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Hybrid and multi-field variational principles for geometrically exact three-dimensional beams

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Abstract

This paper addresses the development of several alternative novel hybrid/multi-field variational formulations of the geometrically exact three-dimensional elastostatic beam boundary-value problem. In the framework of the complementary energy-based formulations, a Legendre transformation is used to introduce the complementary energy density in the variational statements as a function of stresses only. The corresponding variational principles are shown to feature stationarity within the framework of the boundary-value problem. Both weak and linearized weak forms of the principles are presented. The main features of the principles are highlighted, giving special emphasis to their relationships from both theoretical and computational standpoints.

Keywords: Three-dimensional beams, geometrically exact analysis, complementary energy, hybrid and multi-field variational principles.

1. Introduction

Variational principles constitute the core of the development of numerical methods in solid mechanics. The utility of such principles is two-fold: first, they provide a very convenient method for the derivation of the governing equations and natural boundary conditions for complex problems and, second, they provide the mathematical foundation required to produce consistent approximate theories. It is in this second role that variational methods have been most useful in solid mechanics.

Many different variational principles can be constructed depending on the equations enforced in weak form (Washizu, 1982). Perhaps the most remarkable variational principle in solid mechanics is the one-field *principle of stationary total potential energy*, which states that, among all kinematically admissible displacement fields, those that satisfy the equilibrium conditions, lead to a stationary value of the total

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potential energy functional. This principle is, very often and implicitly, taken as a particular case of the principle of virtual work (after specialization to the case of hyperelastic materials and conservative external loads), to develop variationally consistent displacement-based finite element formulations. Approximate solutions derived from these formulations need not to fulfill exactly either the internal differential equilibrium equations or the equilibrium equations on the boundary.

A wide variety of multi-field variational principles (also known as mixed variational principles) can be obtained through the generalization of the principle of stationary total potential energy by means of the classical method of Lagrange multipliers. As it is well known, the principle of stationary total potential energy leads to equilibrium configurations by varying displacements. Hence, in approximation techniques, such as the finite element method, approximate displacement fields yield approximate equilibrium configurations. Although this may lead to sufficiently accurate displacement fields, the corresponding stress fields may be highly erroneous. This occurs because the accuracy of the approximate displacement field rapidly deteriorates when differentiations are required to compute other results, such as the stresses or the deformations. However, multi-field variational principles enable arbitrary variation of multiple fields and, therefore, different types of variables can be handled independently in approximation procedures to control simultaneously multiple categories. In the field of the theory of linear elasticity, the first variational principle of this type is due to (Hellinger, 1914) and (Reissner, 1950), being therefore often referred to as the Hellinger-Reissner principle. It involves stresses and displacements as fundamental unknown variables. A generalization of this principle is known in the literature as the Hu-Washizu principle, which owes its name to the works of (Hu, 1955) and (Washizu, 1955). It depends on three functional arguments: stresses, strains and displacements (Washizu, 1982). Hybrid principles, which are based on modified variational principles with relaxed continuity requirements across element boundaries, have also been proposed in the literature. All these principles are, however, only valid in the framework of the theory of linear elasticity and few studies have been devoted to the extension and generalization of these principles to the geometrically nonlinear setting.

The implementation of the geometrically exact beam theory through the finite element method has been performed mostly by resorting to displacement/rotation-based variational formulations, such as the principle of virtual work or the principle of stationary total potential energy. Nevertheless, alternative variational formulations and associated finite element models relying on different types of variables, such as stresses and strains, can also be established.

Iura and Atluri were the first to propose a functional obtained in a consistent fashion from a general 3-field variational principle for the geometrically exact three-dimensional beam theory (Iura and Atluri, 1988, 1989). The functional was established in terms of the cross-sectional stress-resultants, the displacements of the axis, the rotations of the cross sections and the strain measures of the beam. Pimenta and Yojo also presented a similar 3-field variational principle (Pimenta and Yojo, 1993). Formulations based on modified principles of virtual work were developed by (Jelenic and Saje, 1995) and (Zupan and Saje, 2003a, 2004): the former proposing a formulation based on the generalized principle of virtual work involving only the rotational degrees-of-freedom of the structural model and the latter presenting a modified principle of virtual work expressed only in terms of the strain vectors. Dealing with the geometrically exact three-dimensional beam theory as well, but assuming also material nonlinearities, (Nukala and White, 2004) proposed a 2-field variational principle of the Hellinger-Reissner type which involves, as fundamental unknowns, displacements, rotations and stresses.

It is the purpose of this work to present several alternative novel hybrid/multi-field variational princi-

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ples for the quasi-static geometrically exact (Reissner-Simo) analysis of three-dimensional elastic beams. Departing from the principle of stationary total potential energy, a generalized (3+1)-field principle is constructed. Afterwards, making use of the generalized principle, a 3-field principle of Hu-washizu type and also a 2-field complementary principle of Hellinger-Reissner type will be derived. Finally, two hybrid principles of the Hellinger-Reissner type and also a 2-field and a (2+1)-field principle of complementary energy will be obtained. Legendre's transformation is used to introduce the complementary energy density in the complementary variational statements as a function of stresses only. The principles will be shown to be stationary principles within the framework of the geometrically exact three-dimensional beam boundary-value problem. The main features of the principles are highlighted, giving special emphasis to their relationships from both theoretical and computational points of view.

The outline of the paper is as follows: In section 2 we introduce the geometrically exact three-dimensional beam theory, giving afterwards in section 3 some definitions required to address this theory. The equilibrium boundary-value problem is established in section 4. Before proceeding with the derivation of a consistent system of strain-displacement (compatibility) relations, which is performed in section 6 using the well known principle of virtual work, for the sake of compactness, an assembled description of the statical variables is presented first in section 5. The constitutive equations, established in two alternative (and inverse) forms, are presented in section 7. The global boundary-value problem is stated in section 8. Several hybrid and multi-field variational formulations of the boundary-value problem are developed in section 9 along with their corresponding second-order Gâteaux derivatives which can be used to obtain Newton type numerical solution procedures. Finally, some conclusions are given in section 10. The definitions of the most important operators are given in the Appendix.

In the direct notation, bold face lower-case letters are used to denote vectors, and bold face upper-case letters to denote matrices. The components of vectors and matrices are denoted by light-face letters, as well as the ordinary scalars. Cartesian bases are implicitly assumed. Skew-symmetric tensors and the correspondent axial vector use the same letter, $e.g.$, the skew-symmetric tensor associated to vector $\boldsymbol{a}$ is represented by A, $i.e.$, Skew$(\boldsymbol{a}) = A$. While the scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$, its vector product is represented by $\boldsymbol{a} \times \boldsymbol{b} = a_i b_j \epsilon_{ijk}$. Finally, assuming ν as a vectorial space associated with the three-dimensional Euclidean space $\mathcal{R}^3$, we define the tensorial product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$, $\forall \boldsymbol{c} \in \nu$. $\|(\cdot)\|$ represents the Euclidean norm of $(\cdot)$.

2. The Geometrically Exact Three-Dimensional Beam Theory

The geometrically exact three-dimensional beam theory owes its origins to the pioneering works of (Reissner, 1973, 1981). By considering certain kinematic assumptions, which account for shear deformations and moderately large rotations, Reissner obtained expressions for the strain measures of both planar and three-dimensional beams which involve not only the displacements of the beam axis, but also the rotations of the beam cross sections. (Simo, 1985) and (Simo and Vu-Quoc, 1986), making use of convenient parameterizations of the rotations, recast Reissner's theory in a form valid for any magnitude of the rotations (and displacements).

The theory was later extended by (Simo and Vu-Quoc, 1991) to account for the cross-sectional warping phenomena. In fact, it was in this work that the denomination *geometrically exact beam* was firstly proposed.

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Since the pioneering works cited above, considerable progress has been made on the geometrically exact analysis of three-dimensional framed structures, see *e.g.* the works presented by (Cardona and Geradin, 1988), (Simo and Vu-Quoc, 1991), (Pimenta and Yojo, 1993), (Simo et al., 1995), (Ibrahimbegovic et al., 1995; Ibrahimbegovic and Mikdad, 2000; Ibrahimbegovic et al., 2000; Ibrahimbegovic and Knopf-Lenoir, 2003), (Saje et al., 1998), (Crisfield and Jelenic, 1999; Jelenic and Crisfield, 1999), (Planinc and Saje, 1999), (Iura and Atluri, 1988, 1989; Quadrelli and Atluri, 1996, 1998; Atluri et al., 2001), (Betsch and Steinmann, 2002), (Zupan and Saje, 2003b), (Kapania and Li, 2003b,a), (Romero, 2004), (Mata et al., 2007), (Makinen, 2007), (Lens and Cardona, 2008), (Ghosh and Roy, 2008) and (Pimenta et al., 2008).

In the framework of the geometrically exact three-dimensional beam theory, the geometry of a beam is usually described by the line of centroids of the cross sections and also by the family of the cross sections. As the cross sections are not necessarily normal to the line of centroids at the current state, the configuration space of the beam consists of not only the linear space of position vectors of the line of centroids, but also the non-linear space of the cross sections rotations. In this work, the rotations of the cross sections are described through the Euler-Rodrigues formula which is defined in terms of the total rotation vector. Several alternative parameterizations have however been employed in the literature, such as, among others, parameterizations based on Euler angles or quaternion parameters. Efforts have also been made to overcome the singularities inherent to the various parameterizations. A review of different parameterization techniques for finite rotations in space and related computational issues can be found for instance in (Ibrahimbegovic, 1997).

As a geometrically exact three-dimensional beam theory which includes not only deformation in bending but also shear and axial deformations, several other three/two-dimensional geometrically nonlinear beam theories can be obtained after specialization. Therefore, all variational principles presented in this work can be viewed as generalizations of a wide range of variational principles proposed in the literature for different beam theories, such as Euler-Bernoulli and Timoshenko beam theories.

3. Definitions and Preliminaries

Let us consider the three-dimensional beam depicted in Figure 1. Only initially straight beam configurations and initially undistorted cross sections are assumed to be dealt with. The geometric shape of the cross sections is assumed to be arbitrary and constant along the beam and, furthermore, the cross sections are assumed to suffer only rigid body motions during deformation. The geometry of the beam in the current configuration is described by the centroidal axis, denoted by C, and the set of orientations of the cross sections $\mathcal{S}$. Note however that, although this assumption is very often employed, as it is on the basis of some simplifications of the constitutive relations, the beam theory can be established using any other axis of the beam. The centroidal axis C is parameterized by $S \in [0, L]$, usually referred to as the arc-length of the reference centroidal axis of the beam, with L denoting the length of the beam in its reference configuration. C is decomposed into an internal part, represented by Ω, and a boundary part, identified by $\Gamma = \Gamma_N \cup \Gamma_D = \{0, L\}$, where Γ_N and Γ_D correspond to the Neumann and Dirichlet boundaries, respectively, such that $\Gamma_N \cap \Gamma_D = \emptyset$. In general, the beam may be subject to distributed forces $\tilde{n}$ and moments $\tilde{m}$ (applied in Ω) assumed to depend on S, concentrated forces $\bar{n}$ and moments $\bar{m}$ (applied on Γ_N) and prescribed displacements $\bar{u}$ and rotations $\bar{\theta}$ (defined on Γ_D). The loads are assumed conservative and to act at the centroidal axis of the beam.

In the reference configuration, the centroid axis of the beam is described by the position vector r. In order

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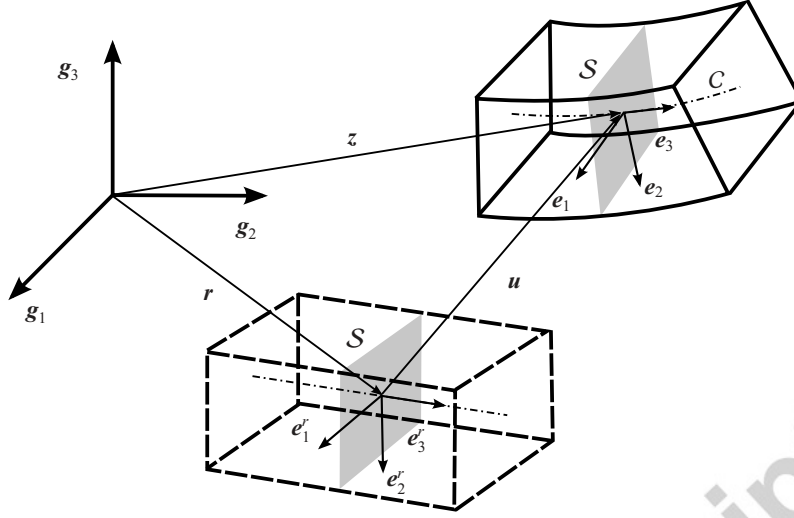
Figure 1: Current and reference configurations of a straight beam

to describe the positions of the cross sections in their reference configuration, a right-handed orthonormal triad of base vectors $\{e_1^r, e_2^r, e_3^r\}$ (the material basis) is introduced. While vectors e_1^r and e_2^r are directed along the principal axes of inertia of the cross section, vector e_3^r is assumed to be normal to the cross section, *i.e.* $e_3^r = e_1^r \times e_2^r$. In addition to the material basis, the spatial basis with orthonormal base vectors $\{e_1, e_2, e_3\}$ is introduced, spanning the physical space in which the beam is embedded. In contrast to the reference configuration, the base vector e_3 need not be directed along the current centroid axis, so that the beam model is capable of representing the first-order shear deformation.

The material (spatial) basis can be regarded as a pull-back (push-forward) mapping of the spatial (material) basis performed by means of the rotation tensor Q as follows

$$e_i^r = Q^T e_i \tag{1a}$$

$$e_i = Q e_i^r \tag{1b}$$

with $i = 1, 2, 3$. The position vector of the current centroidal axis, $z(S)$, and the orientation of the spatial frame attached to the cross section at S, $Q(S)$, fully define the current configuration.

The rotation tensor Q is analytically parameterized through the Euler-Rodrigues formula given by

$$Q = I + h_1(\theta)\Theta + h_2(\theta)\Theta^2 \tag{2}$$

with h_1 and h_2 the following trigonometric functions

$$h_1(\theta) = \frac{\sin \theta}{\theta}$$

$$h_2(\theta) = \frac{1 - \cos \theta}{\theta^2} = \frac{1}{2}\left(\frac{\sin \theta/2}{\theta/2}\right)^2$$

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with $\theta = \|\boldsymbol{\theta}\|$, being $\boldsymbol{\theta}$ the total rotation vector (often called also as the Euler rotation vector). $\boldsymbol{\Theta}$ represents the skew-symmetric tensor associated to $\boldsymbol{\theta}$.

The displacement field at a generic material point lying on the axis of the beam is obtained by

$$\boldsymbol{u} = \boldsymbol{z} - \boldsymbol{r}$$

which, after differentiation with respect to S, leads to

$$\boldsymbol{z}' = \boldsymbol{u}' + \boldsymbol{e}_3^r$$

where $(\cdot)'$ indicates differentiation of $(\cdot)$ with respect to S.

It is worth noting that the true work-conjugate variable of $\boldsymbol{\theta}$ is the pseudo-moment vector $\boldsymbol{\mu} = \boldsymbol{\Gamma}^T \boldsymbol{m}$, rather than the moment vector $\boldsymbol{m}$, with

$$\boldsymbol{\Gamma} = \boldsymbol{I} + h_2(\theta)\boldsymbol{\Theta} + h_3(\theta)\boldsymbol{\Theta}^2$$

where $h_3(\theta)$ is the following trigonometric function

$$h_3(\theta) = \frac{\theta - \sin\theta}{\theta^3}$$

As a consequence, it is $\boldsymbol{\mu}$ which must be prescribed on Γ_N, rather than $\boldsymbol{m}$ as one would expect. For details on this, see *e.g.* (Pimenta, 1993) and (Ritto-Corrêa and Camotim, 2002, 2003).

4. Equilibrium Equations

In the equilibrium of a three-dimensional beam, an infinitesimal element of length dS lying between two cross sections taken normal to the original beam axis is in equilibrium of forces and moments about point P' in the current configuration (see Figure 2), if the following equations hold

$$\boldsymbol{n}' + \tilde{\boldsymbol{n}} = \boldsymbol{0} \tag{4a}$$
$$\boldsymbol{m}' + \boldsymbol{z}' \times \boldsymbol{n} + \tilde{\boldsymbol{m}} = \boldsymbol{0} \tag{4b}$$

On Γ_N the following (Neumann) boundary conditions must hold

$$n\boldsymbol{n} = \bar{\boldsymbol{n}} \tag{5a}$$
$$n\boldsymbol{\mu} = \bar{\boldsymbol{\mu}} \tag{5b}$$

where

$$n = \begin{cases} 1 & \text{if } S = L \\ -1 & \text{if } S = 0 \end{cases}$$

Attending to (1), the material and spatial stress-resultants are shown to obey the following relations

$$\boldsymbol{n} = \boldsymbol{Q}\boldsymbol{n}^r$$
$$\boldsymbol{m} = \boldsymbol{Q}\boldsymbol{m}^r$$
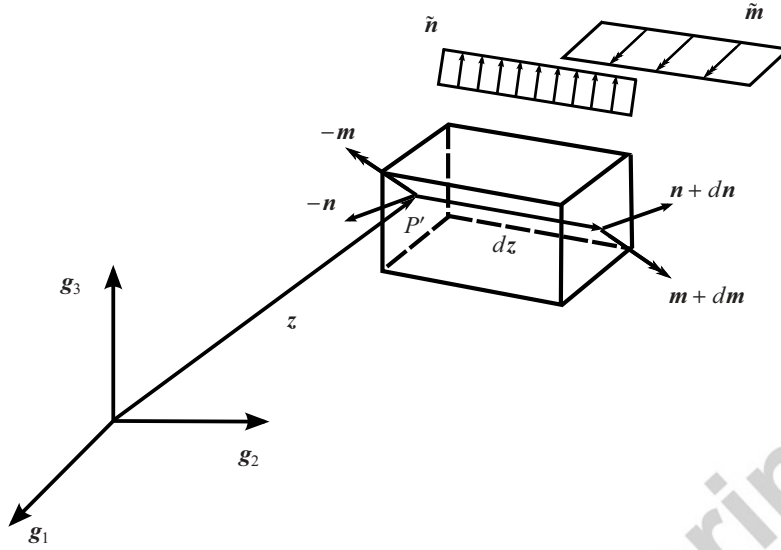
Figure 2: Equilibrium of an infinitesimal length of beam

Using these relations, the equilibrium differential equations (4) can be recast as

$$KQn^r + Qn^{r\prime} + \tilde{n} = 0, \text{ in } \Omega \tag{7a}$$

$$KQm^r + Qm^{r\prime} + z^\prime \times (Qn^r) + \tilde{m} = 0, \text{ in } \Omega \tag{7b}$$

with

$$K = Q^\prime Q^T = \text{Skew}(\Gamma \theta^\prime)$$

5. Assembled Description of Statics

Before proceeding with the derivation of the kinematical differential equations, let us first, for the sake of convenience, introduce an assembled form of the equilibrium boundary-value problem, so that a more compact form of kinematics can be obtained through the application of the principle of virtual work.

Towards this end, let us introduce the following generalized vectors

$$\sigma^r = \begin{bmatrix} n^r \\ m^r \end{bmatrix}, \quad \tilde{q} = \begin{bmatrix} \tilde{n} \\ \tilde{\mu} \end{bmatrix}, \quad \bar{q} = \begin{bmatrix} \bar{n} \\ \bar{\mu} \end{bmatrix}, \quad d = \begin{bmatrix} u \\ \theta \end{bmatrix}$$

representing the generalized vector of stress-resultants, the generalized vector of external distributed loads, the generalized vector of external concentrated loads and the generalized vector of displacements, respectively. Let us also introduce the following generalized operators

$$\Lambda = \begin{bmatrix} Q & O \\ O & Q \end{bmatrix}, \quad H = \begin{bmatrix} Q & O \\ O & \Gamma \end{bmatrix}$$

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Making use of these new generalized vectors and operators, the material form of the equilibrium boundary-value problem, given by equation (7), is reduced to the following synthetic form

$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \ \text{in } \Omega \tag{8a}$$

$$n\boldsymbol{H}\boldsymbol{\sigma}^r = \overline{\boldsymbol{q}}, \ \text{on } \Gamma_N \tag{8b}$$

where $\mathcal{T}_e^r(\boldsymbol{d})$ represents the material equilibrium operator given by

$$\mathcal{T}_e^r(\boldsymbol{d}) = \boldsymbol{\Psi}_e \boldsymbol{\Delta}_e \boldsymbol{\Lambda}$$

with

$$\boldsymbol{\Psi}_e = \left[\begin{array}{ccc} \boldsymbol{I} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Gamma}^T \boldsymbol{Z}' & \boldsymbol{\Gamma}^T \end{array} \right], \ \boldsymbol{\Delta}_e = \left[\begin{array}{cc} \boldsymbol{I}\frac{\partial}{\partial S} & \boldsymbol{O} \\ \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}\frac{\partial}{\partial S} \end{array} \right]$$

where $\boldsymbol{Z}'$ represents the skew-symmetric tensor associated with vector $\boldsymbol{z}'$.

6. Assembled Description of Kinematics

The purpose of this section is to deduce the strain-displacement relationships using the principle of virtual work (weak form of equilibrium).

Let us introduce a trial function space defined as

$$\mathcal{U}_k = \{\boldsymbol{d} \in \mathcal{H}^1(\Omega) | \ \boldsymbol{d} = \overline{\boldsymbol{d}} \ \text{on } \Gamma_D\}$$

with $\overline{\boldsymbol{d}}$ the generalized vector of prescribed displacements on the Dirichlet boundary Γ_D, given by

$$\overline{\boldsymbol{d}} = \left[\begin{array}{c} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{\theta}} \end{array} \right]$$

and a test function space defined as

$$\mathcal{V}_k = \{\delta\boldsymbol{d} \in \mathcal{H}^1(\Omega) | \ \delta\boldsymbol{d} = \boldsymbol{0} \ \text{on } \Gamma_D\}$$

where $\mathcal{H}^1(\Omega)$ represents the standard Sobolev space. Spaces $\mathcal{U}_k$ and $\mathcal{V}_k$ are usually referred to as the *kinematically admissible space* and the *homogeneous kinematically admissible space* (or, generally, the *space of admissible variations*), respectively.

The weak solution to the equilibrium boundary-value problem is a vector $\boldsymbol{d} \in \mathcal{U}_k$ such that, for all $\delta\boldsymbol{d} \in \mathcal{V}_k$, the following variational equation holds

$$\int_\Omega [\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}] \cdot \delta\boldsymbol{d} \, d\Omega + [(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_N} = 0, \ \forall\delta\boldsymbol{d} \in \mathcal{V}_k$$

which, after integrating by parts, leads to

$$-\int_\Omega \boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} \, d\Omega + \int_\Omega \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d} \, d\Omega + [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N} = 0, \ \forall\delta\boldsymbol{d} \in \mathcal{V}_k \tag{9}$$

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where $\mathcal{T}_c^r(d)$ represents the material form of the tangent compatibility operator defined as

$$\mathcal{T}_c^r(d) = \mathbf{\Lambda}^T \mathbf{\Psi}_c \mathbf{\Delta}_c \tag{10}$$

with

$$\mathbf{\Psi}_c = \begin{bmatrix} I & O & Z'\mathbf{\Gamma} \\ O & \mathbf{\Gamma} & \mathbf{\Gamma}' \end{bmatrix}, \ \mathbf{\Delta}_c = \begin{bmatrix} I\frac{\partial}{\partial S} & O \\ O & I\frac{\partial}{\partial S} \\ O & I \end{bmatrix}$$

Note that the operator $\mathcal{T}_c^r$ can be regarded as the adjoint operator of the equilibrium operator $\mathcal{T}_e^r$.

The variational (weak) form (9) is equivalent to the principle of virtual work which states that, if vector $d \in \mathcal{U}_k$ is an equilibrium configuration, then, for any given virtual displacement $\delta d \in \mathcal{V}_k$, the external virtual work δW_{ext} equals the internal virtual work δW_{int}, *i.e.*

$$\delta W_{ext} = \delta W_{int}, \ \forall \delta d \in \mathcal{V}_k \tag{11}$$

with

$$\delta W_{int} = \int_\Omega \boldsymbol{\sigma}^r \cdot \delta \boldsymbol{\varepsilon}^r \ d\Omega$$

$$\delta W_{ext} = \int_\Omega \tilde{\boldsymbol{q}} \cdot \delta d \ d\Omega + [\overline{\boldsymbol{q}} \cdot \delta d]_{\Gamma_N}$$

where

$$\delta \boldsymbol{\varepsilon}^r = \mathcal{T}_c^r(d)\delta d \tag{13}$$

Vector $\delta \boldsymbol{\varepsilon}^r$ can be identified as the generalized virtual material strain vector which is energy-conjugate to the generalized stress-resultant vector $\boldsymbol{\sigma}^r$, and it can be decomposed into two vectors, $\delta\boldsymbol{\eta}^r$ and $\delta\boldsymbol{\kappa}^r$, as follows

$$\delta \boldsymbol{\varepsilon}^r = \begin{bmatrix} \delta\boldsymbol{\eta}^r \\ \delta\boldsymbol{\kappa}^r \end{bmatrix}$$

representing the translational and rotational parts of $\delta \boldsymbol{\varepsilon}^r$ which are energy-conjugate to the stress-resultants $\boldsymbol{n}^r$ and $\boldsymbol{m}^r$, respectively, and which are defined as

$$\delta\boldsymbol{\eta}^r = \boldsymbol{Q}^T \delta \boldsymbol{u}' + \boldsymbol{Q}^T \boldsymbol{Z}' \mathbf{\Gamma} \delta\boldsymbol{\theta} \tag{14a}$$

$$\delta\boldsymbol{\kappa}^r = \boldsymbol{Q}^T (\mathbf{\Gamma}\delta\boldsymbol{\theta})' \tag{14b}$$

Recalling that the reference configuration has been assumed to be undeformed, and after some algebraic manipulations, integration of equations (14) leads to the following strain-displacement relationship

$$\boldsymbol{\varepsilon}^r(d) = \begin{bmatrix} \boldsymbol{\eta}^r \\ \boldsymbol{\kappa}^r \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}^T(\boldsymbol{u}' + \boldsymbol{e}_3^r) - \boldsymbol{e}_3^r \\ \mathbf{\Gamma}^T \boldsymbol{\theta}' \end{bmatrix} \tag{15}$$

It can be easily shown that these strain measures are objective, in the sense that they remain invariant under superposed rigid body motions on the beam (see, *e.g.*, (Crisfield and Jelenic, 1999)).

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7. Constitutive Relations

For an elastic material behavior, a strain energy density per unit reference length $W(\boldsymbol{\varepsilon}^r)$ can be defined such that

$$\boldsymbol{\sigma}^r = \frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r} \tag{16}$$

Taking the derivative of the equation given above with respect to the material strain vector, the following constitutive tangent operator is obtained

$$\boldsymbol{C} = \frac{\partial \boldsymbol{\sigma}^r}{\partial \boldsymbol{\varepsilon}^r} = \frac{\partial^2 W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r \partial \boldsymbol{\varepsilon}^r} = \left[\begin{array}{cc} \frac{\partial^2 W}{\partial \boldsymbol{\eta}^r \partial \boldsymbol{\eta}^r} & \frac{\partial^2 W}{\partial \boldsymbol{\eta}^r \partial \boldsymbol{\kappa}^r} \\ \frac{\partial^2 W}{\partial \boldsymbol{\kappa}^r \partial \boldsymbol{\eta}^r} & \frac{\partial^2 W}{\partial \boldsymbol{\kappa}^r \partial \boldsymbol{\kappa}^r} \end{array} \right] \tag{17}$$

Let us assume that W is a differentiable and convex function with respect to vector $\boldsymbol{\varepsilon}^r$. On the basis of these assumptions, equation (16) establishes a one-to-one correspondence between the stress-resultant vector $\boldsymbol{\sigma}^r$ and the strain vector $\boldsymbol{\varepsilon}^r$. Legendre showed that this type of equations may be transformed into a conjugate form by introducing a new function $W_c(\boldsymbol{\sigma}^r)$, defined by

$$W_c(\boldsymbol{\sigma}^r) = -W(\boldsymbol{\varepsilon}^r) + \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r \tag{18}$$

which is called the *complementary energy density*. Differentiation of this equation yields

$$\boldsymbol{\varepsilon}^r = \frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} \tag{19}$$

Taking the derivative with respect to the material stress-resultant vector gives rise to the inverse of the constitutive tangent operator $\boldsymbol{C}$

$$\boldsymbol{D} = \frac{\partial \boldsymbol{\varepsilon}^r}{\partial \boldsymbol{\sigma}^r} = \frac{\partial^2 W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r \partial \boldsymbol{\sigma}^r} = \left[\begin{array}{cc} \frac{\partial^2 W_c}{\partial \boldsymbol{n}^r \partial \boldsymbol{n}^r} & \frac{\partial^2 W_c}{\partial \boldsymbol{n}^r \partial \boldsymbol{m}^r} \\ \frac{\partial^2 W_c}{\partial \boldsymbol{m}^r \partial \boldsymbol{n}^r} & \frac{\partial^2 W_c}{\partial \boldsymbol{m}^r \partial \boldsymbol{m}^r} \end{array} \right] \tag{20}$$

8. Boundary-Value Problem

The strong form of the boundary-value problem governing the response of a geometrically-exact three-dimensional beam with hyperelastic material behavior consists of the following three sets of differential equations to be solved in Ω

$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \ \text{in} \ \Omega \tag{21a}$$

$$\boldsymbol{\sigma}^r - \frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r} = \boldsymbol{0}, \ \text{in} \ \Omega \tag{21b}$$

$$\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \boldsymbol{0}, \ \text{in} \ \Omega \tag{21c}$$

representing equilibrium, constitutive and compatibility conditions, respectively, and, additionally, a set of prescribed boundary conditions on $\Gamma = \Gamma_N \cup \Gamma_D$, subdivided into Neumann (equilibrium) and Dirichlet (compatibility) conditions defined on Γ_N and Γ_D, respectively, as follows

$$n\boldsymbol{H}\boldsymbol{\sigma}^r = \overline{\boldsymbol{q}}, \text{ on } \Gamma_N \tag{22a}$$

$$\boldsymbol{d} = \overline{\boldsymbol{d}}, \text{ on } \Gamma_D \tag{22b}$$

If the strain energy $W(\boldsymbol{\varepsilon}^r)$ is convex, which is indeed the case of a linear elastic material response, the constitutive relations can be established using either the stiffness format (21b) or the flexibility format given by (19). Conditions for the convexity of the strain energy density and a full discussion of the existence of a uniquely defined inverse of the constitutive relations in the general framework of hyperelasticity were given by (Ogden, 1977). Ogden has concluded that for all deformations of practical interest in rubberlike materials, it is reasonable to suppose the existence of a strictly convex strain energy density function.

9. Stationary Variational Principles

9.1. Principle of Total Potential Energy

The total potential energy associated with vector $\boldsymbol{d}$ is the one-field functional $\Pi_P : \mathcal{U}_k(\Omega) \to \mathcal{R}$ given by

$$\Pi_p(\boldsymbol{d}) = U(\boldsymbol{\varepsilon}^r(\boldsymbol{d})) + F(\boldsymbol{d}) \tag{23}$$

where U represents the strain energy, or internal potential energy, given by

$$U(\boldsymbol{d}) = \int_0^L W(\boldsymbol{\varepsilon}^r(\boldsymbol{d})) \, d\Omega$$

and F represents the external potential energy which, on the assumption of constant external pseudo-moments, is defined as follows

$$F(\boldsymbol{d}) = -\int_0^L \tilde{\boldsymbol{q}} \cdot \boldsymbol{d} \, d\Omega - [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}]_{\Gamma_N}$$

While the compatibility and the constitutive equations may be regarded as constraints assigned to the stationarity of the variational problem defined by the total potential energy given above, the equilibrium equations are derived as their Euler-Lagrange conditions.

Vector $\boldsymbol{d}$ is called a stationary point if the first-order Gâteaux derivative of $\Pi_p(\boldsymbol{d})$ at $\boldsymbol{d} \in \mathcal{U}_k$ in any direction $\delta\boldsymbol{d} \in \mathcal{V}_k$ vanishes, that is, if the following condition holds

$$\delta\Pi_p(\boldsymbol{d}; \delta\boldsymbol{d}) = 0, \ \forall \ \delta\boldsymbol{d} \in \mathcal{V}_k \tag{24}$$

wherein $\delta\Pi_p(\boldsymbol{d}; \delta\boldsymbol{d})$ assumes the following form

$$\delta\Pi_p(\boldsymbol{d}; \delta\boldsymbol{d}) = \int_\Omega [\frac{\partial W}{\partial \boldsymbol{\varepsilon}^r} \cdot \delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}; \delta\boldsymbol{d}) - \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d}] \, d\Omega - [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N}$$

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which, upon substitution of the constitutive relation (21b), gives rise to

$$\delta\Pi_p(\boldsymbol{d};\delta\boldsymbol{d}) = \int_\Omega [\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\varepsilon}^r(\boldsymbol{d};\delta\boldsymbol{d}) - \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d}] \, d\Omega - [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N} \tag{25}$$

Note that, the variational form (24) with $\delta\Pi_p(\boldsymbol{d};\delta\boldsymbol{d})$ given by (25) is equivalent to the principle of virtual work stated in (11).

According to (13), the first-order Gâteaux derivative of the total potential energy can be rewritten as

$$\delta\Pi_p(\boldsymbol{d};\delta\boldsymbol{d}) = \int_\Omega [\boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} - \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d}] \, d\Omega - [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N}$$

Finally, after integrating by parts, the equation given above results as

$$\delta\Pi_p(\boldsymbol{d};\delta\boldsymbol{d}) = - \int_\Omega [\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}] \cdot \delta\boldsymbol{d} \, d\Omega + [(n\boldsymbol{H}\boldsymbol{\sigma}^r - \overline{\boldsymbol{q}}) \cdot \delta\boldsymbol{d}]_{\Gamma_N}$$

which shows that the Euler-Lagrange equations of $\delta\Pi_p$ are the equilibrium equations

$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \text{ in } \Omega$$
$$n\boldsymbol{H}\boldsymbol{\sigma}^r - \overline{\boldsymbol{q}} = \boldsymbol{0}, \text{ on } \Gamma_N$$

Hence, it can be concluded that, a geometrically exact three-dimensional beam is in equilibrium if, and only if, its total potential energy takes a stationary value for all admissible displacement fields that satisfy the kinematical boundary conditions. This result is known as the principle of stationary total potential energy.

Assuming non-follower loads, the second-order Gâteaux derivative of the total potential energy defined at $\boldsymbol{d}$ in the directions of $\delta\boldsymbol{d}$ and $\Delta\boldsymbol{d}$ reads

$$\Delta(\delta\Pi_p(\boldsymbol{d};\delta\boldsymbol{d},\Delta\boldsymbol{d})) = \int_\Omega [\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \boldsymbol{\sigma}^r \cdot \Delta(\delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}))] \, d\Omega$$
$$= \int_\Omega [\frac{\partial\boldsymbol{\sigma}^r}{\partial\boldsymbol{\varepsilon}^r}\Delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \boldsymbol{\sigma}^r \cdot \Delta(\delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}))] \, d\Omega$$

which, using equations (10), (13) and (17), can be rewritten as

$$\Delta(\delta\Pi_p(\boldsymbol{d};\delta\boldsymbol{d},\Delta\boldsymbol{d})) = \int_\Omega [\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} \cdot \boldsymbol{C}\mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d} + \boldsymbol{\sigma}^r \cdot \Delta(\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d})] \, d\Omega$$
$$= \int_\Omega (\Delta_c\delta\boldsymbol{d}) \cdot (\boldsymbol{K}_M + \boldsymbol{K}_G^{\boldsymbol{\sigma}^r})(\Delta_c\Delta\boldsymbol{d}) \, d\Omega \tag{27}$$

where operators $\boldsymbol{K}_M$ and $\boldsymbol{K}_G^{\boldsymbol{\sigma}^r}$ are defined as

$$\boldsymbol{K}_M = \boldsymbol{\Psi}_c^T \boldsymbol{\Lambda} \boldsymbol{C} \boldsymbol{\Lambda}^T \boldsymbol{\Psi}_c$$
$$\boldsymbol{K}_G^{\boldsymbol{\sigma}^r} = \frac{\partial(\boldsymbol{\Psi}_c^T \boldsymbol{\Lambda} \boldsymbol{\sigma}^r)}{\partial(\Delta_c\boldsymbol{d})}$$

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with $\boldsymbol{K}_G^{\sigma^r} = \boldsymbol{K}_G(\boldsymbol{d}, \boldsymbol{\sigma}^r)$. The definitions of the operators $\boldsymbol{K}_M$ and $\boldsymbol{K}_G$ are given in the Appendix. As they represent the material and geometrical contributions to the tangent bilinear form, they are usually referred to as the material and geometric tangent operators, respectively.

Note that the bilinear form given by (27) is always symmetric, even far from an equilibrium state, since conservative external loadings and hyperelastic material behavior have been assumed.

It should also be highlighted that, if conservative follower loads had been considered instead of simple spatially fixed loads, the tangent stiffness matrix would have an additional term associated to the linearization of the follower loads.

To conclude, it is worth noting that most of the traditional displacement/rotation-based finite element models developed on the basis of this variational principle are non-objective (Crisfield and Jelenic, 1999; Jelenic and Crisfield, 1999) and, furthermore, lead to numerical solutions which may be affected by shear-locking effects (Ibrahimbegovic, 1995).

9.2. (3+1)-Field Generalized Principle

The principle of stationary total potential energy can be generalized through the well known method of Lagrange multipliers, which allows to introduce the compatibility equations, assumed now as subsidiary conditions, into the framework of the variational expression of the total potential energy. Accordingly, relaxing equations (21c) and (22b) by means of two sets of Lagrange multipliers, $\boldsymbol{s}$ and $\boldsymbol{r}$, defined in Ω and Γ_D, respectively, the following generalized functional $\Pi_g : \mathcal{H}^1(\Omega) \times \mathcal{H}^0(\Omega) \times \mathcal{H}^0(\Omega) \times \mathcal{R}^6(\Gamma_D)$ is obtained

$$\Pi_g(\boldsymbol{d}, \boldsymbol{\varepsilon}^r, \boldsymbol{s}, \boldsymbol{r}) = -\Pi_P(\boldsymbol{d}) + \int_0^L (\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \boldsymbol{s} \, d\Omega + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot \boldsymbol{r}]_{\Gamma_D} \tag{29}$$

which can be recast as

$$\Pi_g(\boldsymbol{d}, \boldsymbol{\varepsilon}^r, \boldsymbol{s}, \boldsymbol{r}) = \int_0^L [-W(\boldsymbol{\varepsilon}^r) + \tilde{\boldsymbol{q}} \cdot \boldsymbol{d} + (\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \boldsymbol{s}] \, d\Omega + [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot \boldsymbol{r}]_{\Gamma_D} \tag{30}$$

where the independent quantities subject to variation are the unknown vector fields $\boldsymbol{d}$, $\boldsymbol{\varepsilon}^r$, $\boldsymbol{s}$ and $\boldsymbol{r}$ with no subsidiary conditions.

The first-order Gâteaux derivative of Π_g writes

$$\delta\Pi_g = \int_0^L [-\frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r} \cdot \delta\boldsymbol{\varepsilon}^r + \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d} + \boldsymbol{s} \cdot (\delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) + (\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{s}] \, d\Omega$$
$$+ [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot \delta\boldsymbol{r}]_{\Gamma_D} + [\boldsymbol{r} \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, after integrating by parts and rearranging the terms, leads to

$$\delta\Pi_g = \int_0^L [(\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{s} + \tilde{\boldsymbol{q}}) \cdot \delta\boldsymbol{d} + (\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{s} + (\boldsymbol{s} - \frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r}) \cdot \delta\boldsymbol{\varepsilon}^r] \, d\Omega$$
$$+ [(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{s}) \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot \delta\boldsymbol{r}]_{\Gamma_D} + [(\boldsymbol{r} - n\boldsymbol{H}\boldsymbol{s}) \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

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Since the variations $\delta\boldsymbol{d}$, $\delta\boldsymbol{\varepsilon}^r$, $\delta\boldsymbol{s}$ and $\delta\boldsymbol{r}$ are completely arbitrary, in order to have $\delta\Pi_g = 0$, each term in parentheses must vanish independently, *i.e.*, the following Euler-Lagrange equations are obtained

$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{s} + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \text{ in } \Omega$$

$$\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \boldsymbol{0}, \text{ in } \Omega$$

$$\boldsymbol{s} - \frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r} = \boldsymbol{0}, \text{ in } \Omega$$

$$\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{s} = \boldsymbol{0}, \text{ on } \Gamma_N$$

$$\boldsymbol{d} - \overline{\boldsymbol{d}} = \boldsymbol{0}, \text{ on } \Gamma_D$$

$$\boldsymbol{r} - n\boldsymbol{H}\boldsymbol{s} = \boldsymbol{0}, \text{ on } \Gamma_D$$

The physical meaning of the Lagrange multipliers $\boldsymbol{s}$ and $\boldsymbol{r}$ are seen to be that of generalized material stress-resultants (defined in Ω) and reactions (defined on Γ_D), *i.e.*

$$\boldsymbol{s} = \boldsymbol{\sigma}^r, \text{ in } \Omega \tag{32a}$$

$$\boldsymbol{r} = n\boldsymbol{H}\boldsymbol{\sigma}^r, \text{ on } \Gamma_D \tag{32b}$$

The second-order Gâteaux derivative of Π_g reads

$$\Delta(\delta\Pi_g) = \int_0^L [-\frac{\partial^2 W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r \partial \boldsymbol{\varepsilon}^r} \Delta\boldsymbol{\varepsilon}^r \cdot \delta\boldsymbol{\varepsilon}^r + \Delta\boldsymbol{s} \cdot (\delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) - \boldsymbol{s} \cdot \Delta(\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) + (\Delta\boldsymbol{\varepsilon}^r - \Delta\boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{s}] \, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot \delta\boldsymbol{r}]_{\Gamma_D} + [\Delta\boldsymbol{r} \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, using equations (10), (13) and (17), can be rewritten as

$$\Delta(\delta\Pi_g) = \int_0^L [-\boldsymbol{C}\Delta\boldsymbol{\varepsilon}^r \cdot \delta\boldsymbol{\varepsilon}^r + \Delta\boldsymbol{s} \cdot (\delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) - (\Delta_c\delta\boldsymbol{d}) \cdot \boldsymbol{K}_G^s(\Delta_c\Delta\boldsymbol{d}) + (\Delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{s}] \, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot \delta\boldsymbol{r}]_{\Gamma_D} + [\Delta\boldsymbol{r} \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

with $\boldsymbol{K}_G^s = \boldsymbol{K}_G(\boldsymbol{d}, \boldsymbol{s})$.

Contrarily to the variational formulation established on the basis of the principle of stationary total potential energy, this formulation can be used as the basis for the development of hybrid-mixed finite element models in which general approximate displacements/rotations are employed with no need to satisfy *a priori* neither the kinematical differential equations defined in the elements nor the compatibility conditions defined on the interelement boundaries. Furthermore, as the strains are taken as independent variables, the shear-locking phenomena can be avoided without resorting to numerical tricks and objective numerical solutions can be obtained. Despite these outcomes, finite element models developed on the basis of this principle cannot in general produce solutions satisfying in strong form either the equilibrium or the constitutive or the compatibility equations of the problem.

9.3. 3-Field Principle of Hu-Washizu Type

On insertion of the results given by (32) into the energy functional (30) gives rise to the 3-field Hu-Washizu functional or, according to (Felippa, 2000), the 3-field Veubeke-Hu-Washizu functional Π_{HW} :

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$\mathcal{H}^0(\Omega) \times \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega) \rightarrow \mathcal{R}$ defined by

$$\Pi_{HW}(\boldsymbol{\sigma}^r, \boldsymbol{\varepsilon}^r, \boldsymbol{d}) = \int_0^L [-W(\boldsymbol{\varepsilon}^r) + \tilde{\boldsymbol{q}} \cdot \boldsymbol{d} + (\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \boldsymbol{\sigma}^r] \, d\Omega + [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot n\boldsymbol{H}\boldsymbol{\sigma}^r]_{\Gamma_D} \tag{33}$$

in which the independent functions subject to variation are indeed $\boldsymbol{\sigma}^r$, $\boldsymbol{\varepsilon}^r$ and $\boldsymbol{d}$.

The first-order Gâteaux derivative of Π_{HW} writes

$$\delta\Pi_{HW} = \int_0^L [-\frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r} \cdot \delta\boldsymbol{\varepsilon}^r + \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d} + \boldsymbol{\sigma}^r \cdot (\delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) + (\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{\sigma}^r] \, d\Omega$$
$$+ [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)]_{\Gamma_D} + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, after integrating by parts and rearranging the terms, leads to

$$\delta\Pi_{HW} = \int_0^L [(\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}) \cdot \delta\boldsymbol{d} + (\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{\sigma}^r + (\boldsymbol{\sigma}^r - \frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r}) \cdot \delta\boldsymbol{\varepsilon}^r] \, d\Omega$$
$$+ [(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)]_{\Gamma_D}$$

Requiring $\delta\Pi_{HW}$ to vanish for independent variations $\delta\boldsymbol{\sigma}^r$, $\delta\boldsymbol{\varepsilon}^r$ and $\delta\boldsymbol{d}$, the following Euler-Lagrange equations are obtained

$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{s} + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \text{ in } \Omega$$
$$\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \boldsymbol{0}, \text{ in } \Omega$$
$$\boldsymbol{\sigma}^r - \frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r} = \boldsymbol{0}, \text{ in } \Omega$$
$$\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r = \boldsymbol{0}, \text{ on } \Gamma_N$$
$$\boldsymbol{d} - \overline{\boldsymbol{d}} = \boldsymbol{0}, \text{ on } \Gamma_D$$

The second-order Gâteaux derivative of Π_{HW} reads

$$\Delta(\delta\Pi_{HW}) = \int_0^L [-\frac{\partial^2 W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r \partial \boldsymbol{\varepsilon}^r} \Delta\boldsymbol{\varepsilon}^r \cdot \delta\boldsymbol{\varepsilon}^r + \Delta\boldsymbol{\sigma}^r \cdot (\delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) - \boldsymbol{\sigma}^r \cdot \Delta(\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) + (\Delta\boldsymbol{\varepsilon}^r - \Delta\boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{\sigma}^r] \, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)]_{\Gamma_D} + [n\Delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, using equations (10), (13) and (17), can be rewritten as

$$\Delta(\delta\Pi_{HW}) = \int_0^L [-\boldsymbol{C}\Delta\boldsymbol{\varepsilon}^r \cdot \delta\boldsymbol{\varepsilon}^r + \Delta\boldsymbol{\sigma}^r \cdot (\delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) - (\Delta_c\delta\boldsymbol{d}) \cdot \boldsymbol{K}_G^{\sigma^r}(\Delta_c\Delta\boldsymbol{d}) + (\Delta\boldsymbol{\varepsilon}^r - \mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r] \, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot n(\boldsymbol{H}\delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\sigma^r}\delta\boldsymbol{d})]_{\Gamma_D} + [n(\boldsymbol{H}\Delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\sigma^r}\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

where operator $\boldsymbol{H}_d^{\sigma^r}$ is defined as

$$\boldsymbol{H}_d^{\sigma^r} = \begin{bmatrix} \boldsymbol{O} & \frac{\partial(\boldsymbol{Q}\boldsymbol{n}^r)}{\partial\boldsymbol{\theta}} \\ \boldsymbol{O} & \frac{\partial(\hat{\boldsymbol{\Gamma}}\boldsymbol{m}^r)}{\partial\boldsymbol{\theta}} \end{bmatrix}$$

with $\boldsymbol{H}_d^{\sigma^r} = \boldsymbol{H}_d(\boldsymbol{\theta}, \boldsymbol{\sigma}^r)$. The definition of operator $\boldsymbol{H}_d$ is given in the Appendix.

Finite element models constructed on the basis of this principle may benefit from the same numerical features of finite element models derived from the generalized principle presented in the preceding section, with the advantage of using a smaller number of approximation variables.

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9.4. 2-Field Principle of Hellinger-Reissner Type

As a prelude to the introduction of the 2-field principle of Hellinger-Reissner, it is worth noting that this principle owes its origins to the idea that, while in the geometrically nonlinear framework the development of a complementary energy principle involving only stress fields could be associated to some obstacles, no such obstacles should exist for an energy principle expressed in terms of stresses and displacements.

By means of equations (32) and the Legendre transformation (18), variables s and r can be eliminated from the generalized energy functional (30), leading to the 2-field complementary Hellinger-Reissner energy $\Pi_{HR} : \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega) \to \mathcal{R}$ defined as

$$\Pi_{HR}(\boldsymbol{\sigma}^r, \boldsymbol{d}) = \int_\Omega [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \tilde{\boldsymbol{q}} \cdot \boldsymbol{d}] \, d\Omega + [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}]_{\Gamma_N} + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot (\boldsymbol{d} - \overline{\boldsymbol{d}})]_{\Gamma_D} \tag{35}$$

Let us consider the first-order Gâteaux derivative of Π_{HR} at $(\boldsymbol{\sigma}^r, \boldsymbol{d})$ in the direction of $(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d}) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega)$

$$\delta\Pi_{HR} = \int_\Omega [\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} + \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d}] \, d\Omega$$
$$+ [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)]_{\Gamma_D} + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, after integrating by parts, leads to

$$\delta\Pi_{HR} = \int_\Omega [(\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{\sigma}^r + (\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}) \cdot \delta\boldsymbol{d}] \, d\Omega + [(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)]_{\Gamma_D}$$

Accordingly, if $\delta\Pi_{HR}$ is required to vanish for independent variations of $\delta\boldsymbol{\sigma}^r$ and $\delta\boldsymbol{d}$, the following Euler-Lagrange equations are obtained

$$\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \boldsymbol{0}, \text{ in } \Omega$$
$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \text{ in } \Omega$$
$$\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r = \boldsymbol{0}, \text{ on } \Gamma_N$$
$$\boldsymbol{d} - \overline{\boldsymbol{d}} = \boldsymbol{0}, \text{ on } \Gamma_D$$

which represent all the equations governing the boundary-value problem.

As it can be seen, neither the stress fields need to satisfy the differential and boundary equilibrium equations, nor the displacement fields need to satisfy the prescribed boundary conditions. Therefore, it can be stated that the present 2-field principle of Hellinger-Reissner type is suitable for the construction of mixed finite element models in which neither the compatibility nor the equilibrium conditions are required to be satisfied *a priori*.

Note also that, the generalized principle as well as the principle of Hu-Washizu can only be transformed into the principle of Hellinger-Reissner through a Legendre transformation when the constitutive equations are invertible. In other words, neither the generalized principle nor the principle of Hu-Washizu require

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inverting the constitutive equation given by (21b), *i.e.*, to axiomatically assume the existence of a complementary energy density $W_c(\boldsymbol{\sigma}^r)$. This restriction makes the former principles more appropriate for general nonlinear materials than the latter.

The second-order Gâteaux derivative of Π_{HR} reads

$$\Delta(\delta\Pi_{HR}) = \int_0^L [\frac{\partial^2 W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r\partial\boldsymbol{\sigma}^r}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} - \boldsymbol{\sigma}^r \cdot \Delta(\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d})]\, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)]_{\Gamma_D} + [n\Delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, using equations (10), (13) and (20), can be rewritten as

$$\Delta(\delta\Pi_{HR}) = \int_0^L [\boldsymbol{D}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - (\mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot (\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) - (\Delta_c\delta\boldsymbol{d}) \cdot \boldsymbol{K}_G^{\boldsymbol{\sigma}^r}(\Delta_c\Delta\boldsymbol{d})]\, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot n(\boldsymbol{H}\delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\boldsymbol{\sigma}^r}\delta\boldsymbol{d})]_{\Gamma_D} + [n(\boldsymbol{H}\Delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\boldsymbol{\sigma}^r}\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

9.5. (2+1)-Field Principle of Hellinger-Reissner Type

By means of equation (32a) and the Legendre transformation (18), the generalized energy functional (30) gives rise to the following (2+1)-field complementary Hellinger-Reissner energy $\Pi_{HR}^1 : \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega) \times \mathcal{R}^6(\Gamma_D) \to \mathcal{R}$ defined as

$$\Pi_{HR}^1(\boldsymbol{\sigma}^r, \boldsymbol{d}, \boldsymbol{r}) = \int_\Omega [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \tilde{\boldsymbol{q}} \cdot \boldsymbol{d}]\, d\Omega + [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}]_{\Gamma_N} + [\boldsymbol{r} \cdot (\boldsymbol{d} - \overline{\boldsymbol{d}})]_{\Gamma_D} \tag{37}$$

which corresponds to a hybrid form of the 2-field Hellinger-Reissner energy (35).

Carrying out its first-order Gâteaux derivative at $(\boldsymbol{\sigma}^r, \boldsymbol{d}, \boldsymbol{r})$ in the direction of $(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d}, \delta\boldsymbol{r}) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega) \times \mathcal{R}^6(\Gamma_D)$ leads to

$$\delta\Pi_{HR}^1 = \int_\Omega [\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} + \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]\, d\Omega$$
$$+ [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot \delta\boldsymbol{r}]_{\Gamma_D} + [\boldsymbol{r} \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, after integrating by parts, gives

$$\delta\Pi_{HR}^1 = \int_\Omega [(\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{\sigma}^r + (\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}) \cdot \delta\boldsymbol{d}]\, d\Omega +$$
$$[(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_N} + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot \delta\boldsymbol{r}]_{\Gamma_D} + [(\boldsymbol{r} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

Finally, requiring $\delta\Pi_{HR}$ to vanish for independent variations of $\delta\boldsymbol{\sigma}^r$, $\delta\boldsymbol{d}$ and $\delta\boldsymbol{r}$, gives rise to the following Euler-Lagrange equations

$$\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \boldsymbol{0}, \text{ in } \Omega$$
$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \boldsymbol{0}, \text{ in } \Omega$$
$$\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r = \boldsymbol{0}, \text{ on } \Gamma_N$$
$$\boldsymbol{d} - \overline{\boldsymbol{d}} = \boldsymbol{0}, \text{ on } \Gamma_D$$
$$\boldsymbol{r} - n\boldsymbol{H}\boldsymbol{\sigma}^r = \boldsymbol{0}, \text{ on } \Gamma_D$$

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Note that, besides the equations already obtained when using the 2-field Hellinger-Reissner variational principle, an additional equation representing the balance between reactions and internal forces on the Dirichlet boundary Γ_D has also been obtained.

The second-order Gâteaux derivative of Π^1_{HR} reads

$$\Delta(\delta\Pi^1_{HR}) = \int_0^L [\frac{\partial^2 W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r\partial\boldsymbol{\sigma}^r}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} - \boldsymbol{\sigma}^r \cdot \Delta(\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d})]\, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot \delta\boldsymbol{r}]_{\Gamma_D} + [\Delta\boldsymbol{r} \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

which, using equations (10), (13) and (20), can be rewritten as

$$\Delta(\delta\Pi^1_{HR}) = \int_0^L [\boldsymbol{D}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - (\mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot (\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d}) - (\Delta_c\delta\boldsymbol{d}) \cdot \boldsymbol{K}_G^{\sigma^r}(\Delta_c\Delta\boldsymbol{d})]\, d\Omega$$
$$+ [\Delta\boldsymbol{d} \cdot \boldsymbol{r}]_{\Gamma_D} + [\boldsymbol{r} \cdot \delta\boldsymbol{d}]_{\Gamma_D}$$

9.6. (2+2)-Field Principle of Hellinger-Reissner Type

Relaxing the continuity requirement of the displacements along the interelement and Neumann boundaries in the (2+1)-field Hellinger-Reissner energy by means of the Lagrange multipliers $\boldsymbol{q}^\Gamma$, a generalized hybrid form of the 2-field (complementary) Hellinger-Reissner energy can be obtained. It consists in a (2+2)-field Hellinger-Reissner energy $\Pi_{HR} : \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega) \times \mathcal{R}^6(\Gamma) \times \mathcal{R}^6(\Gamma) \to \mathcal{R}$ defined as

$$\Pi^2_{HR}(\boldsymbol{\sigma}^r, \boldsymbol{d}, \boldsymbol{d}^\Gamma, \boldsymbol{q}^\Gamma) = \int_\Omega [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \tilde{\boldsymbol{q}} \cdot \boldsymbol{d}]\, d\Omega + [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}^\Gamma]_{\Gamma_N} + [\boldsymbol{q}^\Gamma \cdot (\boldsymbol{d}^\Gamma - \overline{\boldsymbol{d}})]_{\Gamma_D} + [\boldsymbol{q}^\Gamma \cdot (\boldsymbol{d} - \boldsymbol{d}^\Gamma)]_\Gamma$$

and its first-order Gâteaux derivative at $(\boldsymbol{\sigma}^r, \boldsymbol{d}, \boldsymbol{d}^\Gamma, \boldsymbol{q}^\Gamma)$ in the direction of $(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d}, \delta\boldsymbol{d}^\Gamma, \delta\boldsymbol{q}^\Gamma) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega) \times \mathcal{R}^6(\Gamma) \times \mathcal{R}^6(\Gamma)$ reads

$$\delta\Pi^2_{HR} = \int_\Omega [\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} + \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d}]\, d\Omega$$
$$+ [(\boldsymbol{d} - \boldsymbol{d}^\Gamma) \cdot \delta\boldsymbol{q}^\Gamma]_\Gamma + [\boldsymbol{q}^\Gamma \cdot (\delta\boldsymbol{d} - \delta\boldsymbol{d}^\Gamma)]_\Gamma + [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}^\Gamma]_{\Gamma_N} + [(\boldsymbol{d}^\Gamma - \overline{\boldsymbol{d}}) \cdot \delta\boldsymbol{q}^\Gamma]_{\Gamma_D} + [\boldsymbol{q}^\Gamma \cdot \delta\boldsymbol{d}^\Gamma]_{\Gamma_D}$$

which, after integrating by parts, leads to

$$\delta\Pi^2_{HR} = \int_\Omega [(\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{\sigma}^r + (\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}) \cdot \delta\boldsymbol{d}]\, d\Omega +$$
$$+ [(\boldsymbol{q}^\Gamma - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_\Gamma + [(\boldsymbol{d} - \boldsymbol{d}^\Gamma) \cdot \delta\boldsymbol{q}^\Gamma]_\Gamma + [(\overline{\boldsymbol{q}} - \boldsymbol{q}^\Gamma) \cdot \delta\boldsymbol{d}^\Gamma]_{\Gamma_N} + [(\boldsymbol{d}^\Gamma - \overline{\boldsymbol{d}}) \cdot \delta\boldsymbol{q}^\Gamma]_{\Gamma_D}$$

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Finally, it can be easily shown that, requiring $\delta\Pi^2_{HR}$ to vanish for independent variations of $\delta\boldsymbol{\sigma}^r$, $\delta\boldsymbol{d}$, $\delta\boldsymbol{d}^\Gamma$ and $\delta\boldsymbol{q}^\Gamma$, the following Euler-Lagrange equations are obtained

$$\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \mathbf{0}, \text{ in } \Omega$$

$$\mathcal{T}^r_e(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \mathbf{0}, \text{ in } \Omega$$

$$\overline{\boldsymbol{q}} - \boldsymbol{q}^\Gamma = \mathbf{0}, \text{ on } \Gamma_N$$

$$\boldsymbol{d}^\Gamma - \overline{\boldsymbol{d}} = \mathbf{0}, \text{ on } \Gamma_D$$

$$\boldsymbol{q}^\Gamma - n\boldsymbol{H}\boldsymbol{\sigma}^r = \mathbf{0}, \text{ on } \Gamma$$

$$\boldsymbol{d} - \boldsymbol{d}^\Gamma = \mathbf{0}, \text{ on } \Gamma$$

Besides the equations obtained using the (2+1)-field variational principle of Hellinger-Reissner type, additional equations imposing the continuity of displacements across Γ are also present. The Lagrange multipliers $\boldsymbol{q}^\Gamma$ represent spatial nodal forces, whereas vector $\boldsymbol{d}^\Gamma$ collects the nodal displacements defined on Γ.

The second-order Gâteaux derivative of Π^2_{HR} reads

$$\Delta(\delta\Pi^2_{HR}) = \int_0^L [\frac{\partial^2 W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r\partial\boldsymbol{\sigma}^r}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot \mathcal{T}^r_c(\boldsymbol{d})\delta\boldsymbol{d} - \boldsymbol{\sigma}^r \cdot \Delta(\mathcal{T}^r_c(\boldsymbol{d})\delta\boldsymbol{d})]\, d\Omega$$
$$+ [(\Delta\boldsymbol{d} - \Delta\boldsymbol{d}^\Gamma) \cdot \delta\boldsymbol{q}^\Gamma]_\Gamma + [\Delta\boldsymbol{q}^\Gamma \cdot (\delta\boldsymbol{d} - \delta\boldsymbol{d}^\Gamma)]_\Gamma + [\Delta\boldsymbol{d}^\Gamma \cdot \delta\boldsymbol{q}^\Gamma]_{\Gamma_D} + [\Delta\boldsymbol{q}^\Gamma \cdot \delta\boldsymbol{d}^\Gamma]_{\Gamma_D}$$

which, using equations (10), (13) and (20), can be rewritten as

$$\Delta(\delta\Pi^2_{HR}) = \int_0^L [\boldsymbol{D}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - (\mathcal{T}^r_c(\boldsymbol{d})\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot (\mathcal{T}^r_c(\boldsymbol{d})\delta\boldsymbol{d}) - (\boldsymbol{\Delta}_c\delta\boldsymbol{d}) \cdot \boldsymbol{K}^{\sigma^r}_G(\boldsymbol{\Delta}_c\Delta\boldsymbol{d})]\, d\Omega$$
$$+ [(\Delta\boldsymbol{d} - \Delta\boldsymbol{d}^\Gamma) \cdot \delta\boldsymbol{q}^\Gamma]_\Gamma + [\Delta\boldsymbol{q}^\Gamma \cdot (\delta\boldsymbol{d} - \delta\boldsymbol{d}^\Gamma)]_\Gamma + [\Delta\boldsymbol{d}^\Gamma \cdot \delta\boldsymbol{q}^\Gamma]_{\Gamma_D} + [\Delta\boldsymbol{q}^\Gamma \cdot \delta\boldsymbol{d}^\Gamma]_{\Gamma_D}$$

9.7. 2-Field Principle of Total Complementary Energy

Although in the framework of the 2-field variational principle of Hellinger-Reissner type the equilibrium conditions have been obtained as its Euler-Lagrange equations, they can instead be considered as subsidiary conditions required to be satisfied *a priori*. Accordingly, assuming that equilibrium holds, when subjecting (35) to the equilibrium equations by means of the appropriate Lagrange multipliers, the following augmented Lagrangian is obtained

$$L_c = \int_\Omega [W_e(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \tilde{\boldsymbol{q}} \cdot \boldsymbol{d}]\, d\Omega - \int_\Omega (\mathcal{T}^r_e(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}) \cdot \boldsymbol{d}\, d\Omega$$
$$+ [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}]_{\Gamma_N} - [(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \boldsymbol{d}]_{\Gamma_N} + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot (\boldsymbol{d} - \overline{\boldsymbol{d}})]_{\Gamma_D}$$

which, after integrating by parts, leads to the 2-field complementary energy $\Pi_c : \mathcal{U}_s(\Omega) \rightarrow \mathcal{R}$ defined as

$$\Pi_c(\boldsymbol{\sigma}^r, \boldsymbol{d}) = \int_\Omega [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \boldsymbol{\sigma}^r \cdot \mathcal{T}^r_c(\boldsymbol{d})\boldsymbol{d}]\, d\Omega - [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot \overline{\boldsymbol{d}}]_{\Gamma_D} \tag{40}$$

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with $\mathcal{U}_s$ representing the following *statically admissible space*

$$\mathcal{U}_s = \{(\boldsymbol{\sigma}^r, \boldsymbol{d}) \in (\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega))| \; \mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \boldsymbol{0} \text{ in } \Omega \text{ and } n\boldsymbol{H}\boldsymbol{\sigma}^r - \overline{\boldsymbol{q}} = \boldsymbol{0} \text{ on } \Gamma_N\}$$

Note that, besides stress fields, the present total complementary energy functional depends also on displacement fields. Its mixed character is in sharp contrast with the total complementary energy functionals for fully linear problems, which can be regarded as pure principles as they only depend on stress fields.

The idea of constructing complementary energy-based principles for geometrically nonlinear theories, written only in terms of stresses in analogy to potential energy-based principles, has deserved the attention of several renowned mechanicians since the seventies, such as (Koiter, 1975), (Washizu, 1982) and (Reissner, 1987). However, as pointed out by (Reissner, 1987), the fundamental issue one could face up on attempting to formulate a pure complementary energy principle for a geometrically nonlinear problem is the coupling of stresses and deformations inherent to its corresponding set of equilibrium equations.

It can be shown that the pair $(\boldsymbol{\sigma}^r, \boldsymbol{d}) \in \mathcal{U}_s$ is a solution of the boundary-value problem if, and only if, the following condition holds

$$\delta\Pi_c = 0, \;\; \forall(\delta\boldsymbol{\sigma}^r, \boldsymbol{d}) \in \mathcal{V}_s$$

with $\mathcal{V}_s$ representing a *homogeneous statically admissible space* defined by

$$\mathcal{V}_s = \{(\delta\boldsymbol{\sigma}^r, \boldsymbol{d}) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)| \; \mathcal{T}_e^r(\boldsymbol{d})\delta\boldsymbol{\sigma}^r = \boldsymbol{0} \text{ in } \Omega \text{ and } n\boldsymbol{H}\delta\boldsymbol{\sigma}^r = \boldsymbol{0} \text{ on } \Gamma_N\}$$

In fact, taking the first-order Gâteaux derivative of Π_c in the direction of $(\delta\boldsymbol{\sigma}^r, \boldsymbol{d})$ yields

$$\delta\Pi_c = \int_\Omega [\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d})] \cdot \delta\boldsymbol{\sigma}^r \; d\Omega + \int_\Omega \mathcal{T}_c^r(\boldsymbol{d})\boldsymbol{d} \cdot \delta\boldsymbol{\sigma}^r \; d\Omega - [\overline{\boldsymbol{d}} \cdot n\boldsymbol{H}\delta\boldsymbol{\sigma}^r]_{\Gamma_D}$$

After integrating by parts the equation above and recalling that $(\delta\boldsymbol{\sigma}^r, \boldsymbol{d})$ lies in the statically admissible space $\mathcal{V}_s$, *i.e.*, the following conditions hold

$$\mathcal{T}_e^r(\boldsymbol{d})\delta\boldsymbol{\sigma}^r = \boldsymbol{0}, \text{ in } \Omega$$
$$n\boldsymbol{H}\delta\boldsymbol{\sigma}^r = \boldsymbol{0}, \text{ on } \Gamma_N$$

it is finally obtained that

$$\delta\Pi_c = \int_\Omega [\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d})] \cdot \delta\boldsymbol{\sigma}^r \; d\Omega + [(\boldsymbol{d} - \overline{\boldsymbol{d}}) \cdot n\boldsymbol{H}\delta\boldsymbol{\sigma}^r]_{\Gamma_D}$$

Hence, requiring $\delta\Pi_c$ to vanish for independent pairs $(\delta\boldsymbol{\sigma}^r, \boldsymbol{d}) \in \mathcal{V}_s$ and using equation (19), the following Euler-Lagrange equations are obtained

$$\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \boldsymbol{0}, \text{ in } \Omega$$
$$\boldsymbol{d} - \overline{\boldsymbol{d}} = \boldsymbol{0}, \text{ on } \Gamma_D$$

which clearly correspond to the compatibility equations of the boundary-value problem.

Therefore, it can be concluded that a generalized displacement vector $\boldsymbol{d}$ represents a kinematically admissible field if, and only if, its total complementary energy takes a stationary value for all statically admissible stress-resultant fields.

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It can be easily shown that, on taking the variation of Π_c in the direction of $(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d})$, rather than $(\delta\boldsymbol{\sigma}^r, \boldsymbol{d})$, as it has been performed, leads exactly to the same result.

The second-order Gâteaux derivative of Π_c reads

$$\Delta(\delta\Pi_c) = \int_0^L [\frac{\partial^2 W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r\partial\boldsymbol{\sigma}^r}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r + \Delta(\mathcal{T}_c^r(\boldsymbol{d})\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r] \, d\Omega - [\overline{\boldsymbol{d}} \cdot n\Delta\boldsymbol{H}\delta\boldsymbol{\sigma}^r]_{\Gamma_D}$$

which, using equations (10), (13) and (20), can be rewritten as

$$\Delta(\delta\Pi_c) = \int_0^L [\boldsymbol{D}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - (\mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r + (\Delta_c\delta\boldsymbol{d}) \cdot \boldsymbol{K}_G^{\delta\sigma^r}(\Delta_c\Delta\boldsymbol{d})] \, d\Omega - [\overline{\boldsymbol{d}} \cdot n\boldsymbol{H}_d^{\delta\sigma^r}\delta\boldsymbol{\sigma}^r]_{\Gamma_D}$$

with $\boldsymbol{K}_G^{\delta\sigma^r} = \boldsymbol{K}_G(\boldsymbol{d}, \delta\boldsymbol{\sigma}^r)$ and $\boldsymbol{H}_d^{\delta\sigma^r} = \boldsymbol{H}_d(\boldsymbol{\theta}, \delta\boldsymbol{\sigma}^r)$.

Note that, the main advantage of using a complementary variational principle is that it may be employed as the basis for the development of a finite element model capable of producing statically admissible approximate solutions, *i.e.*, solutions satisfying in strong form all the equilibrium equations of the problem. This is of utmost importance in engineering design as well as in total contrast with all the other variational principles from which the derivation of traditional displacement/rotation-based models or hybrid-mixed models lead to approximate stress fields which may be highly erroneous.

9.8. (2+1)-Field Principle of Complementary Energy

If the equilibrium equations are assumed to be relaxed by means of the Lagrangian multiplier method within the framework of the principle of total complementary energy presented above, the following augmented Lagrangian is obtained

$$\Pi_c^g = \int_\Omega [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\boldsymbol{d}) + \mathcal{T}_c^r(\boldsymbol{d})\boldsymbol{d} \cdot \boldsymbol{\sigma}^r] \, d\Omega + \int_\Omega [\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}] \cdot \boldsymbol{d} \, d\Omega$$
$$+ [(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \boldsymbol{d}^\Gamma]_{\Gamma_N} - [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot \overline{\boldsymbol{d}}]_{\Gamma_D}$$

with $\boldsymbol{d}^\Gamma$ representing a Lagrangian multiplier vector defined on Γ_N. This Lagrangian, after integrating by parts, leads to a generalized complementary energy, $\Pi_c^g : \chi(\Omega) \to \mathcal{R}$, given by

$$\Pi_c^g = \int_\Omega [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} \cdot \boldsymbol{d}] \, d\Omega + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot (\boldsymbol{d} - \overline{\boldsymbol{d}})]_{\Gamma_D} + [\overline{\boldsymbol{q}} \cdot \boldsymbol{d}^\Gamma]_{\Gamma_N} + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot (\boldsymbol{d} - \boldsymbol{d}^\Gamma)]_{\Gamma_N} \quad (43)$$

where the functions in class χ consist of pairs $(\boldsymbol{\sigma}^r, \boldsymbol{d}) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega)$ and a real-valued vector $\boldsymbol{d}^\Gamma$ defined on Γ_N.

The first-order Gâteaux derivative of Π_c^g defined at $(\boldsymbol{\sigma}^r, \boldsymbol{d}, \boldsymbol{d}^\Gamma)$ in the direction of $(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d}, \delta\boldsymbol{d}^\Gamma)$ writes

$$\delta\Pi_c^g = \int_\Omega [\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} \cdot \delta\boldsymbol{\sigma}^r - \delta\boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \boldsymbol{\sigma}^r - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) \cdot \delta\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} \cdot \delta\boldsymbol{d}] \, d\Omega$$
$$+ [n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot (\boldsymbol{d} - \boldsymbol{d}^\Gamma)]_{\Gamma_N} + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot (\delta\boldsymbol{d} - \delta\boldsymbol{d}^\Gamma)]_{\Gamma_N}$$
$$+ [n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot (\boldsymbol{d} - \overline{\boldsymbol{d}})]_{\Gamma_D} + [n\boldsymbol{H}\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{d}]_{\Gamma_D} + [\overline{\boldsymbol{q}} \cdot \delta\boldsymbol{d}^\Gamma]_{\Gamma_N} \quad (44)$$

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for arbitrary $(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d}, \delta\boldsymbol{d}^\Gamma)$ in $\chi(\Omega)$.

The variational problem (weak form) is to find $(\boldsymbol{\sigma}^r, \boldsymbol{d}, \boldsymbol{d}^\Gamma)$ such that

$$\delta\Pi_c^g = 0, \ \forall(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d}, \delta\boldsymbol{d}^\Gamma) \in \chi(\Omega) \tag{45}$$

On insertion of (20) and after integrating by parts yields

$$\delta\Pi_c^g = \int_\Omega [(\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d})) \cdot \delta\boldsymbol{\sigma}^r + (\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}}) \cdot \delta\boldsymbol{d}] \, d\Omega$$
$$+ [n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot (\boldsymbol{d} - \boldsymbol{d}^\Gamma)]_{\Gamma_N} + [(\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}^\Gamma]_{\Gamma_N} + [n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot (\boldsymbol{d} - \overline{\boldsymbol{d}})]_{\Gamma_D}$$

which is formally equivalent to the following system of Euler-Lagrange equations

$$\mathcal{T}_e^r(\boldsymbol{d})\boldsymbol{\sigma}^r + \tilde{\boldsymbol{q}} = \boldsymbol{0} \text{ in } \Omega \tag{46a}$$

$$\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\boldsymbol{d}) = \boldsymbol{0} \text{ in } \Omega \tag{46b}$$

$$\overline{\boldsymbol{q}} - n\boldsymbol{H}\boldsymbol{\sigma}^r = \boldsymbol{0} \text{ in } \Gamma_N \tag{46c}$$

$$\boldsymbol{d} - \boldsymbol{d}^\Gamma = \boldsymbol{0} \text{ in } \Gamma_N \tag{46d}$$

$$\boldsymbol{d} - \overline{\boldsymbol{d}} = \boldsymbol{0} \text{ on } \Gamma_D \tag{46e}$$

Clearly, $\boldsymbol{d}^\Gamma$ represents a displacement vector associated with the continuity condition of $\boldsymbol{\sigma}^r$ across Γ_N, which is implied by condition (46c). Since any solution of the geometrically exact beam boundary-value problem also satisfies (46), the variational principle (45) can be used as a basis for the approximations of (46).

It is fundamental to note that the present principle allows us to construct independent approximations of $\boldsymbol{\sigma}^r$ and $\boldsymbol{d}$. Furthermore, not only the compatibility but also the equilibrium equations are imposed in weak form. However, if approximations lying in the statically admissible space are employed, the equilibrium turns out to be fulfilled in strong form and statically admissible solutions are obtained. The principle can be seen, therefore, as a generalization of the principle of stationary total complementary energy. Further developments of this principle on the basis of an equilibrium finite element formulation will be presented in a companion paper (Santos et al., 2009).

The second-order Gâteaux derivative of the generalized complementary energy Π_c^g defined at point $(\boldsymbol{\sigma}^r, \boldsymbol{d}, \boldsymbol{d}^\Gamma)$ in the directions of admissible variations $(\delta\boldsymbol{\sigma}^r, \delta\boldsymbol{d}, \delta\boldsymbol{d}^\Gamma)$ and $(\Delta\boldsymbol{\sigma}^r, \Delta\boldsymbol{d}, \Delta\boldsymbol{d}^\Gamma)$, *i.e.*, lying in $\chi(\Omega)$, leads to

$$\Delta(\delta\Pi_c^g) = \int_\Omega [\frac{\partial^2 W_c(\boldsymbol{\sigma}^r)}{\partial\boldsymbol{\sigma}^r\partial\boldsymbol{\sigma}^r}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - \mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d} \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} - \boldsymbol{\sigma}^r \cdot \Delta(\mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d})] \, d\Omega$$
$$+ [n\Delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot (\delta\boldsymbol{d} - \delta\boldsymbol{d}^\Gamma)]_{\Gamma_N} + [n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot (\Delta\boldsymbol{d} - \Delta\boldsymbol{d}^\Gamma)]_{\Gamma_N} + [n\Delta(\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)) \cdot (\boldsymbol{d} - \boldsymbol{d}^\Gamma)]_{\Gamma_N}$$
$$+ [n\Delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \delta\boldsymbol{d}]_{\Gamma_D} + [n\delta(\boldsymbol{H}\boldsymbol{\sigma}^r) \cdot \Delta\boldsymbol{d}]_{\Gamma_D} + [n\Delta(\delta(\boldsymbol{H}\boldsymbol{\sigma}^r)) \cdot \boldsymbol{d}]_{\Gamma_D}$$

which, using equations (10), (13) and (20), can be rewritten as

$$\Delta(\delta\Pi_c^g) = \int_\Omega \left[\boldsymbol{D}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\sigma}^r - \mathcal{T}_c^r(\boldsymbol{d})\Delta\boldsymbol{d} \cdot \delta\boldsymbol{\sigma}^r - \Delta\boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\boldsymbol{d})\delta\boldsymbol{d} - (\boldsymbol{\Delta}_c\delta\boldsymbol{d}) \cdot \boldsymbol{K}_G^{\sigma^r}(\boldsymbol{\Delta}_c\Delta\boldsymbol{d}) \right] d\Omega$$
$$+ [n(\boldsymbol{H}\Delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\sigma^r}\Delta\boldsymbol{d}) \cdot (\delta\boldsymbol{d} - \delta\boldsymbol{d}^\Gamma)]_{\Gamma_N} + [n(\boldsymbol{H}\delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\sigma^r}\delta\boldsymbol{d}) \cdot (\Delta\boldsymbol{d} - \Delta\boldsymbol{d}^\Gamma)]_{\Gamma_N}$$
$$+ [n(\boldsymbol{G}_{d\sigma}^\Gamma\Delta\boldsymbol{d} \cdot \delta\boldsymbol{\sigma}^r + \boldsymbol{G}_{dd}^\Gamma\Delta\boldsymbol{d} \cdot \delta\boldsymbol{d} + \boldsymbol{G}_{\sigma d}^\Gamma\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{d})]_{\Gamma_N}$$
$$+ [n(\boldsymbol{H}\Delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\sigma^r}\Delta\boldsymbol{d}) \cdot \delta\boldsymbol{d}]_{\Gamma_D} + [n(\boldsymbol{H}\delta\boldsymbol{\sigma}^r + \boldsymbol{H}_d^{\sigma^r}\delta\boldsymbol{d}) \cdot \Delta\boldsymbol{d}]_{\Gamma_D} + [n(\overline{\boldsymbol{G}}_{d\sigma}\Delta\boldsymbol{d} \cdot \delta\boldsymbol{\sigma}^r + \overline{\boldsymbol{G}}_{dd}\Delta\boldsymbol{d} \cdot \delta\boldsymbol{d} + \overline{\boldsymbol{G}}_{\sigma d}\Delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{d})]_{\Gamma_D}$$

with

$$\boldsymbol{G}_{d\sigma}^\Gamma = \frac{\partial(\boldsymbol{H}^T(\boldsymbol{d} - \boldsymbol{d}^\Gamma))}{\partial\boldsymbol{d}}$$
$$\boldsymbol{G}_{\sigma d}^\Gamma = \frac{\partial((\boldsymbol{H}_d^{\sigma^r})^T(\boldsymbol{d} - \boldsymbol{d}^\Gamma))}{\partial\boldsymbol{\sigma}^r}$$
$$\boldsymbol{G}_{dd}^\Gamma = \frac{\partial((\boldsymbol{H}_d^{\sigma^r})^T(\boldsymbol{d} - \boldsymbol{d}^\Gamma))}{\partial\boldsymbol{d}}$$

and

$$\overline{\boldsymbol{G}}_{d\sigma} = \frac{\partial(\boldsymbol{H}^T(\boldsymbol{d} - \overline{\boldsymbol{d}}))}{\partial\boldsymbol{d}}$$
$$\overline{\boldsymbol{G}}_{\sigma d} = \frac{\partial((\boldsymbol{H}_d^{\sigma^r})^T(\boldsymbol{d} - \overline{\boldsymbol{d}}))}{\partial\boldsymbol{\sigma}^r}$$
$$\overline{\boldsymbol{G}}_{dd} = \frac{\partial((\boldsymbol{H}_d^{\sigma^r})^T(\boldsymbol{d} - \overline{\boldsymbol{d}}))}{\partial\boldsymbol{d}}$$

representing indeed geometric tangent operators defined on the boundaries.

10. Conclusions

Various novel hybrid/multi-field stationary principles, which can provide the basis for the variationally consistent development of finite element formulations for geometrically exact (Reissner-Simo) three-dimensional beam models have been established. Their corresponding weak and linearized weak forms have also been presented. The specialization of these principles to other (two or three-dimensional) geometrically nonlinear beam models, such as for instance the geometrically exact Euler-Bernoulli beam model, and several other lower-order geometrically nonlinear beam models can be easily carried out.

Except for the case of the 2-field principle of total complementary energy, in which the equilibrium conditions have to be assumed as their subsidiary conditions, the remaining novel principles are suitable for the construction of either mixed or hybrid-mixed finite element approaches in which neither compatibility nor equilibrium conditions are required to be satisfied *a priori*.

The (2+1)-field principle of complementary energy can be used for the derivation of a finite element formulation providing statically admissible solutions, a new result that is possible due to the generality of that principle.

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By means of the Lagrangian multiplier method, several alternative hybrid/multi-field variational principles with different subsidiary conditions and Euler-Lagrange equations can be formulated departing from the principles presented in this paper.

The main features of the principles and their relationships from both theoretical and computational standpoints have been highlighted.

Acknowledgements

Appendix

The material part of the tangent operator K_M is given by

$$K_M = \begin{bmatrix} QC_{n^r}Q^T & O & QC_{n^r}Q^TZ'\Gamma \\ O & \Gamma^TQC_{m^r}Q^T\Gamma & \Gamma^TQC_{m^r}Q^T\Gamma' \\ -\Gamma^TZ'QC_{n^r}Q^T & \Gamma'^TQC_{m^r}Q^T\Gamma & K_M^{\theta\theta} \end{bmatrix}$$

where

$$K_M^{\theta\theta} = -\Gamma^TZ'QC_{n^r}Q^TZ'\Gamma + \Gamma'^TQC_{m^r}Q^T\Gamma'$$

As for the geometric part of the tangent operator K_G, it reads

$$K_G = \begin{bmatrix} O & O & -N\Gamma \\ O & O & V^T(\theta, m) \\ \Gamma^TN & V(\theta, m) & K_G^{\theta\theta} \end{bmatrix}$$

where $K_G^{\theta\theta}$ is given by

$$K_G^{\theta\theta} = \Gamma^TZ'N\Gamma - V(\theta, z' \times n) + V_\theta(\theta, \theta', m) - W^T(\theta, \theta')M\Gamma$$

with N and M representing the skew-symmetric tensors associated to vectors n and m, respectively. In this operator, while W is defined as

$$W(\theta, \theta') = h_2(\theta)\Theta' + h_3(\theta)(\Theta\Theta' + \Theta'\Theta) + h_4(\theta)(\theta \cdot \theta')\Theta + h_5(\theta)(\theta \cdot \theta')\Theta^2$$

with h_4 and h_5 the following trigonometric functions

$$h_4(\theta) = \frac{\frac{dh_2(\theta)}{d\theta}}{\theta} = \frac{h_1(\theta) - 2h_2(\theta)}{\theta^2} = \frac{\theta\sin\theta - 2(1-\cos\theta)}{\theta^4}$$

$$h_5(\theta) = \frac{\frac{dh_3(\theta)}{d\theta}}{\theta} = \frac{h_2(\theta) - 3h_3(\theta)}{\theta^2} = \frac{3\sin\theta - \theta(2+\cos\theta)}{\theta^5}$$

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V is defined as

$$V(\boldsymbol{\theta}, \boldsymbol{m}) = h_2(\theta)\boldsymbol{M} + h_3(\theta)(\boldsymbol{M}\boldsymbol{\Theta} - 2\boldsymbol{\Theta}\boldsymbol{M}) - h_4(\theta)(\boldsymbol{\Theta}\boldsymbol{m} \otimes \boldsymbol{\theta}) + h_5(\theta)(\boldsymbol{\Theta}^2 \boldsymbol{m} \otimes \boldsymbol{\theta})$$

and $\boldsymbol{V}_\theta$ comes out as

$$V_\theta(\boldsymbol{\theta}, \boldsymbol{\theta}', \boldsymbol{m}) = h_3(\theta)(\boldsymbol{M}\boldsymbol{\Theta}' - 2\boldsymbol{\Theta}'\boldsymbol{M}) - h_4(\theta)(\boldsymbol{\Theta}'\boldsymbol{m} \otimes \boldsymbol{\theta} + \boldsymbol{\Theta}\boldsymbol{m} \otimes \boldsymbol{\theta}' - (\boldsymbol{\theta} \cdot \boldsymbol{\theta}')\boldsymbol{M}) +$$
$$h_5(\theta)((\boldsymbol{\Theta}'\boldsymbol{\Theta} + \boldsymbol{\Theta}\boldsymbol{\Theta}')\boldsymbol{m} \otimes \boldsymbol{\theta} + \boldsymbol{\Theta}^2 \boldsymbol{m} \otimes \boldsymbol{\theta}' + (\boldsymbol{\theta} \cdot \boldsymbol{\theta}')(\boldsymbol{M}\boldsymbol{\Theta} - 2\boldsymbol{\Theta}\boldsymbol{M})) -$$
$$h_6(\theta)(\boldsymbol{\theta} \cdot \boldsymbol{\theta}')(\boldsymbol{\Theta}\boldsymbol{m} \otimes \boldsymbol{\theta}) + h_7(\theta)(\boldsymbol{\theta} \cdot \boldsymbol{\theta}')(\boldsymbol{\Theta}^2 \boldsymbol{m} \otimes \boldsymbol{\theta})$$

with h_6 and h_7 the following trigonometric functions

$$h_6(\theta) = \frac{\frac{dh_4(\theta)}{d\theta}}{\theta} = \frac{h_3(\theta) - h_2(\theta) - 4h_4(\theta)}{\theta^2}$$
$$h_7(\theta) = \frac{\frac{dh_5(\theta)}{d\theta}}{\theta} = \frac{h_4(\theta) - 5h_5(\theta)}{\theta^2}$$

Operator $\boldsymbol{H}_d$ is given by

$$H_d = \begin{bmatrix} \boldsymbol{O} & -\boldsymbol{N}\boldsymbol{\Gamma} \\ \boldsymbol{O} & \boldsymbol{V}^T(\boldsymbol{\theta}, \boldsymbol{m}) \end{bmatrix}$$

Finally, the geometric tangent operators defined on the boundaries are defined as

$$G_{d\sigma}^\Gamma = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{Q}^T(\boldsymbol{U} - \boldsymbol{U}^\Gamma)\boldsymbol{\Gamma} \\ \boldsymbol{O} & \boldsymbol{V}(\boldsymbol{\theta}, \boldsymbol{\theta} - \boldsymbol{\theta}^\Gamma) \end{bmatrix}$$

$$G_{\sigma d}^\Gamma = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ -\boldsymbol{\Gamma}^T(\boldsymbol{U} - \boldsymbol{U}^\Gamma)\boldsymbol{Q} & \boldsymbol{W}^T(\boldsymbol{\theta}, \boldsymbol{\theta} - \boldsymbol{\theta}^\Gamma)\boldsymbol{Q} \end{bmatrix}$$

$$G_{dd}^\Gamma = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{V}(\boldsymbol{\theta}, \boldsymbol{n} \times (\boldsymbol{u} - \boldsymbol{u}^\Gamma)) + \boldsymbol{\Gamma}^T(\boldsymbol{U} - \boldsymbol{U}^\Gamma)\boldsymbol{N}\boldsymbol{\Gamma} + \boldsymbol{V}_\theta(\boldsymbol{\theta}, \boldsymbol{\theta} - \boldsymbol{\theta}^\Gamma, \boldsymbol{m}) - \boldsymbol{W}^T(\boldsymbol{\theta}, \boldsymbol{\theta} - \boldsymbol{\theta}^\Gamma)\boldsymbol{M}\boldsymbol{\Gamma} \end{bmatrix}$$

and

$$\bar{G}_{d\sigma} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{Q}^T(\boldsymbol{U} - \bar{\boldsymbol{U}})\boldsymbol{\Gamma} \\ \boldsymbol{O} & \boldsymbol{V}(\boldsymbol{\theta}, \boldsymbol{\theta} - \bar{\boldsymbol{\theta}}) \end{bmatrix}$$

$$\bar{G}_{\sigma d} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ -\boldsymbol{\Gamma}^T(\boldsymbol{U} - \bar{\boldsymbol{U}})\boldsymbol{Q} & \boldsymbol{W}^T(\boldsymbol{\theta}, \boldsymbol{\theta} - \bar{\boldsymbol{\theta}})\boldsymbol{Q} \end{bmatrix}$$

$$\bar{G}_{dd} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{V}(\boldsymbol{\theta}, \boldsymbol{n} \times (\boldsymbol{u} - \bar{\boldsymbol{u}})) + \boldsymbol{\Gamma}^T(\boldsymbol{U} - \bar{\boldsymbol{U}})\boldsymbol{N}\boldsymbol{\Gamma} + \boldsymbol{V}_\theta(\boldsymbol{\theta}, \boldsymbol{\theta} - \bar{\boldsymbol{\theta}}, \boldsymbol{m}) - \boldsymbol{W}^T(\boldsymbol{\theta}, \boldsymbol{\theta} - \bar{\boldsymbol{\theta}})\boldsymbol{M}\boldsymbol{\Gamma} \end{bmatrix}$$

It is worth noting that $\boldsymbol{G}_{d\sigma}^\Gamma = (\boldsymbol{G}_{\sigma d}^\Gamma)^T$ and $\bar{\boldsymbol{G}}_{d\sigma} = (\bar{\boldsymbol{G}}_{\sigma d})^T$.

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