.0128	0.0354	2.85%	0.76%	8861.78
2000 - 2005	0.0159	0.0173	1.54%	1.79%	9756.18
2006 - 2010	0.0155	0.0148	3.57%	10.86%	14264.89

RMSEs for futures are the percentage differences between actual and fitted futures prices.
RMSEs for options are the differences between actual and fitted implied option volatilities.

Table 4.7: Goodness of the fit of the model - RMSEs

Figure 4.6 displays time series of implied volatilities and the fit to the three-factor model. There was a lot of fluctuation in the implied volatilities over the last 21 years. The model does well in capturing these changes, as well as the special periods of the Gulf War $1990 - 1991$, the Iraq War 2003 and the Global Financial Crisis 2008.

5   Hedging Performance

To gauge the impact of the hump-shaped volatility specification compared to exponential decaying only volatility specification, we assess the hedging performance of option portfolios on crude oil futures by using the hedge ratios implied by the corresponding models. The various factors of the model manifested by the empirical analysis represent different dimensions of risk to which a portfolio of oil derivatives is exposed. In our stochastic volatility modelling framework, the variation in the crude oil forward curve is instigated by random changes of these forward curve volatility factors as well as random changes in a general stochastic volatility factor. By extending the traditional factor hedging method to accommodate the stochastic volatility specification, a set of futures and futures options are used to hedge the risk associated to the forward curve variation. The technical details of the extended factor hedging are presented in Appendix F.

The portfolio that we choose to hedge is a straddle, which is a typical option portfolio that is traded in these markets and is sensitive to volatility. A long straddle consisting of a call and a put with the same strike of 130 and the same maturity of February 2009 is constructed and hedged by using weights implied by the three-factor models in Section 4.[9] The hedging period is from August 1, 2008 to the straddle maturity of February 17, 2009.

A large number of derivative contracts are available to serve as the hedging instruments. Motivated by the presence of unspanned stochastic volatility, we will start by using futures contracts alone, then using a mixture of futures and option contracts. The 3 futures contracts that will be used have maturities of six-months, nine-months and one-year (ie. February 2009, May 2009 and August 2009), chosen due to their liquidity. The three options contracts used as hedging instruments have the same maturities as the (three) futures contracts but different strikes from the target option. Their strikes are $133, 128$ and 132.5 respectively.

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[9]Seeking a representative example of a period in which the market was very volatile, the hedging performance of an option during the financial crisis in 2008 has been selected. The hedging result is not sensitive to the particular straddle chosen.

We apply a small shock to the system, including both the shock to the stochastic volatility component and the shock directly to the future curve. We then calculate the hedging portfolio weight so that the resulting portfolio is delta neutral, delta-vega neutral, or delta-gamma neutral. The portfolio is then rebalanced every forthnight. The daily P&L of the hedged and unhedged positions are computed, by using the root mean squared error (RMSE) to assess the hedging performance. The daily P&L of a perfect hedge should be 0. The RMSE of our hedged position is computed as

$$RMSE_{hedge} = \sqrt{\sum_{day}(P\&L)^2_{day}}.$$

We repeat the procedure 10,000 times for each of the two model specifications (hump shaped volatility and exponential volatility), under each of the three different hedging schemes, with different combinations of hedging instruments. Table 5.8 shows the hedging errors of the best hedged portfolios. The reported R-squared is the percentage of the variation accounted for in the residuals of the unhedged positions.

	RMSE		$R^2(\%)$	
	hump	exp	hump	exp
Unhedged	2.6170	2.6170		
Delta Hedge (3 futures)	2.2152	2.7465	28.3496	0
Delta Hedge (2 futures + 1 option)	2.0811	2.6081	36.7620	0.6790
Delta-Gamma Hedge (3 futures + 3 options)	1.7513	2.4556	55.2170	11.9544
Delta-Vega Hedge (3 futures + 3 options)	1.5301	2.0576	65.8153	38.1821

Table 5.8: Example: Hedging performance of factor hedging for straddles with forthnightly re-balancing

Regarding the best hedged positions, three observations stand out. First, delta hedging is not as effective as delta-gamma and delta-vega hedging, confirming the existence of stochastic volatility. Moreover, the significant improvement from delta hedging to delta-vega hedging highlights the relative importance of the volatility shocks. Second, hedging performance improves when we replace futures with options as the hedging instruments, accentuating the feature of unspanned futures volatility. Third, hedging performance is always better under the hump shaped volatility specification compared to the exponential volatility specification. Under the simple delta hedging scheme, the hedge under the hump volatility specification can explain 28.3% of of the variation of the unhedged residuals, whereas the hedge under the exponential specification can explain virtually none of the variation. R-squared for the hump

volatility specification increases to 65.8% with the more sophisticated delta-vega hedging scheme.

To understand whether this best hedging performance is representative of the hedging performance in general, we investigate the stability of the hedging performance. Table 5.9 shows the standard deviation of the hedging errors when we apply 10,000 different shocks to the system. The hedging performance is quite stable under the hump shaped volatility specification. On the contrary, the exponential volatility specification results in a very wide range of hedging errors. This result clearly favours the use of the humped shape volatility specification.

	Standard Deviation	
	hump	exp
Delta Hedge (3 futures)	0.6823	4.4529
Delta-Gamma Hedge (3 futures + 3 options)	1.2247	30.5192
Delta-Vega Hedge (3 futures + 3 options)	0.1728	11.4623

Table 5.9: Stability of dynamic hedges

6   Conclusion

A multi-factor stochastic volatility model for commodity futures curves within the Heath, Jarrow, and Morton (1992) framework is proposed. The model aims to capture the main characteristics of the volatility structure in commodity futures markets. The model accommodates exogenous stochastic volatility processes that may be partially unspanned by futures contracts. We specify a hump component for the volatility of the futures curves, which can generate a finite dimensional Markovian forward model. The resulting model is highly tractable with quasi-analytical prices for European options on futures contracts.

The model was fitted to an extensive database of crude oil futures prices and option prices traded in the NYSE over 21 years. We find supporting evidence for three volatility factors, two of which exhibit a hump. This provides new evidence on the volatility structure in crude oil futures markets, which has been traditionally modelled with exponentially declining volatility functions. Finally, by using hedge ratios implied by the proposed unspanned hump-shaped stochastic volatility model, the hedging performance of factor hedging schemes is examined. The results favour the proposed model compared to a model with only exponential decaying volatility.

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The current work suggests new developments in commodity market modelling. Firstly, it will be interesting to verify the existence of humps in the volatility structure of other commodities. Our methodology is generic and can be adapted to any commodity futures market. Additionally, the current model can be adjusted to accommodate stochastic convenience yield and stochastic interest rates. This direction has the potential to provide useful insights on the features of convenience yields in commodity markets.

A Proof of Lemma 2.2

Define the process $X(t,T) = \ln F(t,T,\mathbf{V_t})$. Then an application of the Ito's formula derives

$$dX(t,T) = -\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2(t,T,\mathbf{V_t})dt + \sum_{i=1}^{n}\sigma_i(t,T,\mathbf{V_t})dW_i(t). \tag{A.1}$$

By integrating (A.1) we obtain

$$F(t,T,\mathbf{V_t}) = F(0,T)\exp\left[-\frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,T,\mathbf{V_s})ds + \sum_{i=1}^{n}\int_0^t \sigma_i(s,T,\mathbf{V_s})dW_i(s)\right]. \tag{A.2}$$

For $t = T$, (A.2) derives the dynamics of the commodity spot price as

$$S(t,\mathbf{V_t}) = F(0,t)\exp\left[-\frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t,\mathbf{V_s})ds + \sum_{i=1}^{n}\int_0^t \sigma_i(s,t,\mathbf{V_s})dW_i(s)\right], \tag{A.3}$$

or equivalently,

$$\ln S(t,\mathbf{V_t}) = \ln F(0,t) + \left[-\frac{1}{2}\sum_{i=1}^{n}\int_0^t \sigma_i^2(s,t,\mathbf{V_s})ds + \sum_{i=1}^{n}\int_0^t \sigma_i(s,t,\mathbf{V_s})dW_i(s)\right].$$

By differentiating, it follows that $S(t,\mathbf{V_t})$ satisfies the stochastic differential equation (2.7).

B Proof of Lemma 2.5

We consider the process $X(t,T) = \ln F(t,T,\mathbf{V_t})$ and by integrating (A.1) we obtain (A.2). We need to calculate

$$I = \int_0^t \sigma_i(u,T,\mathbf{V_u}^i)dW_i(u) \tag{B.4}$$

$$II = \int_0^t \sigma_i^2(u,T,\mathbf{V_u}^i)du \tag{B.5}$$

We substitute the volatility specifications (2.9) to obtain

$$
\begin{aligned}
I &= \int_0^t (\kappa_{0i} + \kappa_i(T-u)) e^{-\eta_i(T-u)} \sqrt{\mathbf{V_u}^i} dW_i(u) \\
&= \int_0^t (\kappa_{0i} + \kappa_i(T-t+t-u)) e^{-\eta_i(T-t+t-u)} \sqrt{\mathbf{V_u}^i} dW_i(u) \\
&= \beta_{i1}(T-t) \int_0^t e^{-\eta_i(t-u)} \sqrt{\mathbf{V_u}^i} dW_i(u) + \beta_{i2}(T-t) \int_0^t (t-u) e^{-\eta_i(t-u)} \sqrt{\mathbf{V_u}^i} dW_i(u) \\
&= \beta_{i1}(T-t) \upsilon_{i1}(t) + \beta_{i2}(T-t) \upsilon_{i2}(t)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{i1}(T-t) &= (\kappa_{0i} + \kappa_i(T-t)) e^{-\eta_i(T-t)} \\
\beta_{i2}(T-t) &= \kappa_i e^{-\eta_i(T-t)}
\end{aligned}
$$

and the state variables are defined by

$$
\begin{aligned}
\upsilon_{i1}(t) &= \int_0^t e^{-\eta_i(t-u)} \sqrt{\mathbf{V_u}^i} dW_i(u), \\
\upsilon_{i2}(t) &= \int_0^t (t-u) e^{-\eta_i(t-u)} \sqrt{\mathbf{V_u}^i} dW_i(u).
\end{aligned}
\tag{B.6}
$$

Next

$$
\begin{aligned}
II &= \int_0^t \sigma_i^2(u, T, \mathbf{V_u}^i) du \\
&= \int_0^t (\kappa_{0i} + \kappa_i(T-u))^2 e^{-2\eta_i(T-u)} \mathbf{V_u}^i du \\
&= \int_0^t \left(\beta_{i1}(T-t) e^{-\eta_i(t-u)} + \beta_{i2}(T-t)(t-u) e^{-\eta_i(t-u)} \right)^2 \mathbf{V_u}^i du \\
&= \int_0^t (\gamma_{i1}(T-t) + \gamma_{i2}(T-t)(t-u) + \gamma_{i3}(T-t)(t-u)^2) e^{-2\eta_i(t-u)} \mathbf{V_u}^i du \\
&= \gamma_{i1}(T-t) x_i(t) + \gamma_{i2}(T-t) y_i(t) + \gamma_{i3}(T-t) z_i(t)
\end{aligned}
$$

where

$$
\gamma_{i1}(T-t) = \beta_{i1}(T-t)^2, \quad \gamma_{i2}(T-t) = 2\beta_{i1}(T-t)\beta_{i2}(T-t), \quad \gamma_{i3} = \beta_{i2}(T-t)^2.
$$

We define the state variables

$$\mathsf{x}_i(t) = \int_0^t e^{-2\eta_i(t-u)} \mathbf{V_u}^i du,$$

$$\mathsf{y}_i(t) = \int_0^t (t-u) e^{-2\eta_i(t-u)} \mathbf{V_u}^i du, \tag{B.7}$$

$$\mathsf{z}_i(t) = \int_0^t (t-u)^2 e^{-2\eta_i(t-u)} \mathbf{V_u}^i du.$$

Hence by differentiating we find that

$$d\mathsf{x}_i(t) = (-2\eta_i \mathsf{x}_i(t) + \mathbf{V_t}^i) dt,$$

$$d\mathsf{y}_i(t) = (-2\eta_i \mathsf{y}_i(t) + \mathsf{x}_i(t)) dt,$$

$$d\mathsf{z}_i(t) = (-2\eta_i \mathsf{z}_i(t) + 2\mathsf{y}_i(t)) dt.$$

C   Characteristic Function

We consider the characteristic function

$$\phi(t; v, T_o, T) =: \mathbb{E}_t[\exp\{v \ln F(T_o, T)\}]$$
$$= \mathbb{E}_t[\mathbb{E}_{T_0}[\exp\{v \ln F(T_o, T)\}]] = \mathbb{E}_t[\phi(T_0; v, T_o, T)].$$

Therefore the process $\mathbf{k}(t) = \phi(t; v, T_o, T)$ is a martingale under the risk-neutral measure. Given that $\mathbf{k}(t)$ should be of the form (2.20), an application of Ito's lemma yields that

$$\frac{d\mathbf{k}(t)}{\mathbf{k}(t)} = \left(\frac{dM(t)}{dt} + \sum_{i=1}^n \frac{dN_i(t)}{dt} \mathbf{V_t}^i \right) dt + \sum_{i=1}^n N_i(t) d\mathbf{V_t}^i + v \frac{dF(t,T)}{F(t,T)}$$

$$+ \frac{1}{2} \sum_{i=1}^n N_i^2(t)(d\mathbf{V_t}^i)^2 + \frac{v^2 - v}{2} \left(\frac{dF(t,T)}{F(t,T)} \right)^2 + v \sum_{i=1}^n N_i(t) d\mathbf{V_t}^i \frac{dF(t,T)}{F(t,T)} \tag{C.8}$$

$$+ \sum_{j \neq i=1}^n N_i(t) N_j(t) d\mathbf{V_t}^i d\mathbf{V_t}^j.$$

The drift of this sde should be zero, thus

$$0 = \frac{dM(t)}{dt} + \sum_{i=1}^{n} \frac{dN_i(t)}{dt} \mathbf{V_t}^i + \sum_{i=1}^{n} N_i(t) \mu_i^V (\nu_i^V - \mathbf{V_t}^i)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} N_i^2(t)(\varepsilon_i^V)^2 \mathbf{V_t}^i + \frac{v^2 - v}{2} \sum_{i=1}^{n} \left((\kappa_{0i} + \kappa_i(T-t))e^{-\eta_i(T-t)} \sqrt{\mathbf{V_t}^i} \right)^2 \quad \text{(C.9)}$$

$$+ v \sum_{i=1}^{n} N_i(t)\varepsilon_i^V \sqrt{\mathbf{V_t}^i} \rho_i(\kappa_{0i} + \kappa_i(T-t))e^{-\eta_i(T-t)} \sqrt{\mathbf{V_t}^i}.$$

By using $\varphi_i = (\kappa_{0i} + \kappa_i(T-t))e^{-\eta_i(T-t)}$ then from (C.9) we obtain the ODE (2.21) and (2.22) for $M(t)$ and $N_i(t)$ respectively.

D   Appendix: Extended Kalman Filter

D.1   The extended Kalman filter

Our model consists of 2 sets of equations. The first one is the system equation that describes the evolution of the state variables, namely

$$X_{t+1} = \Phi_0 + \Phi_X X_t + w_{t+1}, \ w_{t+1} \sim iid \, N(0, Q_t), \quad \text{(D.10)}$$

whereas the second one is the observation equation that links the state variables with the market-observable variables and is of the form

$$z_t = h(X_t) + u_t \ u_t \sim iid \, N(0, \Omega). \quad \text{(D.11)}$$

It is noted that the h function is nonlinear here.

Let $\hat{X}_t = E_t[X_t]$ and $\hat{X}_{t|t-1} = E_{t-1}[X_t]$ denote the expectations of X_t at t and $t-1$ respectively and let P_t and $P_{t|t-1}$ denote the corresponding estimation error covariance matrices. Linearizing the $h-$ function around $\hat{X}_{t|t-1}$ we obtain,

$$z_t = (h(\hat{X}_{t|t-1}) - H_t' \hat{X}_{t|t-1}) + H_t' X_t + u_t, \ u_t \sim iid \, N(0, \Omega), \quad \text{(D.12)}$$

where

$$H_t' = \frac{\partial h(X_t)}{\partial X_t'}|_{X_t = \hat{X}_{t|t-1}}. \quad \text{(D.13)}$$

The Kalman filter yields

$$\hat{X}_{t+1|t} = \Phi_0 + \Phi_X \hat{X}_t, \tag{D.14}$$

$$P_{t+1|t} = \Phi_X P_t \Phi_X' + Q_t, \tag{D.15}$$

and

$$\hat{X}_{t+1} = \hat{X}_{t+1|t} + P_{t+1|t} H_t' F_t^{-1} \epsilon_t, \tag{D.16}$$

$$P_{t+1} = P_{t+1|t} - P_{t+1|t} H_t' F_t^{-1} H_t P_{t+1|t}, \tag{D.17}$$

where

$$\epsilon_t = z_{t+1} - h(\hat{X}_{t+1|t}), \tag{D.18}$$

$$F_t = H_t P_{t+1|t} H_t' + \Omega. \tag{D.19}$$

The log-likelihood function is constructed as

$$\log L = -\frac{1}{2}\log(2\pi)\sum_{t=1}^{T} N_t - \frac{1}{2}\sum_{t=1}^{T}\log|F_t| - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t' F_t^{-1}\epsilon_t. \tag{D.20}$$

D.2 The system equation

The dynamics of the state vector under the physical measure can be written as

$$dX_t^i = (\Psi_i - \mathcal{K}_i X_t^i)dt + \sqrt{V_t^i}\Sigma_i dW_i^P(t)$$

where $X_t^i = (x_i(t), y_i(t), z_i(t), \phi_i(t), \psi_i(t), \mathbf{V_t}^i)'$, and $W_i^P(t) = (W_i^1(t), W_i^2(t))'$, and

$$\Psi_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mu_i^V \nu_i^V \end{pmatrix}, \mathcal{K}_i = \begin{pmatrix} 2\eta_i & 0 & 0 & 0 & 0 & -1 \\ -1 & 2\eta_i & 0 & 0 & 0 & 0 \\ 0 & -2 & 2\eta_i & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_i & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \eta_i \\ 0 & 0 & 0 & 0 & 0 & \mu_i^V \end{pmatrix}, \Sigma_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \varepsilon_i^V \end{pmatrix} \cdot R_i$$

where R_i is the correlation matrix for the Wiener processes, i.e. $dW_i^P(t)dW_i^P(t)' = R_i dt$ with

$$R_i = \begin{pmatrix} 1 & 0 \\ \rho_i & \sqrt{1-\rho_i^2} \end{pmatrix}.$$

Applying Ito's Lemma to $e^{\mathcal{K}_i t}X_t^i$, we have

$$
\begin{aligned}
d(e^{\mathcal{K}_i t}X_t^i) &= e^{\mathcal{K}_i t}\mathcal{K}_i X_t^i dt + e^{\mathcal{K}_i t}dX_t^i \\
&= e^{\mathcal{K}_i t}\Psi_i dt + e^{\mathcal{K}_i t}\sqrt{V_t^i}\Sigma dW^P(t).
\end{aligned}
\tag{D.21}
$$

It follows that $X_s^i, s > t$ is given as

$$X_s^i = e^{-\mathcal{K}_i(s-t)}X_t^i + \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i du + \int_t^s e^{-\mathcal{K}_i(s-u)}\sqrt{V_u^i}\Sigma_i dW_i^P(u)$$

The conditional mean of X_s^i, given time t information, is given by

$$E_t[X_s^i] = \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i du + e^{-\mathcal{K}_i(s-t)}X_t^i.
\tag{D.22}$$

and the conditional covariance matrix of X_s^i, given time-t information, is given by

$$
\begin{aligned}
Cov_t[X_s^i] &= E_t\left[\left(\int_t^s e^{-\mathcal{K}_i(s-u)}\sqrt{V_u^i}\Sigma_i dW^P(u)\right)\left(\int_t^s e^{-\mathcal{K}_i(s-u)}\sqrt{V_u^i}\Sigma_i dW^P(u)\right)'\right] \\
&= \int_t^s E_t[V_u^i]e^{-\mathcal{K}_i(s-u)}\Sigma_i R_i \Sigma_i' e^{-\mathcal{K}_i'(s-u)}du \\
&= \int_t^s \left(1 - e^{-\mu_i^V(u-t)}\right)\nu_i^V e^{-\mathcal{K}_i(s-u)}\Sigma_i R_i \Sigma_i' e^{-\mathcal{K}_i'(s-u)}du \\
&+ \left(\int_t^s e^{-\mu_i^V(u-t)}e^{-\mathcal{K}_i(s-u)}\Sigma_i R_i \Sigma_i' e^{-\mathcal{K}_i'(s-u)}du\right)V_t^i.
\end{aligned}
\tag{D.23}
$$

Putting the three factors together, we obtain

$$X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix}, W^P(t) = \begin{pmatrix} W_1^P(t) \\ W_2^P(t) \\ W_3^P(t) \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{K}_2 & 0 \\ 0 & 0 & \mathcal{K}_3 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix},$$

$$Cov_t[X_s] = \begin{pmatrix} Cov_t[X_s^1] & 0 & 0 \\ 0 & Cov_t[X_s^2] & 0 \\ 0 & 0 & Cov_t[X_s^3] \end{pmatrix}$$

The system equation, therefore, can be written in discrete form as

$$X_{t+1} = \Phi_0 + \Phi_X X_t + w_{t+1}, \ w_{t+1} \sim iid \, N(0, Q_t), \tag{D.24}$$

where

$$\Phi_0 = \int_t^{t+dt} e^{-\mathcal{K}(t+dt-u)} \Psi du, \ \Phi_X = e^{-\mathcal{K}dt},$$

and Q_t can be derived directly from (D.23).

E   Models with exponential decaying volatility

E.1   Volatility functions

Proposition: If the volatility function has the form $\sigma_i(t, T, V_t^i) = \kappa_{0i} \exp\left(-\eta_i(T-t)\right) \sqrt{V_t^i}$ then the logarithm of the time$-t$ instantaneous futures prices at time T, $\ln F(t, T)$, is given by

$$\ln F(t, T, V_t) = \ln F(0, T, V_0) + \sum_{i=1}^{3} \left(\beta_{i1}(T-t)x_i(t) - \frac{1}{2}\beta_{i2}(T-t)y_i(t) \right) \tag{E.25}$$

where $x_i(t), y_i(t)$ evolve according to

$$dx_i(t) = -\eta_i x_i(t)dt + \sqrt{V_t^i}dW_i(t), \tag{E.26}$$

$$dy_i(t) = (-2\eta_i y_i(t) + V_t^i)dt, \tag{E.27}$$

$$\tag{E.28}$$

subject to $x_i(0) = y_i(0) = 0$. We also have, for $i = 1, 2, 3$,

$$\beta_{i1}(T-t) = \kappa_{0i}e^{-\eta_i(T-t)}, \tag{E.29}$$

$$\beta_{i2}(T-t) = \kappa_{0i}^2 e^{-2\eta_i(T-t)}. \tag{E.30}$$

$$\tag{E.31}$$

Proof. Similar to the proofs of the case with a hump.

E.2 Transition density

The dynamic of the state vector under the physical measure can be written as

$$dX_t^i = (\Psi_i - \mathcal{K}_i X_t^i)dt + \sqrt{V_t^i}\Sigma_i dW_i^P(t)$$

where, $X_t^i = (x_i(t), y_i(t), V_t^i)'$, and $W_i^P(t) = (W_i^1(t), W_i^2(t))'$, and

$$\Psi_i = \begin{pmatrix} 0 \\ 0 \\ \mu_i^V \nu_i^V \end{pmatrix}, \ \mathcal{K}_i = \begin{pmatrix} \eta_i & 0 & 0 \\ 0 & 2\eta_i & -1 \\ 0 & 0 & \mu_i^V \end{pmatrix}, \Sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \varepsilon_i^V \rho & \varepsilon_i^V \sqrt{1-\rho^2} \end{pmatrix},$$

where R_i is the correlation matrix for the Wiener processes, i.e. $dW_i^P(t)dW_i^P(t)' = R_i dt$
and

$$R_i = \begin{pmatrix} 1 & 0 \\ \rho_i & \sqrt{1-\rho_i^2} \end{pmatrix}.$$

Applying Ito's Lemma to $e^{\mathcal{K}_i t}X_t^i$, we have

$$\begin{aligned} d(e^{\mathcal{K}_i t}X_t^i) &= e^{\mathcal{K}_i t}\mathcal{K}_i X_t^i dt + e^{\mathcal{K}_i t}dX_t^i \\ &= e^{\mathcal{K}_i t}\Psi_i dt + e^{\mathcal{K}_i t}\sqrt{V_t^i}\Sigma dW^P(t). \end{aligned} \tag{E.32}$$

It follows that $X_s^i, s > t$ is given as

$$X_s^i = e^{-\mathcal{K}_i(s-t)}X_t^i + \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i du + \int_t^s e^{-\mathcal{K}_i(s-u)}\sqrt{V_u^i}\Sigma_i dW_i^P(u).$$

The conditional mean of X_s^i, given time t information, is given by

$$E_t[X_s^i] = \int_t^s e^{-\mathcal{K}_i(s-u)}\Psi_i du + e^{-\mathcal{K}_i(s-t)}X_t^i. \tag{E.33}$$

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and the conditional covariance matrix of X_s^i, given time-t information, is given by

$$
\begin{aligned}
Cov_t[X_s^i] &= E_t\left[\left(\int_t^s e^{-\mathcal{K}_i(s-u)}\sqrt{V_u^i}\Sigma_i dW^P(u)\right)\left(\int_t^s e^{-\mathcal{K}_i(s-u)}\sqrt{V_u^i}\Sigma_i dW^P(u)\right)'\right] \\
&= \int_t^s E_t[V_u^i]e^{-\mathcal{K}_i(s-u)}\Sigma_i R_i \Sigma_i' e^{-\mathcal{K}_i'(s-u)}du \\
&= \int_t^s \left(1 - e^{-\mu_i^V(u-t)}\right)\nu_i^V e^{-\mathcal{K}_i(s-u)}\Sigma_i R_i \Sigma_i' e^{-\mathcal{K}_i'(s-u)}du \\
&\quad + \left(\int_t^s e^{-\mu_i^V(u-t)}e^{-\mathcal{K}_i(s-u)}\Sigma_i R_i \Sigma_i' e^{-\mathcal{K}_i'(s-u)}du\right)V_t^i. \quad\quad\text{(E.34)}
\end{aligned}
$$

Putting three factors together, we would get

$$
X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix}, W^P(t) = \begin{pmatrix} W_1^P(t) \\ W_2^P(t) \\ W_3^P(t) \end{pmatrix},
$$

$$
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{K}_2 & 0 \\ 0 & 0 & \mathcal{K}_3 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix},
$$

$$
Cov_t[X_s] = \begin{pmatrix} Cov_t[X_s^1] & 0 & 0 \\ 0 & Cov_t[X_s^2] & 0 \\ 0 & 0 & Cov_t[X_s^3] \end{pmatrix}
$$

E.3 Estimation Results

The parameter estimates for the two-factor exponential volatility model are presented in Table 5.10 and Table 5.11. Parameter estimates for the three-factor exponential volatility volatility model can be found in Table 5.12 and Table 5.13.

		1990 - 1994		1995-1999
	$i=1$	$i=2$	$i=1$	$i=2$
κ_{0i}	0.7650	0.7692	0.2673	1.2553
	(0.0101)	(0.0087)	(0.0026)	(0.0126)
η_i	0.1475	1.3558	0.1378	1.0869
	(0.0006)	(0.0109)	(0.0012)	(0.0122)
μ_i	0.0100	1.1412	0.6317	3.2399
	(0.0003)	(0.0118)	(0.0052)	(0.0219)
ε_i^V	0.8359	1.5542	2	0.8097
	(0.0087)	(0.0131)	(0.0203)	(0.0074)
ρ_i	-0.1071	0.0734	0.0061	0.0346
	(0.0025)	(0.0003)	(0.0001)	(0.0002)
λ_i^V	0.1098	-0.2736	-0.2044	-0.4682
	(0.0012)	(0.0020)	(0.0022)	(0.0067)
λ_i	0.1155	-1.5835	-0.2507	-1.5830
	(0.0027)	(0.0218)	(0.0015)	(0.0157)
F	3.0887		2.9849	
	(0.0227)		(0.0226)	
σ_f	0.0010		0.0010	
	(0.0000)		(0.0001)	
σ_o	0.0100		0.0100	
	(0.0005)		(0.0004)	
$\log L$	-71508.76		-118968.91	

Table 5.10: The two-factor exponential volatility model for the periods 1990 - 1994 and 1995 - 1999.

		2000 - 2005		2006 - 2010
	$i = 1$	$i = 2$	$i = 1$	$i = 2$
κ_{0i}	1.7606	1.5413	2.1136	0.9585
	(0.0201)	(0.0187)	(0.0162)	(0.0096)
η_i	0.0189	1.0204	0.0100	0.9195
	(0.0005)	(0.0099)	(0.0010)	(0.0102)
μ_i	0.0100	3.8547	0.0100	1.3699
	(0.0012)	(0.0287)	(0.0020)	(0.0118)
ε_i^V	0.3520	2	0.3167	2
	(0.0037)	(0.0151)	(0.0033)	(0.0174)
ρ_i	-0.2311	0.0570	-0.1622	-0.1193
	(0.0021)	(0.0004)	(0.0011)	(0.0010)
λ_i^V	0.2568	0.0176	0.3226	0.4541
	(0.0020)	(0.0007)	(0.0022)	(0.0047)
λ_i	0.1021	-1.5832	-0.1757	-1.5832
	(0.0021)	(0.0118)	(0.0014)	(0.0147)
F		2.6574		4.6764
		(0.0217)		(0.0326)
σ_f		0.0010		0.0010
		(0.0000)		(0.0001)
σ_o		0.0560		0.0618
		(0.0006)		(0.0007)
$\log L$		-166933.26		-155215.79

Table 5.11: The two-factor exponential volatility model for the periods 2000 - 2005 and 2006 - 2010.

	1990 - 1994			1995-1999		
	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$
κ_{0i}	1.2195	0.7024	0.5306	0.7549	0.6545	0.3073
	(0.0101)	(0.0087)	(0.0066)	(0.0062)	(0.0058)	(0.0024)
η_i	0.0672	1.0433	0.0638	0.2418	1.2081	0.1812
	(0.0006)	(0.0099)	(0.0006)	(0.0022)	(0.0158)	(0.0019)
μ_i	0.0100	0.8102	0.4636	0.1456	3.0024	0.4493
	(0.0002)	(0.0088)	(0.0052)	(0.0011)	(0.0301)	(0.0039)
ε_i^V	0.9529	1.6467	0.0100	0.6071	1.4203	2
	(0.0100)	(0.0131)	(0.0003)	(0.0054)	(0.0112)	(0.0201)
ρ_i	0.1908	-0.0891	-0.9630	-0.0081	0.0120	0.0500
	(0.0025)	(0.0003)	(0.0089)	(0.0001)	(0.0002)	(0.0006)
λ_i^V	0.9168	0.1016	0.0608	0.6161	0.9896	0.5258
	(0.0112)	(0.0020)	(0.0002)	(0.0067)	(0.0094)	(0.0048)
λ_i	-0.2883	-2.1251	-4	-0.5820	-1.4319	-0.1304
	(0.0027)	(0.0218)	(0.0315)	(0.0057)	(0.0132)	(0.0017)
F		3.3139			3.0540	
		(0.0227)			(0.0209)	
σ_f		0.0010			0.0010	
		(0.0000)			(0.0001)	
σ_o		0.0100			0.0100	
		(0.0005)			(0.0004)	
$\log L$		-63696.40			-89114.50	

Table 5.12: The three-factor exponential volatility model for the periods 1990 - 1994 and 1995 - 1999.

	2000 - 2005			2006 - 2010		
	$i = 1$	$i = 2$	$i = 3$	$i = 1$	$i = 2$	$i = 3$
κ_{0i}	0.9220	2.2277	0.3365	1.0144	0.8733	0.3308
	(0.0087)	(0.0211)	(0.0027)	(0.0109)	(0.0071)	(0.0039)
η_i	0.0304	0.9992	0.0307	0.0100	0.4187	0.0143
	(0.0002)	(0.0101)	(0.0003)	(0.0005)	(0.0034)	(0.0010)
μ_i^V	0.0100	0.6694	3.5999	0.0100	2.1468	0.0100
	(0.0011)	(0.0056)	(0.0308)	(0.0001)	(0.0112)	(0.0001)
ε_i^V	0.7205	2	1.2997	0.8725	2	2
	(0.0073)	(0.0174)	(0.0125)	(0.0076)	(0.0199)	(0.0178)
ρ_i	-0.4674	0.0969	0.6958	-0.6028	0.3760	0.0708
	(0.0033)	(0.0008)	(0.0050)	(0.0047)	(0.0032)	(0.0002)
λ_i^V	-0.2296	-0.4694	0.1256	-4	4	0.4564
	(0.0011)	(0.0045)	(0.0030)	(0.0302)	(0.0328)	(0.0051)
λ_i	0.0168	1.7566	-3.9994	0.1188	4	3.9994
	(0.0010)	(0.0164)	(0.0267)	(0.0023)	(0.0299)	(0.0269)
F		2.8234			9.9989	
		(0.0117)			(0.0615)	
σ_f		0.0010			0.0010	
		(0.0000)			(0.0000)	
σ_O		0.0384			0.2384	
		(0.0003)			(0.0012)	
$\log L$		-58217.07			-51227.23	

Table 5.13: The three-factor exponential volatility model for the periods 2000 - 2005 and 2006 - 2010.

F   Factor hedging for a multi-factor stochastic volatility model

Factor hedging is a broad hedging method that allows one to hedge simultaneously the multiple factors impacting the forward curve of commodities and subsequently the value of commodity derivative portfolios. By considering the n factor stochastic volatility model developed in Section 2, the forward curve should be shocked by each of the n forward curve volatility processes. However, by using a stochastic volatility model, initially an appropriate shock to the variance process is applied, see equation (2.2),

$$\Delta \mathbf{V_t}^i = a_i^V(t, \mathbf{V_t})\Delta t + \sigma_i^V(t, \mathbf{V_t})\Delta W_i^V; \ i = 1, \ldots, n. \tag{F.35}$$

Then, a shock to each volatility factor of the multi-factor model (2.1) is applied, namely for $i = 1, \ldots, n$,

$$\Delta F_i(t, T, \mathbf{V_t}) = F(t, T, \mathbf{V_t})\sigma_i(t, T, \mathbf{V_t})\Delta W_i, \tag{F.36}$$

where ΔW_i is specified through its correlation structure with ΔW_i^V, as given by (2.3). By allowing for both positive and negative changes, the corresponding shocks to the forward curve are obtained. The size of the shocks ΔW_i and ΔW_i^V should be chosen to give a typical movement of the curve and the variance over the hedging period, respectively. If Υ denotes the value of a portfolio, then the changes $\Delta \Upsilon_i$ in the value of the portfolio between the downward and upward shifts of the forward curve for each volatility factor i are computed as

$$\Delta \Upsilon_i = \Upsilon(F_{i,U}(t, T, \mathbf{V_t})) - \Upsilon(F_{i,D}(t, T, \mathbf{V_t})); \; i = 1, \ldots, n, \tag{F.37}$$

where the subscript U indicates an up movement of the forward curve embedding the impact of the change in the variance and subscript D indicates the corresponding down movement of the forward curve.

F.1 Delta Hedging

For an n factor model, factor delta hedging necessitates n hedging instruments. The hedging instruments could be futures contracts or options contracts, but with different maturities. We denote their values by $\Psi(t, T_j)$ for $j = 1, \ldots, n$. By selecting appropriate positions $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ in these hedging instruments such that, for each factor, the change in the hedged portfolio Υ_H is zero, the following conditions are obtained, for $i = 1, \ldots, n$,

$$\Delta \Upsilon_{H,i} = \Delta \Upsilon_i + \delta_1 \Delta \Psi_i(t, T_1) + \delta_2 \Delta \Psi_i(t, T_2) + \ldots + \delta_n \Delta \Psi_i(t, T_n) = 0. \tag{F.38}$$

The system of equations (F.38) is a system of n linear equations with n unknowns and the $\delta_i, i = 1, \ldots, n$, that can be easily obtained explicitly. The $\Delta \Psi_i(t, T_j)$ can be specified as follows; if the hedging instrument is a futures contract with maturity T_j then from (F.36)

$$\Delta \Psi_i(t, T_j) = \Psi(t, T_j)\sigma_i(t, T_j, \mathbf{V_t})\Delta W_i. \tag{F.39}$$

If the hedging instrument is an option on a futures contract with value $F(t, T_j, \mathbf{V_t})$ then $\Delta \Psi_i(t, T_j) = \Psi(F_{i,U}(t, T_j, \mathbf{V_t})) - \Psi(F_{i,D}(t, T_j, \mathbf{V_t}))$. The conditions (F.38) eliminate only risk generated by small changes in the forward curve without directly accounting for the impact of the changes in the volatility, which are crucial in the stochastic volatility setup of

our model.

F.2 Delta-Vega Hedging

In order to account also for the variation in the volatility process, n additional hedging instruments are required to make the portfolio Υ simultaneously delta-vega neutral. Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ denote the positions held in these hedging instruments that have values of $\Lambda(t, T_j, \mathbf{V}_t)$ for $j = 1, \ldots, n$. The positions β are selected such that, for each factor, the overall change of the hedged portfolio is zero, after applying a shock ΔW_i^V to the variance process. Thus the following conditions should hold for $i = 1, \ldots, n$

$$\Delta\Upsilon_{H,i} = \Delta\Upsilon_i + \beta_1\Delta\Lambda_i(t, T_1) + \beta_2\Delta\Lambda_i(t, T_2) + \ldots + \beta_n\Delta\Lambda_i(t, T_n) = 0, \quad \text{(F.40)}$$

where

$$\Delta\Lambda_i(t, T_j) = \Lambda_i(t, T_j, \mathbf{V}_t^U) - \Lambda_i(t, T_j, \mathbf{V}_t^D). \quad \text{(F.41)}$$

The change $\Delta\Upsilon_i$ as a result of this shock is computed by equation (F.37). By initially solving equation (F.40), the position β in the n hedging instruments is determined. For these positions, the portfolio combining Υ and the n hedging instruments, by construction, has a vega of zero but in general, a non-zero residual delta. We can neutralise the delta of the combined portfolio by taking positions in n additional hedging instruments to satisfy condition (F.38) in which $\Delta\Upsilon_i$ is now the changes of the combined portfolio for each factor i.

F.3 Delta-Gamma Hedging

Sensitivity to large price changes can be controlled by gamma hedging. For the portfolio Υ to be gamma neutral, n hedging instruments are required and more specifically n options (as the gamma of a forward or futures contract is zero). The positions $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ in these hedging instruments, with values $\Phi(t, T_j, \mathbf{V}_t)$ for $j = 1, \ldots, n$, are selected such that the gamma of the hedged portfolio is zero with respect to each factor i, that is

$$\Gamma\Upsilon_{H,i} = \Gamma\Upsilon_i + \gamma_1\Gamma\Phi_i(t, T_1) + \gamma_2\Gamma\Phi_i(t, T_2) + \ldots + \gamma_n\Gamma\Phi_i(t, T_n) = 0, \quad \text{(F.42)}$$

where for $i = 1, \ldots, n$,

$$\Gamma \Upsilon_i = \Upsilon(F_{i,U}(t, T, \mathbf{V}_U)) - 2 \times \Upsilon(F_i(t, T, \mathbf{V})) + \Upsilon(F_{i,D}(t, T, \mathbf{V}_D)); \qquad \text{(F.43)}$$

$$\Gamma \Phi_i = \Phi(F_{i,U}(t, T, \mathbf{V}_U)) - 2 \times \Phi(F_i(t, T, \mathbf{V})) + \Phi(F_{i,D}(t, T, \mathbf{V}_D)). \qquad \text{(F.44)}$$

For these positions, the portfolio combining Υ and the n hedging instruments have a non-zero residual delta. For the portfolio Υ to be simultaneously delta-gamma neutral, we must neutralise also the delta of the combined portfolio by taking positions in n additional hedging instruments to satisfy condition (F.38), as it was done for vega-delta hedging, in which $\Delta \Upsilon_i$ is now the changes of the combined portfolio for each factor i.

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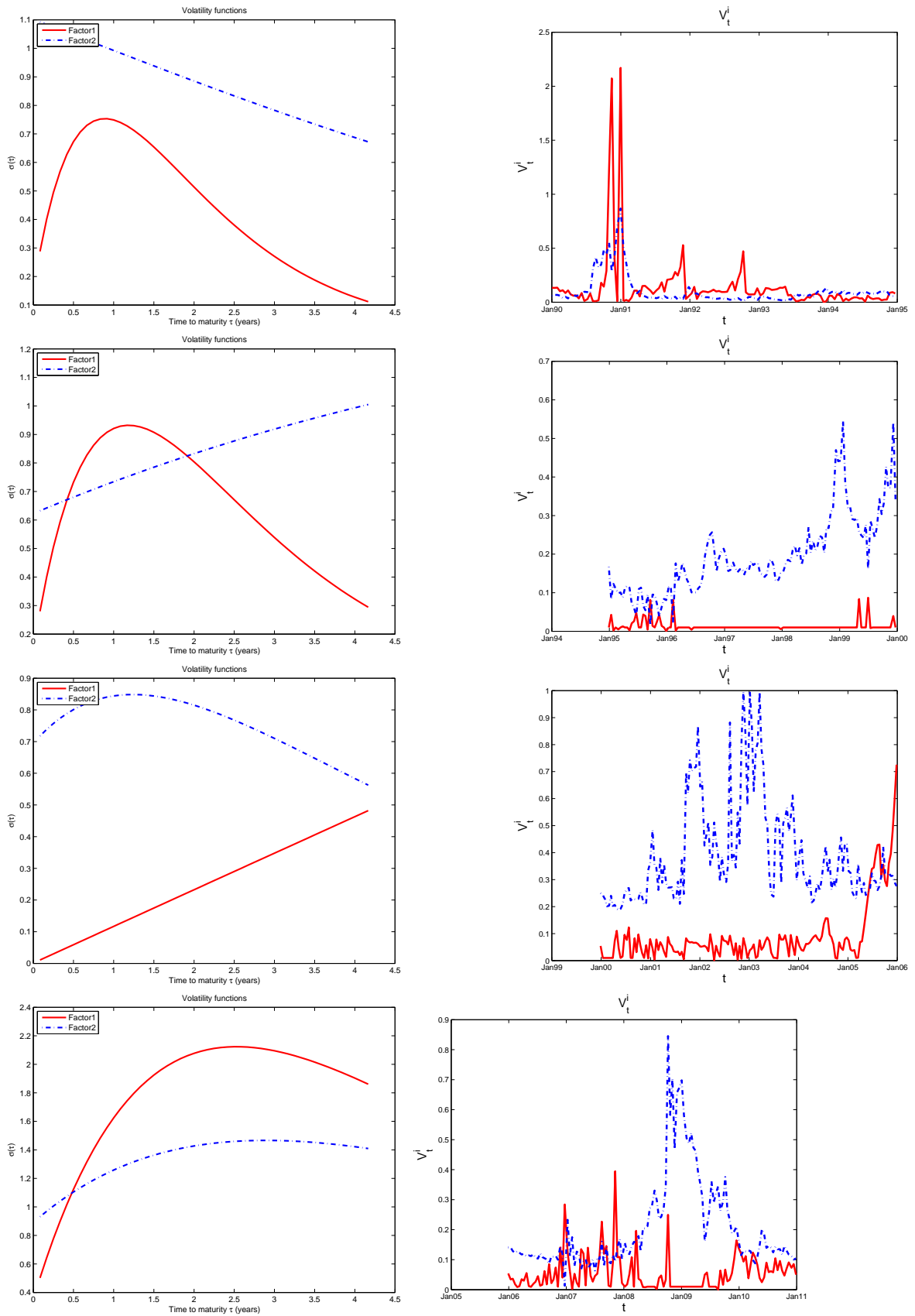
Figure 4.3: $v_i(T - t)$ and $\mathbf{V_t}^i$ for the two-factor model - Top panel: January 1990 to December 1994; Second panel: January 1995 to December 1999; Third panel: January 2000 to December 2005; Bottom Panel: January 2006 to December 2010
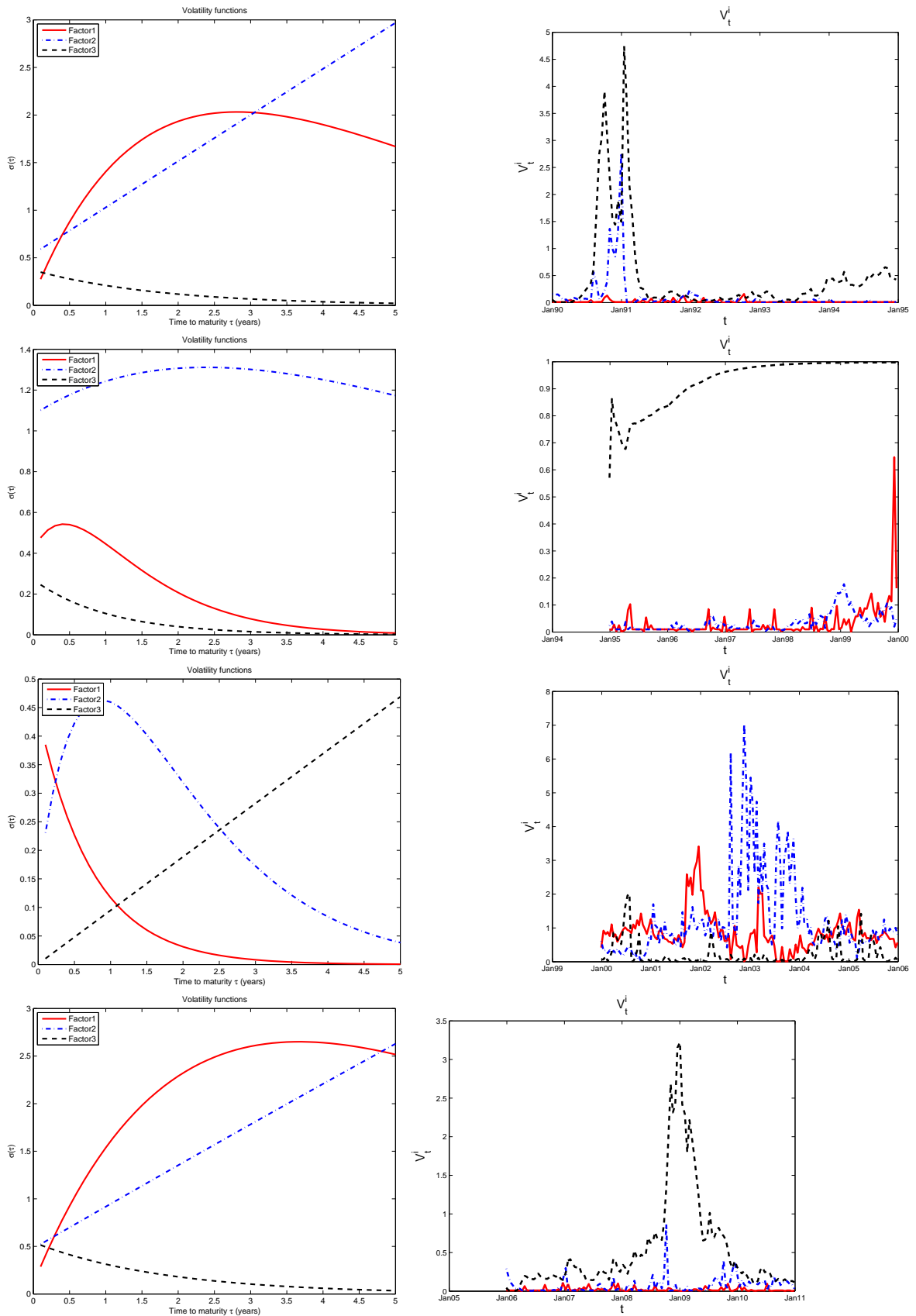
Figure 4.4: $v_i(T - t)$ and $\mathbf{V_t}^i$ for the three-factor model - Top panel: January 1990 to December 1994; Second panel: January 1995 to December 1999; Third panel: January 2000 to December 2005; Bottom Panel: January 2006 to December 2010
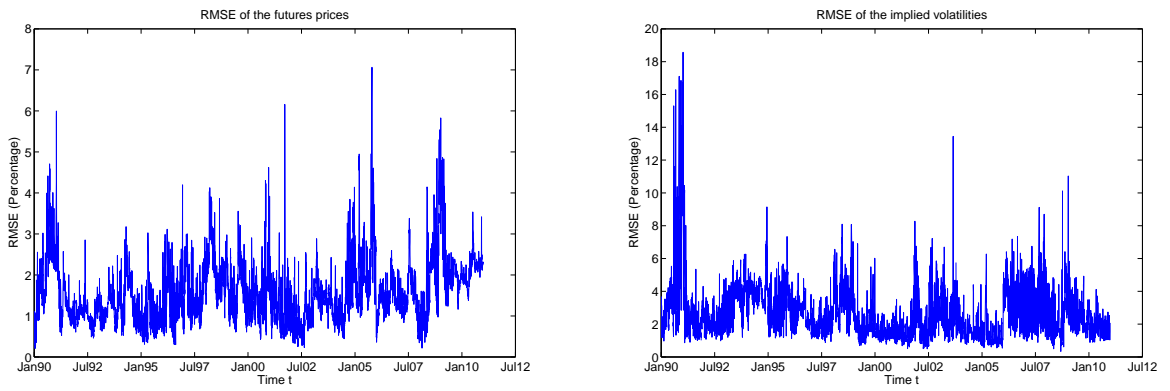
Figure 4.5: RMSEs of the percentage differences between actual and fitted futures prices as well as of the difference between actual and fitted implied option volatilities for a three-factor model from January 1990 to December 2010
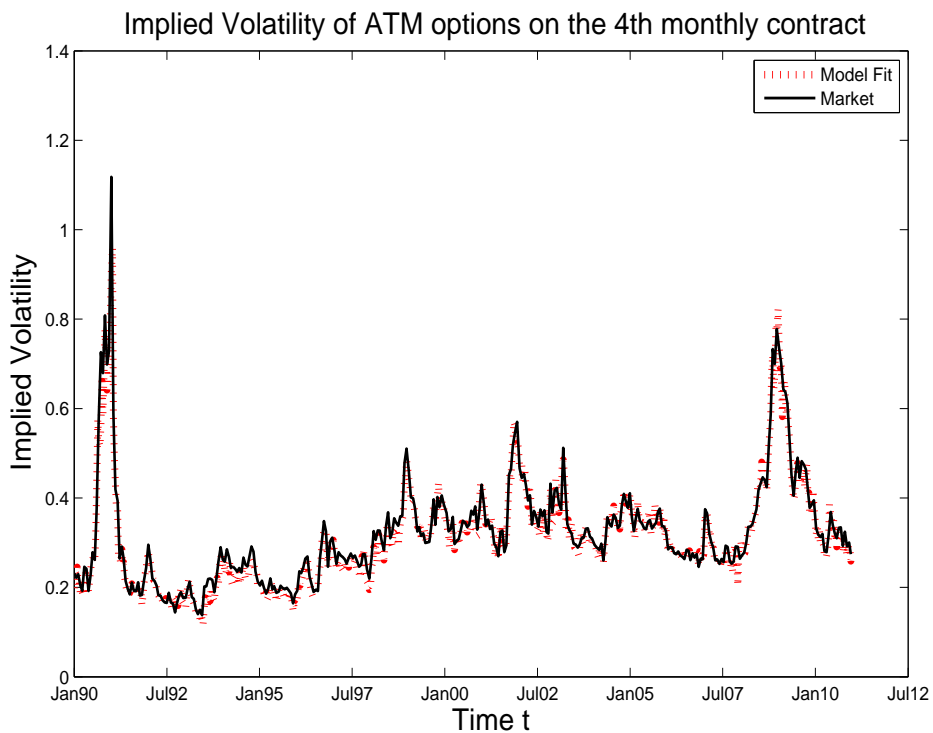

Figure 4.6: Time series of implied volatilities and the fit to the three-factor model

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