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Hybrid Ćirić Type Graphic (Υ, Λ)-Contraction Mappings with Applications to Electric Circuit and Fractional Differential Equations

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Abstract: In this paper, we initiate the notion of Ćirić type rational graphic (Υ, Λ)-contraction pair mappings and provide some new related common fixed point results on partial b-metric spaces endowed with a directed graph G. We also give examples to illustrate our main results. Moreover, we present some applications on electric circuit equations and fractional differential equations.

Keywords: fixed point; a directed graph; Ćirić type graphic (Υ, Λ)-contraction pair of mappings; electric circuit equations; fractional differential equations

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1. Introduction and Preliminaries

The Banach principle [1] has been improved and generalized by several researchers for different kinds of contractions in various spaces. One of these generalizations corresponding to an (Υ, Λ)-contraction, has been established by [2]. Recently, Ameer et al. [3] introduced common fixed point results for generalized multivalued $(\alpha_K^*, \Upsilon, \Lambda)$-contractions in α_K-complete partial b-metric spaces. Ameer et al. [4,5] introduced common fixed point results for generalized multivalued (Υ, Λ)-contractions in complete metric, b-metric spaces. Ameer et al. [6] initiated the notion of rational $(\Upsilon, \Lambda, \Re)$-contractive pair of mappings (where $\Re$ is a binary relation) and established new common fixed point results for these mappings in complete metric spaces. On the other hand, Bakhtin [7] investigated the concept of b-metric spaces. Subsequently, Czerwik [8] initiated the study of fixed point results in b-metric spaces and proved an analogue of Banach's fixed point theorem. Matthews [9] gave the concept of a partial metric space and proved and Banach fixed point result. Shukla [10] extended the notion of a partial metric to a partial b-metric. Afterwards, numerous research articles have been dealt with fixed point theorems for various classes of single-valued and multi-valued operators in b-metric and partial b-metric spaces (see, for example, [3,11–26]). In this article, we shall investigate fixed points of Ćirić type rational graphic (Υ, Λ)-contraction pair mappings on partial b-metric spaces endowed with a directed graph G.

Bakhtin [7] and Czerwik [8] generalized the notion of a metric as follows:

Definition 1 ([7,8])**.** *Let M be a nonempty set and $s \geq 1$ be a real number. A mapping $d : M \times M \to [0, \infty)$ is said to be a b-metric if for all $\zeta_1, \zeta_2, \sigma \in M$,*

(b_1) $d(\zeta_1, \zeta_2) = 0$ *if and only if* $\zeta_1 = \zeta_2$;

(b_2) $d(\zeta_1, \zeta_2) = d(\zeta_2, \zeta_1)$;
(b_3) $d(\zeta_1, \zeta_2) \leq s[d(\zeta_1, \sigma) + d(\sigma, \zeta_2)]$.

The pair (M, d) is called a b-metric space (with coefficient s).

Matthews [9] generalized the notion of a metric as follows:

Definition 2 ([9]). *Let M be a nonempty set. A mapping $P : M \times M \to [0, \infty)$ is said to be a partial metric if for all $\zeta_1, \zeta_2, \sigma \in M$, P satisfies following axioms;*

(P_1) $P(\zeta_1, \zeta_1) = P(\zeta_1, \zeta_2) = P(\zeta_2, \zeta_2)$ *if and only if* $\zeta_1 = \zeta_2$;
(P_2) $P(\zeta_1, \zeta_1) \leq P(\zeta_1, \zeta_2)$;
(P_3) $P(\zeta_1, \zeta_2) = P(\zeta_2, \zeta_1)$;
(P_4) $P(\zeta_1, \zeta_2) \leq P(\zeta_1, \sigma) + P(\sigma, \zeta_2) - P(\sigma, \sigma)$.

The pair (M, P) is called a partial metric space.

Shukla [10] generalized the notion of a partial metric as follows:

Definition 3 ([10]). *Let M be a nonempty set and $s \geq 1$ a real number. A mapping $P_b : M \times M \to [0, \infty)$ is said to be a partial b-metric if for all $\zeta_1, \zeta_2, \sigma \in M$, P_b satisfies the following axioms:*

(P_1) $P_b(\zeta_1, \zeta_1) = P_b(\zeta_1, \zeta_2) = P_b(\zeta_2, \zeta_2)$ *if and only if* $\zeta_1 = \zeta_2$;
(P_2) $P_b(\zeta_1, \zeta_1) \leq P_b(\zeta_1, \zeta_2)$;
(P_3) $P_b(\zeta_1, \zeta_2) = P_b(\zeta_2, \zeta_1)$;
(P_4) $P_b(\zeta_1, \zeta_2) \leq s[P_b(\zeta_1, \sigma) + P_b(\sigma, \zeta_2)] - P_b(\sigma, \sigma)$.

The pair (M, P_b) is called a partial b-metric space (with coefficient s).

Remark 1. *The self distance $P_b(\zeta_1, \zeta_1)$, referring to the size or weight of ζ_1, is a feature used to describe the amount of information contained in M.*

Remark 2. *Obviously, every partial metric space is a partial b-metric space with coefficient $s = 1$, and every b-metric space is a partial b-metric space with zero self-distance. However, the converse of this fact need not hold.*

Definition 4 ([10]). *Let (M, P_b) be a partial b-metric space with coefficient $s \geq 1$. Let $\{\zeta_n\}$ be a sequence in M and $\zeta_1 \in M$. Then*

(i) $\{\zeta_n\}$ *is said to be convergent to* ζ^* *if* $\lim_{n\to\infty} P_b(\zeta_n, \zeta^*) = P_b(\zeta^*, \zeta^*)$.
(ii) $\{\zeta_n\}$ *is said to be Cauchy sequence if* $\lim_{n,m\to\infty} P_b(\zeta_n, \zeta_m)$ *exists and is finite.*
(iii) (M, P_b) *is said to be complete if every Cauchy sequence is convergent in* M.

Lemma 1 ([10]). *Let (ω, P_b, K) be a partial b-metric space.*

(1) Every Cauchy sequence in (ω, d_{P_b}) is also Cauchy in (ω, P_b, K) and vice versa;
(2) (ω, P_b, K) is complete if and only if (ω, d_{P_b}) is a complete metric space;
(3) The sequence $\{\zeta_n\}$ is convergent to some $v \in \omega$ if and only if

$$\lim_{n\to\infty} P_b(\zeta_n, v) = P_b(v, v) = \lim_{n,m\to\infty} P_b(\zeta_n, \zeta_m).$$

Denote a metric space by MS.

Definition 5 ([27]). *Let (M,d) be a MS. $T: M \to M$ is called an F-contraction self-mapping, if there exist $\tau > 0$ and $F \in F$ so that*

$$\forall \zeta, \eta \in M,\ d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F(d(T(\zeta), T(\eta))) \leq F(d(\zeta, \eta)),$$

where F is the family of functions $F: (0,\infty) \to (-\infty,\infty)$ such that

$(F1)$ F is strictly increasing;
$(F2)$ For each sequence $\{\alpha_n\}_{n=1}^{\infty} \subset (0,\infty)$,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \Longleftrightarrow \lim_{n\to\infty} \alpha_n = 0;$$

$(F3)$ There exists $\gamma \in (0,1)$ such that $\lim_{t\longrightarrow 0^+} t^{\gamma} F(t) = 0$.

Theorem 1 ([27]). *Let (M,d) be a complete MS and $T: M \to M$ be an F- contraction mapping. Then T possesses a unique fixed point $\zeta^* \in M$.*

Piri and Kumam [28] modified the set of functions $F \in F$.

Definition 6 ([28]). *Let (M,d) be a MS. $T: M \to M$ is said to be a F-contraction self-mapping if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\forall \zeta, \eta \in M, d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F(d(T(\zeta), T(\eta))) \leq F(d(\zeta, \eta)),$$

where $\mathcal{F}$ is the set of functions $F: (0,\infty) \to (-\infty,\infty)$ satisfying the following conditions:

$(F1)$ F is strictly increasing, i.e., for all $\zeta, \eta \in \mathbb{R}^+$ with $\zeta < \eta$, $F(\zeta) < F(\eta)$;
$(F2)$ For each positive real sequence $\{\alpha_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} F(\alpha_n) = -\infty \text{ if and only if } \lim_{n\to\infty} \alpha_n = 0;$$

$(F3)$ F is continuous.
On the other hand, recently Jleli and Samet [29,30] initiated the concept of θ-contractions.

Definition 7. *Let (M,d) be a MS. A mapping $T: M \to M$ is said to be a θ-contraction, if there exist $\theta \in \Theta$ and a real constant $k \in (0,1)$ such that*

$$\zeta, \eta \in M, d(T(\zeta), T(\eta)) \neq 0 \Longrightarrow \theta(d(T(\zeta), T(\eta))) \leq [\theta(d(\zeta, \eta))]^k,$$

where Θ is the set of functions $\theta: (0,\infty) \longrightarrow (1,\infty)$ such that:
$(\Theta 1)$ θ is non-decreasing;
$(\Theta 2)$ for each positive sequence $\{t_n\}$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+;$$

$(\Theta 3)$ there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\longrightarrow 0^+} \frac{\theta(t)-1}{t^r} = \ell$;
$(\Theta 4)$ θ is continuous.

The main result of Jleli and Samet [29] is

Theorem 2 ([29]). *Let (M,d) be a complete MS. Let $T: M \to M$ be a θ-contraction mapping. Then there exists a unique fixed point of T.*

As in [2], the family of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ verifies:
$(\Theta 1)'$ θ is non-decreasing;
$(\Theta 2)'$ for each positive sequence $\{t_n\}$, $\inf_{t_n \in (0,\infty)} \theta(t_n) = 1$;
$(\Theta 3)'$ θ is continuous, is denoted by Ξ.

Theorem 3 ([2]). *Let $T : \omega \to \omega$ be a self-mapping on the complete MS (ω, d). The following statements are equivalent:*
(i) T is a θ-contraction mapping with $\theta \in \Xi$;
(ii) T is an F-contraction mapping with $F \in \mathcal{F}$.

As in [31], a function $\Upsilon : (0, \infty) \longrightarrow (0, \infty)$ satisfies:
(*i*) Υ is monotone increasing, that is, $t_1 < t_2 \Longrightarrow \Upsilon(t_1) \leq \Upsilon(t_2)$;
(*ii*) $\lim_{n \to \infty} \Upsilon^n(t) = 0$ for all $t > 0$, where Υ^n stands for the nth iterate of Υ, is called a comparison function. Clearly, if Υ is a comparison function, then $\Upsilon(t) < t$ for each $t > 0$.

Lemma 2 ([2]). *Let $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ be a continuous non-decreasing function such that $\inf_{T \in (0,\infty)} \phi(T) = 0$. Let $\{t_k\}_k$ be a positive sequence. So*

$$\lim_{k \to \infty} \Lambda(t_k) = 0 \text{ if and only if } \lim_{k \to \infty} t_k = 0.$$

Example 1 ([31]). *The following functions* $\Upsilon : (0, \infty) \longrightarrow (0, \infty)$ *are comparison functions*:
(i) $\Upsilon(t) =$ *at with* $0 < a < 1$, *for each* $t > 0$*;*
(ii) $\Upsilon(t) = \frac{t}{t+1}$, *for each* $t > 0$.

Denote by Φ the set of functions $\Lambda : (0, \infty) \longrightarrow (0, \infty)$ verifying:
$(\Phi 1)$ Λ is non-decreasing;
$(\Phi 2)$ for each positive sequence $\{t_n\}$,

$$\lim_{n \to \infty} \Lambda(t_n) = 0 \text{ if and only if } \lim_{n \to \infty} t_n = 0;$$

$(\Phi 3)$ Λ is continuous. Liu et al. [2] initiated the concept of (Υ, Λ)-Suzuki contractions.

Definition 8. *Let (M, d) be a MS. A mapping $T : M \to M$ is said to be a (Υ, Λ)-Suzuki contraction, if there exist comparison functions Υ and $\Lambda \in \Phi$ such that, for all $\zeta, \eta \in M$ with $T(\zeta) \neq T(\eta)$,*

$$\frac{1}{2} d(\zeta, T(\zeta)) < d(\zeta, \eta) \Longrightarrow \Lambda(d(T(\zeta), T(\eta))) \leq \Upsilon[\Lambda(U(\zeta, \eta))],$$

where

$$U(\zeta, \eta) = \max\left\{ d(\zeta, \eta), d(\zeta, T(\zeta)), d(\eta, T(\eta)), \frac{d(\zeta, T(\eta)) + d(\eta, T(\zeta))}{2} \right\}.$$

Moreover, let (M, P_b) be a partial metric space, and Δ denotes the diagonal of $M \times M$. Let G be a directed graph, which has no parallel edges such that the set $V(G)$ of its vertices coincides with M, and $E(G) \subseteq M \times M$ contains all loops (i.e., $\Delta \subseteq E(G)$). Hence, G is identify by the pair $(V(G), E(G))$. Denote by G^{-1} the graph obtained from G by reversing the direction of its edges. That is,

$$E\left(G^{-1}\right) = \{(\zeta, \eta) \in M \times M : (\eta, \zeta) \in E(G)\}.$$

It is more adaptable to treat $\widetilde{G}$ a directed graph for which the set of its edges is symmetric. Under this convention, we have that

$$E(\widetilde{G}) = E(G) \cup E\left(G^{-1}\right).$$

In $V(G)$, we define the relation R in the following way: for $\zeta, \eta \in V(G)$, we have $\zeta R \eta$ if and only if there is a path in G from ζ to η. If G is such that $E(G)$ is symmetric, then for $\zeta \in V(G)$, the equivalence class $[\zeta]_{\widetilde{G}}$ in $V(G)$ defined by the relation R is $V(G_\zeta)$. Recall that if ϕ:$M \to M$ is an operator; then, by $Fix(\phi)$ we denote the set of all fixed points of ϕ. Let

$$M_\phi := \{\zeta \in M\} : (\zeta, \phi(\zeta)) \in E(G).$$

Property: A graph is said to satisfy property $(\mathbf{E}^*)$ if for any sequence $\{\zeta_n\}$ in $V(G)$ with $\zeta_n \to \zeta$ as $n \to \infty$, $(\zeta_n, \zeta_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ implies that there is a subsequence $\left\{\zeta_{n(k)}\right\}$ of $\{\zeta_n\}$ with an edge between $\zeta_{n(k)}$ and ζ for $k \in \mathbb{N}$. Throughout this paper, G is a weighted graph such that the weight of each vertex ζ is $P_b(\zeta, \eta)$, and the weight of each edge (ζ, η) is $P_b(\zeta, \eta)$. Since (M, P_b) is a partial b-metric space, the weight assigned to each vertex ζ need not to be zero, and whenever a zero weight is assigned to some edge (ζ, η), it reduces to a loop (ζ, ζ).

2. Main Results

We start with the following definition.

Definition 9. *Let (M, P_b) be a partial b-metric space endowed with a directed graph G, $s > 1$ and ϕ, ψ be self-mappings of M. We say that the pair (ϕ, ψ) is a Ćirić type rational graphic (Υ, Λ)-contraction pair, if:*
(1) For every vertex $u \in G$, we have $(u, \phi(u)), (u, \psi(u)) \in E(G)$;
(2) There exists a comparison function Υ and $\Lambda \in \Phi$ such that for all $\zeta_1, \zeta_2 \in M$, with $(\zeta_1, \zeta_2) \in E(G)$ and $\phi(\zeta_1) \neq \psi(\zeta_2)$, we have

$$\Lambda\left(sP_b\left(\phi\left(\zeta_1\right), \psi\left(\zeta_2\right)\right)\right) \leq \Upsilon\left[\Lambda\left(A_s\left(\zeta_1, \zeta_2\right)\right)\right], \tag{1}$$

where,

$$A_s(\zeta_1, \zeta_2) = \max\left\{\begin{array}{c} P_b(\zeta_1, \zeta_2), P_b(\zeta_1, \phi(\zeta_1)), \\ P_b(\zeta_2, \psi(\zeta_2)), \frac{P_b(\zeta_1, \psi(\zeta_2)) + P_b(\phi(\zeta_1), \zeta_2)}{2s^2}, \\ \frac{P_b(\psi(\zeta_2), \zeta_2)(1 + P_b(\zeta_1, \phi(\zeta_1)))}{1 + P_b(\zeta_1, \zeta_2)}. \end{array}\right\}.$$

Remark 3. *If $\phi = \psi$, then we say that ϕ is a Ćirić type rational graphic (Υ, Λ)-contraction.*

Our first main result is the following.

Theorem 4. *Let (M, P_b) be a complete partial b-metric space endowed with a directed graph G. Let $\phi, \psi : M \to M$ be maps such that (ϕ, ψ) is a Ćirić type rational graphic (Υ, Λ)-contraction pair. If Υ is continuous, then the following assertions hold:*
(a) $Fix(\phi) \neq \emptyset$ or $Fix(\psi) \neq \emptyset$ if and only if $Fix(\phi) \cap Fix(\psi) \neq \emptyset$;
(b) If $\zeta^* \in Fix(\phi) \cap Fix(\psi)$, then the weight assigned to the vertex ζ^* is 0;
(c) $Fix(\phi) \cap Fix(\psi) \neq \emptyset$, provided that G satisfies property $(\mathbf{E}^*)$;
(d) $Fix(\phi) \cap Fix(\psi)$ is a complete set if and only if $Fix(\phi) \cap Fix(\psi)$ is a singleton set.

Proof. (a) Let $Fix(\phi) \neq \emptyset$, so there exists $\zeta^* \in Fix(\phi)$. Then there is an edge between ζ^* and $\phi(\zeta^*)$, so $(\zeta^*, \phi(\zeta^*)) \in E(G)$. Now, we shall prove that $\zeta^* \in Fix(\psi)$; that is, the weight assigned to the edge $(\zeta^*, \psi(\zeta^*))$ is zero. Assume, on the contrary, that a non zero weight is assigned to the edge $(\zeta^*, \psi(\zeta^*))$.

As $(\zeta^*, \psi(\zeta^*)) \in E(G)$ and (ϕ, ψ) is a Ćirić type rational graphic (Υ, Λ)-contraction pair, from (1), we have

$$\begin{aligned}\Lambda\left(P_b\left(\zeta^*, \psi\left(\zeta^*\right)\right)\right) &\leq \Lambda\left(sP_b\left(\phi\left(\zeta^*\right), \psi\left(\zeta^*\right)\right)\right) \\ &\leq \Upsilon\left[\Lambda\left(A_s\left(\zeta^*, \zeta^*\right)\right)\right],\end{aligned}$$

where

$$\begin{aligned}A_s\left(\zeta^*, \zeta^*\right) &= \max\left\{\begin{array}{c} P_b\left(\zeta^*, \zeta^*\right), P_b\left(\zeta^*, \phi\left(\zeta^*\right)\right), \\ P_b\left(\zeta^*, \psi\left(\zeta^*\right)\right), \frac{P_b(\zeta^*, \psi(\zeta^*)) + P_b(\zeta^*, \phi(\zeta^*))}{2s^2}, \\ \frac{P_b(\zeta^*, \psi(\zeta^*))(1 + P_b(\zeta^*, \phi(\zeta^*)))}{1 + P_b(\zeta^*, \zeta^*)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} P_b\left(\zeta^*, \zeta^*\right), P_b\left(\zeta^*, \phi\left(\zeta^*\right)\right), \\ P_b\left(\zeta^*, \psi\left(\zeta^*\right)\right), \frac{P_b(\zeta^*, \psi(\zeta^*)) + P_b(\zeta^*, \phi(\zeta^*))}{2s^2}, \\ \frac{P_b(\zeta^*, \psi(\zeta^*))(1 + P_b(\zeta^*, \phi(\zeta^*)))}{1 + P_b(\zeta^*, \zeta^*)} \end{array}\right\} \\ &= P_b\left(\zeta^*, \phi\left(\zeta^*\right)\right).\end{aligned}$$

Thus,

$$\Lambda\left(P_b\left(\zeta^*, \psi\left(\zeta^*\right)\right)\right) \leq\leq \Upsilon\left[\Lambda\left(P_b\left(\zeta^*, \phi\left(\zeta^*\right)\right)\right)\right] < \Lambda\left(P_b\left(\zeta^*, \phi\left(\zeta^*\right)\right)\right).$$

It is a contradiction. Hence, the weight assigned to the edge $(\zeta^*, \psi(\zeta^*))$ is zero; that is, $\zeta^* = \psi(\zeta^*)$. Thus,

$$\zeta^* \in Fix(\phi) \cap Fix(\psi),$$

Therefore,

$$Fix(\phi) \cap Fix(\psi) \neq \emptyset.$$

Conversely, let $Fix(\phi) \cap Fix(\psi) \neq \emptyset$. So there exists $\zeta^* \in M$ such that $\zeta^* \in Fix(\phi) \cap Fix(\psi)$, and then $\zeta^* \in Fix(\phi)$ and $\zeta^* \in Fix(\psi)$. Thus, the proof of (a) is ended.

(b) Let $\zeta^* \in Fix(\phi) \cap Fix(\psi)$. Suppose on the contrary that the weight assigned to the vertex ζ^* is nonzero. As $(\zeta^*, \zeta^*) \in E(G)$ and (ϕ, ψ) is a Ćirić type rational graphic (Υ, Λ)-contraction pair, we get

$$\begin{aligned}\Lambda\left(P_b\left(\zeta^*, \zeta^*\right)\right) &\leq \Lambda\left(sP_b\left(\phi\left(\zeta^*\right), \psi\left(\zeta^*\right)\right)\right) \\ &\leq \Upsilon\left[\Lambda\left(A_s\left(\zeta^*, \zeta^*\right)\right)\right],\end{aligned}$$

where

$$\begin{aligned}A_s\left(\zeta^*, \zeta^*\right) &= \max\left\{\begin{array}{c} P_b\left(\zeta^*, \zeta^*\right), P_b\left(\zeta^*, \phi\left(\zeta^*\right)\right), \\ P_b\left(\zeta^*, \psi\left(\zeta^*\right)\right), \frac{P_b(\zeta^*, \psi(\zeta^*)) + P_b(\zeta^*, \phi(\zeta^*))}{2s^2}, \\ \frac{P_b(\zeta^*, \psi(\zeta^*))(1 + P_b(\zeta^*, \phi(\zeta^*)))}{1 + P_b(\zeta^*, \zeta^*)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} P_b\left(\zeta^*, \zeta^*\right), P_b\left(\zeta^*, \zeta^*\right), \\ P_b\left(\zeta^*, \zeta^*\right), \frac{P_b(\zeta^*, \zeta^*) + P_b(\zeta^*, \zeta^*)}{2s^2}, \\ \frac{P_b(\zeta^*, \zeta^*)(1 + P_b(\zeta^*, \zeta^*))}{1 + P_b(\zeta^*, \zeta^*)} \end{array}\right\} \\ &= P_b\left(\zeta^*, \zeta^*\right).\end{aligned}$$

It implies that

$$\Lambda\left(P_b\left(\zeta^*, \zeta^*\right)\right) \leq \Upsilon\left[\Lambda\left(P_b\left(\zeta^*, \zeta^*\right)\right)\right] < \Lambda\left(P_b\left(\zeta^*, \zeta^*\right)\right),$$

which is a contradiction. Therefore, the weight assigned to the edge (ζ^*, ζ^*) is zero. The proof of (b) is completed.

(c) Let $\zeta_0 \in M$. If $\zeta^* \in Fix(\phi)$ or $\zeta_0 \in Fix(\psi)$, then from (a) the proof is finished. Assume that $\zeta_0 \notin Fix(\phi)$; then $\phi(\zeta_0) \neq \zeta_0$. Since there is an edge between $\phi(\zeta_0)$ and ζ_0, that is, $(\zeta_0, \phi(\zeta_0)) \in E(G)$, this implies that there is $\phi(\zeta_0) = \zeta_1 \in M$ such that $(\zeta_0, \zeta_1) \in E(G)$. Similarly, $(\zeta_1, \psi(\zeta_1)) \in E(G)$ implies $(\zeta_1, \zeta_2) \in E(G)$. Continuing this process, we can construct the sequence $\{\zeta_n\} \in M$ such that $(\zeta_n, \zeta_{n+1}) \in E(G)$ is defined by

$$\phi(\zeta_{2n}) = \zeta_{2n+1} \text{ and } \zeta_{2n+2} = \psi\zeta_{2n+1} \text{ for all } n \in \mathbb{N}.$$

If the weight assigned to the edge$(\zeta_{2m}, \zeta_{2m+1})$ is zero for some $m \in \mathbb{N}$, then $\zeta_{2m} = \zeta_{2m+1} = \phi(\zeta_{2m})$, which implies $\zeta_{2m} \in Fix(\phi)$, and from (a), $\zeta_{2m} \in Fix(\phi) \cap Fix(\psi)$. Then there is nothing to prove. Assume that the weight assigned to the edge $(\zeta_{2n}, \zeta_{2n+1})$ is non zero for all $n \in \mathbb{N}$; that is, $\zeta_{2n} \neq \zeta_{2n+1}$ for all $n \in \mathbb{N}$. By (1), we get

$$\begin{aligned}
\Lambda(P_b(\zeta_{2n+1}, \zeta_{2n})) &\leq \Lambda(sP_b(\zeta_{2n+1}, \zeta_{2n})) \\
&= \Lambda(sP_b(\phi(\zeta_{2n}), \psi(\zeta_{2n-1}))) \\
&\leq \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} P_b(\zeta_{2n}, \zeta_{2n-1}), P_b(\zeta_{2n}, \phi(\zeta_{2n})), \\ P_b(\zeta_{2n-1}, \psi(\zeta_{2n-1})), \\ \frac{P_b(\zeta_{2n}, \psi(\zeta_{2n-1})) + P_b(\zeta_{2n-1}, \phi(\zeta_{2n}))}{2s^2}, \\ \frac{P_b(\zeta_{2n}, \phi(\zeta_{2n}))(1 + P_b(\zeta_{2n-1}, \psi(\zeta_{2n-1})))}{1 + P_b(\zeta_{2n}, \zeta_{2n-1})} \end{array}\right\}\right)\right) \\
&= \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} P_b(\zeta_{2n}, \zeta_{2n-1}), P_b(\zeta_{2n}, \zeta_{2n+1}), \\ P_b(\zeta_{2n-1}, \zeta_{2n}), \frac{P_b(\zeta_{2n}, \zeta_{2n}) + P_b(\zeta_{2n-1}, \zeta_{2n+1})}{2s^2}, \\ \frac{P_b(\zeta_{2n}, \zeta_{2n+1})(1 + P_b(\zeta_{2n-1}, \zeta_{2n}))}{1 + P_b(\zeta_{2n}, \zeta_{2n-1})} \end{array}\right\}\right)\right) \\
&\leq \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} P_b(\zeta_{2n}, \zeta_{2n-1}), P_b(\zeta_{2n}, \zeta_{2n+1}), \\ P_b(\zeta_{2n-1}, \zeta_{2n}), \frac{P_b(\zeta_{2n}, \zeta_{2n}) + P_b(\zeta_{2n}, \zeta_{2n+1})}{2s}, \\ \frac{P_b(\zeta_{2n-1}, \zeta_{2n})(1 + P_b(\zeta_{2n}, \zeta_{2n+1}))}{P_b(\zeta_{2n}, \zeta_{2n-1})}, \\ \frac{P_b(\zeta_{2n}, \zeta_{2n+1})(1 + P_b(\zeta_{2n-1}, \zeta_{2n}))}{P_b(\zeta_{2n}, \zeta_{2n+1})} \end{array}\right\}\right)\right) \\
&= \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} P_b(\zeta_{2n}, \zeta_{2n-1}), P_b(\zeta_{2n}, \zeta_{2n+1}), \\ P_b(\zeta_{2n-1}, \zeta_{2n}), \frac{P_b(\zeta_{2n}, \zeta_{2n}) + P_b(\zeta_{2n}, \zeta_{2n+1})}{2s^2}, \\ \frac{P_b(\zeta_{2n-1}, \zeta_{2n})(1 + P_b(\zeta_{2n}, \zeta_{2n+1}))}{P_b(\zeta_{2n}, \zeta_{2n-1})} \end{array}\right\}\right)\right) \\
&= \Upsilon(\Lambda(\max\{P_b(\zeta_{2n}, \zeta_{2n-1}), P_b(\zeta_{2n}, \zeta_{2n+1})\})).
\end{aligned} \tag{2}$$

If $\max\{P_b(\zeta_{2n}, \zeta_{2n-1}), P_b(\zeta_{2n}, \zeta_{2n+1})\} = P_b(\zeta_{2n}, \zeta_{2n+1})$; then, from (1) we have,

$$\Lambda(P_b(\zeta_{2n+1}, \zeta_{2n})) \leq \Upsilon(\Lambda(P_b(\zeta_{2n}, \zeta_{2n+1}))) < \Lambda(P_b(\zeta_{2n}, \zeta_{2n+1})),$$

which is a contradiction. Hence, $\max\{P_b(\zeta_{2n}, \zeta_{2n-1}), P_b(\zeta_{2n}, \zeta_{2n+1})\} = P_b(\zeta_{2n-1}, \zeta_{2n})$ and

$$\Lambda(P_b(\zeta_{2n+1}, \zeta_{2n})) \leq \Upsilon(\Lambda(P_b(\zeta_{2n-1}, \zeta_{2n}))), \text{ for all } n \in \mathbb{N}. \tag{3}$$

It yields that

$$\Lambda(P_b(\zeta_{2n+1}, \zeta_{2n})) < \Lambda(P_b(\zeta_{2n-1}, \zeta_{2n})), \text{ for all } n \in \mathbb{N}.$$

Due to property ($\Phi 1$), we get

$$P_b(\zeta_{2n+1}, \zeta_{2n}) < P_b(\zeta_{2n-1}, \zeta_{2n}), \text{ for all } n \in \mathbb{N}. \tag{4}$$

Analogously, one can find that

$$P_b(\zeta_{2n+2}, \zeta_{2n+1}) < P_b(\zeta_{2n}, \zeta_{2n+1}), \text{ for all } n \in \mathbb{N}. \tag{5}$$

The Equations (4) and (5) yield that $\{P_b(\zeta_{2n+1},\zeta_{2n})\}$ is a decreasing sequence. From (5), we have

$$\begin{aligned}\Lambda(P_b(\zeta_{2n},\zeta_{2n+1})) &\leq \Upsilon(\Lambda(P_b(\zeta_{2n},\zeta_{2n+1}))) \leq \Upsilon^2(\Lambda(P_b(\zeta_{2n-1},\zeta_{2n})))\\ &\leq \ldots \leq \Upsilon^{2n}(\Lambda(P_b(\zeta_0,\zeta_1))).\end{aligned} \tag{6}$$

Similarly, one gets

$$\begin{aligned}\Lambda(P_b(\zeta_{2n+1},\zeta_{2n+2})) &\leq \Upsilon(\Lambda(P_b(\zeta_{2n+1},\zeta_{2n+2}))) \leq \Upsilon^2(\Lambda(P_b(\zeta_{2n},\zeta_{2n+1})))\\ &\leq \ldots \leq \Upsilon^{2n+1}(\Lambda(P_b(\zeta_0,\zeta_1))).\end{aligned} \tag{7}$$

Letting $n \longrightarrow \infty$ in (6) and (7), we get

$$0 \leq \lim_{n\longrightarrow\infty}\Lambda(P_b(\zeta_n,\zeta_{n+1})) \leq \lim_{n\longrightarrow\infty}\Upsilon^n(\Lambda(P_b(\zeta_0,\zeta_1))) = 0;$$

that is,

$$\lim_{n\longrightarrow\infty}\Lambda(P_b(\zeta_n,\zeta_{n+1})) = 0.$$

From $(\Phi 2)$ and Lemma 2, we get

$$\lim_{n\longrightarrow\infty}P_b(\zeta_n,\zeta_{n+1}) = 0. \tag{8}$$

Further, from $(P_b 2)$ we have

$$\lim_{n\longrightarrow\infty}P_b(\zeta_n,\zeta_n) = 0. \tag{9}$$

We will prove that $\{\zeta_n\}$ is P_b-Cauchy. We argue by contradiction. Assume that there exist $\varepsilon > 0$ and a sequence $\left\{\hat{h}_n\right\}_{n=1}^{\infty}$ and $\{\hat{\jmath}_n\}_{n=1}^{\infty}$ of natural numbers such for all $n \in \mathbb{N}$, $\hat{h}_n > \hat{\jmath}_n > n$ with $P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right) \geq \varepsilon$, $P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)-1}\right) < \varepsilon$. Therefore,

$$\begin{aligned}\varepsilon &\leq P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right) \leq s\left[P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)-1}\right) + P_b\left(\zeta_{\hat{\jmath}(n)-1},\zeta_{\hat{\jmath}(n)}\right)\right] - P_b(\zeta_{\hat{\jmath}(n)-1},\zeta_{\hat{\jmath}(n)-1})\\ &\leq s\left[P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)-1}\right) + P_b\left(\zeta_{\hat{\jmath}(n)-1},\zeta_{\hat{\jmath}(n)}\right)\right]\\ &< s\varepsilon + sP_b\left(\zeta_{\hat{\jmath}(n)-1},\zeta_{\hat{\jmath}(n)}\right).\end{aligned} \tag{10}$$

Letting $n \to \infty$ in (10), we get

$$\varepsilon \leq \lim_{n\to\infty}P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right) < s\varepsilon. \tag{11}$$

From triangular inequality, we have

$$\begin{aligned}P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right) &\leq s[P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)}\right)] - P_b\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{h}(n)+1}\right)\\ &\leq s[P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)}\right)],\end{aligned} \tag{12}$$

and

$$\begin{aligned}P_b\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{\jmath}(n)}\right) &\leq s[P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)] - P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)}\right)\\ &\leq s[P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right) + P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{\jmath}(n)}\right)].\end{aligned} \tag{13}$$

By taking upper limit as $n \to \infty$ in (12) and applying (8) together with (11),

$$\varepsilon \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right) \leq s\left(\lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right)\right).$$

Again, by taking the upper limit as $n \to \infty$ in (13), we get

$$\frac{\varepsilon}{s} < \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) \leq s\left(\lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)}\right)\right) \leq s^2\varepsilon.$$

Thus

$$\frac{\varepsilon}{s} \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) \leq s^2\varepsilon. \tag{14}$$

Similarly,

$$\frac{\varepsilon}{s} \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)+1}\right) \leq s^2\varepsilon. \tag{15}$$

By triangular inequality, we have

$$\begin{aligned} P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right) &\leq s[P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) + P_b\left(\zeta_{\hat{j}(n)+1}, \zeta_{\hat{j}(n)}\right)] - P_b\left(\zeta_{\hat{j}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \\ &\leq s[P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) + P_b\left(\zeta_{\hat{j}(n)+1}, \zeta_{\hat{j}(n)}\right)]. \end{aligned} \tag{16}$$

Letting $n \to \infty$ in (16) and using the inequalities (8) and (14), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right). \tag{17}$$

Following the above process, we find

$$\lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \leq s^3\varepsilon. \tag{18}$$

From (17) and (18), we get

$$\frac{\varepsilon}{s^2} \leq \lim_{n\to\infty} \sup P_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)+1}\right) \leq s^3\varepsilon. \tag{19}$$

From (8) and (11), we can choose a positive integer $n_0 \geq 1$ such that for all $n \geq n_0$, from (1), we get

$$\begin{aligned} 0 &< \Lambda\left(sP_b\left(\zeta_{\hat{h}(n)+1}, \zeta_{\hat{j}(n)}\right)\right) \leq \Lambda\left(sP_b\left(\phi\left(\zeta_{\hat{h}(n)}\right), \psi\left(\zeta_{\hat{j}(n)-1}\right)\right)\right) \\ &\leq \Upsilon\left(\Lambda\left(A_{P_b}\left(\zeta_{\hat{h}(n)}, \zeta_{\hat{j}(n)-1}\right)\right)\right), \text{ for all } n \geq n_0, \end{aligned}$$

where

$$
\begin{aligned}
A_{P_b}\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)-1}\right) &= \max\left\{\begin{array}{c} P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)-1}\right), P_b\left(\zeta_{\hat{h}(n)},\psi\left(\zeta_{\hat{h}(n)}\right)\right), \\ P_b\left(\zeta_{\hat{j}(n)-1},\phi\left(\zeta_{\hat{j}(n)-1}\right)\right), \\ \frac{P_b\left(\zeta_{\hat{h}(n)},\phi\left(\zeta_{\hat{j}(n)-1}\right)\right)+P_b\left(\zeta_{\hat{j}(n)-1},\psi\left(\zeta_{\hat{h}(n)}\right)\right)}{2s^2}, \\ \frac{P_b\left(\psi\left(\zeta_{\hat{j}(n)-1}\right),\zeta_{\hat{j}(n)-1}\right)\left(1+P_b\left(\zeta_{\hat{h}(n)},\phi\left(\zeta_{\hat{h}(n)}\right)\right)\right)}{1+P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)-1}\right)} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)-1}\right), P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right), \\ P_b\left(\zeta_{\hat{j}(n)-1},\zeta_{\hat{j}(n)}\right), \\ \frac{P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)}\right)+P_b\left(\zeta_{\hat{j}(n)-1},\zeta_{\hat{h}(n)+1}\right)}{2s^2}, \\ \frac{P_b\left(\zeta_{\hat{j}(n)},\zeta_{\hat{j}(n)-1}\right)\left(1+P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{h}(n)+1}\right)\right)}{1+P_b\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)-1}\right)} \end{array}\right\}.
\end{aligned}
$$

Taking the upper limit as $n \to \infty$ and using (8), (11), (14) and (15), we get

$$
\begin{aligned}
\frac{\varepsilon}{s} &\leq \lim_{n\to\infty} \sup A_s\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)-1}\right) \leq \max\left\{\varepsilon, \frac{s\varepsilon+s^2\varepsilon}{2s^2}\right\} \\
&\leq \max\left\{\varepsilon, \frac{s^2\varepsilon+s^2\varepsilon}{2s^2}\right\} = \max\left\{\varepsilon, \frac{2s^2\varepsilon}{2s^2}\right\} = \max\{\varepsilon,\varepsilon\} = \varepsilon.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Lambda(\varepsilon) &= \Lambda(s.\frac{\varepsilon}{s}) \leq \Lambda\left(s\lim_{n\to\infty}\sup P_b\left(\zeta_{\hat{h}(n)+1},\zeta_{\hat{j}(n)}\right)\right) \leq \lim_{n\to\infty} \Upsilon\left(\Lambda\left(A_s\left(\zeta_{\hat{h}(n)},\zeta_{\hat{j}(n)-1}\right)\right)\right) \\
&\leq \Upsilon\left(\Lambda(\varepsilon)\right) < \Lambda(\varepsilon).
\end{aligned}
$$

It is a contradiction. Therefore, $\{\zeta_n\}$ is Cauchy. Since $(M, P_b,)$ is a complete partial b-metric space, by Lemma 1, (M, d_{P_b}) is a complete b-metric space. Therefore, the sequence $\{\zeta_n\}$ converges to some $\zeta^* \in (M, d_{P_b})$. Again, by Lemma 1, there exists $\zeta^* \in M$ such that

$$\lim_{n\to\infty} d_{P_b}\left(\zeta_n,\zeta^*\right) = 0 \tag{20}$$

if and only if

$$\lim_{n\to\infty} P_b\left(\zeta_n,\zeta^*\right) = \lim_{n\to\infty} P_b\left(\zeta^*,\zeta^*\right) = \lim_{n\to\infty} P_b\left(\zeta_n,\zeta_m\right). \tag{21}$$

Now, we show that $\zeta^* \in Fix(\phi)$, so the weight assigned to the edge $(\zeta^*, \phi(\zeta^*))$ is zero. Suppose that $P_b(\zeta^*, \phi(\zeta^*)) > 0$. If $\zeta_{2n+1} \in V(G)$, $n \in \mathbb{N}$, then we get $(\zeta_{2n+1}, \zeta_{2n+2}) = (\zeta_{2n+1}, \psi(\zeta_{2n+1})) \in E(G)$. By property (**E***), there is a subsequence $\left\{\zeta_{2n(k)+1}\right\}$ of $\{\zeta_{2n+1}\}$ with an edge between $\zeta_{2n(k)+1}$ and ζ^* for $k \in \mathbb{N}$. Using (1), one gets

$$\begin{aligned}
\Lambda\left(P_b\left(\phi\left(\zeta^*\right),\zeta_{2n(k)+2}\right)\right) &\leq \Lambda\left(sP_b\left(\phi\left(\zeta^*\right),\psi\left(\zeta_{2n(k)+1}\right)\right)\right) \\
&\leq \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} P_b\left(\zeta^*,\zeta_{2n(k)+1}\right),P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right), \\ P_b\left(\zeta_{2n(k)+1},\psi\left(\zeta_{2n(k)+1}\right)\right), \\ \frac{P_b\left(\zeta^*,\psi\left(\zeta_{2n(k)+1}\right)\right)+P_b\left(\zeta_{2n(k)+1},\phi\left(\zeta^*\right)\right)}{2s^2}, \\ \frac{P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right)\left(1+P_b\left(\zeta_{2n(k)+1},\psi\left(\zeta_{2n(k)+1}\right)\right)\right)}{1+P_b\left(\zeta^*,\zeta_{2n(k)+1}\right)} \end{array}\right\}\right)\right) \\
&= \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} P_b\left(\zeta^*,\zeta_{2n(k)+1}\right),P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right), \\ P_b\left(\zeta_{2n(k)+1},\zeta_{2n(k)+2}\right), \\ \frac{P_b\left(\zeta^*,\zeta_{2n(k)+2}\right)+P_b\left(\zeta_{2n(k)+1},\phi\left(\zeta^*\right)\right)}{2s^2}, \\ \frac{P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right)\left(1+P_b\left(\zeta_{2n(k)+1},\zeta_{2n(k)+2}\right)\right)}{1+P_b\left(\zeta^*,\zeta_{2n(k)+1}\right)} \end{array}\right\}\right)\right) \\
&< \Lambda\left(\max\left\{\begin{array}{c} P_b\left(\zeta^*,\zeta_{2n(k)+1}\right),P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right), \\ P_b\left(\zeta_{2n(k)+1},\zeta_{2n(k)+2}\right), \\ \frac{P_b\left(\zeta^*,\zeta_{2n(k)+2}\right)+P_b\left(\zeta_{2n(k)+1},\phi\left(\zeta^*\right)\right)}{2s^2}, \\ \frac{P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right)\left(1+P_b\left(\zeta_{2n(k)+1},\zeta_{2n(k)+2}\right)\right)}{1+P_b\left(\zeta^*,\zeta_{2n(k)+1}\right)} \end{array}\right\}\right)
\end{aligned} \tag{22}$$

Taking the upper limit as $k \to \infty$ in (22) and using the continuity of Λ, we have

$$\Lambda\left(P_b\left(\phi\left(\zeta^*\right),\zeta^*\right)\right) < \Lambda\left(P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right)\right),$$

a contradiction. Therefore, the assigned weight of the edge $\left(\zeta^*,\phi\left(\zeta^*\right)\right)$ is zero; that is, $\zeta^* \in Fix\left(\phi\right)$. Similarly, $\zeta^* \in Fix\left(\psi\right)$. Hence, $\zeta^* \in Fix\left(\phi\right) \cap Fix\left(\psi\right)$. The proof of (c) is completed.

(d) First, we assume that $Fix\left(\phi\right) \cap Fix\left(\psi\right)$ is complete. We shall prove that $Fix\left(\phi\right) \cap Fix\left(\psi\right)$ is a singleton. On the contrary, suppose that there exists $\zeta^*, \eta^* \in Fix\left(\phi\right) \cap Fix\left(\psi\right)$ such that $\zeta^* \neq \eta^*$. As $\left(\zeta^*,\eta^*\right) \in E\left(G\right)$, so from (1), we have

$$\begin{aligned}
\Lambda\left(P_b\left(\zeta^*,\eta^*\right)\right) &\leq \Lambda\left(sP_b\left(\phi\left(\zeta^*\right),\psi\left(\eta^*\right)\right)\right) \\
&\leq \Upsilon\left(\Lambda\left(\max\left\{\begin{array}{c} P_b\left(\zeta^*,\eta^*\right),P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right), \\ P_b\left(\eta^*,\psi\left(\eta^*\right)\right), \\ \frac{P_b\left(\zeta^*,\psi\left(\eta^*\right)\right)+P_b\left(\eta^*,\phi\left(\zeta^*\right)\right)}{2s^2}, \\ \frac{P_b\left(\zeta^*,\phi\left(\zeta^*\right)\right)\left(1+P_b\left(\eta^*,\psi\left(\eta^*\right)\right)\right)}{1+P_b\left(\zeta^*,\eta^*\right)} \end{array}\right\}\right)\right) \\
&= \Upsilon\left(\Lambda\left(P_b\left(\zeta^*,\eta^*\right)\right)\right) \\
&\quad \Lambda\left(P_b\left(\zeta^*,\eta^*\right)\right).
\end{aligned}$$

It is a contradiction. Thus, $\zeta^* = \eta^*$.

Conversely, assume that $Fix\left(\phi\right) \cap Fix\left(\psi\right)$ is a singleton; then, $Fix\left(\phi\right) \cap Fix\left(\psi\right)$ is complete. □

Example 2. *Let* $M = \{1,2,3,4,5\} = V\left(G\right)$ *and* $P_b : M \times M \to [0,\infty)$ *defined by* $P_b\left(\zeta_1,\zeta_2\right) = \left[\max\left\{\zeta_1,\zeta_2\right\}\right]^2$, *for all* $\zeta_1,\zeta_2 \in M$. *Then* $\left(P_b, M\right)$ *is a complete partial b-metric space with* $s = 2$. *Set*

$$E(G) = \left\{\begin{array}{c} (1,1),(2,2),(3,3),(4,4),(5,5), \\ (2,1),(4,1),(5,1),(3,2), \\ (4,2),(5,2),(4,3),(5,3),(5,4) \end{array}\right\}.$$

Define $\phi, \psi : M \to M$ *by*

$$\phi(\zeta_1) = \begin{cases} 1, & \zeta_1 \in \{1,2,5\}, \\ 2, & \zeta_1 \in \{3,4\}. \end{cases} \quad \text{and } \psi(\zeta_1) = \begin{cases} 1, & \zeta_1 \in \{1,5\}, \\ 2, & \zeta_1 \in \{2,3,4\}. \end{cases}$$

and $\Lambda, \Upsilon : (0,\infty) \longrightarrow (0,\infty)$, *by*

$$\Lambda(t) = te^t, \ t > 0, \Upsilon(t) = \frac{4t}{5}, \ t > 0.$$

It is easy to show that, for every vertex $u \in G$, *we have* $(u, \phi(u)), (u, \psi(u)) \in E(G)$. *Now, for all* $(\zeta_1, \zeta_2) \in M$, *with* $\zeta_1 \neq \zeta_2$,

Hence, by Figure 1, (ϕ, ψ) *is a Ćirić type rational graphic* (Υ, Λ)*-contraction pair. Thus, all the conditions of Theorem 4 are satisfied, and* ϕ *and* ψ *have a unique common fixed point (that is,* 1*). Figure 2 represents the graph with all the possible cases.*

(ζ_1,ζ_2)	$\Lambda(sP_b(\phi(\zeta_1),\psi(\zeta_2)))$	$\Upsilon[(\Lambda(A_s(\zeta_1,\zeta_2))]$
(2,1)	14.77	174.71
(4,1)	23,847.66	113,742,214.66
(5,1)	14.77	1,440,097,986,747.71
(3,2)	23,847.66	58,342.20
(4,2)	23,847.66	113,742,214.66
(5,2)	23,847.66	1,440,097,986,747.71
(4,3)	23,847.66	113,742,214.66
(5,3)	23,847.66	1,440,097,986,747.71
(5,4)	23,847.66	1,440,097,986,747.71

Figure 1. Verification of the contraction (1).

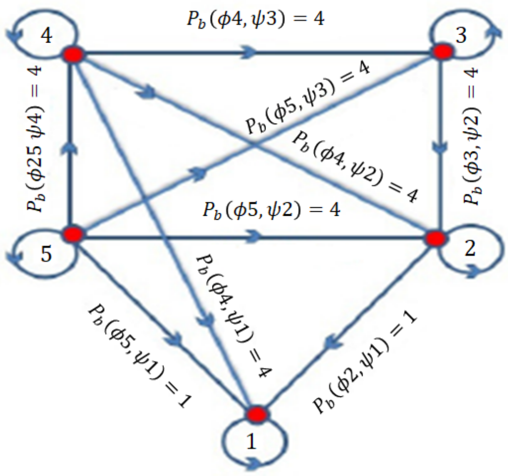

Figure 2. The graph defined in Example 2.

If $\phi = \psi$ in Theorem 4, we obtain the following result.

Corollary 1. *Let* (M, P_b) *be a complete partial b-metric space endowed with a directed graph G and the map* $\phi : M \to M$ *such that* ϕ *is a Ćirić type rational graphic* (Υ, Λ)*-contraction. If* Υ *is continuous, then the*

following assertions hold:
(a) If $\zeta^* \in Fix(\phi)$, then the weight assigned to the vertex ζ^* is 0;
(b) $Fix(\phi) \neq \emptyset$, provided that G satisfies property $(\mathbf{E}^*)$;
(c) $Fix(\phi)$ is a complete set if and only if $Fix(\phi)$ is a singleton set.

If $s = 1$ in Theorem 4, we obtain the following result.

Theorem 5. *Let (M, P) be a complete partial metric space endowed with a directed graph G. Let $\phi, \psi : M \to M$ be maps such that:*
(1) For every vertex $u \in G$, we have $(u, \phi(u)), (u, \psi(u)) \in E(G)$;
(2) There exist a comparison function Υ and $\Lambda \in \Phi$ such that for all $\zeta_1, \zeta_2 \in M$ with $(\zeta_1, \zeta_2) \in E(G)$ and $\phi(\zeta_1) \neq \psi(\zeta_2)$, we have

$$\Lambda(P(\phi(\zeta_1), \psi(\zeta_2))) \leq \Upsilon[\Lambda(A(\zeta_1, \zeta_2))],$$

where

$$A(\zeta_1, \zeta_2) = \max\left\{\begin{array}{c} P(\zeta_1, \zeta_2), P(\zeta_1, \phi(\zeta_1)), \\ P(\zeta_2, \psi(\zeta_2)), \frac{P(\zeta_1, \psi(\zeta_2)) + P_b(\phi(\zeta_1), \zeta_2)}{2s^2}, \\ \frac{P(\psi(\zeta_2), \zeta_2)(1 + P(\zeta_1, \phi(\zeta_1)))}{1 + P(\zeta_1, \zeta_2)}. \end{array}\right\}.$$

If Υ is continuous, then the following assertions hold:
(a) $Fix(\phi) \neq \emptyset$ or $Fix(\psi) \neq \emptyset$ if and only if $Fix(\phi) \cap Fix(\psi) \neq \emptyset$;
(b) If $\zeta^* \in Fix(\phi) \cap Fix(\psi)$, then the weight assigned to the vertex ζ^* is 0;
(c) $Fix(\phi) \cap Fix(\psi) \neq \emptyset$, provided that G satisfies the property $(\mathbf{E}^*)$;
(d) $Fix(\phi) \cap Fix(\psi)$ is complete set if and only if $Fix(\phi) \cap Fix(\psi)$ is a singleton set.

Example 3. *Let $M = [0, 1] = V(G)$ and $P : M \times M \to [0, \infty)$ be defined by $P(\zeta_1, \zeta_2) = \max\{\zeta_1, \zeta_2\}$, for all $\zeta_1, \zeta_2 \in M$. Then, (P, M) is a complete partial metric space. Set*

$$E(G) = \{(\zeta_1, \zeta_2) : \zeta_1, \zeta_2 \in [0, 1]\}.$$

Define $\phi, \psi : M \to M$ by

$$\phi(\zeta_1) = \frac{\zeta_1}{4} \text{ and } \psi(\zeta_1) = \frac{\zeta_1}{5},$$

and $\Lambda, \Upsilon : (0, \infty) \longrightarrow (0, \infty)$, by

$$\Lambda(t) = te^t,\ t > 0,\ \Upsilon(t) = \frac{9t}{10},\ t > 0.$$

It is easy to show that, for every vertex $u \in G$, we have $(u, \phi(u)), (u, \psi(u)) \in E(G)$. Now, for all $(\zeta_1, \zeta_2) \in M$, with $\zeta_1 \neq \zeta_2 \neq 0$,

$$\Lambda(P(\phi(\zeta_1), \psi(\zeta_2))) \leq \Upsilon[\Lambda(A(\zeta_1, \zeta_2))].$$

Therefore, (ϕ, ψ) is a Ćirić type rational graphic (Υ, Λ)-contraction pair. Hence, the conditions of Theorem 5 hold. Moreover, 0 is a common fixed point of ϕ and ψ.

3. Some Consequences

Corollary 2. *Let (M, P_b) be a complete partial b-metric space $(s > 1)$ endowed with a directed graph G. Let $\phi, \psi : M \to M$ be maps such that:*
(1) For every vertex $u \in G$, we have $(u, \phi(u)), (u, \psi(u)) \in E(G)$;
(2) There exist $\theta \in \Xi$ and $k \in (0, 1)$ such that for all $\zeta_1, \zeta_2 \in M$, with $(\zeta_1, \zeta_2) \in E(G)$ and $\phi(\zeta_1) \neq \psi(\zeta_2)$, we have

$$\theta(sP_b(\phi(\zeta_1), \psi(\zeta_2))) \leq [\theta(A_s(\zeta_1, \zeta_2))]^k,$$

where

$$A_s(\zeta_1,\zeta_2)=\max\left\{\begin{array}{c} P_b(\zeta_1,\zeta_2),P_b(\zeta_1,\phi(\zeta_1)),\\ P_b(\zeta_2,\psi(\zeta_2)),\frac{P_b(\zeta_1,\psi(\zeta_2))+P_b(\phi(\zeta_1),\zeta_2)}{2s^2},\\ \frac{P_b(\psi(\zeta_2),\zeta_2)(1+P_b(\zeta_1,\phi(\zeta_1)))}{1+P_b(\zeta_1,\zeta_2)}. \end{array}\right\}.$$

Then the following assertions hold:
(a) $Fix(\phi)\neq\emptyset$ or $Fix(\psi)\neq\emptyset$ if and only if $Fix(\phi)\cap Fix(\psi)\neq\emptyset$;
(b) If $\zeta^*\in Fix(\phi)\cap Fix(\psi)$, then the weight assigned to the vertex ζ^* is 0;
(c) $Fix(\phi)\cap Fix(\psi)\neq\emptyset$, provided that G satisfies property $(\mathbf{E}^*)$;
(d) $Fix(\phi)\cap Fix(\psi)$ is a complete set if and only if $Fix(\phi)\cap Fix(\psi)$ is a singleton set.

Proof. It suffices to take in Theorem 4, $\Upsilon(t):=(\ln k)\,t$ and $\Lambda(t)=\ln(\theta)(t):(0,\infty)\longrightarrow(0,\infty)$. □

Corollary 3. *Let (M,P_b) be a complete partial b-metric space $(s>1)$ endowed with a directed graph G. Let $\phi,\psi:M\to M$ be maps such that:*
(1) For every vertex $u\in G$, we have $(u,\phi(u)),(u,\psi(u))\in E(G)$;
(2) There exist $F\in\mathcal{F}$ and $\tau>0$ such that for all $(\zeta_1,\zeta_2)\in M$ with $(\zeta_1,\zeta_2)\in E(G)$ and $d(\phi(\zeta_1),\psi(\zeta_2))>0$, we have

$$\tau+F(sP_b(\phi(\zeta_1),\psi(\zeta_2)))\leq F(A_s(\zeta_1,\zeta_2)),$$

where

$$A_s(\zeta_1,\zeta_2)=\max\left\{\begin{array}{c} P_b(\zeta_1,\zeta_2),P_b(\zeta_1,\phi(\zeta_1)),\\ P_b(\zeta_2,\psi(\zeta_2)),\frac{P_b(\zeta_1,\psi(\zeta_2))+P_b(\phi(\zeta_1),\zeta_2)}{2s^2},\\ \frac{P_b(\psi(\zeta_2),\zeta_2)(1+P_b(\zeta_1,\phi(\zeta_1)))}{1+P_b(\zeta_1,\zeta_2)}. \end{array}\right\}.$$

Then the following assertions hold:
(a) $Fix(\phi)\neq\emptyset$ or $Fix(\psi)\neq\emptyset$ if and only if $Fix(\phi)\cap Fix(\psi)\neq\emptyset$;
(b) If $\zeta^*\in Fix(\phi)\cap Fix(\psi)$, then the weight assigned to the vertex ζ^* is 0;
(c) $Fix(\phi)\cap Fix(\psi)\neq\emptyset$, provided that G satisfies property $(\mathbf{E}^*)$;
(d) $Fix(\phi)\cap Fix(\psi)$ is a complete set if and only if $Fix(\phi)\cap Fix(\psi)$ is a singleton set.

Proof. The result follows from Theorem 4 by taking $\Upsilon(t)=e^{-\tau}t$ and $\Lambda(t)=e^{F(t)}:(0,\infty)\longrightarrow(0,\infty)$. □

Corollary 4. *Let (M,P_b) be a complete partial b-metric space $(s>1)$ endowed with a directed graph G. Let $\phi,\psi:M\to M$ be maps such that:*
(1) For every vertex $u\in G$, we have $(u,\phi(u)),(u,\psi(u))\in E(G)$;
(2) If for all $(\zeta_1,\zeta_2)\in M$, with $(\zeta_1,\zeta_2)\in E(G)$,

$$d(\phi(\zeta_1),\psi(\zeta_2))\leq B(A_s(\zeta_1,\zeta_2)).A_s(\zeta_1,\zeta_2)),$$

where

$$A_s(\zeta_1,\zeta_2)=\max\left\{\begin{array}{c} P_b(\zeta_1,\zeta_2),P_b(\zeta_1,\phi(\zeta_1)),\\ P_b(\zeta_2,\psi(\zeta_2)),\frac{P_b(\zeta_1,\psi(\zeta_2))+P_b(\phi(\zeta_1),\zeta_2)}{2s^2},\\ \frac{P_b(\psi(\zeta_2),\zeta_2)(1+P_b(\zeta_1,\phi(\zeta_1)))}{1+P_b(\zeta_1,\zeta_2)} \end{array}\right\},$$

and $B:[0,\infty)\to[0,\infty)$ is such that $\lim\limits_{r\longrightarrow t^+}B(r)<1$ for each $t\in(0,\infty)$.
Then the following assertions hold:
(a) $Fix(\phi)\neq\emptyset$ or $Fix(\psi)\neq\emptyset$ if and only if $Fix(\phi)\cap Fix(\psi)\neq\emptyset$;
(b) If $\zeta^*\in Fix(\phi)\cap Fix(\psi)$, then the weight assigned to the vertex ζ^* is 0;
(c) $Fix(\phi)\cap Fix(\psi)\neq\emptyset$, provided that G satisfies property $(\mathbf{E}^*)$;
(d) $Fix(\phi)\cap Fix(\psi)$ is a complete set if and only if $Fix(\phi)\cap Fix(\psi)$ is a singleton set.

Proof. It follows from Theorem 4 by taking $\Upsilon(t) := B(t)\,t$ and $\Lambda(t) = t : (0,\infty) \longrightarrow (0,\infty)$. $\square$

Remark 4. *Theorems 4 and 5 generalize and extend results of Liu et al. [2], Jleli and Samet [29] and Wardowski [27] for partial b-metric spaces and partial metric spaces along with a power graphic contraction pair, respectively.*

4. Applications

4.1. Application to Electric Circuit Equations

In this section, we study the solution of the electric circuit equation (see [32]), which is in the second-order differential equation form. The electric circuit (as in Figure 3):

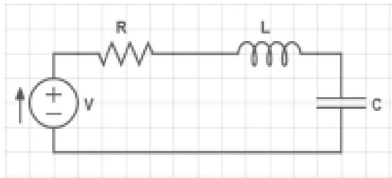

Figure 3. Electric Circuit.

Contains an electromotive force E, a resistor R, an inductor L, a capacitor C, and a voltage V in series. If the current I is the rate of change of charge q with respect to time t, we have $I = \frac{dq}{dt}$ and

$$\begin{aligned} V &= IR, \\ V &= qC, \\ V &= L\frac{dI}{dt}. \end{aligned}$$

By law of Kirchhoffs voltage, the sum of these voltage drops is equal to the supplied voltage; i.e,

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t),$$

or

$$IR + \frac{q}{C} + L\frac{dI}{dt} = V(t),\, q(0) = 0,\ q'(0) = 0. \tag{23}$$

The Green function associated to (23) is given by

$$G(t,s) = \begin{cases} -se^{\tau(s-t)}, \text{ if } 0 \le s \le t \le 1, \\ -te^{\tau(s-t)}, \text{ if } 0 \le t \le s \le 1 \end{cases}$$

where the constant $\tau > 0$ is calculated in terms of R and L.

Let $M = C([0,1])$ be the set of all continuous functions defined on $[0,1]$. The partial b-metric P_b on M is defined by

$$P_b(\zeta_1,\zeta_2) = \max_{0\le t\le 1} |\zeta_1(t) - \zeta_2(t)|^2.$$

Moreover, we define the graph G with the partial ordered relation:

$$\zeta_1,\zeta_2 \in C([0,1]), \zeta_1 \le \zeta_2 \Leftrightarrow \zeta_1(t) \le \zeta_2(t),$$

for all $t \in [0,1]$. Let $E(G) = \{(\zeta_1,\zeta_2) \in M \times M : \zeta_1 \le \zeta_2\}$. Note that (P_b, M) is a complete partial b-metric space with coefficient $s = 2$, including a directed graph G. Clearly, $\Delta = (M \times M) \in E(G)$, and (P_b, M, G) has property $(\mathbf{E}^*)$.

Theorem 6. *Let $\phi : C([0,1]) \to C([0,1])$ of a partial b-metric space $(C([0,1]), P_b)$. Suppose that the following assumptions hold:*
(1) There exists a continuous and non decreasing function K:$[0,1] \times \mathbb{R} \to \mathbb{R}$ such that for all $\zeta_1, \zeta_2 \in C([0,1])$, with $\zeta_1 \leq \zeta_2$,

$$|K(t,\zeta_1) - K(t,\zeta_2)| \leq \tau^2 e^{-\tau} \sqrt{A_s(\zeta_1,\zeta_2)},$$

where

$$A_s(\zeta_1,\zeta_2) = \max \left\{ \begin{array}{c} P_b(\zeta_1,\zeta_2), P_b(\zeta_1,\phi(\zeta_1)), \\ P_b(\zeta_2,\phi(\zeta_2)), \frac{P_b(\zeta_1,\phi(\zeta_2))+P_b(\phi(\zeta_1),\zeta_2)}{2s^2}, \\ \frac{P_b(\phi(\zeta_2),\zeta_2)(1+P_b(\zeta_1,\phi(\zeta_1)))}{1+P_b(\zeta_1,\zeta_2)} \end{array} \right\},$$

where $t \in [0,1]$, and $\tau \geq 1$,
(2) all $\zeta_1 \in C([0,1])$,

$$\zeta_1 \leq \int_0^1 G(t,s)K(t,\zeta_1(s))ds, \text{ for all } t \in [0,1].$$

Then the problem (23) has a unique solution.

Proof. The above problem is equivalent to the integral equation:

$$\zeta_1(t) = \int_0^1 G(t,s)K(t,\zeta_1(s))ds \tag{24}$$

where $t \in [0,1]$. Consider a mapping $\phi : M \to M$ defined by

$$\phi(\zeta_1(t)) = \int_0^t G(t,s)K(t,\zeta_1(s))ds \tag{25}$$

where $t \in [0,1]$. Then, ζ^* is a solution of (24) if and only if ζ^* is a fixed point of ϕ. From Condition (2), it is easy to show that for every $\zeta_1 \in M$, we have $\zeta_1 \leq \phi(\zeta_1)$; i.e., $(u,\phi(u)) \in E(G)$. It follows from Condition (2) that $M_\phi = \{\zeta_1 \in M : \zeta_1 \leq \phi(\zeta_1), \text{i.e.,} (\zeta_1,\phi(\zeta_1)) \in E(G)\} \neq \emptyset$. Let $\zeta_1, \zeta_2 \in M$; then, from Condition (1), we have

$$\begin{aligned} |\phi(\zeta_1(t)) - \phi(\zeta_2(t))| &\leq \int_0^t G(t,s)\,|K(t,\zeta_1(s)) - K(t,\zeta_2(s))|\,ds \\ &\leq \int_0^t G(t,s)\tau^2 e^{-\tau}\sqrt{A_s(\zeta_1,\zeta_2)}ds \\ &\leq \int_0^t \tau^2 e^{-\tau} e^{-2\tau s} e^{2\tau s}\sqrt{A_s(\zeta_1,\zeta_2)}G(t,s)ds \\ &\leq \tau^2 e^{-\tau}\sqrt{A_s(\zeta_1,\zeta_2)}\int_0^t e^{2\tau s}G(t,s)ds \\ &\leq e^{-\tau}\sqrt{A_s(\zeta_1,\zeta_2)}\left[e^{2\tau t}\left(1 - 2t\tau + t\tau e^{-\tau t} - e^{-\tau t}\right)\right]. \end{aligned}$$

Thus,

$$|\phi(\zeta_1(t)) - \phi(\zeta_2(t))|\, e^{-2\tau t} \leq e^{-\tau}\sqrt{A_s(\zeta_1,\zeta_2)}\left[1 - 2t\tau + t\tau e^{-\tau t} - e^{-\tau t}\right].$$

This implies that

$$|\phi(\zeta_1(t)) - \phi(\zeta_2(t))| \leq e^{-\tau}\sqrt{A_s(\zeta_1,\zeta_2)}\left[1 - 2t\tau + t\tau e^{-\tau t} - e^{-\tau t}\right].$$

Since $1-2t\tau+t\tau e^{-\tau t}-e^{-\tau t}\leq 1$, we get that

$$|\phi(\zeta_1(t))-\phi(\zeta_2(t))|\leq e^{-\tau}\sqrt{A_s(\zeta_1,\zeta_2)}.$$

Hence,

$$P_b(\phi(\zeta_1(t)),\phi(\zeta_2(t)))\leq e^{-2\tau}A_s(\zeta_1,\zeta_2).$$

Taking $\Lambda(t)=t$ and $\Upsilon(t)=\frac{2}{e^{2\tau}}$ with $\tau\geq 1$, one gets

$$\begin{aligned}\Lambda(sP_b(\phi(\zeta_1(t)),\phi(\zeta_2(t)))) &= \Lambda(2P_b(\phi(\zeta_1(t)),\phi(\zeta_2(t))))\leq \frac{2}{e^{2\tau}}\Lambda(A_s(\zeta_1,\zeta_2))\\ &= \Upsilon(\Lambda(A_s(\zeta_1,\zeta_2))),\end{aligned}$$

or

$$\Lambda(P_b(\phi(\zeta_1(t)),\phi(\zeta_2(t))))\leq\Upsilon(\Lambda(A_s(\zeta_1,\zeta_2))).$$

Therefore, from Corollary 1 , ϕ has a fixed point. Consequently, the differential equation arising in the electric circuit Equation (23) has a solution. □

4.2. Application to Fractional Differential Equations

We apply the result given by Theorem 4 to study the existence of a solution for a system of nonlinear fractional differential equations (see [33]). Let $M=C([0,1],\mathbb{R})$ be the space of all continuous functions on $[0,1]$. The partial b-metric P_b on M is defined by

$$P_b(r,j)=\max_{t\in[0,1]}|r(t)-j(t)|^2,\ r,j\in\chi.$$

Moreover, we define the graph G with the partial ordered relation:

$$r,j\in C([0,1]),r\leq j\Leftrightarrow r(t)\leq j(t),$$

for all $t\in[0,1]$. Let $E(G)=\{(r,j)\in M\times M: r\leq j\}$. Note that (P_b,M) is a complete partial b-metric space with coefficient $s=2$, including a directed graph G. Clearly, $\Delta=(M\times M)\in E(G)$ and (P_b,M,G) has property $(\mathbf{E}^*)$.

Consider the following system of fractional differential equations:

$$\begin{cases} {}^{C}D^{\alpha}r(t)=K_1(t,r(t))\\ {}^{C}D^{\alpha}j(t)=K_2(t,j(t))\end{cases},\tag{26}$$

with boundary conditions

$$\begin{cases} r(0)=0,\ Ir(1)=r'(0)\\ j(0)=0,\ Qj(1)=j'(0).\end{cases}$$

Note that ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α, defined by

$$\begin{cases} {}^{C}D^{\alpha}K_1(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}K_1^n(s))ds,\\ {}^{C}D^{\alpha}K_2(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}K_2^n(s))ds,\end{cases}$$

where

$$n-1<\alpha<1 \text{ and } n=[\alpha]+1,$$

and $I^{\alpha}K_1$ and $I^{\alpha}K_2$ denote the Riemann–Liouville fractional integral of order α of continuous functions K_1 and K_2, given by

$$\begin{cases} I^{\alpha}K_1(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_1(s)ds, \text{ with } \alpha > 0, \\ Q^{\alpha}K_2(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_2(s)ds, \text{ with } \alpha > 0. \end{cases}$$

The system (23) can be written in the following integral form:

$$\begin{cases} r(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_1(s, r(s))ds \\ +\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_1(u, r(u))duds, \\ j(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_2(s, j(s))ds \\ +\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_2(u, j(u))duds \end{cases}$$

Theorem 7. *Assume that the following conditions hold:*
(i) $K_1, K_2 : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous functions;*
(ii) $K_1(s,.), K_2(s,.) : \mathbb{R} \longrightarrow \mathbb{R}$ *are increasing functions,*
(iii) For all $r, j \in M$, *with* $r \leq j$, *we have*

$$|K_1(s,r) - K_2(s,j)| \leq \frac{e^{-\tau}\Gamma(\alpha+1)}{4}\sqrt{A_s(r,j)},$$

where,

$$A_s(r,j) = \max\left\{ \begin{array}{c} P_b(r,j), P_b(r,\phi(r)), \\ P_b(j,\phi(j)), \frac{P_b(r,\phi(j))+P_b(\phi(r),j)}{2s^2}, \\ \frac{P_b(\phi(j),j)(1+P_b(r,\phi(r)))}{1+P_b(r,j)} \end{array} \right\},$$

(iv) There exist $r_0, j_0 \in C([0,1], \mathbb{R})$ *such that for all* $t \in [0,1]$, *we have*

$$\begin{cases} r_0(t) \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_1(s, r_0(s))ds \\ +\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_1(u, r_0(u))duds, \\ j_0(t) \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_2(s, j_0(s))ds \\ +\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_2(u, j_0(u))duds. \end{cases}$$

Then the system (23) has a solution.

Proof. Define the mappings $\phi, \psi : M \longrightarrow M$ by

$$\begin{cases} \phi(r(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_1(s, r(s))ds \\ +\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_1(u, r(u))duds, \\ \psi(j(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_2(s, j(s))ds \\ +\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_2(u, j(u))duds \end{cases}.$$

Following assumptions (iii) and (iv), we have

$$|\phi(r(t)) - \psi(j(t))| = \left| \begin{array}{c} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_1(s, r(s))ds \\ -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} K_2(s, j(s))ds \\ +\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_1(u, r(u))duds \\ -\frac{2t}{\Gamma(\alpha)}\int_0^1\int_0^s (s-u)^{\alpha-1} K_2(u, j(u))duds \end{array} \right|$$

$$\begin{aligned}
&\leq \left|\int_0^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}\left[K_1(s,r(s))-K_2(s,j(s))\right]ds\right| \\
&\quad +\left|\int_0^1\int_0^s \frac{2}{\Gamma(\alpha)}(s-u)^{\alpha-1}\left[K_1(s,r(u))-K_2(u,j(u)\right]duds\right| \\
&\leq \frac{1}{\Gamma(\alpha)}.\int_0^t (t-s)^{\alpha-1}\left|K_1(s,r(s))-K_2(s,j(s))\right|ds \\
&\quad +\frac{2}{\Gamma(\alpha)}.\int_0^1\int_0^s (s-u)^{\alpha-1}\left|K_1(s,r(u))-K_2(s,j(u))\right|duds \\
&\leq \frac{1}{\Gamma(\alpha)}\frac{e^{-\tau}\Gamma(\alpha+1)}{4}\int_0^t (t-s)^{\alpha-1}\left|r(s)-j(s)\right|ds \\
&\quad +\frac{2}{\Gamma(\alpha)}\frac{e^{-\tau}\Gamma(\alpha+1)}{4}\int_0^1\int_0^s (s-u)^{\alpha-1}\left|r(u)-j(u)\right|duds \\
&\leq \frac{1}{\Gamma(\alpha)}\frac{e^{-\tau}\Gamma(\alpha+1)}{4}\sqrt{A_s(r,j)}.\int_0^t (t-s)^{\alpha-1}ds+ \\
&\quad +\frac{2}{\Gamma(\alpha)}\frac{e^{-\tau}\Gamma(\alpha+1)}{4}.\sqrt{A_s(r,j)}.\int_0^1\int_0^s (s-u)^{\alpha-1}duds
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{e^{-\tau}\Gamma(\alpha).\Gamma(\alpha+1)}{4\Gamma(\alpha).\Gamma(\alpha+1)}\right).\sqrt{A_s(r,j)}+ \\
&\quad 2e^{-\tau}\beta(\alpha+1,1)\frac{\Gamma(\alpha).\Gamma(\alpha+1)}{4\Gamma(\alpha).\Gamma(\alpha+1)}.\sqrt{A_s(r,j)} \qquad (27)\\
&\leq \frac{e^{-\tau}}{4}\sqrt{A_s(r,j)}+\frac{e^{-\tau}}{2}\sqrt{A_s(r,j)}= \\
&\quad \frac{3e^{-\tau}}{4}\sqrt{A_s(r,j)} \qquad (28)
\end{aligned}$$

where β is the beta function. From the inequality (27), we obtain that

$$|\phi(r(t))-\psi(j(t))|\leq \frac{3e^{-\tau}}{4}\sqrt{A_s(r,j)}.$$

Hence,

$$P_b(\phi(r(t)),\psi(j(t)))\leq \frac{9e^{-2\tau}}{16}A_s(r,j).$$

This implies that

$$\begin{aligned}
sP_b(\phi(r(t)),\psi(j(t))) &= 2P_b(\phi(r(t)),\psi(j(t))) \\
&\leq \frac{9}{8e^{2\tau}}P_b(r(t),j(t)) \\
&\leq \frac{9}{8e^{2\tau}}A_s(r(t),j(t)),
\end{aligned}$$

where $\Lambda(t)=t$ and $\Upsilon(t)=\frac{9}{8e^{2\tau}}$, $\tau\geq 1$. Since the above inequality holds for all $r,j\in M$ with $r(t)\leq j(t)$, it is true for any $(r,j)\in E(G)$. Hence, we have

$$\Lambda(sP_b(\phi(r(t)),\psi(j(t))))\leq \Upsilon(\Lambda(A_s(r,j))).$$

Therefore, all hypotheses of Theorem 4 are satisfied. Hence, ϕ and ψ have a common fixed point; that is, the system (26) has at least one solution. □

5. Conclusions

In this paper, we introduced the concept of a Ćirić type rational graphic (Υ, Λ)-contraction pair of mappings and established some new results for such contractions in the context of complete partial b-metric spaces endowed with a directed graph. Moreover, we give some examples in support of main theorems. At the end, we applied our main results to provide solutions of electric circuit equations and also of fractional differential equations. The obtained results generalize several corresponding results in metric spaces.

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