

HYBRID FEEDBACK STABILIZATION OF ROTATIONAL –TRANSLATIONAL ACTUATOR (RTAC) SYSTEM

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SUMMARY

A hybrid feedback control law is proposed for the RTAC system. This hybrid feedback control law is expressed in terms of a continuous feedback part and a part that includes switched parameters determined according to a logic-based switching rule. By appropriate selection of the switching rule, previous theoretical results guarantee that the origin is globally asymptotically stable. Some comments are made about the closed-loop properties, and experiments confirm that good responses are obtained for the case studied.
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1. INTRODUCTION

In this paper we present a novel hybrid feedback control law for global stabilization of the rotational–translational actuator (RTAC) system.^{1–4} This hybrid control approach was originally developed for a class of non-holonomic control systems, but it is applicable to the RTAC system and provides an alternative to other (smooth) feedback laws developed in the literature.^{1,2,4–7}

The RTAC system consists of a platform that can oscillate without damping in the horizontal plane and a rotating eccentric mass which is actuated by a dc motor located on the platform. Feedback laws developed for stabilization of the RTAC system^{1,2,4–7} exploit the nonlinear coupling between the rotational motion of the mass and the translational motion of the mass and the platform to damp out the oscillations of the platform.

After state and control transformations, the RTAC system can be described by a cascade connection of a linear system and a nonlinear, periodic subsystem. This cascade structure is characteristic of underactuated systems in general.⁸ This suggests a possibility of stabilizing the RTAC system with hybrid feedback laws that we originally developed in References 9 and 10 for a class of nonlinear cascade control systems.

Consider a model for the RTAC system^{1,2,4} in normalized dimensionless units:

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = \frac{-x_1 + \varepsilon x_4^2 \sin x_3}{1 - \varepsilon^2 \cos^2 x_3} - \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} u \tag{1b}$$

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$$\dot{x}_3 = x_4 \quad (1c)$$

$$\dot{x}_4 = \frac{\varepsilon \cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3)}{1 - \varepsilon^2 \cos^2 x_3} + \frac{1}{1 - \varepsilon^2 \cos^2 x_3} u \quad (1d)$$

where x_1 is the normalized displacement of the platform from the equilibrium position, $x_2 = \dot{x}_1$, $x_3 = \theta$ is the angle of the rotor and $x_4 = \dot{x}_3$. The control input u is the torque supplied to the eccentric mass. The parameter ε is a constant which quantifies the degree of coupling between translational and rotational motions. A typical value for ε is 0.1. Following Reference 4 we utilize the following state and control transformation:

$$z_1 = x_1 + \varepsilon \sin x_3 \quad (2a)$$

$$z_2 = x_2 + \varepsilon x_4 \cos x_3 \quad (2b)$$

$$y_1 = x_3 \quad (2c)$$

$$y_2 = x_4 \quad (2d)$$

$$v = \frac{1}{1 - \varepsilon^2 \cos^2 y_1} (\varepsilon \cos y_1 (z_1 - (1 + y_2^2) \varepsilon \sin y_1) + u) \quad (2e)$$

Then:

$$\dot{z}_1 = z_2 \quad (3a)$$

$$\dot{z}_2 = -z_1 + \varepsilon \sin y_1 \quad (3b)$$

$$\dot{y}_1 = y_2 \quad (3c)$$

$$\dot{y}_2 = v \quad (3d)$$

Equations (3) have provided the basis for several control design studies^{4,5} that exploit the cascade structure of the equations to construct stabilizing continuously differentiable feedback laws using the integrator backstepping and other approaches. It is also of interest that system (3) can be viewed as an underactuated system in the sense of Reference 8. Our subsequent development also exploits the cascade structure of equations (3), but in a different way. In particular, equations (3) are first expressed in a different form as in References 9 and 10 that is suitable for construction of a non-smooth feedback law, which we refer to as a hybrid feedback law, since it involves both smooth feedback terms and feedback terms with switched parameters.

To demonstrate that the RTAC system falls into the class of cascade nonlinear systems treated in References 9 and 10, consider the following time-periodic state transformation:

$$\hat{z}_1 = z_1 \cos t - z_2 \sin t \quad (4a)$$

$$\hat{z}_2 = z_1 \sin t + z_2 \cos t \quad (4b)$$

Then we obtain a time-periodic nonlinear system:

$$\dot{z}_1 = -\varepsilon \sin t \sin y_1 \quad (5a)$$

$$\dot{z}_2 = \varepsilon \cos t \sin y_1 \quad (5b)$$

$$\dot{y}_1 = y_2 \quad (5c)$$

$$\dot{y}_2 = v \quad (5d)$$

System (5) is in the cascade form treated in References 9 and 10. The variable y_1 (the rotor angle) is referred to as the base variable and its motion is directly affected by the control v . The variables z_1, z_2 are referred to as the fibre variables and the control does not affect them directly, but only through the nonlinear coupling between the base and the fibre variables.

We now describe the intuitive idea behind our hybrid stabilization approach as applied to system (5). If the base variable y_1 is forced to undergo a periodic motion with a period T , the net changes in the fibre variables z_1, z_2 over k cycles are provided by the following expressions:

$$z_1(kT) = z_1(0) - k\varepsilon \int_0^T \sin(y_1(t)) \sin t \, dt \quad (6a)$$

$$z_2(kT) = z_2(0) + k\varepsilon \int_0^T \sin(y_1(t)) \cos t \, dt \quad (6b)$$

If $y_1(t)$, $0 \leq t \leq T$, is selected so that the integrals in (6a) and (6b) are non-zero, a net drift in the fibre variables results. In particular, by an appropriate selection of $y_1(t)$, $0 \leq t \leq T$, we can force $(z_1(kT), z_2(kT))$ to drift, at least for small values of k , towards the origin. For k sufficiently large, $(z_1(kT), z_2(kT))$ may start to drift away from the origin. When this happens, the controller should be switched to induce a smaller amplitude periodic base variable motion so that the fibre variables continue to drift towards the origin. The basic mechanism of switching between time-periodic feedback laws that asymptotically induce appropriate periodic base variable motions that cause a desired drift in the fibre variables is the foundation of our stabilization approach.

The paper is organized as follows. The development of a hybrid feedback controller for stabilization of the RTAC system is the subject of Section 2. Experimental results are described in Section 3. They show that the hybrid controller provides good responses. In Section 4 we discuss additional issues pertinent to hybrid controller development and performance.

2. DEVELOPMENT OF HYBRID FEEDBACK CONTROLLER

The hybrid feedback controller relies on periodic switching between the members of a family of time-periodic feedback laws of the form

$$V(\alpha; y; t) = -y_1 - y_2 - \alpha_1 \sin t + \alpha_2 \cos t \quad (7)$$

where $y = (y_1, y_2)$ and α_1, α_2 are scalar parameters, $\alpha = (\alpha_1, \alpha_2)$. By selecting a specific value for α , we select a specific member of the family (7). The feedback controller is defined as

$$v(t) = V(\alpha^k; y; t), \quad kT \leq t < (k+1)T \quad (8)$$

where $k = 0, 1, 2, \dots, T = 2\pi$, and α^k is determined at the time instant kT based on the sampled values of the fibre variables at this time instant, $\hat{z}_1(kT), \hat{z}_2(kT)$. From expressions (4a) and (4b) we observe that $\hat{z}_1(kT) = z_1(kT)$ and $\hat{z}_2(kT) = z_2(kT)$. For system (1), the feedback law takes the form

$$u(t) = (1 - \varepsilon^2 \cos^2 y_1)(-y_1 - y_2 - \alpha_1^k \sin t + \alpha_2^k \cos t) - \varepsilon \cos y_1(z_1 - (1 + y_2^2)\varepsilon \sin y_1),$$

$$kT \leq t < (k + 1)T \tag{9}$$

Thus controller (9) involves continuous nonlinear feedback terms and periodically switched feedback terms.

Each member of the control family (7) forces y_1 to track asymptotically a periodic steady-state trajectory which is denoted by

$$y_1^{ss}(\alpha; t) = \alpha_1 \cos t + \alpha_2 \sin t \tag{10}$$

The net drift in the fibre variables generated by one cycle of the base variable trajectory (10) is given by

$$G(\alpha) = \begin{bmatrix} G_1(\alpha) \\ G_2(\alpha) \end{bmatrix} = \begin{bmatrix} -\varepsilon \int_0^{2\pi} \sin(\alpha_1 \cos t + \alpha_2 \sin t) \sin t \, dt \\ \varepsilon \int_0^{2\pi} \sin(\alpha_1 \cos t + \alpha_2 \sin t) \cos t \, dt \end{bmatrix} \tag{11}$$

To proceed with the stabilization approach of References 9 and 10, we verify that the map G is open at the origin. This condition can be interpreted as a requirement that the fibre variables can be forced to drift in all possible directions by an appropriate selection of α , no matter how small in magnitude α is restricted to be. The map G satisfies this condition since its Jacobian,

$$\left. \frac{\partial G}{\partial(\alpha_1, \alpha_2)} \right|_{(0,0)} = \begin{bmatrix} 0 & -\varepsilon\pi \\ \varepsilon\pi & 0 \end{bmatrix} \tag{12}$$

is non-singular. The map G is shown in Figure 1.

We now describe a feedback algorithm for selecting α^k at the time instant $t^k = kT$. Let γ be a fixed real number satisfying $0 < \gamma < 1$, $\hat{z} = (\hat{z}_1, \hat{z}_2)$, and $\|\cdot\|$ denote the Euclidean vector norm.

Algorithm 2.1

1. For $k = 0$ (Initialization):
 If $\hat{z}(0) = 0$ and $y(0) = 0$, set $\alpha^0 = 0$;
 Else
 select any $\alpha^0 \neq 0$ such that $\|\alpha^0\| < (\|y(0)\|^2 + \|\hat{z}(0)\|^2)^{1/2}$
2. For $k > 0$: If $\hat{z}(t^k) = 0$, set $\alpha^k = \alpha^{k-1}$.
 Else
 If $G_1(\alpha^{k-1})\hat{z}_1(t^k) + G_2(\alpha^{k-1})\hat{z}_2(t^k) < 0$, set $\alpha^k = \alpha^{k-1}$.
 If $G_1(\alpha^{k-1})\hat{z}_1(t^k) + G_2(\alpha^{k-1})\hat{z}_2(t^k) \geq 0$, select α^k such that
 $\|\alpha^k\| \leq \gamma \|\alpha^{k-1}\|$ and $G(\alpha^k) = -\rho^k \hat{z}(t^k)$ for some $\rho^k > 0$.

The feedback algorithm functions as follows. If initially the system is at the origin, then $\alpha^0 = 0$. Otherwise, a non-zero value is assigned to α^0 . For each $k \geq 1$, the selection of α^k is based on $\hat{z}(t^k)$,

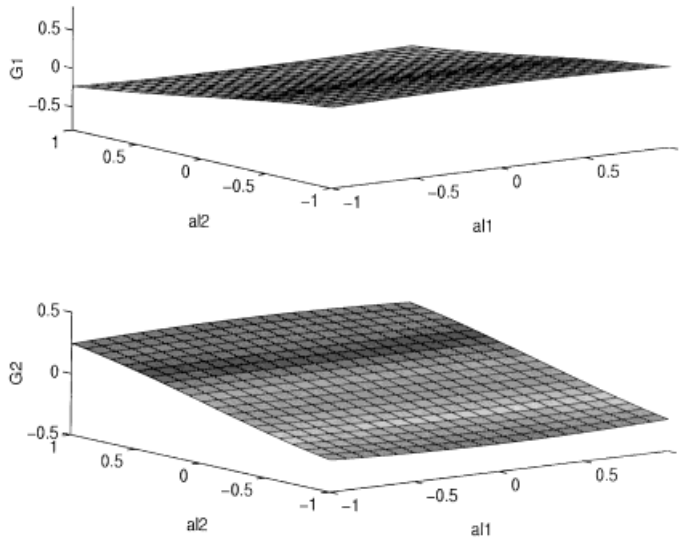


Figure 1. Fibre controllability map

the observed value of the fibre variables, \hat{z} , at the time instant t^k . If $\hat{z}(t^k) = 0$, then $\alpha^k = \alpha^{k-1}$. Suppose $\hat{z}^k \neq 0$. If, based on the steady-state prediction provided by G , the value of α^{k-1} is expected to yield a non-zero net decay in $\|\hat{z}\|$ over the k th period, $kT \leq t \leq (k+1)T$, then $\alpha^k = \alpha^{k-1}$. Otherwise, α^k is selected so that the vector $G(\alpha^k)$ is parallel to $-\hat{z}^k$ and so that the magnitude of α^k is a fraction of the magnitude of α^{k-1} . This selection of α^k guarantees that if the base subsystem (5c), (5d), (8) is in steady state over the k th period, then $\hat{z}^{t^{k+1}} = \hat{z}^{t^k} + G(\alpha^k) = (1 - \rho^k)\hat{z}(t^k)$ and for $\rho^k < 1$, $\|\hat{z}(t^{k+1})\| < \|\hat{z}(t^k)\|$. Note that because of the transients, the actual net change in \hat{z} over the k th period may be different from $G(\alpha^k)$. Since G is open at the origin the required selection of α^k can be always made.

Theorem 2.1 in Reference 9 shows that the feedback law defined according to (7), (8) and Algorithm 2.1 guarantees convergence of all states to the origin for any initial condition.

Actually, a simpler algorithm can be constructed using the results in Reference 9. As shown in Reference 9, global stabilization is guaranteed if the algorithm is based on the approximation of G , \hat{G} , defined by the linear term in the Taylor series expansion of G :

$$\hat{G}(\alpha) = \begin{bmatrix} -\varepsilon\pi\alpha_2 \\ \varepsilon\pi\alpha_1 \end{bmatrix} \tag{13}$$

We arrive at the following algorithm:

Let $v > 0$ be such that for all $\|\alpha\| < v$, $\|\tilde{G}(\alpha)\| < \mu\|\hat{G}(\alpha)\|$ for some $\mu < 1$.

Algorithm 2.2

1. For $k = 0$ (Initialization):
 If $\hat{z}(0) = 0$ and $y(0) = 0$, set $\alpha^0 = 0$;

Else

$$\text{set } \alpha_1^0 = -v\hat{z}_2(0)/\|\hat{z}(0)\|, \alpha_2^0 = v\hat{z}_1(0)/\|\hat{z}(0)\|.$$

2. For $k > 0$: If $\hat{z}(t^k) = 0$, set $\alpha^k = \alpha^{k-1}$.

Else

$$\text{If } -\alpha_2^{k-1}\hat{z}_1(t^k) + \alpha_1^{k-1}\hat{z}_2(t^k) < 0, \text{ set } \alpha^k = \alpha^{k-1}.$$

$$\text{If } -\alpha_2^{k-1}\hat{z}_1(t^k) + \alpha_1^{k-1}\hat{z}_2(t^k) \geq 0, \text{ select}$$

$$\alpha_1^k = -\gamma\hat{z}_2(t^k)\|\alpha^{k-1}\|/\|\hat{z}(t^k)\|$$

$$\alpha_2^k = \gamma\hat{z}_1(t^k)\|\alpha^{k-1}\|/\|\hat{z}(t^k)\|$$

Theorem 2.2 in Reference 9 ensures global convergence of all the states to the origin for the feedback controller defined according to (7), (8) and Algorithm 2.2.

3. EXPERIMENTAL RESULTS

The hybrid feedback controller defined according to (7), (8) and Algorithm 2.2 with $\gamma = 0.7$ has been tested on the experimental RTAC testbed developed by Bupp and coworkers.³ Typical experimental responses are shown in Figures 2–4. Observe the transients of the rotor angle from one periodic cycle to another that result in platform stabilization. Both in terms of speed of response and magnitude of control input the responses are comparable to the best ones reported in Reference 3. Our experience is that the proposed hybrid controller is easy to implement in hardware, easy to tune and provides good closed-loop properties.

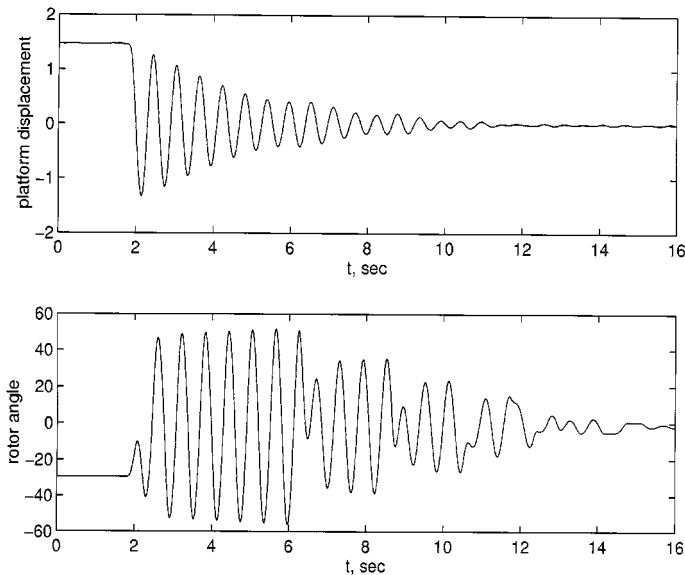


Figure 2. Time histories of the platform displacement (inches) and rotor angle (degrees). The controller is initialized at approximately $t = 2$ s

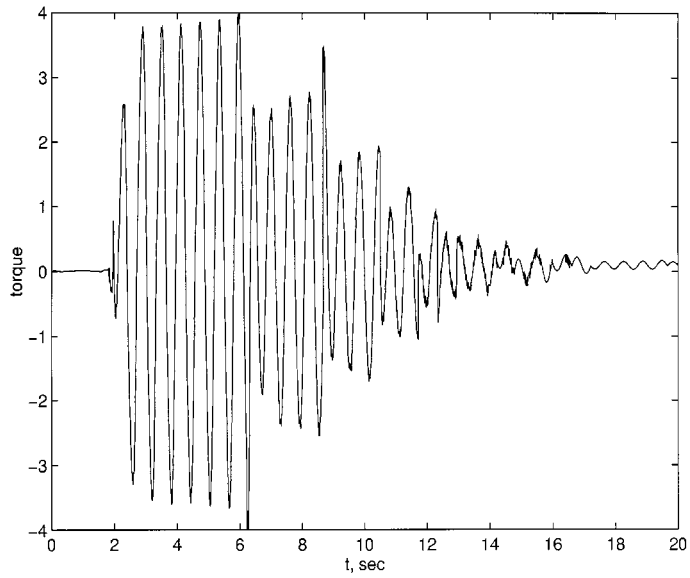


Figure 3. Time history of the torque (oz-inches)

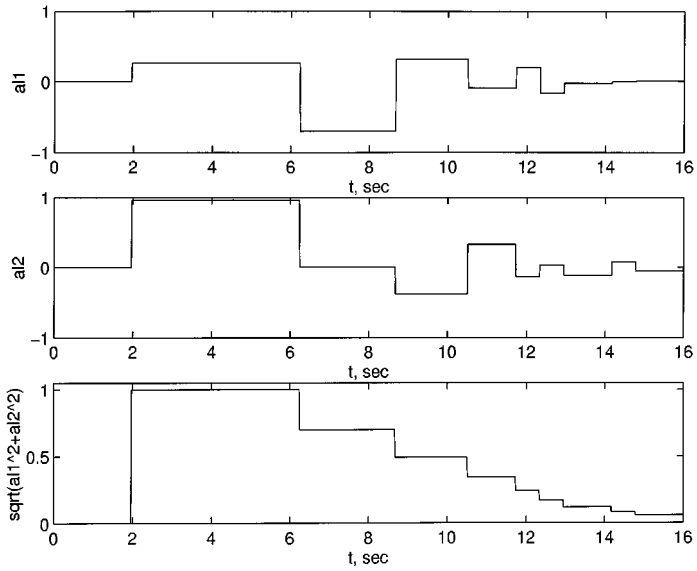


Figure 4. Time history of the control parameters α_1 and α_2

4. REMARKS

Remark 4.1

We have formally indicated, based on our previous theoretical results, that the equilibrium of the closed loop is globally asymptotically stable. This result is important, but other properties, relating to implementation and closed-loop performance, determine the engineering significance of the proposed hybrid control law. Due to the hybrid features of the closed loop, classical methods of analysis are difficult; and formal, strong results for robustness, disturbance attenuation and other closed-loop properties are not currently available. The experimental results indicate that stability and good response properties are maintained for the disturbances and parameter variations that occur in the experimental testbed.

Remark 4.2

Many other choices of the feedback family (7) that result in stabilization are possible. For example, a choice

$$V(\alpha; y; t) = -y_1 - y_2 - \alpha_1 \sin t + \alpha_2 \cos t - \alpha_2 \cos(2t)$$

ensures the continuity of the control signal v .^{*} A selection of the feedback family as

$$V(\alpha; y; t) = -y_1 - y_2 + \alpha_1 + 1.5\alpha_2 \sin(2t) + \alpha_1 \sin t + \alpha_2 \cos t - \alpha_2 \cos(2t)$$

ensures that

$$y_1^{ss}(\alpha; t) = \alpha_1(1 - \cos t) + \alpha_2 \sin t - 0.5\alpha_2 \sin(2t)$$

Hence, $y_1^{ss}(\alpha; kT) = y_1^{ss}(\alpha; kT) = 0$ for all α . This property allows to invoke Theorem 2.3 in Reference 9 that assures that the states converge to the origin at exponential rates. The conditions of Theorem 2.3 in Reference 9 are only sufficient and exponential convergence rates are often exhibited even if the conditions of Theorem 2.3 are not satisfied. For example, this can be observed from the experimental responses in the previous section.

Remark 4.3

Suppose that the initial rotor angle $y(0)$ and the initial rotor velocity $\dot{y}(0)$ are sufficiently small. Then by restricting the magnitude of α^0 , it is possible to satisfy certain state constraints on $y(t)$ and $\dot{y}(t)$. For example, it may be possible to keep the rotor angle within a certain open interval that contains zero.

Remark 4.4

Algorithms 2.1 and 2.2 require calculation of α^k at a time-instant $t^k = kT$, based on the observed value of \dot{z} at the same time instant. Since this calculation must be instantaneous, it may not be feasible from an implementation viewpoint. There are various ways around this difficulty. For example, $\dot{z}(kT)$ can be replaced in Algorithm 2.1 or in Algorithm 2.2 by $\dot{z}((k - \tau)T)$, $0 \leq \tau < 1$. In this case, τT seconds are available to compute α^k . From the results in Reference 9 it follows that stability is maintained for Algorithm 2.2 if α^0 is restricted to be sufficiently small.

^{*}An alternative approach to ensure continuity of v is to augment an integrator to the base dynamics and perform the control design for the augmented system.

Remark 4.5

Stability is not destroyed even when the switchings in Algorithm 2.1 are not implemented exactly at the time instants $t^k = kT$ but take place at time instants $(k + \tau_k)T$, where $0 \leq \tau_k < 1$; see Reference 9.

5. CONCLUSIONS

The proposed hybrid feedback control, illustrated through application of the RTAC example, represents a fundamentally new approach to nonlinear control design. Although the hybrid control design approach and the arguments that support its development are likely to be new to most researchers, the authors believe that the design approach is conceptually simple and effective. The hybrid control structure is consistent with standard digital control implementations, employing nonlinear analog feedback loops and nonlinear digital feedback loops with periodic hold functions.

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