Hurwitz numbers: on the edge between combinatorics and geometry

Sergei K. Lando *

Abstract. Hurwitz numbers were introduced by A. Hurwitz in the end of the nineteenth century. They enumerate ramified coverings of two-dimensional surfaces. They also have many other manifestations: as connection coefficients in symmetric groups, as numbers enumerating certain classes of graphs, as Gromov–Witten invariants of complex curves. Hurwitz numbers belong to a tribe of numerical sequences that penetrate the whole body of mathematics, like multinomial coefficients. They are indexed by partitions, or, more generally, by tuples of partitions, which does not allow one to overview all of them simultaneously. Instead, we usually deal with some of their specific subsequences. The Cayley numbers N^{N-1} enumerating rooted trees on N marked vertices is may be the simplest such instance. The corresponding exponential generating series has been considered by Euler and he gave it the name of Lambert function. Certain series of Hurwitz numbers can be expressed by nice explicit formulas, and the corresponding generating functions provide solutions to integrable hierarchies of mathematical physics. The paper surveys recent progress in understanding Hurwitz numbers.

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1. Hurwitz numbers

Since their introduction by A. Hurwitz in the end of the 19th century [23, 24], the numbers experienced attraction of prominent mathematicians, like H. Weyl, as well as long periods of neglect. During these periods, the efforts of A. Mednykh (see e.g., [39]) were rare attempts to improve our understanding of their nature. The situation changed dramatically in the beginning of 1990's, when the reviving of interest has been strongly supported by demands from mathematical physics, group theory, and algebraic geometry simultaneously. The present paper is devoted to a description of the progress made in the last couple of decades. This progress is a result of joint efforts of many people all over the world.

In this section we give the definition of Hurwitz numbers and discuss some of their combinatorial aspects.

1.1. Simple and general Hurwitz numbers.

Let S_N denote the symmetric group consisting of permutations of N elements $\{1, 2, \ldots, N\}$. Any permutation $\sigma \in S_N$ can be represented as a product of transpositions, and there are many such representations. For a given m, we are interested in enumeration of m-tuples of transpositions $\eta_1, \ldots, \eta_m$ whose product is a given permutation σ,

$$\sigma = \eta_m \circ \cdots \circ \eta_1.$$

The following statements are clear:

- the number of such representations depends on the cyclic type of the permutation σ rather than on the permutation itself;

- there is a minimal number $m_{\min} = m_{\min}(\sigma)$ for which such a representation exists, and this minimal number is $N - c(\sigma)$, where $c(\sigma)$ is the number of cycles in σ. Indeed, the minimal number of transpositions whose product is a cycle of length l is $l - 1$;

- all values of m for which the number of representations is nonzero have the same parity, which coincides with the parity of the permutation σ.

Now we are ready to give a precise definition of a simple Hurwitz number.

Definition 1. Let μ be a partition, $\mu \vdash |\mu|$. The *simple Hurwitz number* $h^{\circ}_{m;\mu}$ is defined as

$$h^{\circ}_{m;\mu} = \frac{1}{|\mu|!} \left| \{(\eta_1, \ldots, \eta_m), \eta_i \in C_2(S_{|\mu|}) | \eta_m \circ \cdots \circ \eta_1 \in C_\mu(S_{|\mu|})\} \right|.$$

Here $C_2(S_{|\mu|})$ denotes the set of all transpositions in $S_{|\mu|}$, and $C_\mu(S_{|\mu|})$ is the set of all permutations of cyclic type $\mu \vdash |\mu|$ in $S_{|\mu|}$, so that, in particular, $C_2(S_{|\mu|}) = C_{1^{|\mu|-2}2^1}(S_{|\mu|})$. The *connected simple Hurwitz number* $h_{m;\mu}$ is defined in a similar way, but we take into account only m-tuples of transpositions such that the subgroup $\langle \eta_1, \ldots, \eta_m \rangle \subseteq S_{|\mu|}$ they generate acts transitively on the set $\{1, \ldots, |\mu|\}$.

The terminology has a topological origin and will be explained later. Below, we denote partitions in one of the two equivalent ways: either as a sequence of decreasing parts, $\mu = (\mu_1, \mu_2, \dots)$, where $\mu_1 \geq \mu_2 \geq \dots$, with only finitely many nonzero parts, or in the multiplicative form $1^{k_1} 2^{k_2} \dots$, where k_i denotes the multiplicity of the part i in the partition, all but finitely many multiplicities being 0 (and the corresponding parts omitted in the notation).

In slightly different terms, Hurwitz numbers enumerate ordered factorizations of permutations of given cyclic type into transpositions, while connected Hurwitz numbers enumerate those factorizations that are transitive.

Hurwitz numbers are not necessarily integers. This is true even for the simplest case,

$$h^{\circ}_{1;2^1} = h_{1;2^1} = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

More generally, for a tuple $\mu_1, \dots, \mu_m$ of partitions of N, one can consider general Hurwitz numbers enumerating representations of the identity permutation as the product of the form $\sigma_m \circ \cdots \circ \sigma_1$, where each permutation σ_i has the cyclic type μ_i, $1 \leq i \leq m$. (For simple Hurwitz numbers, all the permutations but one are transpositions, and the last permutation is σ^{-1}, whose cyclic type coincides with that of σ). The *general Hurwitz number* is defined as the number of m-tuples of permutations $\sigma_1, \dots, \sigma_m$ of given cyclic types whose product is the identity permutation, divided by $N!$. *Connected Hurwitz numbers* are defined similarly, but with the restriction that the subgroup $\langle \sigma_1, \dots, \sigma_m \rangle \subseteq S_N$ generated by the permutations σ_i must act transitively. We do not introduce notation for general Hurwitz numbers, since we are not going to use them in our survey.

It is also worth mentioning other kinds of Hurwitz numbers, like real Hurwitz numbers (see e.g., [1]) or tropical Hurwitz numbers [5], but we are not going to discuss them in detail.

1.2. Topological interpretation.
Hurwitz numbers naturally arise in the enumeration problem for ramified coverings of the 2-sphere. Below, a surface means an oriented two-dimensional manifold. A continuous mapping $\beta : E_1 \to E_2$ of two surfaces is called a *covering* if it is an orientation preserving local homeomorphism, that is, for each point $t \in E_2$ there is a disk neighborhood $U = U(t) \subset E_2$ such that its total preimage $\beta^{-1}(U) \subset E_1$ is a disjoint union of disks, and the restriction of β to each of these disks is an orientation preserving homeomorphism. If E_2 is connected, then the number of disks in the preimage of any disk neighborhood U is the same whatever is the point t, and this number (which may well be infinite) is called the *degree* of the covering.

From the point of view of topology, a smooth projective complex curve is a compact surface. Every nonconstant holomorphic mapping $\beta : E_1 \to E_2$ of two complex curves E_1, E_2 is a *ramified covering*, meaning that it becomes a covering after puncturing E_2 at finitely many points and E_1 at their preimages under β. Locally, at a neighborhood of each point in E_1, a ramified covering looks like $z \mapsto z^k$, for an appropriate choice of complex local coordinates in the source and the target. For all but finitely many points in E_1, the value of k is 1, and it is

greater than 1 for some preimages of the punctures. It is called the *degree* of the preimage.

For any point $t \in E_2$, the sum of the degrees of all its preimages is the same, and it is called the *degree* of the ramified covering. In other words, the degrees of the preimages of any point form a partition of the degree of the covering. For a ramified covering of degree N, all partitions different from 1^N constitute the *ramification type* of the covering. We say that a ramification point in the target surface E_2 is *non-degenerate* if the corresponding partition is $1^{N-2}2^1$, that is, if there is one preimage of degree 2, and $N - 2$ preimages at which the mapping is unramified. Otherwise, the ramification point is said to be *degenerate*. Below, we shall consider finite ramified coverings of the 2-sphere S^2 by compact oriented two-dimensional surfaces.

Consider the ramified covering $z \mapsto z^k$ of the unit disk by the unit disk. As a nonzero point in the target disk goes around 0 and returns to its initial position, its k preimages experience a cyclic permutation of length k. This property allows one to associate to a ramified covering of the sphere a tuple of permutations.

Let $\beta : E \to S^2$ be a ramified covering of degree N, and let $t_1, \ldots, t_m \in S^2$ be all its points of ramification. Pick a point $t \in S^2$ distinct from all t_i and connect it with the points t_i by smooth nonintersecting segments, whose cyclic order at t coincides with the numbering. Now make each segment into a narrow path γ_i around the ramification point in the positive direction. Then the path γ_i induces a permutation σ_i of the fiber $\beta^{-1}(t)$. The cyclic type of the permutation σ_i coincides with the partition given by the degrees of the preimages in $\beta^{-1}(t_i)$, and the product $\sigma_m \circ \cdots \circ \sigma_1$ is the identity permutation of the fiber $\beta^{-1}(t)$, since the concatentation of the paths $\gamma_m \circ \cdots \circ \gamma_1$ is contractible in the punctured sphere $S^2 \setminus \{t_1, \ldots, t_m\}$.

The m-tuple of permutations of the fiber determines the covering uniquely, up to a homeomorphism of the domain. By numbering the preimages $\beta^{-1}(t)$ of the generic point from 1 to N, we can make each permutation σ_i into a permutation of the set $\{1, 2, \ldots, N\}$. Since there are $N!$ possible numberings, we conclude that Hurwitz numbers enumerate ramified coverings of the 2-sphere, with prescribed ramification types. The covering surface is connected if and only if the subgroup of S_N generated by the permutations σ_i acts transitively on the fiber $\beta^{-1}(t)$, which justifies the definition of connected Hurwitz numbers.

Let $E \to S^2$ be a ramified covering. The Riemann–Hurwitz formula allows one to recover the Euler characteristic $\chi(E)$ of the covering surface E from the ramification type. We shall use this formula only for the case of simple Hurwitz numbers, where it acquires the form

$$\chi(E) = N + c(\mu) - m.$$

Here μ is a partition of $N = |\mu|$, $c(\mu)$ is the number of parts in the partition, and m is the number of transpositions. If the covering surface is connected, then its Euler characteristic is $\chi(E) = 2 - 2g$, where g is the genus of the surface. Hence the number m of points of simple ramification can be considered as a substitute for the genus of the covering surface.

1.3. Cut-and-join equation of Goulden and Jackson. Collect the
simple Hurwitz numbers into two generating functions:

$$H^\circ(u; p_1, p_2, \dots) = \sum_{m=1}^{\infty} \sum_{\mu} h^\circ_{m;\mu} p_{\mu_1} p_{\mu_2} \dots \frac{u^m}{m!}; \qquad (1)$$

$$H(u; p_1, p_2, \dots) = \sum_{m=1}^{\infty} \sum_{\mu} h_{m;\mu} p_{\mu_1} p_{\mu_2} \dots \frac{u^m}{m!}, \qquad (2)$$

where in each case μ runs over the set of all partitions of all numbers. These generating functions depend on infinitely many variables and are formal: we do not put any convergence requirements on them.

A very general combinatorial construction relating connected and disconnected objects justifies the following relationship between these two generating functions:

We have $H^\circ = \exp(H)$.

This assertion allows one to translate statements about simple Hurwitz numbers into statements about connected simple Hurwitz numbers and vice versa.

The following result explains many properties of the Hurwitz numbers.

Theorem 1.1 (cut-and-join equation, [14]). *The generating function H° for simple Hurwitz numbers satisfies the following partial differential equation:*

$$\frac{\partial H^\circ}{\partial u} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i+j=n} \left((i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right) H^\circ. \qquad (3)$$

Before explaining why the statement is true, let us note that the cut-and-join equation provides an explicit formula for the generating function H°. Expand it in a power series in u,

$$H^\circ(u; p_1, p_2, \dots) = \sum_{m=0}^{\infty} H^\circ_{(m)}(p_1, p_2, \dots) \frac{u^m}{m!}.$$

Then the cut-and-join equation can be rewritten as the recurrence

$$H^\circ_{(m+1)} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i+j=n} \left((i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right) H^\circ_{(m)} = A H^\circ_{(m)}.$$

Note that the differential operator A on the right is well known in mathematical physics under the name of Calogero–Moser operator. Starting with $H^\circ_{(0)} = e^{p_1}$, we immediately obtain the first few terms of the expansion:

$$H^\circ(u; p_1, p_2, \dots) = e^{p_1} \left(1 + \frac{1}{2} p_2 \frac{u}{1!} + (p_1^2 + \frac{1}{2} p_2^2 + p_3) \frac{u^2}{2!} + \dots \right).$$

The application of the operator A to the function $H^\circ_{(m)}$ always produces finitely many nonzero terms, although the operator itself contains infinitely many of them.

The reason is that the function $H^\circ_{(m)}$ has the form e^{p_1} times a polynomial in $p_1, \ldots,$ p_m, and its derivatives over each p_k with $k > m$ vanish.

Now let us explain why the cut-and-join equation is true. It describes what happens if one of the transpositions in the decomposition of a given permutation is glued with the distinguished permutation, that is, we replace the representation

$$\sigma = \eta_m \circ \eta_{m-1} \circ \cdots \circ \eta_1$$

by the representation

$$\eta_m \circ \sigma = \eta_{m-1} \circ \cdots \circ \eta_1$$

(here we make use of the fact that η_m^2 is the identity permutation). Decreasing of the number of transpositions on the right by one reflects the derivation with respect to u on the left of the cut-and-join equation (3), since this procedure diminishes the degree of u by 1.

Multiplication by a transposition η_m can affect the permutation σ in one of the two different ways: either η_m exchanges two elements belonging to the same cycle of σ, or the elements it exchanges belong to distinct cycles. In the first case, a cycle in σ is split into two cycles the sum of whose lengths coincides with the length of the initial one. In the second case, conversely, two cycles are glued into a single cycle of length equal to the sum of the lengths of the two. Each of the two summands on the right of the cut-and-join equation is in charge of the corresponding possibility. The coefficients reflect the number of ways to choose two elements to be transposed by η_m: for each of the $i + j$ elements in a cycle of length $i + j$ an appropriate pair can be chosen in a unique way (if we fix the cyclic order), while in two cycles, of length i and j, respectively, there are ij choices for a pair whose transposition glues them together.

1.4. Certain formulas for rational Hurwitz numbers.
Hurwitz numbers are said to be *rational* if the number of transpositions in the decomposition is the minimal possible one. The terminology comes from the fact that these numbers enumerate ramified coverings of the sphere by the sphere, that is, rational functions. Thus, rational Hurwitz numbers are, in a sense, the simplest species of Hurwitz numbers, and there are a number of explicit formulas for them.

The first such formula is the one due to Hurwitz (1891), for rational connected simple Hurwitz numbers.

Theorem 1.2 ([23]). *We have*

$$h_{|\mu|+n-2;\mu} = \frac{(|\mu| + n - 2)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} |\mu|^{n-3},$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is a partition of $|\mu| = \mu_1 + \cdots + \mu_n$, and $|\mathrm{Aut}(\mu)|$ is the order of the automorphism group of the partition (for $\mu = 1^{k_1} \ldots N^{k_N}$, we have $|\mathrm{Aut}(\mu)| = k_1! \ldots k_N!$).

Here $|\mu| + n - 2$ is the minimal number of transpositions (generating a permutation group acting transitively) in a product that can produce a permutation of

cyclic type μ. In fact, Hurwitz did not publish the proof of his formula stating that it is too long for a journal paper. The formula was rediscovered in [14], after the problem has been revived in quantum chromodynamics models [7, 22]. A reconstruction of Hurwitz's presumable proof is given in [50]. The ELSV formula, see below, provides an alternative geometric proof [10].

Another instance of formulas for rational Hurwitz numbers is the following

Theorem 1.3 ([14]). *The number of factorizations of a cyclic permutation in S_N into a product of permutations of cyclic types $\nu_1, \ldots, \nu_m$, $\nu_i \vdash N$, is*

$$N^{m-1} \frac{(c(\nu_1) - 1)!}{|\mathrm{Aut}(\nu_1)|} \cdots \frac{(c(\nu_m) - 1)!}{|\mathrm{Aut}(\nu_m)|},$$

where $c(\nu)$ denotes the number of parts in a partition ν.

The proof in [14] is purely combinatorial. Once again, the geometric proof was given in [34].

The formula due to Bousquet-Mélou and Schaeffer enumerates decompositions of a given permutation into a product of a given number of permutations, whatever are their types. It reads as follows.

Theorem 1.4 ([4]). *Denote by $G_\mu(m)$ the number of m-tuples of permutations whose product is a permutation of cyclic type μ, divided by $N!$, $\mu \vdash N$. We have*

$$G_\mu(m) = m \frac{((m-1)N - 1)!}{((m-1)N - c(\mu) + 2)!} \prod_i \binom{m\mu_i - 1}{\mu_i} \mu_i,$$

where $c(\mu)$ is the number of parts in μ.

The original proof got a simplification in [19]. Similarly to the previous two formulas, this one also must have a geometric proof, which is still lacking.

2. Integrable hierarchies for Hurwitz numbers

The Kadomtsev–Petviashvili (below, KP, for brevity) hierarchy is a completely integrable system of partial differential equations playing an important role in mathematical physics. The main goal of the present section is to discuss the following statement.

Theorem 2.1. *The generating function $H(u; p_1, p_2, \ldots)$ for connected simple Hurwitz numbers is a 1-parameter family of solutions to the KP hierarchy.*

In this form the theorem was first stated in [27], but it is implicitly contained in Okounkov's paper [41]. In fact, Okounkov proves a slightly more complicated theorem stating that the generating function for *double* Hurwitz numbers (those, enumerating ramified coverings of the sphere with two points of degenerate ramification) produces a solution to the Toda lattice integrable hierarchy, which was previously conjectured by R. Pandharipande [46].

The theorem above has numerous applications, both on combinatorial and geometric side. In particular, it produces nontrivial recurrence relations on Hurwitz numbers, which mix numbers of different genera.

A general theory of KP equations, due to Sato, interprets solutions to these equations as semi-infinite planes, that is, points in the semi-infinite Grassmannian. We present a brief overlook of Sato's construction. Proving that a given function is a solution, is thus reduced to identification of the semi-infinite plane corresponding to this function. There is no need, in particular, to know the explicit form of the equations. We make such an identification for the function $H(u; p_1, p_2, \dots)$ from a purely combinatorial point of view, without references to their geometric nature.

2.1. Grassmannian embeddings and Plücker equations.

Consider the Grassmannian $G(2, 4)$ of vector 2-planes in the 4-space $V \equiv \mathbb{C}^4$. Any 2-plane in V can be represented by the wedge product $\beta_1 \wedge \beta_2$ of any pair β_1, β_2 of linearly independent vectors in the plane. This wedge product is well defined up to a constant factor; it determines the 2-plane uniquely and thus defines an embedding of $G(2, 4)$ into the projectivization of the wedge square of V, $G(2, 4) \hookrightarrow P\Lambda^2 V$. An immediate generalization of this construction produces an embedding of any Grassmannian $G(k, n)$ of k-planes in n-space V into the projectivization $P\Lambda^k V$.

The *Plücker equations* are the equations of the image of this embedding. Note that the dimension of $G(k, n)$ is $k(n - k)$, while the dimension of $P\Lambda^k V$ is $\binom{n}{k} - 1$, whence, generally speaking, the image of the embedding does not coincide with the whole projectivized wedge product $P\Lambda^k V$. For example, the image of the embedding of $G(2, 4)$ into $P\Lambda^2 V$ is a hypersurface in the 5-dimensional projective space.

Let us find the equation of this hypersurface. Pick a basis e_1, e_2, e_3, e_4 in V. Then $\Lambda^2 V$ is endowed with the natural basis $\beta_{ij} = e_i \wedge e_j$, $1 \leq i < j \leq 4$, and the corresponding natural coordinate system y_{ij}. The image of the embedding of the Grassmannian consists of decomposable vectors. By definition of the wedge product, for a pair of vectors (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), the image of the plane spanned by these two vectors has the projective coordinates

$$y_{ij} = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix} = a_i b_j - a_j b_i.$$

An immediate calculation shows that these coordinates satisfy the homogeneous equation

$$y_{12} y_{34} - y_{13} y_{24} + y_{14} y_{23} = 0,$$

and this is the Plücker equation of the image.

For general values of n and k, the Plücker equations still are quadratic equations. In other words, *the ideal in the ring of polynomials consisting of polynomials vanishing on the image of the Plücker embedding is generated by quadratic polynomials.*

2.2. Space of Laurent series.

Take for the space V the infinite dimensional vector space of formal Laurent series in one variable. Elements of this space

have the form $c_{-k}z^{-k}+c_{-k+1}z^{-k+1}+\ldots$. The powers z^k, $k = \ldots, -2, -1, 0, 1, 2, \ldots$ form the *standard* basis in V. By definition, the *semi-infinite wedge product* $\Lambda^{\frac{\infty}{2}}V$ is the vector space freely spanned by the vectors

$$v_\mu = z^{m_1} \wedge z^{m_2} \wedge z^{m_3} \wedge \ldots, \qquad m_1 < m_2 < m_3 < \ldots, \qquad m_i = \mu_i - i,$$

where μ is a partition, $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$, $\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots$, and all but finitely many parts are 0. In particular, $m_i = -i$ for all i large enough.

The *vacuum vector*

$$v_\emptyset = z^{-1} \wedge z^{-2} \wedge z^{-3} \wedge \ldots$$

corresponds to the empty partition. Similarly, we have

$$v_{1^1} = z^0 \wedge z^{-2} \wedge z^{-3} \wedge \ldots, \quad v_{2^1} = z^1 \wedge z^{-2} \wedge z^{-3} \wedge \ldots, \quad v_{1^2} = z^0 \wedge z^{-1} \wedge z^{-3} \wedge \ldots,$$

and so on.

2.3. The boson-fermion correspondence.

Numbering basis vectors in the semi-infinite wedge product $\Lambda^{\frac{\infty}{2}}V$ (the space of *fermions*) by partitions establishes a natural vector space isomorphism (the *boson-fermion correspondence*) between this space and the vector space of power series in infinitely many variables $p_1, p_2, \ldots$ (the space of *bosons*). This isomorphism takes a basis vector v_μ to the Schur polynomial $s_\mu = s_\mu(p_1, p_2, \ldots)$. The latter is a quasihomogeneous polynomial, of degree $|\mu|$, in the variables p_i, with the degree of p_i set to be i.

The *Schur polynomial* corresponding to a one-part partition is defined by the expansion

$$s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4 + \cdots = e^{p_1 z + p_2 \frac{z^2}{2} + p_3 \frac{z^3}{3} + \cdots},$$

and for a general partition κ it is given by the determinant

$$s_\kappa = \det \; \|s_{\kappa_j - j + i}\|. \tag{4}$$

The indices i, j here run over the set $\{1, 2, \ldots, n\}$ for n large enough, and since $\kappa_i = 0$ for i sufficiently large, the determinant, hence s_κ, is independent of n. Here are a few first Schur polynomials:

$$s_0 = 1, \qquad s_{1^1} = p_1, \qquad s_{2^1} = \frac{1}{2}(p_1^2 + p_2), \qquad s_{3^1} = \frac{1}{6}(p_1^3 + 3p_1 p_2 + 2p_3),$$

$$s_{1^2} = \frac{1}{2}(p_1^2 - p_2), \qquad s_{1^1 2^1} = \frac{1}{3}(p_1^3 - p_3), \qquad s_{1^3} = \frac{1}{6}(p_1^3 - 3p_1 p_2 + 2p_3).$$

2.4. Semi-infinite Grassmannian and the KP equations.

The *semi-infinite Grassmannian* $G(\frac{\infty}{2}, \infty)$ consists of decomposable vectors in $P\Lambda^{\frac{\infty}{2}}V$, that is, of vectors of the form

$$\beta_1(z) \wedge \beta_2(z) \wedge \beta_3(z) \wedge \ldots,$$

where each β_i is a Laurent power series in z and, for i large enough, the leading term in the expansion of β_i is z^{-i}:

$$\beta_i(z) = z^{-i} + c_{i1}z^{-i+1} + c_{i2}z^{-i+2} + \ldots.$$

Definition 2. The *Hirota equations* are the Plücker equations of the embedding of the semi-infinite Grassmannian in the projectivized semi-infinite wedge product $P\Lambda^{\frac{\infty}{2}}V$. Solutions to the Hirota equations (that is, semi-infinite planes) are called τ-*functions* for the KP hierarchy.

As polynomial equations for the coefficients of the expansions of τ-functions, the Hirota equations can be treated as partial differential equations for the functions themselves. Being Plücker equations, the Hirota equations are quadratic in τ.

Definition 3. The form the Hirota equations take for the logarithms of τ-functions under the boson-fermion correspondence is called the *Kadomtsev–Petviashvili*, or *KP, equations.*

In other words, any solution to the KP equations can be obtained as the result of the following procedure:

- take a semi-infinite plane $\beta_1(z) \wedge \beta_2(z) \wedge \dots$ in V;

- by expanding, rewrite the corresponding point in the semi-infinite Grassmannian as a linear combination of the basis vectors v_κ and normalize so as the coefficient of $v_\emptyset$ becomes 1;

- replace in this linear combination each vector v_κ by the corresponding Schur polynomial $s_\kappa(p_1, p_2, \dots)$, which produces a series in infinitely many variables $p_1, p_2, \dots$;

- take the logarithm of the resulting series.

An infinite sequence of homogeneous generators can be chosen for the KP equations, involving derivatives over extending sets of variables. For example, the first KP equation for an unknown function $W = W(p_1, p_2, \dots)$ looks like

$$\frac{\partial^2 W}{\partial p_2^2} = \frac{\partial^2 W}{\partial p_1 \partial p_3} - \frac{1}{2}\left(\frac{\partial^2 W}{\partial p_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 W}{\partial p_1^4}$$

(it contains derivatives only over p_1, p_2, p_3, and is homogeneous, in a natural sense).

2.5. Action of the diagonal matrices. Linear transformations of the vector space V of Laurent polynomials induce linear transformations of the semi-infinite wedge product $\Lambda^{\frac{\infty}{2}}V$. Since linear transformations of V take planes in V to planes, the induced transformations preserve the embedded Grassmannian. In this section we consider the action of those transformations that can be represented by diagonal matrices in the basis $\{z^k\}$ in V, $k \in \mathbb{Z}$: these are the only transformations we need in the study of simple Hurwitz numbers. By obvious reasons, the induced action on $\Lambda^{\frac{\infty}{2}}V$, written in the basis v_κ, also is diagonal.

Example 2.2. Consider the linear transformation $V \to V$ which multiplies z^{-1} by a constant a preserving all the other basis vectors. Clearly, the action of this transformation on $\Lambda^{\frac{\infty}{2}}V$, written in the basis v_κ, multiplies by a each basis vector

containing z^{-1} in its decomposition ($v_\emptyset$, v_{1^2}, and so on), and preserves all other basis vectors (v_{1^1}, v_{2^1}, and so on). The requirement that z^{-1} enters the decomposition of a vector v_κ means that the partition κ contains a part κ_i such that $\kappa_i - i = -1$. Note that any partition can have at most one such part, since the parts κ_i follow in a decreasing order, while the sequence i grows strictly.

An important consequence of this example is that the eigenvalue of the action on $\Lambda^{\frac{\infty}{2}}V$ of a diagonal matrix on V corresponding to the eigenvector v_κ *depends symmetrically on the differences* $\kappa_i - i$. In other words, it belongs to the ring of so-called shifted symmetric functions.

Definition 4. A function on partitions $\kappa = (\kappa_1, \kappa_2, \dots)$ is said to be *shifted symemtric* if it is symmetric under permutations of the shifted parts $\kappa_i - i$.

Let us stress once again that the parts $\kappa_1, \kappa_2, \dots$ of the partition κ go in the nonincreasing order, $\kappa_1 \geq \kappa_2 \geq \dots$, and all but finitely many of them are 0. The definition of a shifted symmetric function bases heavily on this order.

The space of shifted-symmetric functions depending on infinitely many variables is the projective limit Γ of the spaces Γ_k of shifted symmetric functions depending on k variables. (In [42], the algebra Γ is denoted by Λ^*. We use a different notation in order to prevent confusion with the wedge products and the Hodge bundle below). The limit is taken with respect to the projections $\Gamma_{k+1} \to \Gamma_k$ obtained by setting the last argument equal to 0. All complex-valued shifted symmetric functions form an algebra. This algebra was introduced and thoroughly studied in [30]. The reason for introducing it is that the characters of certain natural elements in the centers of group algebras of symmetric groups are shifted symmetric.

Now, we have a naturally defined action on $\Lambda^{\frac{\infty}{2}}V$ of any diagonal matrix $z^k \mapsto a_k z^k$, $a_k \neq 0$, with *finitely many entries* a_k *with negative indices different from* 1. Indeed, were there infinitely many such coefficients, in order to compute the action of the corresponding matrix on a basis vector, say $v_\emptyset$, we would have to compute the product of infinitely many entries. Fortunately, the action on the *projectivized* space $P\Lambda^{\frac{\infty}{2}}V$, which is the main subject of our interest, can be extended to the action of diagonal matrices with infinitely many entries a_k with negative indices different from 1: since we are interested in the action on the projectivized space, only the ratio of the eigenvalues of the basis eigenvectors matters, and this ratio is well defined for an arbitrary diagonal matrix.

Indeed, any two basis vectors $v_\kappa, v_\mu \in \Lambda^{\frac{\infty}{2}}V$ have a common tail: their decompositions are different in the beginning, but coincide after some position, say K. Hence the ratio of the corresponding eigenvalues is just $\frac{a_{\kappa_1-1}\dots a_{\kappa_K-K}}{a_{\mu_1-1}\dots a_{\mu_K-K}}$. That is, we must define the action of a diagonal matrix on $\Lambda^{\frac{\infty}{2}}V$ in a way that preserves this ratio of eigenvalues. Thus the result depends only on the eigenvalue of the vacuum vector $v_\emptyset$, which can be chosen arbitrarily. The most natural normalization is to choose this eigenvalue to be 1. This yields the following induced action on $\Lambda^{\frac{\infty}{2}}V$

of a diagonal matrix (a_k) on V:

$$v_\kappa \mapsto \left(\prod_{i=1}^{\infty} \frac{a_{\kappa_i - i}}{a_{-i}} \right) v_\kappa.$$

The product in the brackets is well defined, since all but finitely many factors are 1. The action of the torus of diagonal matrices on the projectivized seminifinite external product of V is just the inductive limit of the actions of the tori T_K consisting of diagonal matrices with diagonal entries a_i equal to 1 for $i = -(K + 1), -(K + 2), \dots$.

Since the action of the infinite dimensional torus $\bigoplus_{i \in \mathbb{Z}} (\mathbb{C}^*)_i$ on the projectivized semi-infinite wedge product is well defined, it also defines an action of the corresponding Lie algebra. The latter action also is diagonal, and a diagonal matrix $(\alpha_i)_{i \in \mathbb{Z}}$ (with not necessarily nonzero entries) belonging to the Lie algebra acts on a basis vector v_κ by

$$v_\kappa \mapsto \left(\sum_{j=1}^{\infty} (\alpha_{\kappa_j - j} - \alpha_{-j}) \right) v_\kappa.$$

2.6. Symmetric group representations. In this section, we prove Theorem 2.1 stating that the generating series $H(u; p_1, p_2, \dots)$ for simple Hurwitz numbers is a solution to the KP hierarchy for each value of the parameter u. This statement is true for $u = 0$, since $H(0; p_1, p_2, \dots) = p_1$. For general value of u, the statement follows from the fact that $\exp(H)$ is an integral curve of a vector field in $P\Lambda^{\frac{\infty}{2}} V$ tangent to the semi-infinite Grassmannian. This vector field is induced by a linear transformation $V \to V$, which is diagonal in the standard basis z^k. Namely, this is the transformation $z^k \mapsto (k - \frac{1}{2})^2 z^k$.

Let $\mathbb{C}[S_N]$ be the $N!$-dimensional group algebra of the symmetric group. For each partition κ of N, denote by $C_\kappa \in \mathbb{C}[S_N]$ the sum of all permutations in S_N having the cyclic type κ. We will use a special notation C_1 for the class C_{1^N} of the unit permutation, which is the unit of the algebra $\mathbb{C}[S_N]$, and C_2 for the sum $C_{1^{N-2} 2^1}$ of all transpositions. For any κ, the element C_κ is a central element in $\mathbb{C}[S_N]$. These elements span the center of $\mathbb{C}[S_N]$.

The simple Hurwitz numbers have the following natural interpretation as connection coefficients in symmetric groups. Take the mth power C_2^m of the class $C_2 \in \mathbb{C}[S_N]$ and expand it as a linear combination of the basis classes. Then the coefficient of C_μ in this expansion is equal to the number of ways to represent a given permutation of cyclic type μ as a product of m transpositions. In other words,

$$C_2^m = N! \sum_{\mu \vdash N} h_{m;\mu}^{\circ} \frac{C_\mu}{|C_\mu|},$$

where $|C_\mu|$ is the number of elements in the corresponding conjugacy class.

Example 2.3. For $N = 3$ and $m = 4$, we have

$$C_2^4 = 27 C_1 + 27 C_3,$$

whence

$$h^\circ_{4;3^1} = h_{4;3^1} = \frac{2 \cdot 27}{6} = 9.$$

(Let us explain how the coefficient 27 of the class C_1 in the above formula is obtained. Each of the 27 products of three transpositions in S_3 is a transposition. Taking for the fourth transposition one of the two transpositions different from the product we obtain 54 cyclic permutations, that is, the element C_3, which is the sum of the two cyclic permutations, taken with multiplicity 27).

It is convenient to interpret the above relation by assigning the monomial $|C_\mu|p_\mu = |C_\mu|p_{\mu_1}p_{\mu_2}\dots$ to the element C_μ. This correspondence provides an isomorphism between the center of $\mathbb{C}[S_N]$ and the vector space of weighted homogeneous polynomials of degree N in the variables $p_1, p_2, \dots$. Under this isomorphism, we have

$$C_2^m = N! \sum_{\mu \vdash N} h^\circ_{m;\mu} p_\mu.$$

Therefore,

$$e^{C_2 u} = N! \sum_{m=0}^{\infty} \sum_{\mu \vdash N} h^\circ_{m;\mu} p_\mu \frac{u^m}{m!}.$$

In order to compute the action of the element C_2 and that of its exponent, we observe that an element of $\mathbb{C}[S_N]$ is central iff it acts as a scalar on any irreducible representation. In particular, the central elements $\chi_\mu \in \mathbb{C}[S_N]$ which act with the trace 1 in the irreducible representation V_μ and trivially in all other representations form yet another basis in the center of $\mathbb{C}[S_N]$. The elements C_2 and $e^{C_2 u}$, being central, act diagonally in this basis:

$$C_2 : \chi_\mu \mapsto f_2(\mu)\chi_\mu, \qquad e^{C_2 u} : \chi_\mu \mapsto e^{f_2(\mu)u}\chi_\mu,$$

with f_2 given by

$$f_2(\mu) = \frac{1}{2} \sum_{i=1}^{\infty} \left((\mu_i + \frac{1}{2} - i)^2 - (\frac{1}{2} - i)^2 \right).$$

Under the isomorphism above, the element χ_μ is taken exactly to the corresponding Schur function by (yet another) its definition. The equivalence of the two definitions of the Schur function is a standard fact known as the Frobenius theorem; the proof can be found, for example, in [47]. Expanding the function $H^\circ(0; p_1, \dots) = e^{p_1}$ in the basis of Schur polynomials,

$$e^{p_1} = \sum_{\mu} s_\mu(1, 0, 0, \dots) s_\mu(p),$$

we obtain finally

$$H^\circ(u; p_1, p_2, \dots) = \sum_{\mu} s_\mu(1, 0, 0, \dots) s_\mu(p) e^{f_2(\mu)u}.$$

This explicit formula for simple Hurwitz numbers goes back to Burnside. Similarly to formulas in Sec. 1.3 it also can be used for computation of particular simple Hurwitz numbers. Note that the above isomorphism between the center of $\mathbb{C}[S_N]$ and the space of degree N polynomials in the variables p_i takes the multiplication by C_2 to the cut-and-join (or Calogero–Moser) operator A of Sec. 1.3. We conclude that the cut-and-join operator is diagonal in the basis of Schur polynomials. The specific form of the eigenvalue function f_2 shows that this diagonal operator is induced by the diagonal operator $z^k \mapsto (k - 1/2)^2 z^k$ on the space V of Laurent polynomials. This proves Theorem 2.1.

2.7. Application: enumeration of maps and hypermaps.
Informally, a map is a graph drawn on a two-dimensional surface in such a way that its edges do not intersect and self-intersect and its complement is a disjoint union of discs (faces). Maps are studied by topological graph theory, see e.g. [34]. Enumeration of maps of various kinds is a classical problem, nowadays finding numerous applications in quantum field theory. In this section we explain how the study of Hurwitz numbers helps to make enumeration results for maps more precise.

From the point of view of the present paper, the most convenient definition of a map is that in terms of permutation groups.

Definition 5. Pick a finite set D. Then a *map with the set of half-edges D on an oriented surface* is a triple of permutations α, φ, σ of D possessing the following properties:

- α is an involution without fixed points;

- the product $\varphi\alpha\sigma$ is the identity permuation.

The group $G = \langle \alpha, \varphi, \sigma \rangle$ of permutations of D generated by the permutations α, φ, σ is called the *cartographic group* of the map. A map is said to be *connected* if its cartographic group acts on the set D transitively.

For a graph drawn on an oriented surface, D is the set of half-edges, or flags, the permutation α exchanges the ends of each edge, φ rotates the half-edges along the faces in the positive direction, and σ rotates the half-edges around the vertices in the positive direction. Obviously, α is an involution without fixed points, and it is easy to check that the product of these three permutations is indeed the identity permutation. A map is connected iff the underlying surface is.

The number of edges in a map is half the number of elements in D or, which is the same, the number of cycles in the permutation α. The number of vertices in a map is the number of cycles in σ, and the degrees of the vertices are the lengths of the cycles. Similarly, the number of faces is the number of cycles in φ, and the degrees of the faces are the lengths of the cycles.

The notion of *hypermap* is a generalization of that of map. In the definition of a hypermap, we get rid of the assumption that α is an involution without fixed points, thus reestablishing the symmetry between the three permutations.

It is clear now that enumeration of maps or hypermaps of various kinds can be reduced to enumeration of triples of permutations possessing certain specific properties, and enumerative methods described above can be applied.

Denote by $R_\kappa^{(n,m)}$ the number of rooted connected maps with n edges, m faces, and the degrees of the vertices given by the partition κ of $2n$.

Methods close to those in the proof of Theorem 2.1 give the following statement.

Theorem 2.4 ([16]). *The generating series*

$$R(w, z; p_1, p_2, \dots) = \sum_{n,m \geq 1} \sum_{\kappa \vdash 2n} \frac{R_\kappa^{(n,m)}}{2n} p_\kappa w^m z^n,$$

(where, for a given partition $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots)$, p_κ denotes the monomial $p_\kappa = p_{\kappa_1} p_{\kappa_2} p_{\kappa_3} \dots$) is a 2-parameter family of solutions to the KP-hierarchy.

The series R in the theorem can be specialized to include only cubic maps — those whose all vertex degrees are 3. By duality, this is the same as enumerating rooted triangulations of arbitrary genus. The KP equations then can be reduced to produce recurrence relations for the number of rooted triangulations.

Denote the number of rooted triangulations of a genus g surface with $2n$ faces by $T(n, g)$. Then the recurrence relation has the following form. Introduce notation

$$S = \left\{ (n, g) \in \mathbb{Z} \times \mathbb{Z} \mid \quad n \geq -1, \quad 0 \leq g \leq \frac{n+1}{2} \right\}.$$

Theorem 2.5 ([16]). *We have*

$$T(n, g) = \frac{1}{3n + 2} t(n, g),$$

where $t(n, g)$ is defined by the quadratic recurrence

$$t(n, g) = \frac{4(3n + 2)}{n + 1} \left(n(3n - 2)t(n - 2, g - 1) + \sum t(i, h)t(j, k) \right),$$

for $(n, g) \in S \setminus \{(-1, 0), (0, 0)\}$, where the summation is carried over $(i, h) \in S$, $(j, k) \in S$ with $i + j = n - 2$ and $h + k = g$, subject to the initial conditions

$$t(-1, 0) = \frac{1}{2}, \qquad t(n, g) = 0 \text{ for } (n, g) \notin S.$$

The recurrence relation of the theorem allowed Bender, Gao and Richmond to solve a long-standing problem of finding the exact formula for the constant factor in the leading term in the asymptotics of the number of rooted triangulations, as the number of triangles tends to infinity.

Theorem 2.6 ([3]). *The number of rooted triangulations of a genus g surface with $2n$ faces has the asymptotics*

$$T(n, g) \sim 3 \times 6^{(g-1)/2} t_g n^{5(g-1)/2} (12\sqrt{3})^n \text{ as } n \to \infty;$$

here the constant t_g has the form

$$t_g = 8 \frac{[1/5]_g [4/5]_{g-1}}{\Gamma(\frac{5g-1}{2})} \left(\frac{25}{96}\right) u_g,$$

where $[x]_k$ denotes the rising factorial $x(x+1)\dots(x+k-1)$, and the constant u_g is defined by the initial condition $u_1 = 1/10$ and the quadratic recurrence relation

$$u_g = u_{g-1} + \sum_{h=1}^{g-1} \frac{1}{R_1(g,h) R_2(g,h)} u_h u_{g-h} \ for \ g \geq 2,$$

where
$$R_1(g,h) = \frac{[1/5]_g}{[1/5]_h}[1/5]_{g-h}, \quad R_2(g,h) = \frac{[4/5]_{g-1}}{[4/5]_{h-1}}[4/5]_{g-h-1}.$$

The first few values of the constant t_g are

$$t_0 = \frac{2}{\sqrt{\pi}}, \quad t_1 = \frac{1}{24}, \quad t_2 = \frac{7}{4320\sqrt{\pi}}.$$

This constant enters many other asymptotics as well.

3. Intersection theory on moduli spaces of complex curves

The importance of Hurwitz numbers in modern research is mainly due to their connections with the geometry of the moduli space of curves. These connections go back to the work of A. Hurwitz in the end of the 19th century, and found numerous remarkable instances in the last decade.

3.1. The ELSV formula. Let $\overline{\mathcal{M}}_{g;n}$ denote the moduli space of stable genus g complex curves with n pairwise distinct marked points. This is the Delinge–Mumford compactification [8] of the moduli space $\mathcal{M}_{g;n}$ of stable non-singular genus g curves with n marked points. The *stability condition* means that the group of automorphisms of the curve preserving the marked points is finite. For smooth curves, this is equivalent to the following numerical restrictions: either $g \geq 2$, or $g = 1, n \geq 1$, or $g = 0, n \geq 3$. The only singularities of the singular curves are transversal double self-intersections (nodes), and the marked points are not allowed to coincide with the nodes. Both $\mathcal{M}_{g;n}$ and $\overline{\mathcal{M}}_{g;n}$ are smooth complex orbifolds of dimension $3g - 3 + n$.

The natural "forgetting morphism" $\mathcal{M}_{g;n+1} \to \mathcal{M}_{g;n}$ extends to a forgetting morphism of the compactifications, $\overline{\mathcal{M}}_{g;n+1} \to \overline{\mathcal{M}}_{g;n}$. The composition of forgetting morphisms forgets more than one marked point.

To the i th marked point, the line bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g;n}$ is associated; the fiber of this bundle is the cotangent line to the curve at the point. Let ψ_i denote the

first Chern class of $\mathcal{L}_i$, $\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g;n})$, $i = 1, \ldots, n$. The *Hodge bundle* Λ over $\overline{\mathcal{M}}_{g;n}$ is the pull-back, under the forgetting morphism, of the rank g vector bundle over $\overline{\mathcal{M}}_{g;0}$ whose fiber is the vector space of holomorphic 1-forms over the curve. (For $g = 1$, the space $\overline{\mathcal{M}}_{g;0}$ must be replaced by $\overline{\mathcal{M}}_{g;1}$, and for $g = 0$, the Hodge bundle is of rank zero). The characteristic classes of the Hodge bundle are denoted by $c(\Lambda) = 1 + \lambda_1 + \cdots + \lambda_g$, $\lambda_i \in H^{2i}(\overline{\mathcal{M}}_{g;n})$.

A formula due to Ekedahl, Lando, Shapiro, and Vainshtein, now standardly referred to as the *ELSV-formula*, expresses simple Hurwitz numbers in terms of intersection indices of the above characteristic classes over the moduli spaces of stable curves:

$$h_{m;\kappa} = \frac{m!}{|\mathrm{Aut}(\kappa)|} \prod_{i=1}^{n} \frac{\kappa_i^{\kappa_i}}{\kappa_i!} \int_{\overline{\mathcal{M}}_{g;n}} \frac{c(\Lambda^\vee)}{(1 - \kappa_1\psi_1)\ldots(1 - \kappa_n\psi_n)}, \qquad (5)$$

where κ is a partition of $K = |\kappa|$, $\kappa = (\kappa_1, \ldots, \kappa_n)$, $m = 2g - 2 + K + n$ is the number of transpositions, and $c(\Lambda^\vee) = 1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g$ is the total Chern class of the dual Hodge bundle. This formula, together with a brief description of the idea of the proof, has been announced in [9] (with an erroneous sign in the numerator of the integrand). A complete proof was given in [10], and meanwhile another proof appeared in [21]. A special case of (5), that for $\kappa = 1^n$, has been simultaneously and independently discovered in [13].

The formula is understood in the following way: after expanding the denominator as a power series in the classes ψ_i, select the monomials of degree $\dim \overline{\mathcal{M}}_{g;n} = 3g - 3 + n$ in the product and integrate them against the fundamental class of $\overline{\mathcal{M}}_{g;n}$. The result will be a rational number.

The ELSV formula generalizes, to higher genera, Hurwitz's formula (see Theorem 1.2) valid for $g = 0$. In its own turn, it admits a generalization known as the *Mariño–Vafa formula* conjectured in [38] and proved in [37].

In spite of the geometric nature of the ELSV formula, it produces immediate combinatorial consequences. An example is given by the following result, which has been conjectured in [17].

Theorem 3.1 ([10]). *For given g, n, the number*

$$h_{m;\kappa} \frac{|\mathrm{Aut}(\kappa)|}{m!} \prod_{i=1}^{n} \frac{\kappa_i!}{\kappa_i^{\kappa_i}}$$

is a symmetric polynomial in κ_i, of degree $3g - 3 + n$, with the least monomial degree being $2g - 3 + n$.

Although the statement is purely combinatorial, no direct proof of it is known. Double Hurwitz numbers demonstrate a similar behavior. Namely, they are piecewise polynomial [18, 49].

3.2. Linear Hodge integrals as coefficients of a solution to KP.
The right-hand side of the ELSV formula is a linear combination of the intersection

numbers of the form

$$\ell_{j;m_1,\ldots,m_n} = \int_{\overline{\mathcal{M}}_{g;n}} \lambda_j \psi_1^{m_1} \ldots \psi_n^{m_n}.$$

Expressions of this kind are called *linear Hodge integrals*, meaning that they include the Chern classes λ_j of the Hodge bundle, which enter the monomial linearly. Note that the data $(j; m_1, \ldots, m_n)$ determine the genus g uniquely according to the dimension count $3g - 3 + n = j + m_1 + \cdots + m_n$. Similarly to the case of Hurwitz numbers, one can organize the linear Hodge integrals in the generating function

$$L(u; q_1, q_2, \ldots) = \sum_{j,\mu} (-1)^j \ell_{j;m_1,\ldots,m_n} u^{2j} q_{m_1} \ldots q_{m_n}, \tag{6}$$

known as the *enriched Gromov–Witten potential* of a point [17].

In a recent paper, M. Kazarian has shown that this generating function can be easily transformed into a solution of the KP hierarchy. Namely, denote by $G(u; p_1, p_2, \ldots)$ the result of the following substitution to the series L:

$$
\begin{aligned}
q_0 &= p_1, \\
q_1 &= u^2 p_1 + 2u p_2 + p_3, \\
q_2 &= u^4 p_1 + 6u^3 p_2 + 12u^2 p_3 + 10u p_4 + 3p_5, \\
q_3 &= u^6 p_1 + 14u^5 p_2 + 61u^4 p_3 + 124u^3 p_4 + 131u^2 p_5 + 70u p_6 + 15p_7, \\
&\ldots
\end{aligned}
$$

Here the polynomials on the right-hand side are given by the recurrence

$$q_{k+1} = \sum_{m \geq 1} m(u^2 p_m + 2u p_{m+1} + p_{m+2}) \frac{\partial}{\partial p_m} q_k.$$

Theorem 3.2 ([26]). *The function $G(u; p_1, p_2, \ldots)$ is a solution to the KP hierarchy (identically in u).*

The proof of the theorem uses the ELSV formula (5) and the fact that the generating series for the simple Hurwitz numbers is a solution to KP (Theorem 2.1). Note that in the present case, the infinitesimal transformation of the space V of Laurent series corresponding to the solution in question is no longer diagonal. Instead, it is three-diagonal.

3.3. Witten's conjecture. The celebrated Witten conjecture [51] concerns computation of the intersection indices of the ψ-classes over the moduli spaces of curves. Namely, denote

$$\langle \tau_{m_1} \ldots \tau_{m_n} \rangle = \int_{\overline{\mathcal{M}}_{g;n}} \psi_1^{m_1} \ldots \psi_n^{\psi_n},$$

where the genus g can be computed from the dimensional count dim $\overline{\mathcal{M}}_{g;n} = 3g - 3 + n = m_1 + \cdots + m_n$. Collect these intersection indices into the generating series in infinitely many variables t_i,

$$
\begin{aligned}
F(t_0, \dots) &= \sum \frac{\langle \tau_{m_1} \dots \tau_{m_n} \rangle}{n!} t_{m_1} \dots t_{m_n} \\
&= \frac{1}{24} t_1 + \frac{1}{6} t_0^3 + \frac{1}{48} t_1^2 + \frac{1}{24} t_0 t_2 + \frac{1}{6} t_0^3 t_1 + \frac{1}{1152} t_4 + \frac{1}{72} t_1^3 + \frac{1}{12} t_0 t_1 t_2 \\
&\quad + \frac{1}{48} t_0^2 t_3 + \frac{1}{6} t_0^3 t_1^2 + \frac{1}{24} t_0^4 t_2 + \frac{29}{5760} t_2 t_3 + \frac{1}{384} t_1 t_4 + \frac{1}{1152} t_0 t_5 + \dots
\end{aligned}
$$

Witten's conjecture states that

The function F satisfies the KdV hierarchy of partial differential equations. In particular, its second derivative $U = \partial^2 F / \partial t_0$ is a solution to the KdV equation,

$$
\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}. \tag{7}
$$

The KdV equation (7) can be considered as a recurrence relation allowing one to compute the intersection indices of the ψ-classes for arbitrary genus recursively from their values for $g = 0$ and $g = 1$, which are known since Witten's pioneering work [51].

Since its appearance in 1991, the conjecture has got several proofs, including those due to Kontsevich [32], Okounkov and Pandharipande [43], Mirzakhani [40], Kazarian and Lando [27], Kim and Liu [31].

Witten's conjecture is an immediate consequence [26] of Theorem 3.2. Indeed, the solutions of the KdV hierarchy are exactly those solutions of KP that depend only on variables with odd indices. After setting $u = 0$ in G, one obtains a power series in variables p_{2i-1} with odd indices, which is therefore a solution to the KdV hierarchy. The coefficients of this series are $\ell_{0;m_1,\dots,m_n} = \langle \tau_{m_1} \dots \tau_{m_n} \rangle$. It turns into F after rescaling $p_{2i+1} = t_i/(2i-1)!!$. In contrast to most of the other proofs, this one guarantees the whole KdV hierarchy for F, while usually one obtains only the first KdV equation and needs the additional string equation to generate the hierarchy.

4. Further developments and perspectives

The variety of Hurwitz numbers is not exhausted by simple and double Hurwitz numbers. Other species include general Hurwitz numbers, enumerating factorizations into permutations of arbitrary cyclic type, not necessarily transpositions, and r-Hurwitz numbers, where transpositions are replaced by certain "completed r-cycles". In all cases, Hurwitz numbers remain closely related to the geometry of moduli spaces, and both are far from being well understood. In this section we describe briefly possible directions of further research.

4.1. Completed cycles. The center of the group algebra $\mathbb{C}[S_N]$ of the symmetric group S_N is generated by the classes $C_\kappa(S_N)$, where κ is a partition of N. The class $C_\kappa(S_N)$ is the sum of all permutations with the cyclic type κ. For example, $C_{1^{N-2}2^1}(S_N)$ is the sum of all transpositions in S_N.

It is convenient, however, to introduce certain classes in the centers of group algebras for all symmetric groups simultaneously. Let κ be a partition. For an arbitrary integer N, choose $|\kappa|$ elements out of $\{1, \ldots, N\}$ and consider in $\mathbb{C}[S_N]$ the sum of all permutations of these $|\kappa|$ elements, of cyclic type κ, all the other $N - |\kappa|$ elements being fixed. Denote by $\widetilde{C}_\kappa$ the element in the center of $\mathbb{C}[S_N]$ which is the sum of all such permutations, for all $\binom{N}{|\kappa|}$ choices of the $|\kappa|$ elements out of N. (If $|\kappa| > N$, then $\widetilde{C}_\kappa = 0 \in \mathbb{C}[S_N]$; if $|\kappa| = N$, then $\widetilde{C}_\kappa = C_\kappa(S_N)$).

For example, the class $\widetilde{C}_{1^1}$ can be understood as the sum of identity permutations, with a distinguished element in each permutation. In other words, the class $\widetilde{C}_{1^1}$ is the same as the class $N\widetilde{C}_\emptyset = NC_{1^N}(S_N)$. Similarly, the class $\widetilde{C}_{1^2}$ coincides with the class $\frac{N(N-1)}{2}\widetilde{C}_\emptyset$: there are $\binom{N}{2} = \frac{N(N-1)}{2}$ ways to pick two elements in the identity permutation.

The classes $\widetilde{C}_\kappa$ have the following advantage when compared to the classes $C_\kappa(S_N)$: the products of the classes $\widetilde{C}_\kappa$ can be expressed as universal linear combinations of these classes, *which are independent of the order N of the symmetric group.* For example, the equation

$$\widetilde{C}_{2^1}\widetilde{C}_{1^2} = \widetilde{C}_{2^1} + 2\widetilde{C}_{1^1 2^1} + \widetilde{C}_{1^2 2^1}$$

is valid in the center of the group algebra $\mathbb{C}[S_N]$ of any symmetric group S_N, for arbitrary N.

Universality means that there is a natural inclusion of the center of $\mathbb{C}[S_N]$ into that of $\mathbb{C}[S_{N+1}]$ for any N. Tending N to infinity, we obtain a universal center of the group algebra, which can be identified with the infinite dimensional vector space freely spanned by the elements $\widetilde{C}_\kappa$, for arbitrary partitions κ. This space also is endowed with an algebra structure.

This algebra is isomorphic to the algebra Γ of shifted symmetric functions defined in Sec. 2.5. As a vector space, the latter algebra is spanned by the functions f_κ indexed by partitions and defined as follows. A central element $\widetilde{C}_\kappa \in \mathbb{C}[S_{|\mu|}]$ acts on the irreducible representation V_μ of the symmetric group by multiplication by a scalar; by definition, we set $f_\kappa(\mu)$ to be equal to this scalar. The *Frobenius characteristic mapping* $\widetilde{C}_\kappa \mapsto f_\kappa$ establishes an isomorphism between the two algebras.

4.2. r-Hurwitz numbers and generalized Witten's conjecture.

Simple Hurwitz numbers count decompositions of a given permutation into a product of transpositions. It is a natural idea to generalize them by replacing transpositions by permutations in other specific classes. For example, why not consider 3-cycles $\widetilde{C}_3$? However, such a straightforward approach fails. Namely, enumerative formulas for decompositions of a given permutation into a product of 3-cycles lose elegance, when compared to that for Hurwitz numbers, and their relationship with

both mathematical physics and geometry is broken. The same is true for r-cycles for any $r \geq 3$. Fortunately, consistency can be restored by replacing r-cycles $\widetilde{C}_r$ with certain linear combinations of the classes $\widetilde{C}_\kappa$, for certain partitions κ.

Definition 6 ([44]). The *completed r-cycle* $\overline{C}_r$ is the preimage under the Frobenius characteristic mapping of the r th power function

$$(\mu_1, \mu_2, \dots) \mapsto \frac{1}{r} \sum_{i=1}^\infty \left((\mu_i - i + \frac{1}{2})^r - (\frac{1}{2} - i)^r \right).$$

We have explained the reasons why the r th power function must be of such a form in Sec. 2.5 (we use a normalization differing from that in [44] by a constant).

Let us give formulas for few first completed cycles among which we know that the completed 2-cycle simply coincides with the ordinary 2-cycle:

$$
\begin{aligned}
\overline{C}_1 &= \widetilde{C}_{1^1} \\
\overline{C}_2 &= \widetilde{C}_{2^1} \\
\overline{C}_3 &= \widetilde{C}_{3^1} + \widetilde{C}_{1^2} + \frac{1}{12}\widetilde{C}_{1^1} \\
\overline{C}_4 &= \widetilde{C}_{4^1} + 2\widetilde{C}_{1^1 2^1} + \frac{5}{4}\widetilde{C}_{2^1}.
\end{aligned}
$$

These formulas explain the origin of the term "completed cycle": the expansion of a class $\overline{C}_r$ as a linear combination of the classes $\widetilde{C}_\kappa$ starts with the class of the r-cycle $\widetilde{C}_{r^1}$, and then terms of smaller order follow. Explicit formulas for the coefficients on the right of the expressions for all completed cycles can be found in [44].

Now we can define the generalized Hurwitz numbers.

Definition 7. The *simple r-Hurwitz number* for an integer $m \vdash N$ and a partition μ is the normalized coefficient of $\widetilde{C}_\mu$ in the m th power of the completed r-cycle,

$$h_{m;\mu}^{(r)\circ} = \frac{|C_\mu|}{N!}[\widetilde{C}_\mu](\overline{C}_r)^m.$$

The simple r-Hurwitz numbers are collected into the generating function

$$H^{(r)\circ}(u; p_1, p_2, \dots) = \sum_{m=0}^\infty \sum_\mu h_{m;\mu}^{(r)\circ} p_{\mu_1} p_{\mu_2} \dots \frac{u^m}{m!},$$

and its logarithm $H^{(r)}(u; p_1, p_2, \dots) = \log H^{(r)\circ}(u; p_1, p_2, \dots)$ is the generating function for *connected simple r-Hurwitz numbers*.

The definition of the r-Hurwitz numbers and explanation in Sec. 2.5 immediately imply

Theorem 4.1. *The function $H^{(r)}(u; p_1, p_2, \dots)$ is a one-parameter family of solutions to the KP hierarchy of partial differential equations.*

Indeed, this one-parameter family is induced by the infinitesimal diagonal transformation of the vector space V of Laurent polynomials taking the vector z^k to $\frac{1}{r}(k-\frac{1}{2})^r z^k$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$.

A similar theorem is valid for generating functions defined by any finite linear combination of completed cycles. In this case the eigenvalues $\frac{1}{r}(k-\frac{1}{2})^r$ are replaced by an appropriate polynomial in k, which can be arbitrary.

The relationship of r-Hurwitz numbers defined by means of the completed cycles to the geometry of moduli spaces of $(r-1)$-spin structures on algebraic curves is less clear at the moment, and this question is a subject of further investigation.

D. Zvonkine conjectured (private communication) that the simple r-Hurwitz numbers can be expressed in terms of the geometry of moduli spaces of $(r-1)$-spin structures on algebraic curves by an r-analogue of the ELSV-formula. Such a formula could lead, at least in principle, to an alternative proof of the generalized Witten conjecture [51], concerning intersection indices of ψ-classes on the moduli spaces of so-called r-spin curves. At the moment, only one proof of the conjecture is known, see [12], and it proceeds in a very different way.

4.3. Geometry of Hurwitz spaces and universal characteristic classes.
The Hurwitz numbers are related to the geometry of moduli spaces of curves through the geometry of *Hurwitz spaces*. The latter are moduli spaces of meromorphic functions on complex curves. Without giving precise definitions, we just explain the main features of the picture. Each Hurwitz space is fibered over the corresponding moduli space of curves — the fibration proceeds by forgetting the function, and this forgetting mapping relates the geometry of the two spaces in question. In a sense, Hurwitz spaces (and, more generally, spaces of stable mappings) are more natural than moduli spaces of curves.

Each Hurwitz space is also stratified according to the degeneration of the functions. A stratum is formed by the locus of functions with prescribed singularities. On the other hand, the action of the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers on the target curve $\mathbb{C}P^1$ is lifted to the Hurwitz spaces. A Hurwitz number (either simple or a more general one) can be computed as the degree of the closure of the corresponding stratum with respect to the above action. This argument votes for the study of the stratification of Hurwitz spaces.

In the simplest case of polynomials, such a study has been carried out in [35]. In [2, 36, 28, 29], a more general case of rational functions is treated. The study applies methods of global singularity theory started by R. Thom and extended recently by M. Kazarian to the case of multisingularities (see, e.g. [32]). These methods produce universal expressions for the locus of prescribed singularities of an arbitrary generic mapping of two complex manifolds in terms of the characteristic classes of the mapping. When applied to the Hurwitz spaces, these methods yield expressions for the loci of functions with prescribed singularities, which lead to explicit formulas for the corresponding Hurwitz numbers.

The classical Thom approach, as well as its generalization by Kazarian, is applicable to the case of mappings with isolated singularities only. For spaces of stable mappings, this requirement proves to be too restrictive, since they inevitably

contain mappings with nonisolated singularities, namely, those contracting certain irreducible components of the curve to a single point in the target space. Sample computations show, however, that main results can be extended to the nonisolated case as well. The corresponding construction is not elaborated yet in the desired generality.

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Department of Mathematics
State University — Higher School of Economics
7 Vavilova Moscow 117312 Russia

Independent University of Moscow
Laboratiore J.-V.Poncelet
Institute for System Research RAS

E-mail: lando@hse.ru