Hurwitz Spaces of Genus 2 Covers of Elliptic Curves

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Abstract:

Let E be an elliptic curve over a field K of characteristic $\neq 2$ and let $N > 1$ be an integer prime to $\mathrm{char}(K)$. The purpose of this paper is to construct the moduli space $H_{E/K,N}$ which "classifies" the set $\mathrm{Cov}_{E/K,N}(K)$ of genus 2 covers of E of degree N which are nomalized in a certain sense.

More precisely, it is shown here that the assignment $L \mapsto \mathrm{Cov}_{E/K,N}(L)$ (where L/K is any field extension) extends in a natural way to a *Hurwitz functor*

$$\mathcal{H}_{E/K,N} : \underline{Sch}_{/K} \to \underline{Sets}$$

which is analogous to the Hurwitz functors considered by Fulton[Fu] (where the base is $\mathbf{P}^1$), and that we have

Theorem. *If $N \geq 3$, then the functor $\mathcal{H}_{E/K,N}$ is finely represented by a smooth, affine and geometrically connected curve $H_{E/K,N}/K$ which is an open subset of a certain twist $X_{E/K,N,-1}$ of the modular curve $X(N)$ of level N; in particular, $H_{E,N} \otimes \overline{K}$ is isomorphic to an open subset of $X(N)_{/\overline{K}}$, where $\overline{K}$ denotes the algebraic closure of K.*

Since the above "twisted modular curve" $X_{E/K,N,-1}/K$ is the moduli space which classifies pairs (E', ψ) where E'/K is an elliptic curve and $\psi : E[N] \xrightarrow{\sim} E'[N]$ is a K-rational anti-isometry of the N-torsion subgroups of E and E', the above theorem may be viewed as a refinement and extension of the "basic construction" of genus 2 curves (with elliptic differentials) presented in Frey/Kani[FK] in which it was shown how to construct a genus 2 cover of E/K from the data (E', ψ).