HURWITZ CLASS NUMBERS WITH LEVEL AND MODULAR CORRESPONDENCES

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ABSTRACT. In this paper, we prove Hurwitz-Eichler type formulas for Hurwitz class numbers with each level M when the modular curve $X_0(M)$ has genus zero. A key idea is to calculate intersection numbers of modular correspondences with the level in two different ways. A generalization of Atkin-Lehner involutions for $\Gamma_0(M)$ and its subgroup $\Gamma_0^{(M')}(M)$ is introduced to calculate intersection multiplicities of modular correspondences at cusps.

1. INTRODUCTION AND STATEMENT OF RESULTS

For a positive integer M, D with $D \equiv 0, 3 \bmod 4$, let us define

$$H^M(D) := \sum_{[Q] \in \mathcal{Q}^M_{-D,>0}/\Gamma_0(M)} \frac{2}{\#\Gamma_0(M)_Q}$$

and call it the Dth Hurwitz class number of level M. Here, for integers a, b, c, let us write a quadratic form $[a, b, c] := aX^2 + bXY + cY^2$ whose discriminant is $\operatorname{disc} Q := b^2 - 4ac$ and let

$$\mathcal{Q}^M_{-D,>0} := \{Q = [Ma, b, c] \mid a, b, c \in \mathbb{Z}, a > 0, \operatorname{disc} Q = -D\}.$$

The group

$$\Gamma_0(M) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod M \right\}$$

acts on $\mathcal{Q}^M_{-D,>0}$ by

$$\left(Q \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)(X, Y) := Q(aX + bY, cX + dY), \quad Q \in \mathcal{Q}^M_{-D,>0}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M).$$

We denote by $\Gamma_0(M)_Q$ the stabilizer of $Q \in \mathcal{Q}^M_{-D,>0}$ under this action. To compute $\mathcal{Q}^M_{-D,>0}/\Gamma_0(M)$ is equivalent to understand imaginary quadratic points with discriminant $-D$ on a suitable fundamental domain for the modular curve $Y_0(M) := \Gamma_0(M)\backslash\mathbb{H}$ and the related reduction theory as well. This will be carried out in Section 9.

When $M = 1$, we put $H(D) := H^1(D)$. For a positive integer N which is not a square, the following relation is known as Hurwitz-Eichler relation:

$$\sum_{x \in \mathbb{Z},\ x^2 < 4N} H(4N - x^2) = \sum_{ad = N} \max\{a, d\}. \tag{1.1}$$

Eichler's original proof is found in [7]. Another proof will be found in [10] by calculating intersection multiplicities at cusps and intersection number of certain algebraic cycles on $\mathbb{P}^1 \times \mathbb{P}^1$ which are called modular correspondences.

In this paper, we consider an analog of the relation (1.1) for $M > 1$ such that the genus of the modular curve $X_0(M)$ is zero. Our main result is the following theorem:

Theorem 1.1. *Let M be $2 \le M \le 10$ or $M \in \{12, 13, 16, 18, 25\}$. Let N be a positive integer which is coprime to M and is not a square. It holds that*

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(i)
$$\sum_{x\in\mathbb{Z},\ x^2<4N} H^M\left(4N-x^2\right) = \sum_{ad=N} |a-d|$$
if either of the following conditions is satisfied:
(a) $M \in \{2,3,5,7,13\}$,
(b) $M = 9$ *and* $N \equiv -1 \bmod 3$,
(c) $M = 25$ *and* $N \equiv \pm 2 \bmod 5$,

(ii)
$$\sum_{x\in\mathbb{Z},\ x^2<4N} H^4\left(4N-x^2\right) = 2\sum_{ad=N,a>d} (a-2d)$$
if $M = 4$,

(iii)
$$\sum_{x\in\mathbb{Z},\ x^2<4N} H^M\left(4N-x^2\right) = 2\sum_{ad=N,a>d} (a-3d)$$
if either of the following conditions is satisfied:
(a) $M \in \{6,8,10\}$,
(b) $M = 9$ *and* $N \equiv 1 \bmod 3$,
(c) $M = 16$ *and* $N \equiv -1 \bmod 4$,
(d) $M = 18$ *and* $N \equiv -1 \bmod 6$,

(iv)
$$\sum_{x\in\mathbb{Z},\ x^2<4N} H^{12}\left(4N-x^2\right) = 2\sum_{ad=N,a>d} (a-5d)$$
if either of the following conditions is satisfied:
(a) $M = 12$,
(b) $M = 16$ *and* $N \equiv 1 \bmod 4$,

(v)
$$\sum_{x\in\mathbb{Z},\ x^2<4N} H^{18}\left(4N-x^2\right) = 2\sum_{ad=N,a>d} (a-7d)$$
if $M = 18$ *and* $N \equiv 1 \bmod 6$,

(vi)
$$\sum_{x\in\mathbb{Z},\ x^2<4N} H^{25}\left(4N-x^2\right) = \sum_{ad=N} |a-d| - 8\sum_{ad=N,a>d,a\equiv d \bmod 5} d$$
if $M = 25$ *and* $N \equiv \pm 1 \bmod 5$.

To prove Theorem 1.1, we calculate both sides in two ways as is done in the proof of [10, Corollary 1.1]. In our calculation, we use the modular correspondence $T_N^{\Gamma_0(M)}$ with level M and degree N, which is our main theme.

We state a definition of modular correspondences. Let
$$\mathbb{H} := \{\tau = x + y\sqrt{-1} \mid x, y \in \mathbb{R}, y > 0\}$$
be the complex upper half-plane. For a positive integer M, we define the modular curves of level M as
$$Y_0(M) := \Gamma_0(M)\backslash\mathbb{H}, \quad X_0(M) := \Gamma_0(M)\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}).$$
They admit the structure as Riemann surfaces and it turns out that $X_0(M)$ is compact. Each element in $\Gamma_0(M)\backslash(\mathbb{Q} \cup \{i\infty\})$ is called a cusp.

In this paper, we assume that the modular curve $X_0(M)$ has genus zero. It is well-known that such M is $1 \le M \le 10$ or $M = 12, 13, 16, 18, 25$ ([13, Section 3]). For a positive integer N coprime to M, the modular correspondence of degree N with respect to $\Gamma_0(M)$ introduced in [12] is defined by
$$T_N^{\Gamma_0(M)} := \bigcup_{A=\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)\in \mathrm{M}_2(\mathbb{Z}),\, ad=N,\, 0\le b<d} \{(\Gamma_0(M)\tau, \Gamma_0(M)A(\tau)) \in X_0(M) \times X_0(M)\}. \tag{1.2}$$

It turns out that the modular correspondence $T_N^{\Gamma_0(M)}$ is an algebraic cycle in $X_0(M)\times X_0(M)$ by [12, Theorem 2.9]. The modular correspondence $T_N^{\Gamma_0(M)}$ and the diagonal set

$$\Delta := \{(\Gamma_0(M)\tau, \Gamma_0(M)\tau) \in X_0(M)\times X_0(M)\}$$

intersect properly when N is not a square and the intersection number of them on $Y_0(M)\times Y_0(M)$ coincides with the left-hand side in Theorem 1.1 by [12, Theorem 1.2].

On the other hand, we can calculate the intersection number on $Y_0(M)\times Y_0(M)$ by subtracting it on $X_0(M)\times X_0(M)\setminus Y_0(M)\times Y_0(M)$ from it on $X_0(M)\times X_0(M)$. The result coincides with the right-hand side in Theorem 1.1 by the following theorem.

Theorem 1.2. *Let $M\in\{2,3,5,6,7,8,10,12,13\}$ and N be a positive integer coprime to M which is not a square. Then the following holds.*

(i) *The intersection number of Δ and $T_N^{\Gamma_0(M)}$ is*

$$(\Delta\cdot T_N^{\Gamma_0(M)})_{X_0(M)\times X_0(M)} = 2\sum_{d\mid N} d.$$

(ii) *The intersection multiplicity of Δ and $T_N^{\Gamma_0(M)}$ at a pair (s,s) of cusp s is*

$$(\Delta\cdot T_N^{\Gamma_0(M)})_{(s,s)} = 2\sum_{ad=N, a>d} d.$$

(iii) *We have*

$$(\Delta\cdot T_N^{\Gamma_0(M)})_{Y_0(M)\times Y_0(M)} = 2\sum_{ad=N, a>d} (a-(c_0(M)-1)d)$$

where

$$c_0(M) := \#\{\textit{cusps in } X_0(M)\} = \begin{cases} 2 & \textit{if } M\in\{2,3,5,7,13\},\\ 3 & \textit{if } M=4,\\ 4 & \textit{if } M\in\{6,8,10\},\\ 6 & \textit{if } M=12.\end{cases}$$

In the above theorem, we treat only the case when $M\in\{2,3,5,6,7,8,10,12,13\}$ for simplicity. In other remaining cases, we state similar results in Section 8.

The most important part of Theorem 1.2 is to calculate the intersection multiplicities at cusps in (ii). Although we can achieve it by using Atkin-Lehner involutions, they exist less than cusps for some levels and thus we introduce generalized Atkin-Lehner involutions.

Recently, Brunier-Schwagenscheidt gave various interesting formulas involving our generalized Hurwitz class numbers in a different context [3, Example 4.2]. As the second author of loc.cit. commented to the author, it would be interesting to study any relation between our work and theirs.

This paper will be organized as follows. In Section 2, we summarize known results for our modular correspondences $T_N^{\Gamma_0(M)}$. In Section 3, we introduce a subgroup $G_0(M)\subset \mathrm{GL}_2(\mathbb{Q})$ which contains the above matrix A. In Section 4, for each cusp s, we give explicitly a matrix $W\in \mathrm{GL}_2(\mathbb{Q})\cap M_2(\mathbb{Z})$ which satisfies $W(i\infty)=s$ and normalize $\Gamma_0(M)$. When M is square-free, each cusp is represented by the image of $i\infty$ under an Atkin-Lehner involution. However, this is not true when M is not square-free. then there exists a cusp such that there does not exist an For this reason, we introduce a generalization of Atkin-Lehner involutions. In Section 5, we give a condition whether such matrices introduced in Section 4 normalize $\Gamma_0(M)$. In Section 6, we classify cusps. We calculate the intersection multiplicities at cusps in Section 7 and prove Theorem 1.1 in Section 8. In Section 9, we give explicit computation of the Hurwitz class number $H^M(D)$ for M when $2\le M\le 10$ or $M=13$. In Section 10, we give some examples of Theorem 1.1 for small N and conjecture for a square N.

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2. Known results

In this section, we summarize results for modular correspondences in [12]. Let M be a positive integer such that the modular curve $X_0(M)$ has genus zero. In this case, there exists the unique isomorphism $t\colon X_0(M) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$ satisfying $\mathrm{div}(t) = (0) - (\infty)$ and having an expansion

$$t(\tau) = q^{-1} + c_0 + c_1 q + \cdots \in q^{-1} + \mathbb{Z}[[q]]$$

with $q := e^{2\pi\sqrt{-1}\tau}$ for $M > 1$. Such t is given as an explicit products of the Dedekind eta function in [12, Table 1] which refers to [11, Subsection 3.1]. For $M = 1$, we put $t\colon X_0(1) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$ as the j-invariant.

The modular correspondence $T_N^{\Gamma_0(M)}$ defined in (1.2) is an algebraic cycle in $X_0(M) \times X_0(M)$ by the following theorem. We remark that the definition of the modular correspondence $T_N^{\Gamma_0(M)}$ in (1.2) differs from the original definition in [12] but it is essentially the same.

Theorem 2.1 ([12, Theorem 2.9])**.** *For a positive integer N coprime to M, the image of the modular correspondence $T_N^{\Gamma_0(M)}$ under the map $t \times t\colon X_0(M) \times X_0(M) \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is the zero set of a polynomial $\Phi_N^{\Gamma_0(M)}(X, Y) \in \mathbb{Z}[X, Y]$.*

The polynomial $\Phi_N^{\Gamma_0(M)}(X, Y)$ is called the modular polynomial of level M and degree N.

The following theorem states that the intersection number of two modular correspondences on $Y_0(M) \times Y_0(M)$ is equal to the left-hand side in Theorem 1.1.

Theorem 2.2 ([12, Theorem 1.2])**.** *For positive integers N_1 and N_2 coprime to M, two algebraic cycles $T_{N_1}^{\Gamma_0(M)}$ and $T_{N_2}^{\Gamma_0(M)}$ intersect properly if and only if the integer N_1N_2 is not a square. Moreover, in this case, the intersection number on $Y_0(M) \times Y_0(M)$ is given as*

$$\begin{aligned}(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{Y_0(M)\times Y_0(M)} &:= \sum_{(x_0,y_0)\in Y_0(M)\times Y_0(M)} (T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{(x_0,y_0)} \\ &= \sum_{x\in\mathbb{Z},\ x^2<4N_1N_2} \sum_{d\mid(N_1,N_2,x)} d \cdot H^M\left(\frac{4N_1N_2 - x^2}{d^2}\right).\end{aligned}$$

In particular, for a non-square positive integer N coprime to M, we have

$$(T_1^{\Gamma_0(M)} \cdot T_N^{\Gamma_0(M)})_{Y_0(M)\times Y_0(M)} = \sum_{x\in\mathbb{Z},\ x^2<4N} H^M\left(4N - x^2\right).$$

Here we remark that $T_1^{\Gamma_0(M)}$ is equal to the diagonal set Δ by (1.2).

3. A subgroup $G_0(M)$ in $\mathrm{GL}_2(\mathbb{Q})$

In this section, we introduce and study the subgroup $G_0(M)$ of $\mathrm{GL}_2(\mathbb{Q})$ which plays an important role in studying intersections of modular correspondences at cusps.

Let M be a positive integer.

Definition 3.1. *Set*

$$\mathbb{Z}_{(M)} := \left\{\frac{a}{b} \in \mathbb{Q} \;\middle|\; a, b \in \mathbb{Z}, (b, M) = 1\right\},$$

$$\mathrm{GL}_2(\mathbb{Z}_{(M)}) := \left\{\gamma \in M_2(\mathbb{Z}_{(M)}) \mid \det\gamma \in \mathbb{Z}_{(M)}^\times\right\},$$

and

$$G_0(M) := \left\{ \gamma \in \mathrm{GL}_2(\mathbb{Z}_{(M)}) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod M\mathbb{Z}_{(M)} \right\}.$$

If M is a prime number p, then $\mathbb{Z}_{(p)}$ is a localization of $\mathbb{Z}$ at the prime ideal $(p) := p\mathbb{Z}$. We also remark that $\Gamma_0(M) = G_0(M) \cap \mathrm{SL}_2(\mathbb{Z})$.

Our aim in this section is to prove the following proposition which plays an important role in studying the action of $G_0(M)$ on cusps in the next section.

Proposition 3.2. *It holds that* $G_0(M) = \Gamma_0(M)G_0(M)_{i\infty}$.

Before giving a proof, we prepare the followings.

Definition 3.3. *For rational numbers a and b, we have a unique rational number $g \in \mathbb{Q}_{\geq 0}$ such that $a\mathbb{Z} + b\mathbb{Z} = g\mathbb{Z}$. We put $(a, b) := g$ and call it the greatest common divisor of a and b.*

If $g \neq 0$, then $a/g, b/g \in \mathbb{Z}$. In the case when both a and b are integers, the above (a, b) is the usual greatest common divisor of a and b.

The following property of the greatest common divisor is quite elementary.

Lemma 3.4. *For rational numbers $a \in \mathbb{Z}_{(M)}^\times$ and $b \in \mathbb{Z}_{(M)}$, we have $(a, b) \in \mathbb{Z}_{(M)}^\times$.*

Proof. Let $g := (a, b) \in \mathbb{Z}_{(M)}$. Since $g^{-1}a \in \mathbb{Z}$, we have $g \in \mathbb{Z}_{(M)}^\times$. □

Proof of Proposition 3.2. For a matrix

$$A = \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in G_0(M),$$

we have $a \in \mathbb{Z}_{(M)}^\times$ since $D := ad - Mbc \in \mathbb{Z}_{(M)}^\times$. By Lemma 3.4, we have $(a, c) \in \mathbb{Z}_{(M)}^\times$. Thus there exists a matrix

$$\gamma = \begin{pmatrix} a/(a,c) & * \\ Mc/(a,c) & * \end{pmatrix} \in \Gamma_0(M).$$

We have $A(i\infty) = \gamma(i\infty)$. □

4. Cusps and Atkin-Lehner involutions

In this section, for each cusp s in $X_0(M)$, we consider whether there exists a matrix $W \in \mathrm{SL}_2(\mathbb{R})$ which satisfies $W(i\infty) = s$ and normalize both of $\Gamma_0(M)$ and $G_0(M)$. Since all cusps are expressed as the form m/M with an integer $0 \leq m < M$, we need a matrix $W \in \mathrm{SL}_2(\mathbb{R})$ with the form

$$\frac{1}{\sqrt{D}} \begin{pmatrix} m & u \\ M & v \end{pmatrix}.$$

Typical such matrices are Atkin-Lehner involutions.

Definition 4.1. *For a positive divisor m of a positive integer M such that $(m, M/m) = 1$, there exist integers u, v such that $mv - Mu/m = 1$. We denote*

$$W_m = W_m^M := \frac{1}{\sqrt{m}} \begin{pmatrix} m & u \\ M & mv \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

For $m = 0$, we set

$$W_0 = W_0^M := \frac{1}{\sqrt{M}} \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}). \tag{4.1}$$

We call them ***Atkin-Lehner involutions****.*

We can check that Atkin-Lehner involutions normalize $\Gamma_0(M)$ and $G_0(M)$ by direct calculation.

If M is square-free, one can find an Atkin-Lehner involution W such that $W(i\infty) = s$ for each cusp $s \in X_0(M)$. However, this is not true if M is not square-free.

For this reason, we introduce the following generalization of Atkin-Lehner involutions.

Definition 4.2. *For a positive integer M and an integer m, let $D := (M, m^2)$. Take u and v such that $(M,m)mv - Mu = D$ and sert We set*

$$W_m = W_m^M := \frac{1}{\sqrt{D}} \begin{pmatrix} m & u \\ M & (M,m)v \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

*It is called a **generalized Atkin-Lehner involution**.*

In the case when m is a positive divisor of M such that $(m, M/m) = 1$, W_m is an Atkin-Lehner involution.

In general, generalized Atkin-Lehner involutions do not normalize neither $\Gamma_0(M)$ nor $G_0(M)$. For example, if $M = 25$ and $(m, 25) = 5$, then it turns out that W_m^M do not normalize neither $\Gamma_0(M)$ nor $G_0(M)$ in Section 5. To calculate the intersection multiplicity of the modular correspondences at a pair of cusps in such a case, we define the following subgroup of $\Gamma_0(M)$.

Definition 4.3. *For a positive integer M and its positive divisor M', put*

$$\Gamma_0^{(M')}(M) := \left\{ \begin{pmatrix} a & * \\ * & d \end{pmatrix} \in \Gamma_0(M) \;\middle|\; a \equiv d \bmod M' \right\}.$$

This group is a congruence subgroup of level M since $\Gamma(M) \subset \Gamma_0^{(M')}(M)$. We put $X_0^{(M')}(M) := \Gamma_0^{(M')}(M) \backslash (\mathbb{Q} \cup \{i\infty\}) \cup \mathbb{H}$ be the associated modular curve.

For example, $\Gamma_0^{(1)}(M) = \Gamma_0(M)$.

To conclude this section, we enumerate cusps m/M and generalized Atkin-Lehner involutions W_m^M when the genus of the modular curve $X_0(M)$ is zero, that is exactly when $1 \le M \le 10$ or $M \in \{12, 13, 16, 18, 25\}$.

Here we remark that $i\infty$ and $1/M$ are $\Gamma_0(M)$-equivalent. When $M = 1$, $X_0(1)$ has only one cusp $i\infty$ and $X_0(M)$ has two cusps $i\infty$ and 0 for a prime number M. For a composite number M, $X_0(M)$ has one or more cusps except for $i\infty, 0$.

In the case when M is a composite number, we list the cusps in $X_0(M)$ in Table 1 and Atkin-Lehner involutions W_m^M for $m \ne 1, M$ in Table 2. In the case when M is not a prime number nor a product of two prime numbers, that is, $M \in \{4, 8, 9, 12, 16, 18, 25\}$, in Table 3 we compile generalized Atkin-Lehner involutions for cusps $s = m/M$ in $X_0(M)$ which is not an image of $i\infty$ under any Atkin-Lehner involution.

Table 1. Cusps except for $i\infty$ and 0 in $X_0(M)$ with a composite number M

M	Cusps
4	$\frac{1}{2}$
6	$\frac{1}{2}, \frac{1}{3}$
8	$\frac{1}{2}, \frac{1}{4}$
9	$\frac{1}{3}, \frac{2}{3}$
10	$\frac{1}{2}, \frac{1}{5}$

M	Cusps
12	$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$
16	$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}$
18	$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{9}$
25	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$

5. The normalizer of $\Gamma_0^{(M')}(M)$

In this section, we study the normalizers of $\Gamma_0(M)$, $G_0(M)$, and the congruence subgroup $\Gamma_0^{(M')}(M)$ in $\mathrm{SL}_2(\mathbb{Z})$ and give conditions whether generalized Atkin-Lehner involutions normalize

Table 2. Atkin-Lehner involutions for $m \neq 1, M$

M	A cusp $s = m/M$	An Atkin-Lehner involution $W_m = W_m^M$
6	$\frac{1}{2} = \frac{3}{6}, \frac{1}{3} = \frac{2}{6}$	$W_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \quad W_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}$
10	$\frac{1}{2} = \frac{5}{10}, \frac{1}{5} = \frac{2}{10}$	$W_5 = \frac{1}{\sqrt{5}}\begin{pmatrix} 5 & 2 \\ 10 & 5 \end{pmatrix}, \quad W_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 \\ 10 & 6 \end{pmatrix}$
12	$\frac{1}{3} = \frac{4}{12}, \frac{1}{4} = \frac{3}{12}$	$W_4 = \frac{1}{2}\begin{pmatrix} 4 & 1 \\ 12 & 4 \end{pmatrix}, \quad W_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & 2 \\ 12 & 9 \end{pmatrix}$
18	$\frac{1}{2} = \frac{9}{18}, \frac{1}{9} = \frac{2}{18}$	$W_9 = \frac{1}{3}\begin{pmatrix} 9 & 4 \\ 18 & 9 \end{pmatrix}, \quad W_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 1 \\ 18 & 10 \end{pmatrix}$

Table 3. Generalized Atkin-Lehner involutions for cusps $\notin \{0, \infty, \frac{1}{p^e}, \frac{1}{q^f}\}$ when $M = p^e q^f$

M	A cusp $s = m/M$	A generalized Atkin-Lehner involution $W_m = W_m^M$
4	$\frac{1}{2} = \frac{2}{4}$	$W_2 = \frac{1}{2}\begin{pmatrix} 2 & 2 \\ 4 & 6 \end{pmatrix}$
8	$\frac{1}{2} = \frac{4}{8}, \frac{1}{4} = \frac{2}{8}$	$W_4 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 4 & 1 \\ 8 & 4 \end{pmatrix}, \quad W_2 = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 8 & 6 \end{pmatrix}$
9	$\frac{1}{3} = \frac{3}{9}, \frac{2}{3} = \frac{6}{9}$	$W_3 = \frac{1}{3}\begin{pmatrix} 3 & 2 \\ 9 & 9 \end{pmatrix}, \quad W_6 = \frac{1}{3}\begin{pmatrix} 6 & 1 \\ 9 & 3 \end{pmatrix}$
12	$\frac{1}{2} = \frac{6}{12}, \frac{1}{6} = \frac{2}{12}$	$W_6 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 6 & 2 \\ 12 & 6 \end{pmatrix}, \quad W_2 = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 12 & 8 \end{pmatrix}$
16	$\frac{1}{2} = \frac{8}{16}, \frac{1}{4} = \frac{4}{16},$	$W_8 = \frac{1}{4}\begin{pmatrix} 8 & 1 \\ 16 & 4 \end{pmatrix}, \quad W_4 = \frac{1}{4}\begin{pmatrix} 4 & 1 \\ 16 & 8 \end{pmatrix},$
	$\frac{3}{4} = \frac{12}{16}, \frac{1}{8} = \frac{2}{16}$	$W_{12} = \frac{1}{4}\begin{pmatrix} 12 & 2 \\ 16 & 4 \end{pmatrix}, \quad W_2 = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 16 & 10 \end{pmatrix}$
18	$\frac{1}{3} = \frac{6}{18}, \frac{2}{3} = \frac{12}{18},$	$W_6 = \frac{1}{3\sqrt{2}}\begin{pmatrix} 6 & 1 \\ 18 & 6 \end{pmatrix}, \quad W_{12} = \frac{1}{3\sqrt{2}}\begin{pmatrix} 12 & 3 \\ 18 & 6 \end{pmatrix},$
	$\frac{1}{6} = \frac{3}{18}, \frac{5}{6} = \frac{15}{18}$	$W_3 = \frac{1}{3}\begin{pmatrix} 3 & 1 \\ 18 & 9 \end{pmatrix}, \quad W_{15} = \frac{1}{3}\begin{pmatrix} 15 & 2 \\ 18 & 3 \end{pmatrix}$
25	$\frac{1}{5} = \frac{5}{25}, \frac{2}{5} = \frac{10}{25},$	$W_5 = \frac{1}{5}\begin{pmatrix} 5 & 1 \\ 25 & 10 \end{pmatrix}, \quad W_{10} = \frac{1}{5}\begin{pmatrix} 10 & 1 \\ 25 & 5 \end{pmatrix},$
	$\frac{3}{5} = \frac{15}{25}, \frac{4}{5} = \frac{20}{25}$	$W_{15} = \frac{1}{5}\begin{pmatrix} 15 & 2 \\ 25 & 5 \end{pmatrix}, \quad W_{20} = \frac{1}{5}\begin{pmatrix} 20 & 3 \\ 25 & 5 \end{pmatrix}$

them. Throughout this section, we fix positive integers M, f and M_0 such that $M = f^2 M_0$ and M_0 is square-free.

Firstly, we prepare the following subgroup of $\mathrm{SL}_2(\mathbb{R})$ which turns out to be the normalizer of $\Gamma_0^{(M')}(M)$ later in this section.

Definition 5.1. *For a positive integer h with $h^2 \mid M$, that is, $h \mid f$, define*

$$\Gamma_0^{*,h}(M) := \left\{ \frac{1}{\sqrt{e}} \begin{pmatrix} ep & q/h \\ Mr/h & es \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \;\middle|\; \begin{array}{l} e \in \mathbb{Z}_{>0}, e \mid M/h^2, \\ p, q, r, s \in \mathbb{Z} \end{array} \right\},$$

$$G_0^{*,h}(M) := \left\{ \frac{1}{\sqrt{e}} \begin{pmatrix} ep & q/h \\ Mr/h & es \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \;\middle|\; \begin{array}{l} e \in \mathbb{Z}_{>0}, e \mid M/h^2, \\ p, q, r, s \in \mathbb{Z}_{(M)} \end{array} \right\}.$$

We give several remarks for $\Gamma_0^{*,h}(M)$.

Remark 5.2. (i) The symbol $\Gamma_0^{*,h}(M)$ is introduced in [15, Definition 1.7].

(ii) The subset $\Gamma_0^{*,h}(M)$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$ by [15, Proposition 1.2 (i)] and the same argument shows that $G_0^{*,h}(M)$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$.

(iii) For a fixed integer h with $h^2 \mid M$ and a matrix

$$\frac{1}{\sqrt{e}}\begin{pmatrix} ep & q/h \\ Mr/h & es \end{pmatrix} \in \Gamma_0^{*,h}(M),$$

the positive integer e is uniquely determined and called eterminant in [1].

(iv) The group $\Gamma_0^{*,1}(M)$ is generated by $\Gamma_0(M)$ and Atkin-Lehner involutions and it is usually written as $\Gamma_0^*(M)$.

The following states whether generalized Atkin-Lehner involutions are in $\Gamma_0^{*,h}(M)$.

Lemma 5.3. *For a positive integer m and a generalized Atkin-Lehner involution W_m^M, we have $W_m^M \in \Gamma_0^{*,(f,m)}(M)$.*

Proof. Let

$$W_m^M = \frac{1}{\sqrt{D}}\begin{pmatrix} m & u \\ M & (M,m)v \end{pmatrix}, \quad D := (M, m^2).$$

Here we put $e := D/(f,m)^2, m' := m/(f,m)$. Then we have

$$W_m^M = \frac{1}{\sqrt{e}}\begin{pmatrix} m' & u/(f,m) \\ M/(f,m) & (M/(f,m),m')v \end{pmatrix}$$

and $e = (M_0 f^2/(f,m)^2, m'^2)$. Since $f^2/(f,m)^2$ and m'^2 are coprime and M_0 is square-free, we have $e = (M_0, m')$. Thus $e \mid (M/(f,m), m')$ and $W_m^M \in \Gamma_0^{*,(f,m)}(M)$. □

Here we remark that Lemma 5.3 does not cover the fact that Atkin-Lehner involutions are in $\Gamma_0^*(M)$.

Secondly, we compare $\Gamma_0^{*,h}(M)$ and $\Gamma_0^{*,h'}(M)$.

Lemma 5.4. *If $h \mid h'$ then $\Gamma_0^{*,h}(M) \subset \Gamma_0^{*,h'}(M)$ and $G_0^{*,h}(M) \subset G_0^{*,h'}(M)$.*

Proof. We prove only the first statement since the second statement can be proved by the following argument.

For a matrix

$$W = \frac{1}{\sqrt{e}}\begin{pmatrix} ep & q/h \\ Mr/h & es \end{pmatrix} \in \Gamma_0^{*,h}(M),$$

put positive integers g, e_0 such that $e = g^2 e_0$ and e_0 is square-free. Let $g' := (g, h'/h)$ and $e' := e/g'^2$. Since

$$W = \frac{1}{\sqrt{e'}}\begin{pmatrix} e'g'p & q(h'/gh)/h' \\ Mr(h'/gh)/h' & e'g's \end{pmatrix},$$

it suffices to show $e' \mid M/h'^2$.

Because $g^2 e_0 = e \mid M/h^2 = M_0(f/h)^2$ and M_0 is square-free, we have $g \mid f/h$. Thus $e_0 = e/g^2 \mid M/g^2h^2 = M_0(f/gh)^2$. We have $e_0 \mid M_0 f/gh$ since e_0 is square-free. Here $\det W = 1$ implies

$$1 = \left(e, \frac{M}{h^2 e}\right) = \left(g^2 e_0, \frac{M_0 f/gh}{e_0}\frac{f}{gh}\right)$$

and thus $(e_0, f/gh) = 1$. Therefore we have $e_0 \mid M_0$.

Since $g \mid f/h$, we have $g/g' \mid f/g'h = (f/h')(h'/g'h)$. By the definition of g', we have $(g/g', h'/g'h) = 1$ and thus $g/g' \mid f/h'$.

As a result, we have $e' = e_0(g/g')^2 \mid M_0(f/h')^2 = M/h'^2$. □

Lemma 5.5. *For a positive integer h with $h^2 \mid M$ and a matrix*

$$W = \frac{1}{\sqrt{e}}\begin{pmatrix} ep & q/h \\ Mr/h & es \end{pmatrix} \in \Gamma_0^{*,h}(M),$$

let $g := (h,p,s)(h,q,r)$ and $h' := h/g$. Then we have $W \in \Gamma_0^{*,h'}(M)$.

The same result holds for $G_0^{*,h}(M)$.

Proof. Since $\det W = 1$, (h,p,s) and (h,q,r) are coprime and thus $g \mid h$. Let

$$e' := e(h,p,s)^2, p' := p/(h,p,s)^2, q' := q/(h,q,r)^2, r' := r/(h,q,r)^2, s' := s/(h,p,s)^2.$$

Then we have

$$W = \frac{1}{\sqrt{e'}} \begin{pmatrix} e'p' & q'/h' \\ Mr'/h' & e's' \end{pmatrix}$$

and $e' \mid Mg^2/h^2 = M/h'^2$. Thus we have $W \in \Gamma_0^{*,h'}(M)$.

The statement for $G_0^{*,h}(M)$ is proved by the same argument. □

Thirdly, we determine the normalizers.

The first statement in the following proposition is proved in [15, Lemma 2.1, Proposition 2.3]. We give other proof.

Proposition 5.6. *We have*

$$\left\{W \in \mathrm{SL}_2(\mathbb{R}) \;\middle|\; W^{-1}\Gamma_1(M)W \subset \Gamma_0(M)\right\} = \Gamma_0^{*,f}(M),$$
$$\left\{W \in \mathrm{SL}_2(\mathbb{R}) \;\middle|\; W^{-1}\Gamma_1(M)W \subset G_0(M)\right\} = G_0^{*,f}(M).$$

Proof. We prove only the first statement. The second statement can be proved similarly by replacing $\mathbb{Z}$ with $\mathbb{Z}_{(M)}$ in the following argument.

Let

$$W = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

be an element of the left-hand side. Then

$$W^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} W = \begin{pmatrix} 1+rs & s^2 \\ -r^2 & 1-rs \end{pmatrix},$$
$$W^{-1}\begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix} W = \begin{pmatrix} 1-Mpq & -Mq^2 \\ Mp^2 & 1+Mpq \end{pmatrix},$$
$$W^{-1}\begin{pmatrix} 1 & 1 \\ M & 1+M \end{pmatrix} W = \begin{pmatrix} 1+rs-Mq(p+r) & s^2-Mq(q+s) \\ -r^2+Mp(p+r) & 1-rs+Mp(q+s) \end{pmatrix}$$

are in $\Gamma_0(M)$. Hence $p^2, Mq^2, r^2/M, s^2, Mpq, pr, rs \in \mathbb{Z}$. Therefore we can rewrite

$$W = \frac{1}{\sqrt{e}} \begin{pmatrix} ep & q/f \\ Mr/f & es \end{pmatrix}$$

with a positive square-free integer e and rational numbers p, q, r, s which satisfy

$$ep^2, \frac{M}{f^2 e}q^2, \frac{M}{f^2 e}r^2, es^2 \in \mathbb{Z}.$$

Since e and $M_0 = M/f^2$ are square-free, p, q, r, s are integers. Also we have $Mqr/f^2e = 1-eps \in \mathbb{Z}$. Thus $e \mid M(q^2, qr, r^2)/f^2 = M_0(q,r)^2$. Since M_0 is square-free, we have $e \mid M_0(q,r)$.

Let $g := (e,q,r)$. We will show $g = 1$. Since $\det W = 1$, we have $e = (e^2, M(q,r)/f^2) = (e^2, M_0 g(q,r)/g)$. Here $(q,r)/g$ is coprime to e and thus $e = (e^2, M_0 g) = g(g(e/g)^2, M_0)$. Since M_0 is square-free, we have $e = g(e, M_0)$. Therefore $e/g \mid M_0$ and $(g, M_0/(e/g)) = 1$. Hence $gM_0/(e/g) = g^2M_0/e$ is square-free. Since e is square-free, we have $g = 1$.

By combining the above discussion, we have $e \mid M_0$ and thus $W \in \Gamma_0^{*,f}(M)$.

Conversely, suppose

$$W = \frac{1}{\sqrt{e}} \begin{pmatrix} ep & q/f \\ Mr/f & es \end{pmatrix} \in \Gamma_0^{*,f}(M).$$

For any matrix $\gamma = \left(\begin{smallmatrix} a & b \\ Mc & d \end{smallmatrix}\right) \in \Gamma_1(M)$, we have

$$(5.1)\qquad \begin{aligned} W^{-1}\gamma W &= \begin{pmatrix} A & B \\ MC & D \end{pmatrix}, \\ A &:= eaps + \frac{M}{f}(brs - cpq) - \frac{M}{f^2e}dqr, \\ B &:= -\frac{(a-d)qs}{f} - \frac{M}{f^2e}cq^2 + ebp^2, \\ C &:= -\frac{(a-d)pr}{M} - \frac{M}{f^2e}br^2 + ecs^2, \\ D &:= -\frac{M}{f^2e}aqr - \frac{M}{f}(brs - cpq) + edps. \end{aligned}$$

Thus we obtain $W^{-1}\gamma W \in \Gamma_0(M)$. □

To determine the normalizers of $\Gamma_0(M)$ and $G_0(M)$, we prepare a lemma from elementary number theory.

Lemma 5.7. *For a positive integer M, let $\varepsilon(M)$ be the greatest common divisor of $a-d$ for all integers a and d such that $\left(\begin{smallmatrix} a & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(M)$. Similarly, let $e(M)$ be the greatest common divisor of $a-d$ for all integers a and d such that $\left(\begin{smallmatrix} a & * \\ * & d \end{smallmatrix}\right) \in G_0(M) \cap M_2(\mathbb{Z})$. Then we have*

$$\varepsilon(M) = (M, 24), \quad e(M) = (M, 2).$$

Thus we have $\Gamma_0(M) = \Gamma_0^{(M')}(M)$ where $M' := (M, 24)$.

Proof. Since the map

$$\Gamma_0(M) \to (\mathbb{Z}/M\mathbb{Z})^\times, \begin{pmatrix} a & * \\ * & * \end{pmatrix} \mapsto a \bmod M$$

is surjective, we have

$$\varepsilon(M) = (a - d \mid ad \equiv 1 \bmod M), \quad e(M) = (a - d \mid a, d \in \mathbb{Z} \cap \mathbb{Z}_{(M)}^\times).$$

It suffices to show when M is a power of a prime number since $\varepsilon(M), e(M)$ divides M.

If $M = 16$, then a pair $(\overline{a}, \overline{d}) \in ((\mathbb{Z}/16\mathbb{Z})^\times)^2$ such that $ad \equiv 1 \bmod 16$ is $\pm(\overline{1}, \overline{1})$, $\pm(\overline{3}, \overline{11})$ or $\pm(\overline{7}, \overline{7})$. This implies that $\varepsilon(16) = (16, 1-1, 11-3, 7-7) = 8$. Thus if M is a power of 2, then $\varepsilon(M) = (M, 24)$.

If $M = 9$, then then a pair $(\overline{a}, \overline{d}) \in ((\mathbb{Z}/9\mathbb{Z})^\times)^2$ such that $ad \equiv 1 \bmod 9$ is $\pm(\overline{1}, \overline{1})$ or $\pm(\overline{2}, \overline{5})$ and we have $\varepsilon(9) = 3$. Thus if M is a power of 3, then $\varepsilon(M) = (M, 24)$.

Suppose M is a power of a prime number $p \geq 5$. For an integer a such that $2a \equiv 1 \bmod p$, since $4 \not\equiv 1 \bmod p$ we have $a \not\equiv 2 \bmod p$. Therefore, we find $\varepsilon(p) \mid (p, a-2) = 1$ and $\varepsilon(M) = (M, 24)$.

The same argument works for $e(M)$. □

For a group G and its subgroup H, we denote by $N_G(H)$ the normalizer of H in G.

By Proposition 5.6, (5.1) and Lemma 5.7, we have the following proposition whose the first statement is stated without a proof in [1], [2, Theorem 1], [5, Section 3] and a proof is found in [15, Corollary 3.2].

Proposition 5.8. *We have the followings.*

(i) $N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M)) = \Gamma_0^{*,(f,24)}(M)$.

(ii) $N_{\mathrm{SL}_2(\mathbb{R})}(G_0(M)) = \Gamma_0^{*,(f,2)}(M)$.

(iii) *For a positive divisor M' of M, we have $N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0^{(M')}(M)) = \Gamma_0^{*,(f,M',2M/M')}(M)$.*

Proof. (i). Since both sides are in $\Gamma_0^{*,f}(M)$ by Lemma 5.4 and Proposition 5.6, it is enough to show that for $W \in \Gamma_0^{*,f}(M)$, W normalizes $\Gamma_0(M)$ if and only if $W \in \Gamma_0^{*,(f,24)}(M)$. By Lemma

5.5, we can assume

$$W = \frac{1}{\sqrt{e}} \begin{pmatrix} ep & q/h \\ Mr/h & es \end{pmatrix} \in \Gamma_0^{*,h}(M), \quad h \mid f, \quad (h,p,s) = (h,q,r) = 1.$$

In this case, W normalizes $\Gamma_0(M)$ if and only if $h \mid (a-d)(pr, qs)$ for any $a, d \in \mathbb{Z}$ with $ad \equiv 1 \bmod M$ by (5.1). By Lemma 5.7, this is equivalent to $h \mid (M, 24)(pr, qs)$.

Since $\det W = 1$, we have $(p,q) = (p,r) = (q,s) = 1$. Thus

$$\begin{aligned} (p,s) &= (p,q)(p,s) = ((p,q,s)p, qs) = (p, qs), \\ (q,r) &= (q,r)(s,r) = ((q,r,s)r, qs) = (r, qs), \\ (p,qs)(r,qs) &= (pr, (p,r,qs)qs) = (pr, qs). \end{aligned}$$

and we obtain $(pr, qs) = (p,s)(q,r)$. Since $(h,p,s) = (h,q,r) = 1$, W normalizes $\Gamma_0(M)$ if and only if $h \mid (M, 24)$. Since $f \mid h$, this is equivalent to $h \mid (f, 24)$, that is, $W \in \Gamma_0^{*,(f,24)}(M)$ by Lemma 5.4.

(ii) can be proved by the same argument in (i).

(iii). Let

$$W = \frac{1}{\sqrt{e}} \begin{pmatrix} ep & q/h \\ Mr/h & es \end{pmatrix} \in \Gamma_0^{*,h}(M)$$

with $h \mid f, (h,p,s) = (h,q,r) = 1$. For a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ Mc & d \end{smallmatrix}\right) \in \Gamma_0^{(M')}(M)$, let

$$W^{-1}\gamma W = \begin{pmatrix} A & B \\ MC & D \end{pmatrix}.$$

By (5.1), $B, C \in \mathbb{Z}$ for any $\gamma \in \Gamma_0^{(M')}(M)$ if and only if $h \mid M'(pr, qs) = M'(p,s)(q,r)$. Since $(h,p,s) = (h,q,r) = 1$ and $f \mid h$, this is equivalent to $h \mid (f, M')$.

By (5.1), $A \equiv D \bmod M'$ for any $\gamma \in \Gamma_0^{(M')}(M)$ if and only if

$$0 \equiv A - D \equiv 2\frac{M}{h}(brs - cpq) \bmod M'$$

for any $\gamma \in \Gamma_0^{(M')}(M)$, which is equivalent to $h \mid (pr, qs)2M/M' = (p,s)(q,r)2M/M'$. This is equivalent to $h \mid (f, 2M/M')$. □

Finally, we give conditions whether generalized Atkin-Lehner involutions normalize $\Gamma_0(M)$, $G_0(M)$, or $\Gamma_0^{(M')}(M)$.

Proposition 5.9. *For a positive integer m and a generalized Atkin-Lehner involution W_m^M, the followings hold.*

(i) *$W_m^M \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M))$ if and only if $(f,m) \mid (f, 24)$.*

(ii) *$W_m^M \in N_{\mathrm{SL}_2(\mathbb{R})}(G_0(M))$ if and only if $(f,m) \mid (f, 2)$.*

(iii) *For $M' := M/(f,m)$, we have $W_m^M \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0^{(M')}(M))$.*

Proof. These follow from Lemma 5.3, Lemma 5.4, and Proposition 5.8. □

In the case when the modular curve $X_0(M)$ has genus zero, we have the following.

Proposition 5.10. *Let $s \in X_0(M)$ be a cusp expressed as $s = m/M = W_m^M(i\infty)$ with an integer $0 \leq m < M$. Let W_m^M be a generalized Atkin-Lehner involution.*

(i) *If $(M,s) \notin \{(25,1/5),(25,2/5),(25,3/5),(25,4/5)\}$, then we have $W_m^M \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M))$.*

(ii) *If*

$$(M,s) \notin \left\{ \begin{matrix} (9,1/3),(9,2/3), \\ (16,1/2),(16,1/4),(16,3/4),(16,1/8), \\ (18,1/3),(18,2/3),(18,1/6),(18,5/6), \\ (25,1/5),(25,2/5),(25,3/5),(25,4/5), \end{matrix} \right\},$$

then we have $W_m^M \in N_{\mathrm{SL}_2(\mathbb{R})}(G_0(M))$.

(iii) *If* $M=25$ *and* $s \in \{1/5, 2/5, 3/5, 4/5\}$*, then we have* $W_m^M \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0^{(5)}(25))$*.*

6. The action of $G_0(M)$ on cusps

In this section, we study the action of $G_0(M)$ on cusps to give a condition that a pair of cusps is a point of modular correspondence and calculate the intersection multiplicity.

Firstly, we describe explicitly cusps and the action of $G_0(M)$ on them.

Definition 6.1. *Let* $C_0(M) := \Gamma_0(M)\backslash(\mathbb{Q} \cup \{i\infty\})$ *be the set of cusps in the modular curve* $X_0(M)$ *and we define its subset*

$$C_0(M)_n := \left\{ \Gamma_0(M)\frac{l}{n} \;\middle|\; \bar{l} \in (\mathbb{Z}/(n, M/n)\mathbb{Z})^\times \right\}$$

for a positive divisor n *of* M*.*

We have a decomposition

$$C_0(M) = \coprod_{n|M} C_0(M)_n \tag{6.1}$$

by the following lemma.

Lemma 6.2 ([6, Proposition 3.8.3])**.** *For integers* l, n, l', n' *with* $(l,n) = (l',n') = 1$*, let* $s = l/n, s' = l'/n'$*. Then* $\Gamma_0(M)s = \Gamma_0(M)s'$ *if and only if there exists an integer* d *such that*

$$(d', M) = 1, \quad n' \equiv dn \bmod M, \quad dl' \equiv l \bmod (M, n).$$

By the above lemma, the group $G_0(M)$ acts on $C_0(M)_n$ for a positive divisor n of M. This action is essentially it of $G_0(M)_{i\infty}$ by Proposition 3.2. We describe it in the following Proposition.

Proposition 6.3. *Let* $0 \le m, m' < M$ *be integers,* $s = m/M, s' = m'/M$ *be rational numbers and*

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_0(M) \cap M_2(\mathbb{Z}).$$

Then $\Gamma_0(M)s = \Gamma_0(M)A(s')$ *if and only if*

$$(M, m) = (M, m'), \quad m'ad \equiv m(d, m'a + Mb)^2 \bmod M.$$

Proof. Let $g := (d, m'a + Mb)$. By Lemma 6.2, $\Gamma_0(M)s = \Gamma_0(M)A(s')$ if and only if there exists an integer d' such that

$$(d', M) = 1, \quad \frac{M}{(M,m')}\frac{d}{g} \equiv d'\frac{M}{(M,m)} \bmod M, \quad d'\frac{m'a + Mb}{(M,m')g} \equiv \frac{m}{(M,m)} \bmod \frac{M}{(M,m)}.$$

This is equivalent to the condition in the statement since we can choose $d' = d/g$. □

Secondly, we establish definitions of some kind of cusps and study the action of $G_0(M)$ on them.

Definition 6.4. *We define subsets of* $C_0(M)$ *as*

$$C_0'(M) := \{W(i\infty) \in C_0(M) \mid W \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M))\},$$
$$C_0''(M) := \{W(i\infty) \in C_0(M) \mid W \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M)) \cap N_{\mathrm{SL}_2(\mathbb{R})}(G_0(M))\}.$$

Proposition 6.5. *It holds that*

$$C_0'(M) = \coprod_{n|M,\,(n,M/n)|(24,M/(M,24))} C_0(M)_n, \quad C_0''(M) = \coprod_{n|M,\,(n,M/n)|(2,M/(M,2))} C_0(M)_n.$$

Proof. Let f and M_0 be positive integers such that $M = f^2 M_0$ and M_0 is square-free. For a positive integer m, let $n := M/(M,m)$. It suffices to show that $(f,n) = (n, M/n)$ since $(f,m) = (f,n)$. Since $(n, M/n)^2 \mid n \cdot M/n = M$, we have $(n, M/n) \mid (f,n)$. Since

$$(f,n)^2 \mid M = \frac{M}{n} \frac{n}{(f,n)} (f,n),$$

we have $(f,n) \mid M/n$ and thus $(f,n) \mid (n, M/n)$. By Proposition 5.9, we obtain the statement. □

The actions of $G_0(M)$ on $C_0'(M)$ and $C_0''(M)$ is described as follows by Proposition 6.5.

Corollary 6.6. *The sets p–00–$C_0''(M), C_0'(M) \setminus C_0''(M)$, and $C_0(M) \setminus C_0'(M)$ are stable under the action of $G_0(M)$.*

In the case when the modular curve $X_0(M)$ has genus zero, we have the following by Proposition 5.10.

Corollary 6.7. *Let $1 \leq M \leq 10$ or $M \in \{12, 13, 16, 18, 25\}$.*

(i) *If $M \neq 25$, then $C_0(M) = C_0'(M)$.*
(ii) *If $M \notin \{9, 16, 18, 25\}$, then $C_0(M) = C_0'(M) = C_0''(M)$.*
(iii) *If $M \in \{9, 16, 18, 25\}$, then*

$$\begin{aligned} C_0(9) \smallsetminus C_0''(9) &= \left\{ \frac{1}{3}, \frac{2}{3} \right\}, & C_0''(9) &= \{ i\infty, 0 \}, \\ C_0(16) \smallsetminus C_0''(16) &= \left\{ \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8} \right\}, & C_0''(16) &= \{ i\infty, 0 \}, \\ C_0(18) \smallsetminus C_0''(18) &= \left\{ \frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6} \right\}, & C_0''(18) &= \left\{ i\infty, 0, \frac{1}{2}, \frac{1}{9} \right\}, \\ C_0(25) \smallsetminus C_0''(25) &= \left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right\}, & C_0'(25) &= C_0''(25) = \{ i\infty, 0 \}. \end{aligned}$$

7. Intersection multiplicities at cusps

In the rest of this paper, we assume that the modular curve $X_0(M)$ has genus zero, that is, $1 \leq M \leq 10$ or $M = 12, 13, 16, 18, 25$.

Our goal in this section is to calculate the intersection multiplicity of the modular correspondences at a pair (s, s') of cusps in the modular curve $X_0(M)$.

Firstly, we consider the condition $\Gamma_0(M)s = \Gamma_0(M)A(s')$ for $s, s' \in \mathbb{Q} \cup \{i\infty\}$ and a matrix $A \in G_0(M)$ in (1.2).

Let $t \colon X_0(M) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$ be the isomorphism defined in Section 2.

Proposition 7.1. *Let $0 \leq m, m' < M$ be integers and put*

$$s = m/M, \quad s' = m'/M, \quad D := (M, m^2), \quad D' := (M, m'^2).$$

Let

$$W = W_m^M = \frac{1}{\sqrt{D}} \begin{pmatrix} m & u \\ M & v \end{pmatrix}, \quad W' = W_{m'}^M = \frac{1}{\sqrt{D'}} \begin{pmatrix} m' & * \\ M & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

be generalized Atkin-Lehner involutions and

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_0(M) \cap M_2(\mathbb{Z})$$

be a matrix such that $\Gamma_0(M)s = \Gamma_0(M)A(s')$. Then the order of $t \circ AW'(\tau) - t(s)$ with respect to $q := e^{2\pi\sqrt{-1}\tau}$ is $(d, m'a + Mb)^2/ad$.

Proof. Put

$$W^{-1}AW' = \sqrt{\frac{D}{D'}}\begin{pmatrix} k & * \\ -Ml & * \end{pmatrix}$$

with rational numbers k, l. Then we have

$$k = \frac{v}{D}\left(m'a + Mb\right) - \frac{M}{D}ud \in \mathbb{Z}, \quad l = \frac{m'a - md}{D} + \frac{M}{D}b \in \frac{1}{D}\mathbb{Z}.$$

There exists a matrix

$$\gamma = \begin{pmatrix} k & * \\ -Ml & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Let $g = (k, Ml)$. We have

$$W^{-1}AW' = \sqrt{\frac{D}{D'}}\gamma\begin{pmatrix} g & * \\ 0 & ad/g \end{pmatrix}.$$

We show $g = (d, m'a + Mb)$. Since $k - vl = d$ by direct calculation, we have $g = (k, Md, Ml)$. Since $k \equiv vm'a/D \bmod (M, m)$, we have $(k, M) = 1$. Thus we have

$$g = (k, d, Ml) = (k, d, Dl) = (k, d, m'a + Mb) = (d, m'a + Mb).$$

By assumption there exists a matrix $\delta \in \Gamma_0(M)$ such that $\delta(s) = A(s')$. Since

$$\delta W(i\infty) = \delta(s) = A(s') = AW'(i\infty) = W\gamma\begin{pmatrix} g & * \\ 0 & ad/g \end{pmatrix}(i\infty) = W\gamma(i\infty),$$

we have $\gamma^{-1}W^{-1}\delta W \in \mathrm{SL}_2(\mathbb{Q})_{i\infty}$.

We show that the diagonal elements of $\gamma^{-1}W^{-1}\delta W$ are integers. If $W \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M))$, then $W^{-1}\delta W \in \Gamma_0(M)$ and thus $\gamma^{-1}W^{-1}\delta W \in \mathrm{SL}_2(\mathbb{Z})$. If $W \notin N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M))$, then $M = 25$ and $W \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0^{(5)}(25))$ by Proposition 5.10. By Definition 4.2 and (5.1), a matrix $W^{-1}\delta W$ has integral element except for the $(1, 2)$ entry and its $(1, 2)$ entry is in $5^{-1}\mathbb{Z}$. Since $l \in 5^{-1}\mathbb{Z}$, the diagonal elements of $\gamma^{-1}W^{-1}\delta W$ are integers.

Thus we can express as

$$\gamma^{-1}W^{-1}\delta W = \pm\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Therefore we have

$$t \circ AW' = t \circ W\gamma\begin{pmatrix} g & * \\ 0 & ad/g \end{pmatrix} = t \circ \delta W\begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}\begin{pmatrix} g & * \\ 0 & ad/g \end{pmatrix} = t \circ W\begin{pmatrix} g & * \\ 0 & ad/g \end{pmatrix}.$$

Let n be the order of $t \circ W(\tau) - t(s)$ with respect to q. Then the order of $t \circ AW'(\tau) - t(s)$ with respect to q is ng^2/ad. We need to show $n = 1$.

If $W \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M))$, then by Proposition 5.10 and thus $t \circ W\colon X_0(M) \to \mathbb{P}^1(\mathbb{C})$ is an isomorphism. Therefore $n = 1$.

If $W \notin N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0(M))$, then $M = 25$ and $W \in N_{\mathrm{SL}_2(\mathbb{R})}(\Gamma_0^{(5)}(25))$ by Proposition 5.10. The map $t \circ W\colon X_0^{(5)}(25) \to \mathbb{P}^1(\mathbb{C})$ is the composition of the isomorphism $W\colon X_0^{(5)}(25) \to X_0^{(5)}(25)$, natural projection $X_0^{(5)}(25) \to X_0(25)$ and the isomorphism $t\colon X_0(25) \to \mathbb{P}^1(\mathbb{C})$. The ramification index at $i\infty$ the natural projection $X_0^{(5)}(25) \to X_0(25)$ is

$$[\{\pm I\}\Gamma_0(25)_{i\infty} : \{\pm I\}\Gamma_0^{(5)}(25)_{i\infty}] = 1$$

by [6, Section 3.1] and thus we also have $n = 1$ in this case. This completes the proof. □

Secondly, we consider the condition that a pair of cusps is a point on the modular correspondence.

Definition 7.2. *Let $s, s' \in X_0(M)$ be cusps and*

$$W = \frac{1}{\sqrt{D}} \begin{pmatrix} m & u \\ M & v \end{pmatrix}, W' = \frac{1}{\sqrt{D'}} \begin{pmatrix} m' & u' \\ M & v' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

be generalized Atkin-Lehner involutions such that $s = W(i\infty), s' = W'(i\infty)$. Let N be a positive integer coprime to M. We define $\delta_{s,s'}(N) := 1$ if $(M, m) = (M, m')$ and there exists an integer g such that $m'N \equiv mg^2 \bmod M$. Otherwise we define $\delta_{s,s'}(N) := 0$.

Remark 7.3. In the case when $M \neq 25$, we have $\delta_{s,s'}(N) = 1$ if and only if $D = D'$ and $m \equiv Nm' \bmod (M, m^2)$ since $(M, m^2) \mid 12$ and $\bar{1}$ is the unique square element of $(\mathbb{Z}/12\mathbb{Z})^\times$. In the case when $M = 25$, we have $\delta_{s,s'}(N) = 1$ if and only if $D = D'$ and $s \equiv \pm Ns' \bmod \mathbb{Z}$ by Table 3. Thus $\delta_{s,s'}(N) = 1$ if and only if $s = s' \in C''_0(M)$,

$$\begin{aligned}
&M = 9, && N \equiv 1 \bmod 3, && s = s' \in \left\{\frac{1}{3}, \frac{2}{3}\right\}, \\
&M = 9, && N \equiv 1 \bmod 3, && (s, s') \in \left\{\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right\}, \\
&M = 16, && && s = s' \in \left\{\frac{1}{2}, \frac{1}{8}\right\}, \\
&M = 16, && N \equiv 1 \bmod 4, && s = s' \in \left\{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}\right\}, \\
&M = 16, && N \equiv -1 \bmod 4, && (s, s') \in \left\{\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right\}, \\
&M = 18, && N \equiv 1 \bmod 6, && s = s' \in \left\{\frac{1}{3}, \frac{2}{3}, \frac{1}{6}, \frac{5}{6}\right\}, \\
&M = 18, && N \equiv -1 \bmod 6, && (s, s') \in \left\{\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{6}, \frac{5}{6}\right), \left(\frac{5}{6}, \frac{1}{6}\right)\right\}, \\
&M = 25, && s, s' \in \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}, && s \equiv \pm Ns' \bmod \mathbb{Z}
\end{aligned}$$

by Table 3.

Theorem 7.4. *For a positive integer N coprime to M, the modular correspondence $T_N^{\Gamma_0(M)} \subset X_0(M) \times X_0(M)$ satisfies that*

$$T_N^{\Gamma_0(M)} \subset Y_0(M)^2 \cup \{(s, s') \mid \delta_{s,s'}(N) \neq 0\}.$$

In particular, if $M \notin \{9, 16, 18, 25\}$ then a pair of cusps on $T_N^{\Gamma_0(M)}$ is a form (s, s).

Proof. By (1.2), if $(\tau, \tau') \in X_0(M) \times X_0(M)$ is a point of $T_N^{\Gamma_0(M)}$, then there exist integers a, b, and d such that $ad = N, 0 \leq b < d$ and $\Gamma_0(M)\tau = \Gamma_0(M)\frac{a\tau' + b}{d}$. Thus (τ, τ') is a point on $Y_0(M) \times Y_0(M)$ or a pair of two cusps (s, s').

Suppose that $(\tau, \tau') = (s, s')$ is a pair of two cusps. Since N is coprime to M, we have $A := \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in G_0(M)$. By Proposition 6.3, we have $(M, m) = (M, m')$ and $m'N \equiv m(d, m'a + Mb)^2 \bmod M$ and thus $\delta_{s,s'}(N) = 1$. $\square$

Finally, we calculate the intersection multiplicity at cusps by Proposition 7.1.

Proposition 7.5. *Let N_1, N_2 be positive integers coprime to M. Suppose that N_1N_2 is not a square. Then for two cusps s, s' in $X_0(M)$, we have*

$$(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{(s,s')} = \delta_{s,s'}(N_1)\delta_{s,s'}(N_2) \sum_{a_1d_1 = N_1, a_2d_2 = N_2} \min\{a_1d_2, a_2d_1\}$$

unless $M = 25$ and $s, s' \in \{1/5, 2/5, 3/5, 4/5\}$. If $M = 25$ and $s, s' \in \{1/5, 2/5, 3/5, 4/5\}$, then we have

$$(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{(s,s')} = \delta_{s,s'}(N_1)\delta_{s,s'}(N_2) \sum_{\substack{a_1d_1=N_1, a_2d_2=N_2,\\ a_1s\equiv d_1s', a_2s\equiv d_2s' \bmod \mathbb{Z}}} \min\{a_1d_2, a_2d_1\}.$$

Proof. Let $s = m/M, s' = m'/M'$ with integers $0 \le m, m' < M$ and $W := W_m^M, W' := W_{m'}^M \in \mathrm{SL}_2(\mathbb{R})$ be generalized Atkin-Lehner involutions. Let $\Phi_{N_i}^{\Gamma_0(M)}(X,Y) \in \mathbb{Z}[X,Y]$ be the modular polynomial whose existence is guaranteed by Theorem 2.1. Then the intersection multiplicity at (s, s') is

$$\begin{aligned}
&(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{(s,s')}\\
=&(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{(W(\infty),W'(\infty))}\\
=&\dim_{\mathbb{C}} \mathbb{C}\left[\left[q,q'\right]\right]/(\Phi_{N_i}^{\Gamma_0(M)}(t\circ W(\tau), t\circ W'(\tau')) \mid i=1,2)\\
=&\frac{1}{N_1N_2} \sum_{\substack{A_i \in I_{N_i,\mathrm{mat}}^{\Gamma_0(M)},\\ \Gamma_0(M)s=\Gamma_0(M)A_i(s'), i=1,2}} \dim_{\mathbb{C}} \mathbb{C}\left[\left[q, q'_{N_1N_2}\right]\right] \Big/ \left(t\circ W(\tau), t\circ A_iW'(\tau')) \mid i=1,2\right).
\end{aligned}$$

By Proposition 7.1 in the case when A is the identity matrix, the order of $t \circ W(\tau) - t(s)$ with respect to q is 1. Thus we have

$$\mathbb{C}\left[\left[q, q'_{N_1N_2}\right]\right] = \mathbb{C}\left[\left[t\circ W(\tau), q'_{N_1N_2}\right]\right].$$

Hence the intersection multiplicity is

$$\begin{aligned}
&(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{(s,s')}\\
=&\frac{1}{N_1N_2} \sum_{\substack{A_i \in I_{N_i,\mathrm{mat}}^{\Gamma_0(M)},\\ \Gamma_0(M)s=\Gamma_0(M)A_i(s'), i=1,2}} \dim_{\mathbb{C}} \mathbb{C}\left[\left[q'_{N_1N_2}\right]\right] \Big/ \left(t\circ A_1W'(\tau') - t\circ A_2W'(\tau')\right)\\
=& \sum_{\substack{A_i \in I_{N_i,\mathrm{mat}}^{\Gamma_0(M)},\\ \Gamma_0(M)s=\Gamma_0(M)A_i(s'), i=1,2}} (\text{the order of } t(A_1W'(\tau')) - t(A_2W'(\tau')) \text{ with respect to } q').
\end{aligned}$$

By Proposition 7.1, this is equal to

$$\sum_{\substack{A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i\end{pmatrix} \in I_{N_i,\mathrm{mat}}^{\Gamma_0(M)},\\ \Gamma_0(M)s=\Gamma_0(M)A_i(s'), i=1,2}} \min_{i=1,2}\left\{\frac{(d_i, m'a_i + Mb_i)^2}{N_i}\right\}.$$

For a positive integer N coprime to M and its positive divisor g, set

$$A(N,g) := \#\left\{A = \begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \in I_{N,\mathrm{mat}}^{\Gamma_0(M)} \,\middle|\, g = (d, m'a+Mb)\right\}.$$

By Proposition 6.3, the intersection multiplicity is

$$\delta_{s,s'}(N_1)\delta_{s,s'}(N_2) \sum_{\substack{g_1|N_1, g_2|N_2,\\ m'N_1\equiv mg_1^2, m'N_2\equiv mg_2^2 \bmod M}} A(N_1,g_1)A(N_2,g_2)\min\left\{\frac{g_1^2}{N_1}, \frac{g_2^2}{N_2}\right\}.$$

Here we have

$$A(N,g) = \#\left\{(a,\bar{b},d) \,\middle|\, ad = N, \bar{b} \in \mathbb{Z}/d\mathbb{Z}, g = (d, m'a+Mb)\right\}.$$

Since the plus $m'a$ map $\mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$ induces the bijection between

$$\left\{\bar{b} \in \mathbb{Z}/d\mathbb{Z} \,\middle|\, g = (d, Mb) = (b,d)\right\}$$

and

$$\left\{\bar{b} \in \mathbb{Z}/d\mathbb{Z} \;\middle|\; g = (d, m'a + Mb)\right\},$$

we have

$$\begin{aligned} A(N,g) &= \sum_{g|d|N} \#\{\bar{b} \in \mathbb{Z}/d\mathbb{Z} \mid g = (b,d)\} = \sum_{g|d|N} \#\left(\mathbb{Z}/(d/g)\mathbb{Z}\right)^{\times} \\ &= \sum_{e|N/g} \#\left(\mathbb{Z}/e\mathbb{Z}\right)^{\times} = \frac{N}{g}. \end{aligned}$$

Thus the intersection multiplicity is

$$\begin{aligned} &\delta_{s,s'}(N_1)\delta_{s,s'}(N_2) \sum_{\substack{g_1|N_1, g_2|N_2, \\ m'N_1 \equiv mg_1^2, m'N_2 \equiv mg_2^2 \bmod M}} \frac{N_1}{g_1}\frac{N_2}{g_2} \min\left\{\frac{g_1^2}{N_1}, \frac{g_2^2}{N_2}\right\} \\ =&\delta_{s,s'}(N_1)\delta_{s,s'}(N_2) \sum_{\substack{g_1h_1 = N_1, g_2h_2 = N_2, \\ g_1s \equiv h_1s', g_2s \equiv h_2s' \bmod \mathbb{Z}}} \min\{g_1h_2, g_2h_1\}. \end{aligned}$$

Unless $M = 25$ and $s, s' \in \{1/5, 2/5, 3/5, 4/5\}$, the condition $g_1s \equiv h_1s', g_2s \equiv h_2s' \bmod \mathbb{Z}$ holds for any g_1, h_1, g_2 and h_2. □

8. The class number formulas

In this section, let N_1 and N_2 be positive integers coprime to M and suppose N_1N_2 is not a square.

The intersection number of modular correspondences on $X_0(M) \times X_0(M)$ is calculated as follows.

Lemma 8.1. *We have*

$$(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{X_0(M)\times X_0(M)} = 2\sigma(N_1)\sigma(N_2) = \sum_{a_1d_1 = N_1, a_2d_2 = N_2} (a_1d_2 + a_2d_1).$$

Proof. Since $X_0(M)$ has genus zero, $X_0(M) \times X_0(M)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The intersection number of divisors on the algebraic surface $\mathbb{P}^1 \times \mathbb{P}^1$ only depends on its degrees. The degree of the algebraic cycle $T_{N_i}^{\Gamma_0(M)}$ is $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N_i)]$, which is the same value in the case when $M = 1$ which is treated in [10]. Thus the intersection number on $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{\mathbb{P}^1\times\mathbb{P}^1} = 2\sigma_1(N_1)\sigma_1(N_2)$$

by [10, Lemma 3.1]. The last equality follows from the definition of the divisor function $\sigma(N)$. □

By combining Proposition 7.5 and Lemma 8.1, we can calculate the intersection number of $T_{N_1}^{\Gamma_0(M)}$ and $T_{N_2}^{\Gamma_0(M)}$ on $X_0(M) \times X_0(M)$ as follows.

Theorem 8.2. *Unless $M = 25$ and $N_1 \equiv \pm N_2 \bmod 5$, we have*

$$(T_{N_1}^{\Gamma_0(M)} \cdot T_{N_2}^{\Gamma_0(M)})_{Y_0(M)\times Y_0(M)} = 2 \sum_{a_1d_1 = N_1, a_2d_2 = N_2, a_1d_2 > a_2d_1} (a_1d_2 - \delta_M(N_1, N_2)a_2d_1)$$

where

$$\delta_M(N_1,N_2) := -1 + \sum_{(s,s')\in C_0(M)} \delta_{s,s'}(N_1)\delta_{s,s'}(N_2)$$

$$= \begin{cases} 0 & \textit{if } M=1, \\ 1 & \textit{if } M=2,3,5,7,13, \\ 2 & \textit{if } M=4, \\ 3 & \textit{if } M=6,8,10, \\ 3 & \textit{if } M=9, N_1\equiv N_2 \bmod 3, \\ 1 & \textit{if } M=9, N_1\equiv -N_2 \bmod 3, \\ 5 & \textit{if } M=12, \\ 5 & \textit{if } M=16, N_1\equiv N_2 \bmod 4, \\ 3 & \textit{if } M=16, N_1\not\equiv N_2 \bmod 4, \\ 7 & \textit{if } M=18, N_1\equiv N_2\equiv 1 \bmod 6, \\ 5 & \textit{if } M=18, N_1\equiv N_2\equiv -1 \bmod 6, \\ 3 & \textit{if } M=18, N_1\not\equiv N_2 \bmod 6, \\ 1 & \textit{if } M=25, N_1\not\equiv \pm N_2 \bmod 5. \end{cases}$$

If $M=25$ and $N_1\equiv \pm N_2 \bmod 5$, then

$$\begin{aligned}&(T_{N_1}^{\Gamma_0(25)}\cdot T_{N_2}^{\Gamma_0(25)})_{Y_0(25)\times Y_0(25)} \\ =& \sum_{a_1d_1=N_1,a_2d_2=N_2} |a_1d_2-a_2d_1| - 4 \sum_{\substack{a_1d_1=N_1,a_2d_2=N_2,\\ a_1d_2\equiv a_2d_1 \bmod 5}} \min\{a_1d_2,a_2d_1\}.\end{aligned}$$

In particular, if M is a prime number p, that is, $M=p=2,3,5,7,13$, we have

$$(T_{N_1}^{\Gamma_0(p)}\cdot T_{N_2}^{\Gamma_0(p)})_{Y_0(p)\times Y_0(p)} = \sum_{a_1d_1=N_1,a_2d_2=N_2} |a_1d_2-a_2d_1|.$$

Proof. By Theorem 7.4, the intersection number is

$$\begin{aligned}&(T_{N_1}^{\Gamma_0(M)}\cdot T_{N_2}^{\Gamma_0(M)})_{Y_0(M)^2} \\ =&(T_{N_1}^{\Gamma_0(M)}\cdot T_{N_2}^{\Gamma_0(M)})_{\mathbb{P}^1\times\mathbb{P}^1} - \sum_{(s,s')\in C_0(M)^2} (T_{N_1}^{\Gamma_0(M)}\cdot T_{N_2}^{\Gamma_0(M)})_{(s,s')}.\end{aligned}$$

Unless $M=25$ and $N_1\equiv\pm N_2 \bmod 5$, the intersection number is

$$\begin{aligned}&\sum_{a_1d_1=N_1,a_2d_2=N_2}(a_1d_2+a_2d_1) - (1+\delta_M(N_1,N_2))\sum_{a_1d_1=N_1,a_2d_2=N_2}\min\{a_1d_2,a_2d_1\} \\ =&2\sum_{a_1d_1=N_1,a_2d_2=N_2,a_1d_2>a_2d_1}(a_1d_2-\delta_M(N_1,N_2)a_2d_1)\end{aligned}$$

by Proposition 7.5. If $M=25$ and $N_1\equiv\pm N_2 \bmod 5$, then the intersection number is

$$\begin{aligned}&(T_{N_1}^{\Gamma_0(25)}\cdot T_{N_2}^{\Gamma_0(25)})_{\mathbb{P}^1\times\mathbb{P}^1} - (T_{N_1}^{\Gamma_0(25)}\cdot T_{N_2}^{\Gamma_0(25)})_{(i\infty,i\infty)} - (T_{N_1}^{\Gamma_0(25)}\cdot T_{N_2}^{\Gamma_0(25)})_{(0,0)} \\ &- \sum_{s,s'\in\{1/5,2/5,3/5,4/5\}} (T_{N_1}^{\Gamma_0(25)}\cdot T_{N_2}^{\Gamma_0(25)})_{(s,s')} \\ =& \sum_{a_1d_1=N_1,a_2d_2=N_2}|a_1d_2-a_2d_1| - \sum_{s,s'\in\{1/5,2/5,3/5,4/5\}}\ \sum_{\substack{a_1d_1=N_1,a_2d_2=N_2,\\ a_1s\equiv d_1s',a_2s\equiv d_2s' \bmod \mathbb{Z}}}\min\{a_1d_2,a_2d_1\} \\ =& \sum_{a_1d_1=N_1,a_2d_2=N_2}|a_1d_2-a_2d_1| - 4\sum_{\substack{a_1d_1=N_1,a_2d_2=N_2,\\ a_1d_2\equiv a_2d_1 \bmod 5}}\min\{a_1d_2,a_2d_1\}.\end{aligned}$$

□

We have the following main Theorem in this paper.

Theorem 8.3. *Unless $M = 25$ and $N_1 \equiv \pm N_2 \bmod 5$, we have*

$$\begin{aligned}&\sum_{x\in\mathbb{Z},\ x^2<4N_1N_2} \sum_{d|(N_1,N_2,x)} d \cdot H^M\left(\frac{4N_1N_2 - x^2}{d^2}\right)\\ &= 2 \sum_{a_1d_1=N_1, a_2d_2=N_2, a_1d_2>a_2d_1} (a_1d_2 - \delta_M(N_1,N_2)a_2d_1)\end{aligned}$$

where $\delta_M(N_1,N_2)$ is defined in Theorem 8.2. If $M = 25$ and $N_1 \equiv \pm N_2 \bmod 5$, then

$$\begin{aligned}&\sum_{x\in\mathbb{Z},\ x^2<4N_1N_2} \sum_{d|(N_1,N_2,x)} d \cdot H^{25}\left(\frac{4N_1N_2 - x^2}{d^2}\right)\\ &= \sum_{a_1d_1=N_1, a_2d_2=N_2} |a_1d_2 - a_2d_1| - 8 \sum_{\substack{a_1d_1=N_1, a_2d_2=N_2, a_1d_2>a_2d_1,\\ a_1d_2\equiv a_2d_1 \bmod 5}} a_2d_1.\end{aligned}$$

Proof. It follows from Theorem 2.2 and Theorem 8.2. □

Theorem 1.1 is the special case in this theorem.

9. Explicit computation of $H^M(D)$ for some M

In this section, we give a method to compute the Hurwitz class number $H^M(D)$ for M when $2 \le M \le 10$ or $M \in \{12, 13, 16, 18, 25\}$.

If the level is a prime number $p \in \{2, 3, 5, 7, 13\}$, then $H^p(D)$ can be calculated from $H(D)$ by the following theorem.

Theorem 9.1 ([4, Lemma 3.2])**.** *For a prime number $p \in \{2, 3, 5, 7, 13\}$ and a positive integer $D \equiv 0, 3 \bmod 4$, we have*

$$H^p(D) = \left(1 + \left(\frac{-D}{p}\right)\right)\left(H(D) + p \cdot H\left(\frac{D}{p^2}\right)\right).$$

Here we define $H(D/p^2) := 0$ if $p^2 \nmid D$.

This theorem is slightly different from the original statement of [4, Lemma 3.2]. See also a proof of [12, Proposition 3.3].

In general, we can calculate the Hurwitz class number $H^M(D)$ by considering a fundamental domain of $\Gamma_0(M)$. The following elementary lemma is useful for computing $H^M(D)$.

Lemma 9.2. *Let $D \equiv 0, 3 \bmod 4$ be a positive integer, $[Ma, b, c] \in \mathcal{Q}^M_{-D,>0}$ with $a, c \ge 1$ and*

$$w_Q := \frac{-b + \sqrt{-D}}{2Ma}.$$

For positive integers m and n, if

$$\left|w_Q \pm \frac{m}{n}\right| \ge \frac{1}{n},$$

then $\pm b \le ka + lc$ where

$$k := \frac{M(m^2-1)}{mn}, \quad l := \frac{n}{m}.$$

Moreover, if $m \ge 2$, $|b| \le ka + lc$, and $r_-c + t_- \le a \le r_+c - t_+$ with $t_+, t_- > 0$ and

$$r_+ := \frac{1}{M}\left(\frac{n}{m-1}\right)^2, \quad r_- := \frac{1}{M}\left(\frac{n}{m+1}\right)^2,$$

then $a \le D/C$ where

$$C := \min\left\{t_+\left(k^2 - \frac{l^2}{r_+(r_+-t_+)}\right), \quad -t_-\left(k^2 - \frac{l^2}{r_-(r_-+t_-)}\right)\right\}.$$

Proof. By direct calculation, the condition

$$\left|w_Q \pm \frac{m}{n}\right| \geq \frac{1}{n}$$

is equivalent to $Mam^2 \mp bmn + cn^2 \geq Ma$, that is, $\pm b \leq ka + lc$.

Suppose $m \geq 2$ and $|b| \leq ka + lc$. Let $f(x) := k^2x + l^2/x$. Then we have

$$D \geq 4Mac - (ka + lc)^2 = ac\left(4M - 2kl - f\left(\frac{a}{c}\right)\right).$$

By direct calculation, $f(x) = 4M - 2kl$ if and only if $x = r_+$ or $x = r_-$. Moreover, if $r_- \leq x \leq l/k$ then $f(x)$ is monotonic decreasing and if $l/k \leq x \leq r_+$ then $f(x)$ is monotonic increasing by elementary calculus. For $t_+, t_- > 0$, we have

$$f\left(r_- + \frac{t_-}{c}\right) = f(r_-) + \frac{t_-}{c}\left(k^2 - \frac{l^2}{r_-(r_- + t_-/c)}\right) \leq f(r_-) + \frac{t_-}{c}\left(k^2 - \frac{l^2}{r_-(r_- + t_-)}\right)$$

and

$$f\left(r_+ - \frac{t_+}{c}\right) = f(r_+) - \frac{t_+}{c}\left(k^2 - \frac{l^2}{r_+(r_+ - t_+/c)}\right) \leq f(r_+) + \frac{t_+}{c}\left(k^2 - \frac{l^2}{r_+(r_+ - t_+)}\right).$$

Hence if $r_-c + t_- \leq a \leq r_+c - t_+$, then

$$D \geq ac\left(4M - 2kl - f\left(\frac{a}{c}\right)\right) \geq aC.$$

$\square$

Here we calculate $H^M(D)$ when M is a composite number by the following lemma.

Lemma 9.3. *Let $D \equiv 0, 3 \bmod 4$ be a positive integer.*

(i) *The set*

$$\left\{[4a, b, c] \in \mathcal{Q}^4_{-D,>0} \;\middle|\; \begin{array}{c} |b| \leq 4\min\{a, c\}, \\ \text{if } |b| = 4\min\{a, c\} \text{ then } 0 \leq b \end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^4_{-D,>0}/\Gamma_0(4)$. Moreover, if $[4a, b, c]$ is an element of this set, then $a, c \leq (D+1)/8$.

(ii) *The set*

$$\left\{[6a, b, c] \in \mathcal{Q}^6_{-D,>0} \;\middle|\; \begin{array}{c} |b| \leq 6\min\{a, c, (2/5)(a+c)\}, \\ \text{if } |b| = 6\min\{a, c, (2/5)(a+c)\} \text{ then } 0 \leq b \end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^6_{-D,>0}/\Gamma_0(6)$. Moreover, if $[6a, b, c]$ is an element of this set, then $a, c \leq (25/24)D$.

(iii) *The set*

$$\left\{[8a, b, c] \in \mathcal{Q}^8_{-D,>0} \;\middle|\; \begin{array}{c} |b| \leq 8a, -8c \leq b \leq 4c, -(8/7)(2a+3c) \leq b, -(4/5)(4a+3c) \leq b \\ \text{if } b \in \{\pm 8a, -8c, 4c, -(8/7)(2a+3c), -(4/5)(4a+3c)\} \\ \text{then } -4a \leq b \end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^8_{-D,>0}/\Gamma_0(8)$. Moreover, if $[8a, b, c]$ is an element of this set, then $a, c \leq (245/96)D$.

(iv) *The set*

$$\left\{[9a, b, c] \in \mathcal{Q}^9_{-D,>0} \;\middle|\; \begin{array}{c} |b| \leq 9\min\{a, c, (2/5)(3a+2c)\}, \\ \text{if } |b| = 9\min\{a, c, (2/5)(3a+2c)\} \text{ then } 0 \leq b \end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^9_{-D,>0}/\Gamma_0(9)$. Moreover, if $[9a, b, c]$ is an element of this set, then $a, c \leq (25/72)D$.

(v) *The set*

$$\left\{[10a,b,c]\in\mathcal{Q}^{10}_{-D,>0}\;\middle|\;\begin{array}{c}|b|\le 10\min\{a,(3/5)c,(1/11)(4a+3c),(2/9)(2a+c)\},\\ \text{if } |b|=10\min\{a,(3/5)c,(1/11)(4a+3c),(2/9)(2a+c)\}\\ \text{then } (20/3)a\le b \text{ or } |b|\le 6a\end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^{10}_{-D,>0}/\Gamma_0(10)$*. Moreover, if* $[10a,b,c]$ *is an element of this set, then* $a,c\le(121/35)D$*.*

(vi) *The set*

$$\left\{[12a,b,c]\in\mathcal{Q}^{12}_{-D,>0}\;\middle|\;\begin{array}{c}|b|\le\min\{12a,(12/5)(2a+c),(24/7)(a+c)\},\\ -12c\le b\le 8c, b\ge\max\{-(12/11)(2a+5c),-(8/9)(3a+5c)\},\\ \text{if } |b|=\min\{12a,(12/5)(2a+c),(24/7)(a+c)\}, b=-12c, b=8c \text{ or}\\ b=\max\{-(12/11)(2a+5c),-(8/9)(3a+5c)\},\ \text{then } -4a\le b\le 12a\end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^{12}_{-D,>0}/\Gamma_0(12)$*. Moreover, if* $[12a,b,c]$ *is an element of this set, then* $a,c\le(1573/240)D$*.*

(vii) *The set*

$$\left\{[16a,b,c]\in\mathcal{Q}^{16}_{-D,>0}\;\middle|\;\begin{array}{c}|b|\le\min\{16a,8c,(8/7)(4a+3c)\}, b\le(4/5)(8a+3c),\\ b\ge\max\{-(48/17)(2a+c),-(16/31)(12a+5c),-(4/9)(16a+5c)\},\\ \text{if } |b|=\min\{16a,8c,(8/7)(4a+3c)\}, b=(4/5)(8a+3c) \text{ or}\\ b=\max\{-(48/17)(2a+c),-(16/31)(12a+5c)\},\\ \text{then } |b|\le 8a, b\ge(32/3)a \text{ or } -12a\le b\le -(32/3)a\end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^{16}_{-D,>0}/\Gamma_0(16)$*. Moreover, if* $[16a,b,c]$ *is an element of this set, then* $a,c\le D$*.*

(viii) *The set*

$$\left\{[18a,b,c]\in\mathcal{Q}^{18}_{-D,>0}\;\middle|\;\begin{array}{c}|b|\le\min\left\{\begin{array}{c}18a,12c,(12/11)(3a+5c),(18/19)(4a+5c),\\(12/7)(3a+2c),(12/5)(3a+c),(72/17)(a+c)\end{array}\right\},\\ \text{if } |b|=\min\left\{\begin{array}{c}18a,12c,(12/11)(3a+5c),(18/19)(4a+5c),\\(12/7)(3a+2c),(12/5)(3a+c),(72/17)(a+c)\end{array}\right\},\\ \text{then } |b|\le 6a,(36/5)a\le b\le 9a, 12a\le|b|<18a \text{ or } b=18a\end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^{18}_{-D,>0}/\Gamma_0(18)$*. Moreover, if* $[18a,b,c]$ *is an element of this set, then* $a,c\le(361/45)D$*.*

(ix) *The set*

$$\left\{[25a,b,c]\in\mathcal{Q}^{25}_{-D,>0}\;\middle|\;\begin{array}{c}|b|\le\min\left\{\begin{array}{c}25a,10c,(10/9)(5a+4c),(2/7)(25a+12c),\\(10/11)(10a+3c),(20/9)(5a+c)\end{array}\right\},\\ \text{if } |b|=\min\left\{\begin{array}{c}25a,10c,(10/9)(5a+4c),(2/7)(25a+12c),\\(10/11)(10a+3c),(20/9)(5a+c)\end{array}\right\},\\ \text{then } |b|\le 10a, 14a\le|b|\le 20a \text{ or } b=25a\end{array}\right\}$$

is a complete system of representatives of $\mathcal{Q}^{25}_{-D,>0}/\Gamma_0(25)$*. Moreover, if* $[25a,b,c]$ *is an element of this set, then* $a,c\le(968/175)D$*.*

Proof. Firstly, we get a complete system of representatives of $\mathcal{Q}^{M}_{-D,>0}/\Gamma_0(M)$ in each case by Lemma 9.2 since we have fundamental domains of $\Gamma_0(4),\Gamma_0(6),\Gamma_0(8),\Gamma_0(9),\Gamma_0(10),\Gamma_0(12),$

$\Gamma_0(16)$, $\Gamma_0(18)$, and $\Gamma_0(25)$ as

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 1/4|\ge 1/4,\\ \text{if } |\mathrm{Re}(\tau)|=1/2 \text{ or } |\tau\pm 1/4|=1/4 \text{ then } \mathrm{Re}(\tau)\le 0\end{array}\right\},$$

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 1/6|\ge 1/6, |\tau\pm 5/12|\ge 5/12,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau\pm 1/6|=1/6 \text{ or } |\tau\pm 5/12|\ge 5/12 \text{ then } \mathrm{Re}(\tau)\le 0\end{array}\right\},$$

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau+1/4|\ge 1/4, |\tau-1/8|\ge 1/8,\\ |\tau-7/24|\ge 1/24, |\tau-5/12|\ge 1/12,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau+1/4|=1/4, |\tau-1/8|=1/8, |\tau-7/24|=1/24\\ \text{or } |\tau-5/12|=1/12 \text{ then } \mathrm{Re}(\tau)\le 1/4\end{array}\right\},$$

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 1/6|\ge 1/6, |\tau\pm 5/12|\ge 5/12,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau\pm 1/6|=1/6 \text{ or } |\tau\pm 5/12|\ge 5/12 \text{ then } \mathrm{Re}(\tau)\le 0\end{array}\right\},$$

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 1/6|\ge 1/6, |\tau\pm 11/30|\ge 1/30, |\tau\pm 9/20|\ge 1/20,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau\pm 1/6|=1/6, |\tau\pm 11/30|=1/30 \text{ or } |\tau\pm 9/20|=1/20\\ \text{then } \mathrm{Re}(\tau)\le -1/3 \text{ or } |\mathrm{Re}(\tau)|\le 3/10\end{array}\right\}, \tag{9.1}$$

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 5/12|\ge 1/12, |\tau\pm 7/24|\ge 1/24, |\tau+1/8|\ge 1/8,\\ |\tau-1/12|\ge 1/12, |\tau-11/60|\ge 1/60, |\tau-9/40|\ge 1/40,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau\pm 5/12|=1/12, |\tau\pm 7/24|=1/24, |\tau+1/8|=1/8,\\ |\tau-1/12|=1/12, |\tau-11/60|=1/60 \text{ or } |\tau-9/40|=1/40\\ \text{then } -1/2\le\mathrm{Re}(\tau)\le 1/6\end{array}\right\},$$

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 1/8|\ge 1/8, |\tau\pm 7/24|\ge 1/24, |\tau+5/12|\ge 1/12,\\ |\tau-17/48|\ge 1/48, |\tau-31/80|\ge 1/80, |\tau-9/20|\ge 1/20,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau\pm 1/8|=1/8, |\tau\pm 7/12|=1/24, |\tau+5/12|=1/12,\\ |\tau-17/48|=1/48, |\tau-31/80|=1/80 \text{ or } |\tau-9/20|=1/20,\\ \text{then } |\mathrm{Re}(\tau)|\le 1/4, \mathrm{Re}(\tau)\le -1/3 \text{ or } 1/3\le\mathrm{Re}(\tau)\le 3/8\end{array}\right\},$$

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 1/12|\ge 1/12, |\tau\pm 11/60|\ge 1/60,\\ |\tau\pm 19/90|\ge 1/90, |\tau\pm 17/72|\ge 1/72, |\tau\pm 7/24|\ge 1/24, |\tau\pm 5/12|\ge 1/12,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau\pm 1/12|=1/12, |\tau\pm 11/60|=1/60,\\ |\tau\pm 19/90|=1/90, |\tau\pm 17/72|=1/72, |\tau\pm 7/24|=1/24 \text{ or } |\tau\pm 5/12|=1/12,\\ \text{then } |\mathrm{Re}(\tau)|\le 1/6, -1/4\le\mathrm{Re}(\tau)\le -1/5, 1/3\le|\mathrm{Re}(\tau)|<1/2 \text{ or } |\mathrm{Re}(\tau)|=-1/2\end{array}\right\},$$

and

$$\left\{\tau\in\mathbb{H}\ \middle|\ \begin{array}{c}|\mathrm{Re}(\tau)|\le 1/2, |\tau\pm 1/10|\ge 1/10, |\tau\pm 9/40|\ge 1/40,\\ |\tau\pm 7/24|\ge 1/24, |\tau\pm 11/30|\ge 1/30, |\tau\pm 9/20|\ge 1/20,\\ \text{if } |\mathrm{Re}(\tau)|=1/2, |\tau\pm 1/10|=1/10, |\tau\pm 9/40|=1/40,\\ |\tau\pm 7/24|=1/24, |\tau\pm 11/30|=1/30 \text{ or } |\tau\pm 9/20|=1/20,\\ \text{then } |\mathrm{Re}(\tau)|\le 1/5, 7/25\le|\mathrm{Re}(\tau)|\le 2/5 \text{ or } |\mathrm{Re}(\tau)|=-1/2\end{array}\right\} \tag{9.2}$$

by using the algorithm in [9] which is based on the theory of Farey symbols in [8] and is implemented for Sage [14] by Chris A. Kurth.

Secondly, we bound a and c in each case.

(i). For a quadratic form $[4a,b,c]$, $(-b+\sqrt{b^2-4ac})/2$ is a point of above fundamental domain if and only if $[4a,b,c]$ is an element of the set in statement. In this case, if $a>c$ then

$$D=-b^2+16ac\ge -16c^2+16ac=16c(a-c)$$

and thus we have $c \le D/16$ and $a - c \le D/16$. Then we obtain $c < a \le D/8$. Similarly, if $a < c$ then we have $a < c \le D/8$. If $a = c$, then $|b| \le 4a - 1$ since $0 < D = -b^2 + 16a^2$. Thus we have $D \ge -(4a-1)^2 + 16a^2 = 8a - 1$.

For other cases, we have the boundings by Lemma 9.2 and similar argument in (i). For (ii), we have:

(a) If $3a < 2c$ or $3c < 2a$, then $a, c \le D/6$.
(b) If $3a = 2c$ or $3c = 2a$, then $a, c \le (D+1)/8$.
(c) If $2a \le 2c < 3a$ or $2c \le 2a < 3c$, then $a, c \le (25/24)D$.

For (iii), we have:

(a) If $2a < c$ or $2c < a$, then $a, c \le (3/16)D$.
(b) If $2a = c$ or $2c = a$, then $a, c \le (D+1)/8$.
(c) If $c < 2a$ and $b \ge 0$, then $a, c \le D/16$.
(d) If $4c < 8a < 9c$ and $b \le 0$, then $a, c \le (25/16)D$.
(e) If $9c < 8a < 16c$ and $b \le 0$, then $a, c \le (245/96)D$.

For (iv), we have:

(a) If $9a < 4c$, then $a, c \le (5/18)D$.
(b) If $9a = 4c$, then $a, c \le (D+1)/4$.
(c) If $4a < 4c < 9a$, then $a, c \le (25/72)D$.
(d) If $a = c$, then $a, c \le (D+1)/6$.
(e) If $c < a$, then $a, c \le D/18$.

For (v), we have:

(a) If $5a < 2c$, then $a, c \le (3/20)D$.
(b) If $5a = 2c$, then $a, c \le (D+1)/8$.
(c) If $(2/5)c < a < (5/8)c$, then $a, c \le (81/40)D$.
(d) If $8a = 5c$, then $a, c \le (D+1)/10$.
(e) If $(5/8)c < a < (9/10)c$, then $a, c \le (121/35)D$.
(f) If $10a = 9c$, then $a, c \le (D+1)/12$.
(g) If $9c < 10a$, then $a, c \le D/4$.

For (vi), we have:

(a) If $3a < c$ or $3c < a$, then $a, c \le D/12$.
(b) If $3a = c$ or $3c = a$, then $a, c \le (D+1)/8$.
(c) If $(1/3)c < a < (3/4)c$, then $a, c \le (25/8)D$.
(d) If $a = (3/4)c$ or $a = (4/3)$, then $a, c \le (D+1)/12$.
(e) If $(3/4)c < a < (4/3)c$, then $a, c \le (49/36)D$.
(f) If $(4/3)c < a < (25/12)c$, then $a, c \le (243/64)D$.
(g) If $a = (25/12)c$, then $a, c \le (3/125)(D+1)$.
(h) If $(25/12)c < a < 3c$, then $a, c \le (1573/240)D$.

For (vii), we have:

(a) If $4a < c$, then $a, c \le (5/64)D$.
(b) If $4a = c$, then $a, c \le (D+1)/8$.
(c) If $(1/4)c < a < (9/16)c$, then $a, c \le (25/32)D$.
(d) If $16a = 9c$, then $a, c \le (D+1)/12$.
(e) If $(9/16)c < a < c$, then $a, c \le (245/192)D$.
(f) If $a = c$, then $a, c \le (D+1)/32$.
(g) If $c < a$, then $a, c \le D/32$.

For (viii), we have:

(a) If $9a < 2c$, then $a, c \le (5/36)D$.
(b) If $9a = 2c$, then $a, c \le (D+1)/8$.
(c) If $(2/9)c < a < (1/2)c$, then $a, c \le (425/72)D$.
(d) If $2a = c$ or $a = 2c$, then $a, c \le (D+1)/24$.

(e) If $(1/2)c < a < (8/9)c$, then $a, c \leq (686/360)D$.
(f) If $9a = 8c$ or $8a = 9c$, then $a, c \leq (D+1)/16$.
(g) If $(8/9)c < a < (9/8)c$, then $a, c \leq (289/64)D$.
(h) If $(9/8)c < a < (25/18)c$, then $a, c \leq (361/45)D$.
(i) If $18a = 25c$, then $a, c \leq (5/72)(D+1)$.
(j) If $(25/18)c < a < 2c$, then $a, c \leq (1573/360)D$.
(k) If $2c < a$, then $a, c \leq D/24$.

For (ix), we have:

(a) If $25a < 4c$, then $a, c \leq (3/50)D$.
(b) If $25a = 4c$, then $a, c \leq (D+1)/80$.
(c) If $(4/25)c < a < (1/4)c$, then $a, c \leq (81/16)D$.
(d) If $4a = c$, then $a, c \leq (D+1)/100$.
(e) If $(1/4)c < a < (9/25)c$, then $a, c \leq (968/175)D$.
(f) If $25a = 9c$, then $a, c \leq (D+1)/120$.
(g) If $9c < 25a < 16c$, then $a, c \leq (245/72)D$.
(h) If $25a = 16c$, then $a, c \leq (D+1)/160$.
(i) If $c < a$, then $a, c \leq D/100$.

$\square$

To compute $H^M(D)$, we need a criterion whether the stabilizer $\Gamma_0(M)_Q$ is non-trivial for a quadratic form $Q \in \mathcal{Q}^M_{-D,>0}$. Such quadratic forms correspond to elliptic points for $\Gamma_0(M)$. By [6, Corollary 3.7.2], $\Gamma_0(M)$ has no elliptic points for $M \in \{4, 6, 8, 9, 12, 16, 18\}$ and has exactly 2 elliptic points of period 2 and no elliptic points of period 3 for $M \in \{10, 25\}$. For $M = 10$, elliptic points in the fundamental domain in (9.1) are $(\pm 3 + \sqrt{-1})/10$ whose corresponding quadratic forms are $[10, \mp 6, 1]$. For $M = 25$, elliptic points in the fundamental domain in (9.2) are $(\pm 7 + \sqrt{-1})/25$ whose corresponding quadratic forms are $[25, \mp 14, 2]$.

We show $H(D)$ and $H^M(D)$ for a positive integer $D \leq 50$ in Tables 4 and 5.

10. Examples

In this section, we give several examples of our formula in Theorem 1.1 and give conjectures for a square N.

To extend our formula in Theorem 1.1 for a square N, we need to define $H^M(0)$.

In the case when the level is 1, put the 0th Hurwitz class number $H(0) := -1/12$. Then Hurwitz-Eichler relation (1.1) holds for a square N:

$$\sum_{x \in \mathbb{Z},\ x^2 \leq 4N} H(4N - x^2) = \sum_{ad=N} \max\{a, d\}.$$

Similarly, we define the Hurwitz class number $H^M(0)$ for M with $2 \leq M \leq 10$ or $M \in \{12, 13, 16, 18, 25\}$ by

$$H^M(0) := -\frac{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)]}{12} = -\frac{M}{12} \prod_{p \mid M} \left(1 + \frac{1}{p}\right). \tag{10.1}$$

Under this definition, we calculate

$$S(N) := \sum_{x \in \mathbb{Z},\ x^2 \leq 4N} H\left(4N - x^2\right), \quad S^M(N) := \sum_{x \in \mathbb{Z},\ x^2 \leq 4N} H^M\left(4N - x^2\right)$$

for a positive integer $N \leq 12$ in Table 6 and Table 7. We can confirm that Theorem 1.1 holds for square-free N coprime to M.

Here we have the following conjecture which treats the case when N is a square.

Conjecture 10.1. *Let M be $2 \leq M \leq 10$ or $M \in \{12, 13, 16, 18, 25\}$ and N be a positive integer coprime to M. We put the number $\delta_M(1, N)$ as in Theorem 8.2 and the Hurwitz class number $H^M(0)$ as in (10.1). Unless $M = 25$ and $N \equiv \pm 1 \bmod 5$, we have*

$$\sum_{x \in \mathbb{Z},\ x^2 \leq 4N} H^M\left(4N - x^2\right) = \sum_{ad=1} \left(\max\{a, d\} - \delta_M(1, N) \min\{a, d\}\right)$$

and if $M = 25$ and $N \equiv \pm 1 \bmod 5$, we have

$$\sum_{x \in \mathbb{Z},\ x^2 \leq 4N} H^{25}\left(4N - x^2\right) = \sum_{ad=N} |a - d| - 4 \sum_{ad=N, a \equiv d \bmod 5} \min\{a, d\}.$$

References

[1] M. Akbas and D. Singerman. The normalizer of $\Gamma_0(N)$ in $\mathrm{PSL}(2, \mathbf{R})$. *Glasgow Math. J.*, 32(3):317–327, 1990.

[2] F. Bars. The group structure of the normalizer of $\Gamma_0(N)$ after Atkin-Lehner. *Comm. Algebra*, 36(6):2160–2170, 2008.

[3] J.H. Bruinier and M. Schwagenscheidt. Theta lifts for lorentzian lattices and coefficients of mock theta functions. *Math. Z.*, 2020.

[4] S. Choi and C. H. Kim. Some remarks on Hurwitz-Kronecker class numbers and traces of singular moduli. *Ramanujan Journal*, 38:579–596, 2015.

[5] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bull. London Math. Soc.*, 11(3):308–339, 1979.

[6] F. Diamond and J. Shurman. *A First Course in Modular Forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 2005.

[7] M. Eichler. On the class of imaginary quadratic fields and the sums of divisors of natural numbers. *J. Indian Math. Soc. (N.S.)*, 19:153–180 (1956), 1955.

[8] R. S. Kulkarni. An arithmetic-geometric method in the study of the subgroups of the modular group. *Amer. J. Math.*, 113(6):1053–1133, 1991.

[9] C. A. Kurth and L. Long. Computations with finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ using Farey symbols. In *Advances in algebra and combinatorics*, pages 225–242. World Sci. Publ., Hackensack, NJ, 2008.

[10] J. Ling. Intersection of modular polynomials. *Proceedings of the American Mathematical Society*, 137(5):1543–1549, 2009.

[11] R. S. Maier. On rationally parametrized modular equations. *J. Ramanujan Math. Soc.*, 24:1–73, 2009.

[12] Y. Murakami. Intersection numbers of modular correspondences for genus zero modular curves. *J. Number Theory*, 209:167–194, 2020.

[13] A. Sebbar. Torsion-free genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{R})$. *Duke Math. J.*, 110:377–396, 2001.

[14] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.1)*, 2020. `https://www.sagemath.org`.

[15] S. Zemel. Normalizers of congruence groups in $\mathrm{SL}_2(\mathbb{R})$ and automorphisms of lattices. *Int. J. Number Theory*, 13(5):1275–1300, 2017.

TABLE 4. $H(D)$ and $H^M(D)$ for positive integers $D \leq 100$.

D	$H(D)$	$H^2(D)$	$H^3(D)$	$H^4(D)$	$H^5(D)$	$H^6(D)$	$H^7(D)$	$H^8(D)$
0	$-1/12$	$-1/4$	$-1/3$	$-1/2$	$-1/2$	-1	$-2/3$	-1
3	1/3	0	1/3	0	0	0	2/3	0
4	1/2	1/2	0	0	1	0	0	0
7	1	2	0	2	0	0	1	2
8	1	1	2	0	0	2	0	0
11	1	0	2	0	2	0	0	0
12	4/3	2	4/3	2	0	2	8/3	0
15	2	4	2	4	2	4	0	4
16	3/2	5/2	0	3	3	0	0	2
19	1	0	0	0	2	0	2	0
20	2	2	4	0	2	4	4	0
23	3	6	6	6	0	12	0	6
24	2	2	2	0	4	2	4	0
27	4/3	0	7/3	0	0	0	8/3	0
28	2	4	0	6	0	0	2	8
31	3	6	0	6	6	0	6	6
32	3	5	6	6	0	10	0	4
35	2	0	4	0	2	0	2	0
36	5/2	5/2	4	0	5	4	0	0
39	4	8	4	8	8	8	0	8
40	2	2	0	0	2	0	4	0
43	1	0	0	0	0	0	0	0
44	4	6	8	6	8	12	0	0
47	5	10	10	10	0	20	10	10
48	10/3	6	10/3	7	0	6	20/3	8
51	2	0	2	0	4	0	0	0
52	2	2	0	0	0	0	4	0
55	4	8	0	8	4	0	8	8
56	4	4	8	0	8	8	4	0
59	3	0	6	0	6	0	6	0
60	4	8	4	12	4	8	0	16
63	5	10	8	10	0	16	5	10
64	7/2	13/2	0	9	7	0	0	10
67	1	0	0	0	0	0	0	0
68	4	4	8	0	0	8	8	0
71	7	14	14	14	14	28	0	14
72	3	3	6	0	0	6	0	0
75	7/3	0	7/3	0	4	0	14/3	0
76	4	6	0	6	8	0	8	0
79	5	10	0	10	10	0	0	10
80	6	10	12	12	6	20	12	8
83	3	0	6	0	0	0	6	0
84	4	4	4	0	8	4	4	0
87	6	12	6	0	0	12	12	12
88	2	2	0	0	0	0	0	0
91	2	0	0	0	4	0	2	0
92	6	12	12	18	0	24	0	24
95	8	16	16	16	8	32	0	16
96	6	10	6	12	12	10	12	8
99	3	0	6	0	6	0	0	0
100	5/2	5/2	0	0	5	0	0	0

TABLE 5. $H^M(D)$ for positive integers $D \leq 100$.

D	$H^9(D)$	$H^{10}(D)$	$H^{12}(D)$	$H^{13}(D)$	$H^{16}(D)$	$H^{18}(D)$	$H^{25}(D)$
0	-1	$-3/2$	-2	$-7/6$	-2	-3	$-5/2$
3	0	0	0	2/3	0	0	0
4	0	1	0	1	0	0	1
7	0	0	0	0	2	0	0
8	2	0	0	0	0	2	0
11	2	0	0	0	0	0	2
12	0	0	2	8/3	0	0	0
15	0	4	4	0	4	0	0
16	0	5	0	3	0	0	3
19	0	0	0	0	0	0	2
20	4	2	0	0	0	4	0
23	6	0	12	6	6	12	0
24	0	4	0	0	0	0	4
27	4	0	0	8/3	0	0	0
28	0	0	0	0	8	0	0
31	0	12	0	0	6	0	6
32	6	0	12	0	0	10	0
35	4	0	0	4	0	0	0
36	6	5	0	5	0	6	5
39	0	16	8	4	8	0	8
40	0	2	0	4	0	0	0
43	0	0	0	2	0	0	0
44	8	12	12	0	0	12	8
47	10	0	20	0	10	20	0
48	0	0	7	20/3	8	0	0
51	0	0	0	4	0	0	4
52	0	0	0	2	0	0	0
55	0	8	0	8	8	0	0
56	8	8	0	8	0	8	8
59	6	0	0	0	0	0	6
60	0	8	12	0	16	0	0
63	12	0	16	0	10	24	0
64	0	13	0	7	12	0	7
67	0	0	0	0	0	0	0
68	8	0	0	8	0	8	0
71	14	28	28	0	14	28	14
72	12	0	0	0	0	12	0
75	0	0	0	14/3	0	0	10
76	0	12	0	0	0	0	8
79	0	20	0	10	0	0	10
80	12	10	24	0	24	20	0
83	6	0	0	0	0	0	0
84	0	8	0	0	0	0	8
87	0	0	12	12	12	0	0
88	0	0	0	4	0	0	0
91	0	0	0	2	0	0	4
92	12	0	36	12	36	24	0
95	16	16	32	16	32	32	0
96	0	20	10	0	12	0	12
99	12	0	0	0	0	0	6
100	0	5	0	5	0	0	15

Table 6. $S(N)$ and $S^M(N)$ for positive integers $N \leq 25$.

N	$S(N)$	$S^2(N)$	$S^3(N)$	$S^4(N)$	$S^5(N)$	$S^6(N)$	$S^7(N)$	$S^8(N)$
1	1	0	0	-1	0	-2	0	-2
2	4	6	2	4	2	2	2	4
3	6	4	10	2	4	6	4	0
4	10	18	6	18	6	10	6	12
5	10	8	8	6	18	4	8	4
6	18	28	30	20	12	46	12	20
7	14	12	12	10	12	8	26	8
8	24	46	18	52	18	34	18	52
9	21	16	40	11	16	30	16	6
10	30	48	24	36	54	36	24	36
11	22	20	20	18	20	16	20	16
12	44	80	74	83	32	134	32	64
13	26	24	24	20	24	20	24	20
14	42	68	36	52	36	56	78	52
15	40	32	68	24	72	52	32	16
16	52	102	42	118	42	82	42	132
17	34	32	32	30	32	28	32	28
18	66	106	126	80	54	202	54	80
19	38	36	36	34	36	32	36	32
20	70	128	56	136	126	100	56	108
21	56	48	96	38	48	80	104	32
22	66	108	60	60	60	96	60	84
23	46	44	44	42	44	40	44	40
24	100	192	170	196	80	326	80	224
25	55	48	48	41	108	34	48	34

Table 7. $S^M(N)$ for positive integers $N \leq 25$.

N	$S^9(N)$	$S^{10}(N)$	$S^{12}(N)$	$S^{13}(N)$	$S^{16}(N)$	$S^{18}(N)$	$S^{25}(N)$
1	-2	-2	-4	0	-4	-6	-4
2	2	2	0	2	4	2	2
3	8	0	2	4	0	4	4
4	-2	10	8	6	8	-6	-2
5	8	14	0	8	0	4	16
6	24	16	32	12	20	36	4
7	8	8	4	12	8	0	12
8	18	34	36	18	44	34	18
9	44	6	20	16	-4	28	4
10	12	86	24	24	36	12	48
11	20	16	12	20	16	16	12
12	60	56	139	32	56	108	32
13	20	20	14	50	16	12	24
14	36	56	40	36	52	56	20
15	56	56	36	32	16	40	64
16	22	82	90	42	132	42	18
17	32	28	24	32	24	28	32
18	144	82	152	54	80	228	54
19	32	32	28	36	32	24	36
20	56	230	104	56	104	100	112
21	80	32	62	48	64	64	40
22	48	96	72	60	64	72	60
23	44	40	36	44	52	40	44
24	140	152	370	80	308	268	40
25	34	94	16	48	44	6	126