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Hybrid inverse problems and internal functionals

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This paper reviews recent results on hybrid inverse problems, which are also called coupled-physics inverse problems of multiwave inverse problems. Inverse problems tend to be most useful in, e.g., medical and geophysical imaging, when they combine high contrast with high resolution. In some settings, a single modality displays either high contrast or high resolution but not both. In favorable situations, physical effects couple one modality with high contrast with another modality with high resolution. The mathematical analysis of such couplings forms the class of hybrid inverse problems.

Hybrid inverse problems typically involve two steps. In a first step, a well-posed problem involving the high-resolution low-contrast modality is solved from knowledge of boundary measurements. In a second step, a quantitative reconstruction of the parameters of interest is performed from knowledge of the point-wise, internal, functionals of the parameters reconstructed during the first step. This paper reviews mathematical techniques that have been developed in recent years to address the second step.

Mathematically, many hybrid inverse problems find interpretations in terms of linear and nonlinear (systems of) equations. In the analysis of such equations, one often needs to verify that qualitative properties of solutions to elliptic linear equations are satisfied, for instance the absence of any critical points. This paper reviews several methods to prove that such qualitative properties hold, including the method based on the construction of complex geometric optics solutions.

1. Introduction

The success of most medical imaging modalities rests on their high, typically submillimeter, resolution. Computerized tomography (CT), magnetic resonance imaging (MRI), and ultrasound imaging (UI) are typical examples of such modalities. In some situations, these modalities fail to exhibit a sufficient contrast between different types of tissues, whereas other modalities, for example based on the optical, elastic, or electrical properties of these tissues, do display such high contrast. Unfortunately, the latter modalities, such as optical tomography (OT), electrical impedance tomography (EIT) and elastographic imaging (EI), involve a highly smoothing measurement operator and are thus typically low-resolution as stand-alone modalities.

Hybrid inverse problems concern the combination of a high contrast modality with a high resolution modality. By combination, we mean the existence of a physical mechanism that couples these two modalities. Several examples of physical couplings are reviewed in Section 2. A different strategy, consisting of fusing data acquired independently for two or more imaging modalities, is referred to as multimodality imaging and is not considered in this paper. Examples of possible physical couplings include: optics or electromagnetism with ultrasound in photoacoustic tomography (PAT), thermoacoustic tomography (TAT) and in ultrasound modulated optical tomography (UMOT), also called acousto-optic tomography (AOT); electrical currents with ultrasound in ultrasound modulated electrical impedance tomography (UMEIT), also called electroacoustic tomography (EAT); electrical currents with magnetic resonance in magnetic resonance EIT (MREIT) or current density impedance imaging (CDII); and elasticity with ultrasound in transient elastography (TE). Some hybrid modalities have been explored experimentally whereas other hybrid modalities have not been tested yet. Some have received quite a bit of mathematical attention whereas other ones are less well understood. While more references will be given throughout the review, we refer the reader at this point to the recent books [Ammari 2008; Scherzer 2011; Wang and Wu 2007] and their references for general information about practical and theoretical aspects of medical imaging.

Reconstructions in hybrid inverse problems typically involve two steps. In a first step, an inverse problem involving the high-resolution-low-contrast modality needs to be solved. In PAT and TAT for instance, this corresponds to reconstructing the initial condition of a wave equation from available boundary measurements. In UMEIT and UMOT, this corresponds in an idealized setting to inverting a Fourier transform that is reminiscent of the reconstructions performed in MRI. In transient elastography, this essentially corresponds to solving an inverse scattering problem in a time-dependent wave equation. In this review, we assume that this first step has been performed.

Our interest is in the second step of the procedure, which consists of reconstructing the coefficients that display high contrasts from the mappings obtained during the first step. These mappings involve internal functionals of the coefficients of interest. Typically, if γ is a coefficient of interest and u is the solution to a partial differential equation involving γ, then the internal "measurements" obtained in the first step take the form $H(x) = \gamma(x)u^j(x)$ for $j = 1, 2$ or $H(x) = \gamma(x)|\nabla u|^j(x)$ again for $j = 1, 2$.

Several questions can then be raised: are the coefficients, e.g., γ, uniquely characterized by the internal measurements $H(x)$? How stable are the reconstructions? If specific boundary conditions are prescribed at the boundary of the domain of interest, how do the answers to the above questions depend on such boundary conditions? The answers to these questions depend on the physical model of interest. However, there are important common features that we would like to present in this review.

One such feature relates to the stability of the reconstructions. Loosely speaking, an inverse problem is well-posed, or at least not severely ill-posed, when singularities in the coefficients of interest propagate into singularities in the available data. The map reconstructed during step 1 provides local, point-wise, information about the coefficients. Singularities of the coefficient do not need to propagate to the domain's boundary and we thus expect resolution of hybrid modalities to be significantly improved compared to the stand-alone high-contrast-low-resolution modalities. This will be verified in the examples reviewed here.

Another feature is the relationship between hybrid inverse problems and nonlinear partial differential equations. Typically, both the coefficient γ and the solution u are unknown. However, for measurements of the form $H(x) = \gamma(x)u^j(x)$, one can eliminate γ in the equation for u using the expression for $H(x)$. This results in a nonlinear equation for $u(x)$. The resulting nonlinear equations often do not display any of the standard features that are amenable to proofs of uniqueness, such as admitting a variational formulation with a strictly convex functional. The main objective is to obtain uniqueness and stability results for such equations, often in the presence of redundant (overdetermined) information.

A third feature shared by many hybrid inverse problems is that their solution strategies often require that the forward solution u satisfy certain qualitative properties, such as for instance the absence of any critical point (points where $\nabla u = 0$). The derivation of qualitative properties such as lower bounds for the modulus of a gradient is a difficult problem. In two dimensions of space, the fact that critical points of elliptic solutions are necessarily isolated is of great help. In higher dimension, such results no longer hold in general. A framework to obtain the requested qualitative behavior of the elliptic solutions is based on the so-called complex geometric optics (CGO) solutions. Such solutions, when they can be constructed, essentially allow us to treat the unknown coefficients as perturbations of known operators, typically the Laplace operator. Using these solutions, we can construct an open set of boundary conditions for which the requested property is guaranteed. This procedure provides a restricted class of boundary conditions for which the solutions to the hybrid inverse problems are shown to be uniquely and stably determined by the internal measurements. From a practical point of view, these mathematical results confirm the physical intuition that the coupling of high contrast and high resolution modalities indeed provides reconstructions that are robust with respect to errors in the measurements.

The rest of this paper is structured as follows. Section 2 is devoted to the modeling of the hybrid inverse problems and the derivation of the internal measurements for the applications considered in this paper, namely: PAT, TAT, UMEIT, UMOT, TE, CDII. The following two sections present recent results of uniqueness and stability obtained for such hybrid inverse problems: Section 3 focuses on internal functionals of the solution u of the forward problem, whereas Section 4 is concerned with internal functionals of the gradient of the solution ∇u. As we mentioned above, these uniqueness and stability results hinge on the forward solutions u to verify some qualitative properties. Section 5 summarizes some of these properties in the two-dimensional case and presents the derivation of such properties in higher spatial dimensions by means of complex geometric optics (CGO) solutions. Some concluding remarks are proposed in Section 6.

2. Physical modeling

High resolution imaging modalities include ultrasound imaging and magnetic resonance imaging. High contrast modalities include optical tomography, electrical impedance tomography, and elastography. This sections briefly presents four couplings between high-contrast and high-resolution modalities: two different methods to couple ultrasound and optics or (low frequency) electromagnetism in PAT/TAT via the photoacoustic effect and in UMOT/UMEIT via ultrasound modulation; the coupling between ultrasound and elastography in transient elastography; and the coupling between electrical impedance tomography and magnetic resonance imaging in CDII/MREIT.

2A. ***The photoacoustic effect.*** The photoacoustic effect may be described as follows. A pulse of radiation is sent into a domain of interest. A fraction of the propagating radiation is absorbed by the medium. This generates a thermal expansion, which is the source of ultrasonic waves. Ultrasound then propagates to the boundary of the domain where ultrasonic transducers measure the pressure field. The physical coupling between the absorbed radiation and the emitted sound is called the photoacoustic effect. This is the premise for the medical imaging technique photoacoustic tomography (PAT).

Two types of radiation are typically considered. In optoacoustic tomography (OAT), near-infrared photons, with wavelengths typically between $600nm$ and $900nm$ are used. The reason for this frequency window is that they are not significantly absorbed by water molecules and thus can propagate relatively deep into tissues. OAT is often simply referred to as PAT and we will follow this convention here. In thermoacoustic tomography (TAT), low-frequency microwaves, with wavelengths on the order of $1m$, are sent into the medium. The rationale for using such frequencies is that they are less absorbed than optical frequencies and thus propagate into deeper tissues.

In both PAT and TAT, the first step of an inversion procedure is the reconstruction of the map of absorbed radiation from the ultrasonic measurements. In both applications, the inversion may be recast as the reconstruction of an initial condition of a wave equation from knowledge of ultrasound measurements. Assuming a domain of infinite extension with nonperturbative measurements to simplify the presentation, ultrasound propagation is modeled by the wave equation

$$\begin{aligned} &\frac{1}{c_s^2(x)}\frac{\partial^2 p}{\partial t^2} - \Delta p = 0, && t > 0,\ \ x \in \mathbb{R}^n, \\ &p(0,x) = H(x) \quad \text{and} \quad \frac{\partial p}{\partial t}(0,x) = 0, && x \in \mathbb{R}^n. \end{aligned} \tag{1}$$

Here c_s is the speed of sound (assumed to be known), n is spatial dimension, and $H(x)$ is the ultrasonic signal generated at time $t = 0$. Measurements are then of the form $p(t,x)$ for $t > 0$ and $x \in \partial X$ at the boundary of a domain X where $H(x)$ is supported.

Note that the effect of propagating radiation is modeled as an *initial* condition at $t = 0$. The reason for this stems from the large difference between light speed (roughly $2.3\ 10^8 m/s$ in water) and sound speed (roughly $1.5\ 10^3 m/s$ in water). When a short pulse of radiation is emitted into the medium, we may assume that it propagates into the medium at a time scale that is very short compared to that of ultrasound. This is a very valid approximation in PAT but is a limiting factor in the (still significantly submillimeter) spatial resolution we expect to obtain in TAT; see [Bal et al. 2010; 2011b], for example.

For additional references to the photoacoustic effect, we refer the reader to the works [Cox et al. 2009a; 2009b; Fisher et al. 2007; Xu and Wang 2006; Xu et al. 2009] and their references. The first step in thermo- and photoacoustics is the reconstruction of the absorbed radiation map $H(x)$ from boundary acoustic wave measurements. There is a vast literature on this inverse source problem in the mathematical and physical literature; we refer the reader to [Ammari et al. 2010; Finch et al. 2004; Haltmeier et al. 2004; Hristova et al. 2008; Kuchment and Kunyansky 2008; Patch and Scherzer 2007; Stefanov and Uhlmann 2009], for example. Serious difficulties may need to be addressed in this first step, such as limited data, spatially varying acoustic sound speed [Ammari et al. 2010; Hristova et al. 2008; Stefanov and Uhlmann 2009], and the effects of acoustic wave attenuation [Kowar and Scherzer 2012]. In this paper, we assume that the absorbed radiation map $H(x)$ has been reconstructed. This provides now *internal* information about the properties of the domain of interest. What we can extract from such information depends on the model of radiation propagation. The resulting inverse problems are called quantitative PAT (QPAT) and quantitative TAT (QTAT) for the different modalities of radiation propagation, respectively.

In the PAT setting with near-infrared photons, arguably the most accurate model for radiation propagation is the radiative transfer equation. We shall not describe this model here and refer the reader to [Bal et al. 2010] for QPAT in this setting and to [Bal 2009] for more general inverse problems for the radiative transfer equation. The models we consider for radiation propagation are as follows.

QPAT modeling. In the diffusive regime, photon (radiation) propagation is modeled by the second-order elliptic equation

$$\begin{aligned} -\nabla\cdot\gamma(x)\nabla u+\sigma(x)u &= 0 \quad \text{in } X \\ u &= f \quad \text{on } \partial X. \end{aligned} \tag{2}$$

To simplify, we assume that Dirichlet conditions are prescribed at the boundary of the domain ∂X. Throughout the paper, we assume that X is a bounded open domain in $\mathbb{R}^n$ with smooth boundary ∂X. The optical coefficients $(\gamma(x),\sigma(x))$ are $\gamma(x)$ the diffusion coefficient and $\sigma(x)$ the absorption coefficient, which are assumed to be bounded from above and below by positive constants.

The information about the coefficients in QPAT takes the following form:

$$H(x)=\Gamma(x)\sigma(x)u(x) \quad \text{a.e. } x\in X. \tag{3}$$

The coefficient $\Gamma(x)$ is the Grüneisen coefficient. It models the strength of the photoacoustic effect, which converts absorption of radiation into emission of ultrasound. The objective of QPAT is to reconstruct (γ,σ,Γ) from knowledge of $H(x)$ in (3) obtained for a given number of illuminations f in (2). This is an example of an internal measurement that is *linear* in the solution $u(x)$ and the absorption coefficient σ. For more on QPAT, see, e.g., [Ren and Bal 2012; Bal and Uhlmann 2010; Cox et al. 2009a; 2009b; Ripoll and Ntziachristos 2005; Zemp 2010] and their references.

QTAT modeling. Low-frequency radiation in QTAT is modeled by the system of Maxwell's equations

$$\begin{aligned} -\nabla\times\nabla\times E + k^2 E + ik\sigma(x)E &= 0 \quad \text{in } X, \\ \nu\times E &= f \quad \text{on } \partial X. \end{aligned} \tag{4}$$

Here, E is the (time-harmonic) electromagnetic field with fixed wavenumber $k = \frac{\omega}{c}$ where ω is the frequency and c the speed of light. We assume that radiation is controlled by the boundary condition $f(x)$ on ∂X. The unknown coefficient is the conductivity (absorption) coefficient $\sigma(x)$. Setting $\Gamma = 1$ to simplify, the map of absorbed electromagnetic radiation is then of the form

$$H(x) = \sigma(x)|E|^2(x). \tag{5}$$

The above system of equations may be simplified by modeling radiation by a scalar quantity $u(x)$. In this setting, radiation is modeled by the Helmholtz equation

$$\begin{aligned} \Delta u + k^2 u + ik\sigma(x)u &= 0 \quad \text{in } X \\ u &= f \quad \text{on } \partial X, \end{aligned} \tag{6}$$

for a given boundary condition $f(x)$. The internal data are then of the form

$$H(x) = \sigma(x)|u|^2(x). \tag{7}$$

For such models, QTAT then consists of reconstructing $\sigma(x)$ from knowledge of $H(x)$. Note that $H(x)$ is now a *quadratic* quantity in the solutions $E(x)$ or $u(x)$. There are relatively few results on QTAT; see [Bal et al. 2011b; Li et al. 2008].

2B. ***The ultrasound modulation effect.*** We consider the elliptic equation

$$\begin{aligned} -\nabla\cdot\gamma(x)\nabla u + \sigma(x)u &= 0 \quad \text{in } X, \\ u &= f \quad \text{on } \partial X. \end{aligned} \tag{8}$$

The objective of ultrasound modulation is to send an acoustic signal through the domain X that modifies the coefficients γ and σ. We assume here that the sound speed is constant and that we are able to generate an acoustic signal that takes the form of the plane wave $p\cos(k\cdot x + \varphi)$ where p is amplitude, k wave-number and φ an additional phase. We assume that the acoustic signal modifies the properties of the diffusion equation and that the effect is small. The coefficients in (8) are thus modified as

$$\gamma_\varepsilon(x) = \gamma(x)(1+\zeta\varepsilon\mathfrak{c}) + O(\varepsilon^2), \quad \sigma_\varepsilon(x) = \sigma(x)(1+\eta\varepsilon\mathfrak{c}) + O(\varepsilon^2), \tag{9}$$

where we have defined $\mathfrak{c} = \mathfrak{c}(x) = \cos(k\cdot x+\varphi)$ and where $\varepsilon = p\Gamma$ is the product of the acoustic amplitude $p \in \mathbb{R}$ and a measure $\Gamma > 0$ of the coupling between the acoustic signal and the modulations of the constitutive parameters in (8). We assume that $\varepsilon \ll 1$. The terms in the expansion are characterized by ζ and η and depend on the specific application.

Let u and v be solutions of (8) with fixed boundary conditions f and h, respectively. When the acoustic field is turned on, the coefficients are modified as described in (9) and we denote by u_ε and v_ε the corresponding solution. Note that $u_{-\varepsilon}$ is the solution obtained by changing the sign of p or equivalently by replacing φ by $\varphi+\pi$.

By standard regular perturbation arguments, we find that $u_\varepsilon = u_0 + \varepsilon u_1 + O(\varepsilon^2)$. Multiplying the equation for u_ε by $v_{-\varepsilon}$ and the equation for $v_{-\varepsilon}$ by u_ε, subtracting the results, and using standard integrations by parts, we obtain that

$$\int_X (\gamma_\varepsilon-\gamma_{-\varepsilon})\nabla u_\varepsilon\cdot\nabla v_{-\varepsilon}+(\sigma_\varepsilon-\sigma_{-\varepsilon})u_\varepsilon v_{-\varepsilon}dx = \int_{\partial X}\gamma_{-\varepsilon}\frac{\partial v_{-\varepsilon}}{\partial\nu}u_\varepsilon-\gamma_\varepsilon\frac{\partial u_\varepsilon}{\partial\nu}v_{-\varepsilon}d\sigma. \tag{10}$$

We assume that $\gamma_\varepsilon\partial_\nu u_\varepsilon$ and $\gamma_\varepsilon\partial_\nu v_\varepsilon$ are measured on ∂X, at least on the support of $v_\varepsilon = h$ and $u_\varepsilon = f$, respectively, for several values ε of interest. The above equation also holds if the Dirichlet boundary conditions are replaced by Neumann boundary conditions. Let us define

$$J_\varepsilon := \frac{1}{2}\int_{\partial X}\gamma_{-\varepsilon}\frac{\partial v_{-\varepsilon}}{\partial\nu}u_\varepsilon-\gamma_\varepsilon\frac{\partial u_\varepsilon}{\partial\nu}v_{-\varepsilon}d\sigma \;=\; \varepsilon J_1+\varepsilon^2 J_2+O(\varepsilon^3). \tag{11}$$

We assume that the real valued functions $J_m = J_m(k,\varphi)$ are known from the physical measurement of the Cauchy data of the form $(u_\varepsilon, \gamma_\varepsilon\partial_\nu u_\varepsilon)$ and $(v_\varepsilon, \gamma_\varepsilon\partial_\nu v_\varepsilon)$ on ∂X.

Equating like powers of ε, we find that at the leading order

$$\int_X \big[\zeta\gamma(x)\nabla u_0\cdot\nabla v_0(x)+\eta\sigma(x)u_0v_0(x)\big]\cos(k\cdot x+\varphi)\,dx = J_1(k,\varphi). \tag{12}$$

Acquiring this for all $k\in\mathbb{R}^n$ and $\varphi = 0, \frac{\pi}{2}$, this yields after inverse Fourier transform:

$$H[u_0,v_0](x) = \zeta\gamma(x)\nabla u_0\cdot\nabla v_0(x)+\eta\sigma(x)u_0v_0(x). \tag{13}$$

In the setting of ultrasound modulated optical tomography (UMOT), the coefficients γ_ε and σ_ε in (9) take the form [Bal and Schotland 2010]

$$\gamma_\varepsilon(x) = \frac{\tilde{\gamma}_\varepsilon}{c_\varepsilon^{n-1}}(x) \quad \text{and} \quad \sigma_\varepsilon(x) = \frac{\tilde{\sigma}_\varepsilon}{c_\varepsilon^{n-1}}(x),$$

where $\tilde{\sigma}_\varepsilon$ is the absorption coefficient, $\tilde{\gamma}_\varepsilon$ is the diffusion coefficient, c_ε is the light speed, and n is spatial dimension. When the pressure field is turned on, the amount of scatterers and absorbers is modified by compression and dilation. Since the diffusion coefficient is inversely proportional to the scattering coefficient, we find that

$$\tilde{\sigma}_\varepsilon(x) = \tilde{\sigma}(x)\big(1+\varepsilon\mathfrak{c}(x)\big), \qquad \frac{1}{\gamma_\varepsilon(x)} = \frac{1}{\gamma(x)}\big(1+\varepsilon\mathfrak{c}(x)\big).$$

The pressure field changes the index of refraction of light as follows $c_\varepsilon(x) = c(x)(1+\psi\varepsilon\mathfrak{c}(x))$, where ψ is a constant (roughly equal to $\frac{1}{3}$ for water). This shows that

$$\zeta = -(1+(n-1)\psi), \quad \eta = 1-(n-1)\psi. \tag{14}$$

In the application of ultrasound modulated electrical impedance tomography (UMEIT), $\gamma(x)$ is a conductivity coefficient and $\sigma = 0$. We then have $\gamma_\varepsilon(x) = \gamma(x)(1+\varepsilon\mathfrak{c}(x))$ with thus $\zeta = 1$ and $\eta = 0$. The objective of UMOT and UMEIT is to reconstruct (part of) the coefficients $(\gamma(x), \sigma(x))$ in the elliptic equation

$$\begin{aligned} -\nabla\cdot\gamma(x)\nabla u + \sigma(x)u &= 0 \quad \text{in } X, \\ u &= f \quad \text{on } \partial X \end{aligned} \tag{15}$$

from measurements of the form

$$H[u_0, v_0](x) = \zeta\gamma(x)\nabla u_0\cdot\nabla v_0(x) + \eta\sigma(x)u_0v_0(x), \tag{16}$$

for one or several values of the illumination $f(x)$ on ∂X.

In a simplified version of UMOT (also called acousto-optic tomography; AOT), $\zeta = 0$ and the measurements are quadratic (or bilinear) in the solutions to the elliptic equation. More challenging mathematically is the case $\zeta = 1$ and $\eta = 0$ where the measurements are quadratic (or bilinear) in the *gradients* of the solution. No theoretical results exist to date in the setting where both ζ and η are nonvanishing.

The effect of ultrasound modulation is difficult to observe experimentally as the coupling coefficient Γ above is rather small. For references on ultrasound modulation in different contexts, see [Ammari et al. 2008; Bal 2012; Bal and Schotland 2010; Capdeboscq et al. 2009; Gebauer and Scherzer 2008; Kuchment and Kunyansky 2011; Zhang and Wang 2004]. These references concern the so-called incoherent regime of wave propagation, while the coherent regime, whose mathematical structure is different, is addressed in the physical literature in, e.g., [Atlan et al. 2005; Kempe et al. 1997; Wang 2004].

2C. ***Transient elastography.*** Transient elastography images the (slow) propagation of shear waves using ultrasound. For more details, see [McLaughlin et al. 2010] and its extended list of references. As shear waves propagate, the resulting displacements can be imaged by ultrafast ultrasound. Consider a scalar approximation of the equations of elasticity

$$\begin{aligned} \nabla\cdot\gamma(x)\nabla u(x,t) &= \rho(x)\partial_{tt}u(x,t), \quad && t\in\mathbb{R},\ x\in X,\\ u(x,t) &= f(x,t), && t\in\mathbb{R},\ x\in\partial X, \end{aligned} \tag{17}$$

where $u(x,t)$ is the (say, downward) displacement, $\gamma(x)$ is one of the Lamé parameters and $\rho(x)$ is density. Using ultrafast ultrasound measurements, the displacement $u(x,t)$ can be imaged. This results in a very simplified model of transient elastography where we aim to reconstruct (γ,ρ) from knowledge of $u(x,t)$; see [McLaughlin et al. 2010] for more complex models. We may slightly generalize the model as follows. Upon taking Fourier transforms in the time domain and accounting for possible dispersive effects of the tissues, we obtain

$$\begin{aligned} \nabla\cdot\gamma(x;\omega)\nabla u(x;\omega)+\omega^2\rho(x;\omega)u(x;\omega) &= 0, \quad \omega\in\mathbb{R},\ x\in X,\\ u(x;\omega) &= f(x;\omega), \quad \omega\in\mathbb{R},\ x\in\partial X. \end{aligned} \tag{18}$$

The inverse transient elastography problem with dispersion effect would then be the reconstruction of $(\gamma(x;\omega),\rho(x;\omega))$ from knowledge of $u(x;\omega)$ corresponding to one or several boundary conditions $f(x;\omega)$ applied at the boundary ∂X. This hybrid inverse problem again involves measurements that are *linear* in the solution u.

2D. ***Current density imaging.*** Magnetic impedance electrical impedance tomography (MREIT) and current density impedance imaging (CDII) are two modalities aiming to reconstruct the conductivity in an equation using magnetic resonance imaging (MRI). The electrical potential u solves the following elliptic equation

$$\begin{aligned} -\nabla\cdot\gamma(x)\nabla u &= 0 \quad \text{in } X,\\ u &= f \quad \text{on } \partial X, \end{aligned} \tag{19}$$

with $\gamma(x)$ the unknown conductivity and f a prescribed voltage at the domain's boundary. The electrical current density $J=-\gamma\nabla u$ satisfies the system of Maxwell's equations

$$\nabla\cdot J=0,\quad J=\frac{1}{\mu_0}\nabla\times B,\qquad x\in X. \tag{20}$$

Here μ_0 is a constant, known, magnetic permeability.

Ideally, the whole field B can be reconstructed from MRI measurements. This provides access to the current density $J(x)$ in the whole domain X. CDI then corresponds to reconstructing γ from knowledge of J. In practice, acquiring B requires rotation of the domain of interest (or of the MRI apparatus) which is not straightforward. MREIT thus assumes knowledge of the third component B_z of the magnetic field for several possible boundary conditions. This provides information about $\gamma(x)$. We do not consider the MREIT inverse problem further and refer the reader to the recent review [Seo and Woo 2011] and its references for additional information.

Several works have considered the problem of the reconstruction of γ in (19) from knowledge of the scalar information $|J|$ rather than the full current J. This inverse problem, referred to as the 1-Laplacian, will be addressed below and compared to the 0-Laplacian that appears in UMEIT and UMOT. For more on MREIT and CDII, we refer the reader to [Kim et al. 2002; Nachman et al. 2007; 2009; 2011] and their references.

3. Reconstructions from functionals of *u*

In this section, we consider internal measurements $H(x)$ of the form $H(x) = \tau(x)u(x)$ for $\tau(x)$ a function that depends linearly on unknown coefficients such as the diffusion coefficient γ or the absorption coefficient σ in Section 3A and internal measurements $H(x)$ of the form $H(x) = \tau(x)|u(x)|^2$ in Section 3B, where τ again depends linearly on unknown coefficients. Measurements of the first form find applications in quantitative photoacoustic tomography (QPAT) and transient elastography (TE) while measurements of the second form find applications in quantitative thermoacoustic tomography (QTAT) and simplified models of acousto-optics tomography (AOT).

3A. ***Reconstructions from linear functionals in u.*** Recall the elliptic model for photon propagation in tissues:

$$\begin{aligned} -\nabla\cdot\gamma(x)\nabla u+\sigma(x)u &= 0 \quad \text{in } X,\\ u &= f \quad \text{on } \partial X. \end{aligned} \tag{21}$$

The information about the coefficients in QPAT takes the form

$$H(x) = \Gamma(x)\sigma(x)u(x) \quad \text{a.e. } x\in X. \tag{22}$$

The coefficient $\Gamma(x)$ is the Grüneisen coefficient. In many works in QPAT, it is assumed to be constant. We assume here that it is Lipschitz continuous and bounded above and below by positive constants.

Nonunique reconstruction of three coefficients. Let f_1 and f_2 be two Dirichlet conditions on ∂X and u_1 and u_2 be the corresponding solutions to (21). We make the following assumptions:

(i) The coefficients (γ, σ, Γ) are of class $W^{1,\infty}(X)$ and bounded above and below by positive constants. The coefficients (γ, σ, Γ) are known on ∂X.

(ii) The illuminations f_1 and f_2 are positive functions on ∂X and are the traces on ∂X of functions of class $C^3(\bar{X})$.

(iii) the vector field

$$\beta := H_1 \nabla H_2 - H_2 \nabla H_1 = H_1^2 \nabla \frac{H_2}{H_1} = H_1^2 \nabla \frac{u_2}{u_1} = -H_2^2 \nabla \frac{H_1}{H_2} \tag{23}$$

is a vector field in $W^{1,\infty}(X)$ such that $\beta \not\equiv 0$ a.e.

(iii′) Same as (iii) above with

$$|\beta|(x) \geq \alpha_0 > 0 \quad \text{a.e. } x \in X. \tag{24}$$

Beyond the regularity assumptions on (γ, σ, Γ), the domain X, and the boundary conditions f_1 and f_2, the only real assumption we impose is (24). In general, there is no guaranty that the gradient of u_2/u_1 does not vanish. In dimension $d = 2$, a simple condition guarantees that (24) holds. We have the following result [Alessandrini 1986; Nachman et al. 2007]:

Lemma 3.1 [Bal and Ren 2011a]. *Assume that $h = g_2/g_1$ on ∂X is an almost two-to-one function in the sense of* [Nachman et al. 2007], *i.e., a function that is a two-to-one map except possibly at its minimum and at its maximum. Then* (24) *is satisfied.*

In dimension $d \geq 3$, the above result on the (absence of) critical points of elliptic solutions no longer holds. By continuity, we verify that (24) is satisfied for a large class of illuminations when γ is close to a constant and σ is sufficiently small. For arbitrary coefficients (γ, σ) in dimension $d \geq 3$, a proof based on CGO solutions shows that (24) is satisfied for an open set of illuminations; see [Bal and Uhlmann 2010] and Section 5B below. Note also that (24) is a sufficient condition for us to solve the inverse problem of QPAT. In [Alessandrini 1986], a similar problem is addressed in dimension $d = 2$ without assuming a constraint of the form (24).

We first prove a result that provides uniqueness up to a specified transformation.

Theorem 3.2 [Bal and Ren 2011a; Bal and Uhlmann 2010]. *Assume that the hypotheses* (i)–(iii) *hold.*

(a) *$H_1(x)$ and $H_2(x)$ uniquely determine the measurement operator*

$$\mathcal{H} : H^{\frac{1}{2}}(\partial X) \to H^1(X),$$

which to f defined on ∂X associates $\mathcal{H}(f) = H$ in X defined by (3).

(b) *The measurement operator $\mathcal{H}$ uniquely determines the two functionals*

$$\chi(x) := \frac{\sqrt{\gamma}}{\Gamma\sigma}(x), \quad q(x) := -\Big(\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} + \frac{\sigma}{\gamma}\Big)(x). \tag{25}$$

Here Δ is the Laplace operator.

(c) *Knowledge of the two functionals χ and q uniquely determines $H_1(x)$ and $H_2(x)$. In other words, the reconstruction of (γ, σ, Γ) is unique up to transformations that leave (χ, q) invariant.*

The proof of this theorem is given in [Bal and Ren 2011a] under the additional assumption (iii′). The following minor modification allows one to prove the theorem as stated above. The original proof is based on the fact that the equality $\int_X (\rho - 1)^2 |\beta|^2 dx = 0$ implies that $\rho = 1$ a.e. Such a result clearly still holds provided that $\beta \neq 0$ a.e.

Note that the result $\beta \neq 0$ a.e. holds for a very large class of boundary conditions (f_1, f_2). Indeed, β is the solution of (26) below with χ^2 bounded from below by a positive constant. This implies that u_2/u_1 is the solution of an elliptic equation.

Thus, the set $\beta = 0$ corresponds to the set of critical points $\nabla(u_2/u_1) = 0$. When u_2/u_1 is not constant, it is proved that for such elliptic equations, the set of critical points $\nabla(u_2/u_1) = 0$ is of Lebesgue measure zero provided that the coefficients in (21) are sufficiently smooth [Hardt et al. 1999; Robbiano and Salazar 1990]. We thus find that so long that f_1/f_2 is not a constant a.e., then $\beta \neq 0$ a.e. and the two internal functionals (H_1, H_2) uniquely characterize the coefficients (χ, q).

Reconstruction of two coefficients. The above result shows that the unique reconstruction of (γ, σ, Γ) is not possible even from knowledge of the full measurement operator $\mathcal{H}$ defined in Theorem 3.2. Two well-chosen illuminations uniquely determine the functionals (χ, q) and acquiring additional measurements does not provide any new information. However, we can prove that if one coefficient in (γ, σ, Γ) is known, then the other two coefficients are uniquely determined:

Corollary 3.3 [Bal and Ren 2011a]. *Under the hypotheses of Theorem 3.2, let (χ, q) in* (25) *be known.*

(a) *If Γ is known, then (γ, σ) are uniquely determined.*

(b) *If γ is known, then (σ, Γ) are uniquely determined.*

(c) *If σ is known, then (γ, Γ) are uniquely determined.*

The above uniqueness results are *constructive*. In all cases, we need to solve the following transport equation for χ:

$$-\nabla\cdot(\chi^2\beta)=0 \quad \text{in } X, \qquad \chi_{|\partial X} \text{ known on } \partial X, \tag{26}$$

with β the vector field defined in (23). This uniquely defines $\chi > 0$. Then we find that

$$q(x) = -\frac{\Delta(H_1\chi)}{H_1\chi} = -\frac{\Delta(H_2\chi)}{H_2\chi}. \tag{27}$$

This provides explicit reconstructions for (χ, q) from knowledge of (H_1, H_2) when (24) holds.

In case (b), no further equation needs to be solved. In cases (a) and (c), we need to solve an elliptic equation for $\sqrt{\gamma}$, which is the linear equation

$$\begin{aligned}(\Delta+q)\sqrt{\gamma}+\frac{1}{\Gamma\chi}&=0 && \text{in } X,\\ \sqrt{\gamma}_{|\partial X}&=\sqrt{\gamma_{|\partial X}} && \text{on } \partial X,\end{aligned} \tag{28}$$

in (a) and the (uniquely solvable) nonlinear (semilinear) equation

$$\begin{aligned}\sqrt{\gamma}(\Delta+q)\sqrt{\gamma}+\sigma&=0 && \text{in } X,\\ \sqrt{\gamma}_{|\partial X}&=\sqrt{\gamma_{|\partial X}} && \text{on } \partial X,\end{aligned} \tag{29}$$

in (c). These inversion formulas were implemented numerically in [Bal and Ren 2011a]. Moreover, reconstructions are known to be Hölder or Lipschitz stable depending on the metric used in the stability estimate. For instance:

Theorem 3.4 [Bal and Ren 2011a]. *Assume that the hypotheses of Theorem 3.2 and* (iii′) *hold. Let* $H=(H_1,H_2)$ *be the measurements corresponding to the coefficients* (γ,σ,Γ) *for which hypothesis* (iii) *holds. Let* $\tilde{H}=(\tilde{H}_1,\tilde{H}_2)$ *be the measurements corresponding to the same illuminations* (f_1,f_2) *with another set of coefficients* $(\tilde{\gamma},\tilde{\sigma},\tilde{\Gamma})$ *such that* (i) *and* (ii) *still hold. Then we find that*

$$\|\chi-\tilde{\chi}\|_{L^p(X)} \le C\|H-\tilde{H}\|^{\frac{1}{2}}_{(W^{1,\frac{p}{2}}(X))^2} \qquad \textit{for all } 2\le p<\infty. \tag{30}$$

Let us assume, moreover, that $\gamma(x)$ *is of class* $C^3(\bar{X})$. *Then we have that*

$$\|\chi-\tilde{\chi}\|_{L^\infty(X)} \le C\|H-\tilde{H}\|^{\frac{p}{3(d+p)}}_{(L^{\frac{p}{2}}(X))^2} \qquad \textit{for all } 2\le p<\infty. \tag{31}$$

We may for instance choose $p=4$ above to measure the noise level in the measurement H in the square integrable norm when noise is described by its power spectrum in the Fourier domain. This shows that reconstructions in QPAT are Hölder stable, unlike the corresponding reconstructions in optical tomography [Bal 2009; Uhlmann 2009].

An application to transient elastography. We can apply the above results to the time-harmonic reconstruction in a simplified model of transient elastography. Let us assume that γ and ρ are unknown functions of $x \in X$ and $\omega \in \mathbb{R}$. Recall that the displacement solves (18). Assuming that $u(x;\omega)$ is known after step 1 of the reconstruction using the ultrasound measurements, then we are in the setting of Theorem 3.2 with $\Gamma\sigma = 1$. Let us then assume that the two illuminations $f_1(x;\omega)$ and $f_2(x;\omega)$ are chosen such that for u_1 and u_2 the corresponding solutions of (18), we have that (24) holds. We have seen a sufficient condition for this to hold in dimension $n = 2$ in Lemma 3.1 and will present other sufficient conditions in Section 5B, which is devoted to CGO solutions in the setting $n \geq 3$. Then, (25) shows that the reconstructed function χ uniquely determines the Lamé parameter $\gamma(x;\omega)$ and that the reconstructed function q then uniquely determines $\omega^2\rho$ and hence the density parameter $\rho(x;\omega)$. The reconstructions are performed for each frequency ω independently. We may summarize this as follows:

Corollary 3.5. *Under the hypotheses of Theorem 3.2 and the hypotheses described above, let (χ, q) in* (25) *be known. Then $(\gamma(x;\omega), \rho(x;\omega))$ are uniquely determined by two well-chosen measurements. Moreover, the stability results in Theorem 3.4 hold.*

Alternatively, we may assume that in a given range of frequencies, $\gamma(x)$ and $\rho(x)$ are independent of ω. In such a setting, we expect that one measurement $u(x;\omega)$ for two different frequencies will provide sufficient information to reconstruct $(\gamma(x), \rho(x))$. Assume that $u(x;\omega)$ is known for $\omega = \omega_j$, $j = 1, 2$ and define $0 < \alpha = \omega_2^2\omega_1^{-2} \neq 1$. Then straightforward calculations show that

$$\nabla \cdot \gamma\beta_\alpha = 0, \quad \beta_\alpha = (u_1\nabla u_2 - \alpha u_2 \nabla u_1). \tag{32}$$

This provides a transport equation for γ that can be solved stably provided that $|\beta_\alpha| \geq c_0 > 0$, i.e., β_α does not vanish on X. Then, Theorem 3.2 and Theorem 3.4 apply in this setting. Since β_α cannot be written as the ratio of two solutions as in (23) when $\alpha = 1$, the results obtained in Lemma 3.1 do not apply when $\alpha \neq 1$. However, we prove in Section 5B that $|\beta_\alpha| \geq c_0 > 0$ is satisfied for an open set of illuminations constructed by means of CGO solutions for all $\alpha > 0$; see (96) below.

Reconstruction of one coefficient. Let us conclude this section by some comments on the reconstruction of a single coefficient from a measurement linear in u. From an algorithmic point of view, such reconstructions are significantly simpler. Let us consider the framework of Corollary 3.3. When Γ is the only unknown coefficient, then we solve for u in (21) and reconstruct Γ from knowledge of H.

When only σ is unknown, then we solve the elliptic equation for u

$$\begin{aligned} -\nabla\cdot\gamma\nabla u + \frac{H}{\Gamma} &= 0 \quad \text{in } X, \\ u &= g \quad \text{on } \partial X, \end{aligned}$$

and then evaluate $\sigma = H/(\Gamma u)$.

When only γ is unknown with either $H = \sigma u$ in QPAT or with $H = u$ in elastography or in applications to ground water flows [Alessandrini 1986; Richter 1981], then u is known and γ solves the transport equation

$$\begin{aligned} -\nabla\cdot\gamma\nabla u &= S \qquad \text{in } X, \\ \gamma &= \gamma_{|\partial X} \quad \text{on } \partial X, \end{aligned}$$

with S known. Provided that the vector field ∇u does not vanish, the above equation admits a unique solution as in (26). The stability results of Theorem 3.4 then apply. Other stability results based on solving the transport equation by the method of characteristics are presented in [Richter 1981]. In two dimensions of space, the constraint that the vector field ∇u does not vanish can be partially removed. Under appropriate conditions on the oscillations of the illumination g on ∂X, stability results are obtained in [Alessandrini 1986] in cases where ∇u is allowed to vanish.

3B. *Reconstructions from quadratic functionals in u.*

Reconstructions under smallness conditions. The TAT and (simplified) AOT problems are examples of a more general class we define as follows. Let $P(x, D)$ be an operator acting on functions defined in $\mathbb{C}^m$ for $m \in \mathbb{N}^*$ an integer and with values in the same space. Consider the equation

$$\begin{aligned} P(x, D)u &= \sigma(x)u, \quad x \in X, \\ u &= f, \qquad\quad x \in \partial X. \end{aligned} \tag{33}$$

We assume that the above equation admits a unique weak solution in some Hilbert space $\mathcal{H}_1$ for sufficiently smooth illuminations $f(x)$ on ∂X.

For instance, P could be the Helmholtz operator $ik^{-1}(\Delta + k^2)$ seen in the preceding section with $u \in \mathcal{H}_1 := H^1(X;\mathbb{C})$ and $f \in H^{\frac{1}{2}}(\partial X;\mathbb{C})$. Time-harmonic Maxwell's equations can be put in that framework with m and

$$P(x, D) = \frac{1}{ik}(\nabla\times\nabla\times - k^2). \tag{34}$$

We impose an additional constraint on $P(x, D)$ that the equation $P(x, D)u = f$ on X with $u = 0$ on ∂X admits a unique solution in $\mathcal{H} = L^2(X;\mathbb{C}^m)$. For Maxwell's equations, this constraint is satisfied so long as k^2 is not an internal eigenvalue of the Maxwell operator [Dautray and Lions 1990]. This is expressed by the existence of a constant $\alpha > 0$ such that:

$$(P(x, D)u, u)_{\mathcal{H}} \geq \alpha(u, u)_{\mathcal{H}}. \tag{35}$$

We assume that the conductivity σ is bounded from above by a positive constant:

$$0 < \sigma(x) \leq \sigma_M \qquad \text{a.e. } x \in X. \tag{36}$$

We denote by Σ_M the space of functions $\sigma(x)$ such that (36) holds. Measurements are of the form

$$H(x) = \sigma(x)|u|^2,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{C}^m$. Then we have the following result.

Theorem 3.6 [Bal et al. 2011b]. *Let $\sigma_j \in \Sigma_M$ for $j = 1, 2$. Let u_j be the solution to $P(x, D)u_j = \sigma_j u_j$ in X with $u_j = f$ on ∂X for $j = 1, 2$. Define the internal functionals $H_j(x) = \sigma_j(x)|u_j(x)|^2$ on X.*

Assume that σ_M is sufficiently small that $\sigma_M < \alpha$. Then:

(i) (Uniqueness) *If $H_1 = H_2$ a.e. in X, then $\sigma_1(x) = \sigma_2(x)$ a.e. in X where $H_1 = H_2 > 0$.*

(ii) (Stability) *We have the stability estimate*

$$\big\|(\sqrt{\sigma_1} - \sqrt{\sigma_2})w_1\big\|_{\mathcal{H}} \leq C\big\|(\sqrt{H_1} - \sqrt{H_2})w_2\big\|_{\mathcal{H}}, \tag{37}$$

for some universal constant C and for positive weights given by

$$\begin{aligned} w_1^2(x) &= \prod_{j=1,2} \frac{|u_j|}{\sqrt{\sigma}_j}(x), \\ w_2(x) &= \frac{1}{\alpha - \sup\limits_{x\in X} \sqrt{\sigma_1\sigma_2}} \max_{j=1,2} \frac{\sqrt{\sigma}_j}{|u_{j'}|}(x) + \max_{j=1,2} \frac{1}{\sqrt{\sigma}_j}(x). \end{aligned} \tag{38}$$

Here $j' = j'(j)$ is defined as $j'(1) = 2$ and $j'(2) = 1$.

The theorem uses the spectral gap in (35). Some straightforward algebra shows that

$$P(x, D)(u_1 - u_2) = \sqrt{\sigma_1\sigma_2}\big(|u_2|\hat{u}_1 - |u_1|\hat{u}_2\big) + (\sqrt{H_1} - \sqrt{H_2})\bigg(\frac{\sqrt{\sigma_1}}{|u_1|} - \frac{\sqrt{\sigma_2}}{|u_2|}\bigg).$$

Here we have defined $\hat{u} = u/|u|$. Although this does not constitute an equation for $u_1 - u_2$, it turns out that

$$\big||u_2|\hat{u}_1 - |u_1|\hat{u}_2\big| = |u_2 - u_1|.$$

This combined with (35) yields the theorem after some elementary manipulations; see [Bal et al. 2011b].

Reconstructions for the Helmholtz equation. Let us consider the scalar model of TAT. We assume that $\sigma \in H^p(X)$ for $p > n/2$ and construct

$$q(x) = k^2 + ik\sigma(x) \in H^p(X), \quad p > \frac{n}{2}. \tag{39}$$

We assume that $q(x)$ is the restriction to X of the compactly supported function (still called q) $q \in H^p(\mathbb{R}^n)$. The extension is chosen so that $\|q_{|X}\|_{H^p(X)} \leq C\|q\|_{H^p(\mathbb{R}^n)}$ for some constant C independent of q; see [Bal and Uhlmann 2010]. Then (6) may be recast as

$$\begin{aligned} \Delta u + q(x)u &= 0 \quad \text{in } X, \\ u &= f \quad \text{on } \partial X. \end{aligned} \tag{40}$$

The measurements are of the form $H(x) = \sigma(x)|u|^2(x)$.

The inverse problem consists of reconstructing $q(x)$ from knowledge of $H(x)$. Note that $q(x)$ need not be of the form (39). It could be a real-valued potential in a Helmholtz equation as considered in [Triki 2010] with applications in the so-called inverse medium problem. The reconstruction of $q(x)$ in (40) from knowledge of $H(x) = \sigma(x)|u|^2(x)$ has been analyzed in [Bal et al. 2011b; Triki 2010]. We cite two stability results, one global and one local.

Theorem 3.7 [Bal et al. 2011b]. *Let $Y = H^p(X)$ and $Z = H^{p-\frac{1}{2}}(\partial X)$, where $p > n/2$. Let $\mathcal{M}$ be the space of functions in Y with norm bounded by a fixed (arbitrary) $M > 0$. Let σ and $\tilde{\sigma}$ be functions in $\mathcal{M}$. Let $f \in Z$ be a given (complex-valued) illumination and $H(x)$ the measurement given in* (7) *for a solution u of* (6)*. Define $\tilde{H}(x)$ similarly, with $\tilde{\sigma}$ replacing σ in* (7) *and* (6)*.*

Then there is an open set of illuminations f in Z such that $H(x) = \tilde{H}(x)$ in Y implies that $\sigma(x) = \tilde{\sigma}(x)$ in Y. Moreover, there exists a constant C independent of σ and $\tilde{\sigma}$ in $\mathcal{M}$ such that

$$\|\sigma - \tilde{\sigma}\|_Y \leq C\|H - \tilde{H}\|_Y. \tag{41}$$

The theorem is written in terms of σ, which is the parameter of interest in TAT. The same result holds if σ is replaced by $q(x)$ in (41). The reconstruction of σ is also constructive as the application of a Banach fixed point theorem. The proof is based on the construction of complex geometric optics solutions that will be presented in Section 5B.

Theorem 3.8 [Triki 2010]. *Let $q(x) \geq c_0 > 0$ be real-valued, positive, bounded on X and such that 0 is not an eigenvalue of $\Delta + q$ with domain equal to*

$H_0^1(X) \cap H_2(\bar{X})$. *Let* $\tilde{q}$ *satisfy the same hypotheses and let* H *and* $\tilde{H}$ *be the corresponding measurements.*

Then there is a constant $\varepsilon > 0$ *such that if* q *and* $\tilde{q}$ *are* ε*-close in* $L^\infty(X)$ *and if* f *is in an* ε*-dependent open set of (complex-valued) illuminations, then there is a constant* C *such that*

$$\|q - \tilde{q}\|_{L^2(X)} \le C \|H - \tilde{H}\|_{L^2(X)}. \tag{42}$$

Theorems 3.7 and 3.8 show that the TAT and the inverse medium problem are stable inverse problems. This is confirmed by the numerical reconstructions in [Bal et al. 2011b]. The first result is more global but requires more regularity of the coefficients. It is based on the use of complex geometric optics CGO solutions to show that an appropriate functional is contracting in the space of continuous functions. The second result is more local in nature (a global uniqueness result is also proved in [Triki 2010]) but requires less smoothness on the coefficient $q(x)$ and provides a stability estimate in the larger space $L^2(X)$. It also uses CGO solutions to show that the norm of a complex-valued solution to an elliptic equation is bounded from below by a positive constant. In both cases, the CGO solutions have traces at the boundary ∂X and the chosen illumination f needs to be chosen in the vicinity of such traces.

The results obtained in Theorem 3.6 under smallness constraints on σ apply for very general illuminations f. The above two results apply for more general (essentially arbitrary) coefficients but require more severe constraints on the illuminations f.

Nonunique reconstruction in the AOT setting. The results above concern the uniqueness of the reconstruction of the potential in a Helmholtz equation when well-chosen complex-valued boundary conditions are imposed. They also show that the reconstruction of $0 < c_0 \le q(x)$ in $\Delta u + qu = 0$ with real-valued $u = f$ from knowledge of qu^2 is unique. This corresponds to $P(x, D) = -\Delta$. In a simplified version of the acousto-optics problem considered in [Bal and Schotland 2010], it is interesting to look at the problem where $P(x, D) = \Delta$ and where the measurements are given by $H(x) = \sigma(x)u^2(x)$. Here, u is the solution of the elliptic equation $(-\Delta + \sigma)u = 0$ on X with $u = f$ on ∂X. Assuming that f is nonnegative, which is the physically interesting case, we obtain that $|u| = u$ and hence

$$\Delta(u_1 - u_2) = \sqrt{\sigma_1\sigma_2}(u_2 - u_1) + (\sqrt{H_1} - \sqrt{H_2})\left(\frac{\sigma_1}{\sqrt{H_1}} - \frac{\sigma_2}{\sqrt{H_2}}\right).$$

Therefore, as soon as 0 is not an eigenvalue of $\Delta + \sqrt{\sigma_1\sigma_2}$, we obtain that $u_1 = u_2$ and hence that $\sigma_1 = \sigma_2$. For σ_0 such that 0 is not an eigenvalue of

$\Delta + \sigma_0$, we find that for σ_1 and σ_2 sufficiently close to σ_0, then $H_1 = H_2$ implies that $\sigma_1 = \sigma_2$ on the support of $H_1 = H_2$.

However, it is shown in [Bal and Ren 2011b] that two different, positive, absorptions σ_j for $j = 1, 2$, may in some cases provide the same measurement $H = \sigma_j u_j^2$ with $\Delta u_j = \sigma_j u_j$ on X with $u_j = f$ on ∂X and in fact $\sigma_1 = \sigma_2$ on ∂X so that these absorptions cannot be distinguished by their traces on ∂X. This counterexample shows that conditions such as the smallness condition in Theorem 3.6 are necessary in general.

More generally, and following [Bal and Ren 2011b], consider an elliptic problem of the form

$$\begin{aligned} Pu &= \sigma u \quad \text{in } X, \\ u &= f \quad \text{on } \partial X, \end{aligned} \tag{43}$$

and assume that measurements of the form $H(x) = \sigma(x)u^2(x)$ are available. Here, P is a self-adjoint, nonpositive, elliptic operator, which for concreteness we will take of the form $Pu = \nabla\cdot\gamma(x)\nabla u$ with $\gamma(x)$ known, sufficiently smooth, and bounded above and below by positive constants. We assume $f > 0$ and $\gamma > 0$ so that by the maximum principle, $u > 0$ on X. We also assume enough regularity on ∂X and f so that $u \in C^{2,\beta}(\bar{X})$ for some $\beta > 0$ [Gilbarg and Trudinger 1977].

We observe that

$$\begin{aligned} uPu &= H \quad \text{in } X, \\ u &= f \quad \text{on } \partial X, \end{aligned} \tag{44}$$

so that the inverse problem may be recast as a semilinear problem. The nonuniqueness result is derived from [Ambrosetti and Prodi 1972] and in some sense generalizes the observation that $x \mapsto x^2$ admits 0, 1, or 2 (real-valued) solutions depending on the value of x^2. Let us define

$$\phi : C^{2,\beta}(\bar{X}) \to C^{0,\beta}(\bar{X}), \quad u \mapsto \phi(u) = uPu. \tag{45}$$

The singular points of ϕ are calculated from its first-order Fréchet derivative:

$$\phi'(u)v = vPu + uPv. \tag{46}$$

The operator $\phi'(u)$ is not invertible when $\sigma := (Pu)/u$ is such that $P + \lambda\sigma$ admits $\lambda = 1$ as an eigenvalue. Let σ_0 be such that $P + \sigma_0$ is not invertible. We assume that the corresponding eigen-space is one-dimensional and spanned by the eigenvector $\psi > 0$ on ∂X such that $(P + \sigma_0)\psi = 0$ and $\psi = 0$ on ∂X. Let us define u_0 as

$$\begin{aligned} Pu_0 &= \sigma_0 u_0 \quad \text{in } X, \\ u_0 &= f \quad \text{on } \partial X, \\ \sigma_0 &> 0. \end{aligned} \tag{47}$$

Moreover, u_0 is a singular point of $\phi(u)$ with $\phi'(u_0)\psi = 0$. Then *define*

$$u_\delta := u_0 + \delta\psi \quad \text{in } X, \qquad \delta \in (-\delta_0, \delta_0), \tag{48}$$

$$\sigma_\delta := \frac{Pu_\delta}{u_\delta} = \sigma_0 \frac{u_0 - \delta\psi}{u_0 + \delta\psi}, \tag{49}$$

$$H_\delta := \sigma_\delta u_\delta^2 = \sigma_0 u_\delta u_{-\delta} = \sigma_0(u_0^2 - \delta^2\psi^2). \tag{50}$$

We choose δ_0 such that $\sigma_\delta > 0$ a.e. on X for all $\delta \in (-\delta_0, \delta_0)$. Then:

Proposition 3.9 [Bal and Ren 2011b]. *Let u_0 be a singular point and $H_0 = \phi(u_0)$ a critical value of ϕ as above and let ψ be the normalized solution of $\phi'(u_0)\psi = 0$. Let u_δ, σ_δ, and H_δ be defined as in* (48)–(50) *for $0 \neq \delta \in (-\delta_0, \delta_0)$ for δ_0 sufficiently small. Then we verify that*:

$$\sigma_\delta \neq \sigma_{-\delta}, \quad \sigma_\delta > 0, \quad H_\delta = H_{-\delta}, \quad Pu_\delta = \sigma_\delta u_\delta \ \textit{ in } X, \quad u_\delta = f \ \textit{ on } \partial X.$$

This shows the nonuniqueness of the reconstruction of σ from knowledge of $H = \sigma u^2$. Moreover we verify that $\sigma_{\pm\delta}$ agree on ∂X so that this boundary information cannot be used to distinguish between σ_δ and $\sigma_{-\delta}$. The nonuniqueness result is not very restrictive since we have seen that two coefficients, hence one coefficient, may be uniquely reconstructed from *two* well-chosen illuminations in the PAT results. Nonetheless, the above result shows once more that identifiability of the unknown coefficients is not always guaranteed by the availability of internal measurements.

4. Reconstructions from functionals of ∇u

We have seen two models of hybrid inverse problems with measurements involving ∇u. In UMEIT, the measurements are of the form $H(x) = \gamma(x)|\nabla u|^2(x)$ whereas in CDII, they are of the form $H(x) = \gamma(x)|\nabla u|(x)$.

We consider more generally measurements of the form $H(x) = \gamma(x)|\nabla u|^{2-p}$ for u the solution to the elliptic equation

$$\begin{aligned} -\nabla \cdot \gamma(x)\nabla u &= 0 \quad \text{in } X, \\ u &= f \quad \text{on } \partial X. \end{aligned} \tag{51}$$

Since $H(x)$ is linear in $\gamma(x)$, we have formally what appears to be an extension to $p \geq 0$ of the p-Laplacian elliptic equations

$$-\nabla \cdot \frac{H(x)}{|\nabla u|^{2-p}} \nabla u = 0, \tag{52}$$

posed on a bounded, smooth, open domain $X \subset \mathbb{R}^n$, $n \geq 2$, with prescribed Dirichlet conditions, say. When $1 < p < \infty$, the above problem is known to admit a variational formulation with convex functional $J[\nabla u] = \int_X H(x)|\nabla u|^p dx$, which admits a unique minimizer in an appropriate functional setting solution of the above associated Euler–Lagrange equation [Evans 1998].

When $p = 1$, the equation becomes degenerate while for $p < 1$, the equation is in fact hyperbolic. We consider the problem of measurements that are quadratic (or bilinear) in ∇u (with applications to UMEIT and UMOT) in the next two sections. In the following section, we consider the case $p = 1$.

4A. ***Reconstruction from a single power density measurement.*** The presentation follows [Bal 2012]. When $p = 0$ so that measurements are of the form $H(x) = \gamma(x)|\nabla u|^2$, the above 0-Laplacian turns out to be a hyperbolic equation. Anticipating this behavior, we assume the availability of Cauchy data (i.e., u and $\gamma\nu\cdot\nabla u$ with ν the unit outward normal to X) on ∂X rather than simply Dirichlet data. Then (52) with $p = 0$ becomes after some algebra

$$\begin{aligned}(I - 2\widehat{\nabla u} \otimes \widehat{\nabla u}) : \nabla^2 u + \nabla \ln H \cdot \nabla u = 0 \quad \text{in } X,\\ u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = j \quad \text{on } \partial X.\end{aligned} \tag{53}$$

Here $\widehat{\nabla u} = \nabla u / |\nabla u|$. With

$$g^{ij} = g^{ij}(\nabla u) = -\delta^{ij} + 2(\widehat{\nabla u})_i(\widehat{\nabla u})_j \quad \text{and} \quad k^i = -(\nabla \ln H)_i,$$

the above equation is in coordinates

$$\begin{aligned}g^{ij}(\nabla u)\partial^2_{ij} u + k^i \partial_i u = 0 \quad \text{in } X,\\ u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = j \quad \text{on } \partial X.\end{aligned} \tag{54}$$

Since g^{ij} is a definite matrix of signature $(1, n-1)$, (54) is a quasilinear strictly hyperbolic equation. The Cauchy data f and j then need to be provided on a spacelike hyper-surface in order for the hyperbolic problem to be well-posed [Hörmander 1983]. This is the main difficulty in solving (54) with redundant Cauchy boundary conditions.

In general, we cannot hope to reconstruct $u(x)$, and hence $\gamma(x)$ on the whole domain X. The reason is that the direction of "time" in the second-order hyperbolic equation is $\widehat{\nabla u}$(x). The normal $\nu(x)$ at the boundary ∂X will distinguish between the (good) part of ∂X that is "spacelike" and the (bad) part of ∂X that is "timelike". Spacelike surfaces such as $t = 0$ provide stable information to solve the standard wave equation whereas in general it is known that arbitrary singularities can form in a wave equation from information on "timelike" surfaces such as $x = 0$ or $y = 0$ in a three-dimensional setting (where (t, x, y) are local coordinates of X) [Hörmander 1983].

Uniqueness and stability. Let (u,γ) and $(\tilde u,\tilde\gamma)$ be two solutions of the Cauchy problem (54) with measurements (H, f, j) and $(\tilde H, \tilde f, \tilde j)$, where we define the reconstructed conductivities

$$\gamma(x)=\frac{H}{|\nabla u|^2}(x),\quad \tilde\gamma(x)=\frac{\tilde H}{|\nabla\tilde u|^2}(x). \tag{55}$$

Let $v=\tilde u-u$. We find that

$$\nabla\cdot\Big(\frac{H}{|\nabla\tilde u|^2|\nabla u|^2}\big((\nabla u+\nabla\tilde u)\otimes(\nabla u+\nabla\tilde u)-(|\nabla u|^2+|\nabla\tilde u|^2)I\big)\nabla v \\ +\delta H\Big(\frac{\nabla\tilde u}{|\nabla\tilde u|^2}+\frac{\nabla u}{|\nabla u|^2}\Big)\Big)=0.$$

This equation is recast as

$$\begin{aligned} \mathfrak{g}^{ij}(x)\partial^2_{ij}v+\mathfrak{k}^i\partial_i v+\partial_i(l^i\delta H)=0 &\quad \text{in } X,\\ v=\tilde f-f,\quad \frac{\partial v}{\partial\nu}=\tilde j-j &\quad \text{on } \partial X, \end{aligned} \tag{56}$$

for appropriate coefficients $\mathfrak{k}^i$ and l^i, where

$$\begin{aligned} \mathfrak{g}(x)&=\frac{H}{|\nabla\tilde u|^2|\nabla u|^2}\big((\nabla u+\nabla\tilde u)\otimes(\nabla u+\nabla\tilde u)-(|\nabla u|^2+|\nabla\tilde u|^2)I\big)\\ &=\alpha(x)\big(\boldsymbol{e}(x)\otimes\boldsymbol{e}(x)-\beta^2(x)\big(I-\boldsymbol{e}(x)\otimes\boldsymbol{e}(x)\big)\big), \end{aligned} \tag{57}$$

with

$$\boldsymbol{e}(x)=\frac{\nabla u+\nabla\tilde u}{|\nabla u+\nabla\tilde u|}(x),\quad \beta^2(x)=\frac{|\nabla u+\nabla\tilde u|^2}{|\nabla u+\nabla\tilde u|^2-(|\nabla u|^2+|\nabla\tilde u|^2)}(x), \tag{58}$$

and $\alpha(x)$ is the appropriate (scalar) normalization coefficient. For ∇u and $\nabla\tilde u$ sufficiently close so that $\nabla u\cdot\nabla\tilde u>0$, then the above linear equation for v is strictly hyperbolic. We define the Lorentzian metric $\mathfrak{h}=\mathfrak{g}^{-1}$ so that $\mathfrak{h}_{ij}$ are the coordinates of the inverse of the matrix $\mathfrak{g}^{ij}$. We denote by $\langle\cdot,\cdot\rangle$ the bilinear product associated to $\mathfrak{h}$ so that $\langle u,v\rangle=\mathfrak{h}_{ij}u^iv^j$ where the two vectors u and v have coordinates u^i and v^i, respectively. We verify that

$$\mathfrak{h}(x)=\frac{1}{\alpha(x)}\Big(\boldsymbol{e}(x)\otimes\boldsymbol{e}(x)-\frac{1}{\beta^2(x)}\big(I-\boldsymbol{e}(x)\otimes\boldsymbol{e}(x)\big)\Big). \tag{59}$$

The spacelike part Σ_g of ∂X is given by $\mathfrak{h}(\nu,\nu)>0$, i.e., ν is a timelike vector, or equivalently

$$|\nu(x)\cdot\boldsymbol{e}(x)|^2>\frac{1}{1+\beta^2(x)},\quad x\in\partial X. \tag{60}$$

Above, the dot product is with respect to the standard Euclidean metric and ν is a unit vector for the Euclidean metric, not for the metric $\mathfrak{h}$. Let Σ_1 be an open connected component of Σ_g and let $\mathbb{O} = \cup_{0<\tau<s}\Sigma_2(\tau)$ be a domain of influence of Σ_1 swept out by the spacelike surfaces $\Sigma_2(\tau)$; see [Bal 2012; Taylor 1996]. Then we have the following local stability result:

Theorem 4.1 (local uniqueness and stability). *Let u and $\tilde{u}$ be two solutions of* (54). *We assume that $\mathfrak{g}$ constructed in* (57) *is strictly hyperbolic. Let Σ_1 be an open connected component of Σ_g the spacelike component of ∂X and let $\mathbb{O}$ be a domain of influence of Σ_1 constructed as above. Let us define the energy*

$$E(dv) = \langle dv, \nu_2\rangle^2 - \tfrac{1}{2}\langle dv, dv\rangle\langle \nu_2, \nu_2\rangle. \tag{61}$$

Here, dv is the gradient of v in the metric $\mathfrak{h}$ given in coordinates by $\mathfrak{g}^{ij}\partial_j v$. Then

$$\int_{\mathbb{O}} E(dv)\,dx \le C\bigg(\int_{\Sigma_1} |f-\tilde{f}|^2 + |j-\tilde{j}|^2\,d\sigma + \int_{\mathbb{O}} |\nabla\delta H|^2\,dx\bigg), \tag{62}$$

where dx and $d\sigma$ are the standard measures on $\mathbb{O}$ and Σ_1, respectively.

In the Euclidean metric, let $\nu_2(x)$ be the unit vector to $x \in \Sigma_2(\tau)$, define $c(x) := \nu_2(x)\cdot \boldsymbol{e}(x)$ and

$$\theta := \min_{x\in\Sigma_2(\tau)}\bigg(c^2(x) - \frac{1}{1+\beta^2(x)}\bigg). \tag{63}$$

Then

$$\int_{\mathbb{O}} |v^2| + |\nabla v|^2 + (\gamma-\tilde{\gamma})^2\,dx \\ \le \frac{C}{\theta^2}\bigg(\int_{\Sigma_1} |f-\tilde{f}|^2 + |j-\tilde{j}|^2\,d\gamma + \int_{\mathbb{O}} |\nabla\delta H|^2\,dx\bigg), \tag{64}$$

where γ and $\tilde{\gamma}$ are the conductivities in (55). *Provided that $f = \tilde{f}$, $j = \tilde{j}$, and $H = \tilde{H}$, we obtain that $v = 0$ and the uniqueness result $u = \tilde{u}$ and $\gamma = \tilde{\gamma}$.*

The proof is based on adapting energy methods for hyperbolic equations as they are summarized in [Taylor 1996]. The energy $E(dv)$ fails to control dv for null-like or spacelike vectors, i.e., $\mathfrak{h}(dv, dv) \le 0$. The parameter θ measures how timelike the vector dv is on the domain of influence $\mathbb{O}$. As $\mathbb{O}$ approaches the boundary of the domain of influence of Σ_g and θ tends to 0, the energy estimates deteriorate as indicated in (64).

Assuming that the errors on the Cauchy data f and j are negligible, we obtain the following stability estimate for the conductivity

$$\|\gamma-\tilde{\gamma}\|_{L^2(\mathbb{O})} \le \frac{C}{\theta}\|H-\tilde{H}\|_{H^1(X)}. \tag{65}$$

Under additional regularity assumptions on γ, for instance assuming that $H \in H^s(X)$ for $s \geq 2$, we find by standard interpolation that

$$\|\gamma - \tilde{\gamma}\|_{L^2(©)} \leq \frac{C}{\theta} \|H - \tilde{H}\|_{L^2(X)}^{1-\frac{1}{s}} \|H + \tilde{H}\|_{H^s(X)}^{\frac{1}{s}}, \tag{66}$$

We thus obtain Hölder-stable reconstructions in the practical setting of square integrable measurement errors. However, stability is *local*. Only on the domain of influence of the spacelike part of the boundary can we obtain a stable reconstruction. This can be done by solving a nonlinear strictly hyperbolic equation analyzed in [Bal 2012] using techniques summarized in [Hörmander 1997].

Global reconstructions. In the preceding result, the main roadblock to global reconstructions was that the domain of influence of the spacelike part of the boundary was a strict subset of X. There is a simple solution to this problem: simply make sure that the whole boundary is a level set of u and that no critical points of u (where $\nabla u = 0$) exist. Then all of X is in the domain of influence of the spacelike part of ∂X, which is the whole of ∂X. This setting can be made possible independent of the conductivity γ in two dimensions of space but not always in higher dimensions.

Let $n = 2$. We assume that X is an open smooth domain diffeomorphic to an annulus with boundary $\partial X = \partial X_0 \cup \partial X_1$. We assume that $f = 0$ on the external boundary ∂X_0 and that $f = 1$ on the internal boundary ∂X_1. The boundary of X is thus composed of two smooth connected components that are different level sets of the solution u. The solution u to (51) is uniquely defined on X. Then we can show:

Proposition 4.2 [Bal 2012]. *We assume that both the geometry of X and $\gamma(x)$ are sufficiently smooth. Then $|\nabla u|$ is bounded from above and below by positive constants. The level sets $\Sigma_c = \{x \in X : u(x) = c\}$ for $0 < c < 1$ are smooth curves that separate X into two disjoint subdomains.*

The proof is based on the fact that critical points of solutions to elliptic equations in two dimensions are isolated [Alessandrini 1986]. The result extends to higher dimensions provided that $|\nabla u|$ does not vanish with exactly the same proof. In the absence of critical points, we thus obtain that $\boldsymbol{e}(x) = \widehat{\nabla u} = \nu(x)$ so that $\nu(x)$ is clearly a timelike vector. Then the local results of Theorem 4.1 become global results, which yields the following proposition:

Proposition 4.3. *Let X be the geometry described above in dimension $n \geq 2$ and $u(x)$ the solution to* (51). *We assume here that both the geometry and $\gamma(x)$ are sufficiently smooth. We also* assume *that $|\nabla u|$ is bounded from above and below by positive constants. Then the nonlinear* (54) *admits a unique solution* *and the reconstruction of u and of γ is stable in X in the sense described in Theorem 4.1.*

In dimensions $n \geq 3$, we cannot guaranty that u does not have any critical point independent of the conductivity. If the conductivity is close to a constant, then by continuity, u does not have any critical point and the above result applies. This proves the result for sufficiently small perturbations of the case $\gamma(x) = \gamma_0$. In the general case, however, we cannot guaranty that ∇u does not vanish and in fact can produce counterexamples (see [Bal 2012]):

Proposition 4.4 [Bal 2012; Briane et al. 2004; Melas 1993]. *There is an example of a smooth conductivity such that u admits critical points.*

So in dimensions $n \geq 3$, we are not guaranteed that the nonlinear equation will remain strictly hyperbolic. What we can do, however, is again to use the notion of complex geometric optics solutions. We have the result:

Theorem 4.5 [Bal 2012]. *Let γ be extended by $\gamma_0 = 1$ on $\mathbb{R}^n \backslash \tilde{X}$, where $\tilde{X}$ is the domain where γ is not known. We assume that γ is smooth on $\mathbb{R}^n$. Let $\gamma(x) - 1$ be supported without loss of generality on the cube $(0, 1) \times (-\frac{1}{2}, \frac{1}{2})^{n-1}$. Define the domain $X = (0, 1) \times B_{n-1}(a)$, where $B_{n-1}(a)$ is the $(n-1)$-dimensional ball of radius a centered at 0 and where a is sufficiently large that the light cone for the Euclidean metric emerging from $B_{n-1}(a)$ strictly includes $\tilde{X}$.*

There exists an open set of illuminations (f_1, f_2) such that if u_1 and u_2 are the corresponding solutions of (51), *then $\gamma(x)$ is uniquely determined by the measurements*

$$\begin{aligned} H_1(x) &= \gamma(x)|\nabla u_1|^2(x), \\ H_2(x) &= \gamma(x)|\nabla u_2|^2(x), \\ H_3(x) &= \gamma(x)|\nabla(u_1 + u_2)|^2, \end{aligned} \tag{67}$$

with the corresponding Cauchy data (f_1, j_1), (f_2, j_2) and $(f_1 + f_2, j_1 + j_2)$ at $x_1 = 0$.

Let $\tilde{H}_i$ be measurements corresponding to $\tilde{\gamma}$ and let $(\tilde{f}_1, \tilde{j}_1)$ and $(\tilde{f}_2, \tilde{j}_2)$ be the corresponding Cauchy data at $x_1 = 0$. Assume that $\gamma(x)-1$ and $\tilde{\gamma}(x)-1$ are smooth and such that their norm in $H^{\frac{n}{2}+3+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon > 0$ are bounded by M. Then for a constant C that depends on M, we have the global stability result

$$\|\gamma-\tilde{\gamma}\|_{L^2(\tilde{X})} \leq C\left(\|d_C-\tilde{d}_C\|_{(L^2(B_{n-1}(a)))^4} + \sum_{i=1}^{3} \|\nabla H_i - \nabla \tilde{H}_i\|_{L^2(X)} \right). \tag{68}$$

Here, we have defined $d_C = (f_1, j_1, f_2, j_2)$ with $\tilde{d}_C$ being defined similarly.

The three measurements H_i in (67) actually correspond to two physical measurements since H_3 may be determined from the experiments yielding H_1 and H_2, as demonstrated in [Bal 2012; Kuchment and Kunyansky 2011]. The three measurements are constructed so that two independent strictly hyperbolic Lorentzian metrics can be constructed everywhere inside the domain. These metrics are constructed by means of CGO solutions. The boundary conditions f_j have to be close to the traces of the CGO solutions. We thus obtain a global Lipschitz stability result. The price to pay is that the open set of illuminations is not very explicit and may depend on the conductivities one seeks to reconstruct.

For conductivities that are close to a constant, several reconstructions are therefore available. We have seen that geometries of the form of an annulus (with a hole that can be arbitrarily small and arbitrarily close to the boundary where $f = 0$) allowed us to obtain globally stable reconstructions since in such situations, it is relatively easy to avoid the presence of critical points. The method of CGO solutions can be shown to apply for a well-defined set of illuminations since the (harmonic) CGO solutions are explicitly known for the Euclidean metric and of the form $e^{\rho\cdot x}$ for ρ a complex valued vector such that $\rho\cdot\rho = 0$. After linearization in the vicinity of the Euclidean metric, another explicit reconstruction procedure was introduced in [Kuchment and Kunyansky 2011].

4B. ***Reconstructions from multiple power density measurements.*** Rather than reconstructing γ from one given measurement of the form $\gamma(x)|\nabla u|^2$, we can instead acquire several measurements of the form

$$H_{ij}(x) = \gamma(x)\nabla u_i(x)\cdot\nabla u_j(x) \quad \text{in } X, \quad 1 \le i, j \le M, \tag{69}$$

where u_j solves the elliptic problem (51) with f given by f_j for $1 \le j \le M$. The result presented in Theorem 4.5 above provides a positive answer for $M = 2$ when the available internal functionals are augmented by Cauchy data at the boundary of the domain of interest.

Results obtained in [Bal et al. 2011a; Capdeboscq et al. 2009; Monard and Bal 2012] and based on an entirely different procedure and not requiring knowledge of boundary data show that $M = 2\lfloor\frac{n+1}{2}\rfloor$ measurements allow for a global reconstruction of γ, i.e., M for n even and $M+1$ for n odd. Such results were first obtained in [Capdeboscq et al. 2009] in the case $n = 2$ and have been extended with a slightly different presentation to the cases $n = 2$ and $n = 3$ in [Bal et al. 2011a] while the general case $n \ge 2$ is treated in [Monard and Bal 2012]. Let us assume that $n = 3$ for concreteness. Then $H_{ij} = S_i \cdot S_j$, where we have defined

$$S_j(x) = \sqrt{\gamma(x)}\nabla u_j(x), \quad 1 \le j \le M.$$

Let $S=(S_1,S_2,S_3)$ be a matrix of $n=3$ column vectors S_j. Then $S^TS=H$, where S^T is the transpose matrix made of the rows given by the S_j. We do not know S or the S_j, but we know its normal matrix $S^TS=H$. Let T be a matrix such that $R=ST^T$ is a rotation-valued field on X. Two examples are $T=H^{-\frac{1}{2}}$ or the lower-triangular T obtained by the Gram–Schmidt procedure. We thus have information on S. We need additional equations to solve for S, or equivalently R, uniquely. The elliptic equation may be written as $\nabla\cdot\sqrt{\gamma}S_j=0$, or equivalently

$$\nabla\cdot S_j+F\cdot S_j=0,\quad F=\nabla(\log\sqrt{\gamma})=\tfrac{1}{2}\nabla\log\gamma. \tag{70}$$

Now, since $\gamma^{-\frac{1}{2}}S_j$ is a gradient, its curl vanishes and we find that

$$\nabla\times S_j-F\times S_j=0. \tag{71}$$

Here, F is unknown. We first eliminate it from the equations and then find a closed form equation for S or equivalently for R as a field in $SO(n;\mathbb{R})$.

Let T be the aforementioned matrix T, say $T=H^{-\frac{1}{2}}$ with entries t_{ij} for $1\le i,j\le n$. Let t^{ij} be the entries of T^{-1} and define the vector fields

$$V_{ij}:=\nabla(t_{ik})t^{kj},\quad \text{i.e.,}\quad V_{ij}^l=\partial_l(t_{ij})t^{kl},\quad 1\le i,j,l\le n. \tag{72}$$

We then define $R(x)=S(x)T^T(x)\in SO(n;\mathbb{R})$ the matrix whose columns are composed of the column vectors $R_j=S_jT^T$. Then in all dimension $n\ge 2$, we find

Lemma 4.6 [Bal et al. 2011a; Monard and Bal 2012]. *In $n\ge 2$, we have*

$$F=\frac{1}{n}\left(\frac{1}{2}\nabla\log\det H+\sum_{i,j=1}^{n}\big((V_{ij}+V_{ji})\cdot R_i\big)R_j\right). \tag{73}$$

The proof in dimension $n=2,3$ can be found in [Bal et al. 2011a] and in arbitrary dimension in [Monard and Bal 2012].

Note that the determinant of H needs to be positive on the domain X in order for the above expression for F to make sense. It is, however, difficult to ensure that the determinant of several gradients remains positive and there are in fact counterexamples as shown in [Briane et al. 2004]. Here again, complex geometric optics solutions are useful to control the determinant of gradients of elliptic solutions locally and globally using several solutions. We state a global result in the practical setting $n=3$.

Let there be $m\ge 3$ solutions of the elliptic equation and assume that there exists an open covering $\mathbb{O}=\{\Omega_k\}_{1\le k\le N}$ ($X\subset\bigcup_{k=1}^N\Omega_i$), a constant $c_0>0$

and a function $\tau : [1, N] \ni i \mapsto \tau(i) = (\tau(i)_1, \tau(i)_2, \tau(i)_3) \in [1, m]^3$, such that

$$\inf_{x\in\Omega_i} \det(S_{\tau(i)_1}(x), S_{\tau(i)_2}(x), S_{\tau(i)_3}(x)) \geq c_0, \quad 1 \leq i \leq N. \tag{74}$$

Then we have the following result:

Theorem 4.7 (3D global uniqueness and stability). *Let $X \subset \mathbb{R}^3$ be an open convex bounded set, and let two sets of $m \geq 3$ solutions of* (51) *generate measurements $(H, \tilde{H})$ whose components belong to $W^{1,\infty}(X)$. Assume that one can define a couple $(\mathbb{O}, \tau)$ such that* (74) *is satisfied for both sets of solutions S and $\tilde{S}$. Let also $x_0 \in \overline{\Omega}_{i_0} \subset \overline{X}$ and $\gamma(x_0)$, $\tilde{\gamma}(x_0)$, $\{S_{\tau(i_0)_i}(x_0), \tilde{S}_{\tau(i_0)_i}(x_0)\}_{1\leq i\leq 3}$ be given. Let γ and $\tilde{\gamma}$ be the conductivities corresponding to the measurements H and $\tilde{H}$, respectively. Then we have the following stability estimate*:

$$\|\log\gamma - \log\tilde{\gamma}\|_{W^{1,\infty}(X)} \leq C\big(\epsilon_0 + \|H - \tilde{H}\|_{W^{1,\infty}(X)}\big), \tag{75}$$

where ϵ_0 is the error at the initial point x_0

$$\epsilon_0 = |\log\gamma_0 - \log\tilde{\gamma}_0| + \sum_{i=1}^{3} \|S_{\tau(i_0)_i}(x_0) - \tilde{S}_{\tau(i_0)_i}(x_0)\|.$$

This shows that the reconstruction of γ is stable from such redundant measurements. Moreover, the reconstruction is constructive. Indeed, after eliminating F from the equations for R, we find an equation of the form $\nabla R = G(x, R)$, where $G(x, R)$ is polynomial of degree three in the entries of R. This is a redundant equation whose solution, when it exists, is unique and stable with respect to perturbations in G and the conditions at a given point x_0.

That (74) is satisfied can again be proved by means of complex geometric optics solutions, as is briefly mentioned in Section 5B below; see [Bal et al. 2011a].

4C. ***Reconstruction from a single current density measurement.*** Let us now come back to the 1-Laplacian, which is a degenerate elliptic problem. In many cases, this problem admits multiple admissible solutions [Kim et al. 2002]. The inverse problem then cannot be solved uniquely. In some settings, however, uniqueness can be restored [Kim et al. 2002; Nachman et al. 2007; 2009; 2011].

Recall that the measurements are of the form $H(x) = \gamma|\nabla u|$ so that u solves the following degenerate quasilinear equation

$$\nabla\cdot\frac{H(x)}{|\nabla u|}\nabla u = 0 \quad \text{in } X. \tag{76}$$

Different boundary conditions may then be considered. It is shown in [Kim et al. 2002] that the above equation augmented with Neumann boundary conditions of the form

$$\frac{H}{|\nabla u|}\frac{\partial u}{\partial \nu} = h \text{ on } \partial X, \qquad \int_{\partial X} u d\sigma = 0,$$

admits an infinite number of solutions once it admits a solution, and may also admit no solution at all. One possible strategy is to acquire two measurements of the form $H(x) = \gamma|\nabla u|$ corresponding to two prescribed currents. In this setting, it is shown in [Kim et al. 2002] that (appropriately defined) singularities of γ are uniquely determined by the measurements. We refer the reader to the latter reference for the details.

Alternatively, we may augment the above (76) with Dirichlet data. Then the reconstruction of γ was shown to be uniquely determined in [Nachman et al. 2007; 2009; 2011]. Why Dirichlet conditions help to stabilize the equation may be explained as follows. The 1-Laplace equation (76) may be recast as

$$(I - \widehat{\nabla u} \otimes \widehat{\nabla u}) : \nabla^2 u + \nabla \ln H \cdot \nabla u = 0,$$

following similar calculations to those leading to (53). The only difference is the "2" in front of $\widehat{\nabla u} \otimes \widehat{\nabla u}$ replaced by "1", or more generally $2 - p$ for a p-Laplacian. When $p > 1$, the problem remains strictly elliptic. When $p < 1$, the problem is hyperbolic, and when $p = 1$, it is degenerate in the direction $\widehat{\nabla u}$ and elliptic in the transverse directions. We can therefore modify u so that its level sets remain unchanged and still satisfy the above partial differential equation. This modification can also be performed so that Neumann boundary conditions are not changed. This is the procedure used in [Kim et al. 2002] to show the nonuniqueness of the reconstruction for the 1-Laplacian with Neumann boundary conditions.

Dirichlet conditions, however, are modified by changes in the level sets of u. It turns out that even with Dirichlet conditions, several (viscosity) solutions to (76) may be constructed when $H \equiv 1$; see [Nachman et al. 2007; 2011]. However, such solutions involve vanishing gradients on sets of positive measure.

The right formulation for the CDII inverse problem that allows one to avoid vanishing gradients is to recast (76) as the minimization of the functional

$$F[\nabla v] = \int_X H(x)|\nabla v| dx, \tag{77}$$

over $v \in H^1(X)$ with $v = f$ on ∂X. Note that $F[\nabla v]$ is convex although it is not *strictly* convex. Moreover, let γ be the conductivity and $H = \gamma|\nabla u|$ the corresponding measurement. Let then $v \in H^1(X)$ with $v = f$ on ∂X. Then,

$$F[\nabla v] = \int_X \gamma|\nabla u||\nabla v| dx \geq \int_X \gamma \nabla u \cdot \nabla v dx = \int_{\partial X} \sigma \frac{\partial u}{\partial \nu} f ds = F[\nabla u],$$

by standard integrations by parts. This shows that u minimizes F. We have the following result:

Theorem 4.8 [Nachman et al. 2011]. *Let $(f, H) \in C^{1,\alpha}(\partial X) \times C^{\alpha}(\bar{X})$ with $H = \gamma|\nabla u|$ for some $\gamma \in C^{\alpha}(\bar{X})$. Assume that $H(x) > 0$ a.e. in X. Then the minimization of*

$$\operatorname{argmin}\{F[\nabla v] : v \in W^{1,1}(X) \cap C(\bar{X}),\ v_{|\partial X} = f\}, \tag{78}$$

has a unique solution u_0. Moreover $\sigma_0 = H|\nabla u_0|^{-1}$ is the unique conductivity associated to the measurement $H(x)$.

It is known in two dimensions of space that $H(x) > 0$ is satisfied for a large class of boundary conditions $f(x)$; see Lemma 5.2 in the next section. In three dimensions of space, however, critical points of u may arise as observed earlier in this paper; see [Bal 2012]. The CGO solutions that are analyzed in the following section allow us to show that $H(x) > 0$ holds for an open set of illuminations f at the boundary of the domain ∂X; see (95) below.

Unfortunately, no such results exists for real-valued solutions and constraints such as $H(x) > 0$ in dimension $n \geq 3$ will not hold for a given f independent of the conductivity γ.

Several reconstruction algorithms have been devised in [Kim et al. 2002; Nachman et al. 2007; 2009; 2011], to which we refer for additional details. The numerical simulations presented in these papers show that when uniqueness is guaranteed, then the reconstructions are very high resolution and quite robust with respect to noise in the data, as is expected for general hybrid inverse problems.

5. Qualitative properties of forward solutions

5A. ***The case of two spatial dimensions.*** Several explicit reconstructions obtained in hybrid inverse problems require that the solutions to the considered elliptic equations satisfy specific qualitative properties such as the absence of any critical point or the positivity of the determinant of gradients of solutions. Such results can be proved in great generality in dimension $n = 2$ but do not always hold in dimension $n \geq 3$.

In dimension $n = 2$, the critical points of u (points x where $\nabla u(x) = 0$) are necessarily isolated as is shown in [Alessandrini 1986]. From this and techniques of quasiconformal mappings that are also restricted to two dimensions of space, we can show the following results.

Lemma 5.1 [Alessandrini and Nesi 2001]. *Let u_1 and u_2 be the solutions of* (51) *on X simply connected with boundary conditions $f_1 = x_1$ and $f_2 = x_2$ on ∂X, respectively, where $x = (x_1, x_2)$ are Cartesian coordinates on X. Assume* *that γ is sufficiently smooth. Then $(x_1, x_2) \mapsto (u_1, u_2)$ from X to its image is a diffeomorphism. In other words*, $\det(\nabla u_1, \nabla u_2) > 0$ *uniformly on $\overline{X}$.*

This result is useful in the analysis of UMEIT and UMOT in the case of redundant measurements. It is shown in [Briane et al. 2004] that the appropriate extension of this result is false in dimension $n \geq 3$.

We recall that a function continuous on a simple closed contour is almost two-to-one if it is two-to-one except possibly at its maximum and minimum [Nachman et al. 2007]. Then we have, quite similarly to the result in Lemma 3.1 and Proposition 4.2, which also use the results in [Alessandrini 1986], the following:

Lemma 5.2 [Nachman et al. 2007]. *Let X be a simply connected planar domain and let u be solution of* (51) *with f almost two-to-one and σ sufficiently smooth. Then $|\nabla u|$ is bounded from below by a positive constant on $\overline{X}$. Moreover, the level sets of u are open curves inside X with their two end points on ∂X.*

This shows that for a large class of boundary conditions with one maximum and one minimum, the solution u cannot have any critical point in $\bar{X}$. On an annulus with boundaries equal to level sets of u, we saw in Proposition 4.2 that u had no critical points on X in dimension $n = 2$. This was used to show that the normal vector to the level sets of u always forms a timelike vector for the Lorentzian metric defined in (54).

All these results no longer hold in dimension $n \geq 3$. See [Bal 2012; Briane et al. 2004] for counterexamples. In dimension $n \geq 3$, the required qualitative properties cannot be obtained for a given set of illuminations (boundary conditions) independent of the conductivity. However, for conductivities that are bounded (with an arbitrary bound) in an appropriate norm, there are open sets of illuminations that allow us to obtain the required qualitative properties. One way to construct such solutions is by means of the complex geometric optics solutions that are analyzed in the next section.

5B. *Complex geometric optics solutions.*

CGO solutions and Helmholtz equations. Complex geometrical optics (CGO) solutions allow us to treat the potential q in the equation

$$\begin{aligned} (\Delta + q)u &= 0 \quad \text{in } X, \\ u &= f \quad \text{on } \partial X, \end{aligned} \tag{79}$$

as a perturbation of the leading operator Δ. When $q = 0$, CGO solutions are harmonic solutions defined on $\mathbb{R}^n$ and are of the form

$$u_\rho(x) = e^{\rho \cdot x}, \qquad \rho \in \mathbb{C}^n \text{ such that } \rho \cdot \rho = 0.$$

For $\rho = \rho_r + i\rho_i$ with ρ_r and ρ_i vectors in $\mathbb{R}^n$, this means that $|\rho_r|^2 = |\rho_i|^2$ and $\rho_r \cdot \rho_i = 0$.

When $q \not\equiv 0$, CGO solutions are solutions of the following problem

$$\Delta u_\rho + q u_\rho = 0, \qquad u_\rho \sim e^{\rho\cdot x} \text{ as } |x| \to \infty. \tag{80}$$

More precisely, we say that u_ρ is a solution of the above equation with $\rho\cdot\rho = 0$ and the proper behavior at infinity when it is written as

$$u_\rho(x) = e^{\rho\cdot x}\big(1 + \psi_\rho(x)\big), \tag{81}$$

for $\psi_\rho \in L^2_\delta$ a weak solution of

$$\Delta\psi_\rho + 2\rho\cdot\nabla\psi_\rho = -q(1+\psi_\rho). \tag{82}$$

The space L^2_δ for $\delta \in \mathbb{R}$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{L^2_\delta}$ defined as

$$\|u\|_{L^2_\delta} = \left(\int_{\mathbb{R}^n} \langle x\rangle^{2\delta}|u|^2 dx\right)^{\frac{1}{2}}, \qquad \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}. \tag{83}$$

Let $-1 < \delta < 0$ and $q \in L^2_{\delta+1}$ and $\langle x\rangle q \in L^\infty$. One of the main results in [Sylvester and Uhlmann 1987] is that there exists $\eta = \eta(\delta)$ such that the above problem admits a unique solution with $\psi_\rho \in L^2_\delta$ provided that

$$\|\langle x\rangle q\|_{L^\infty} + 1 \le \eta|\rho|.$$

Moreover, $\|\psi_\rho\|_{L^2_\delta} \le C|\rho|^{-1}\|q\|_{L^2_{\delta+1}}$ for some $C = C(\delta)$. In the analysis of many hybrid problems, we need smoother CGO solutions than what was recalled above. We introduce the spaces H^s_δ for $s \ge 0$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{H^s_\delta}$ defined as

$$\|u\|_{H^s_\delta} = \left(\int_{\mathbb{R}^n} \langle x\rangle^{2\delta}|(I-\Delta)^{\frac{s}{2}}u|^2 dx\right)^{\frac{1}{2}}. \tag{84}$$

Here $(I-\Delta)^{\frac{s}{2}}u$ is defined as the inverse Fourier transform of $\langle\xi\rangle^s\hat{u}(\xi)$, where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$.

Proposition 5.3 [Bal and Uhlmann 2010]. *Let $-1 < \delta < 0$ and $k \in \mathbb{N}^*$. Let $q \in H_1^{\frac{n}{2}+k+\varepsilon}$ (hence $q \in H_{\delta+1}^{\frac{n}{2}+k+\varepsilon}$) and let ρ be such that*

$$\|q\|_{H_1^{\frac{n}{2}+k+\varepsilon}} + 1 \le \eta|\rho|. \tag{85}$$

Then ψ_ρ, the unique solution to (82), *belongs to $H_\delta^{\frac{n}{2}+k+\varepsilon}$ and*

$$|\rho|\|\psi_\rho\|_{H_\delta^{\frac{n}{2}+k+\varepsilon}} \le C\|q\|_{H_{\delta+1}^{\frac{n}{2}+k+\varepsilon}}, \tag{86}$$

for a constant C that depends on δ and η.

We also want to obtain estimates for ψ_ρ and u_ρ restricted to the bounded domain X. We have the following result.

Corollary 5.4 [Bal and Uhlmann 2010]. *Let us assume the regularity hypotheses of the previous proposition. Then we find that*

$$|\rho|\|\psi_\rho\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} + \|\psi_\rho\|_{H^{\frac{n}{2}+k+1+\varepsilon}(X)} \le C\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}. \tag{87}$$

These results show that for ρ sufficiently large, ψ_ρ is small compared to 1 in the class $C^k(\bar{X})$ by Sobolev imbedding.

Let $Y = H^p(X)$ and $\mathcal{M}$ the ball in Y of functions with norm bounded by a fixed $M > 0$. Not only do we have that ψ_ρ is small for $|\rho|$ large, but we have the following Lipschitz stability with respect to changes in the potential $q(x)$:

Lemma 5.5 [Bal et al. 2011b]. *Let ψ_ρ be the solution of*

$$\Delta\psi_\rho + 2\rho\cdot\nabla\psi_\rho = -q(1+\psi_\rho), \tag{88}$$

and let $\tilde{\psi}_\rho$ be the solution of the same equation with q replaced by $\tilde{q}$, where $\tilde{q}$ is defined as in (27) *with σ replaced by $\tilde{\sigma}$. We assume that q and $\tilde{q}$ are in $\mathcal{M}$. Then there is a constant C such that for all ρ with $|\rho| \ge |\rho_0|$, we have*

$$\|\psi_\rho - \tilde{\psi}_\rho\|_Y \le \frac{C}{|\rho|}\|\sigma - \tilde{\sigma}\|_Y. \tag{89}$$

This is the property used in [Bal et al. 2011b] to show that σ in the TAT problem (6)–(7) solves the equation

$$\sigma(x) = e^{(\rho+\bar{\rho})\cdot x} H(x) - \mathcal{H}_f[\sigma](x) \text{ on } X,$$

where

$$\mathcal{H}_f[\sigma](x) = \sigma\big(\psi_f + \overline{\psi_f} + \psi_f\overline{\psi_f}(x)\big),$$

is a contraction map for f in an open set of illuminations; see [Bal et al. 2011b]. The result in Theorem 3.7 then follows by a Banach fixed point argument.

CGO solutions and elliptic equations. Consider the more general elliptic equation

$$\begin{aligned} -\nabla\cdot\gamma\nabla u + \sigma u &= 0 \quad \text{in } X, \\ u &= f \quad \text{on } \partial X. \end{aligned} \tag{90}$$

Upon defining $v = \sqrt{\gamma}u$, we find that

$$(\Delta + q)v = 0 \text{ in } X, \quad q = -\frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} - \frac{\sigma}{\gamma}.$$

In other words, we find CGO solutions for (90) defined on $\mathbb{R}^n$ and of the form

$$u_\rho(x) = \frac{1}{\sqrt{\gamma}} e^{\rho\cdot x}\big(1 + \psi_\rho(x)\big), \tag{91}$$

with $|\rho|\psi_\rho(x)$ bounded uniformly provided that γ and σ are sufficiently smooth coefficients.

5C. *Application of CGO solutions to qualitative properties of elliptic solutions.*

Lower bound for the modulus of complex valued solutions. The above results show that for $|\rho|$ sufficiently large, then $|u_\rho|$ is uniformly bounded from below by a positive constant on compact domains. Note that u_ρ is complex valued and that its real and imaginary parts oscillate very rapidly. Indeed,

$$e^{\rho\cdot x} = e^{\rho_r\cdot x}\big(\cos(\rho_i \cdot x) + i\,\sin(\rho_i \cdot x)\big),$$

which is rapidly increasing in the direction ρ_r and rapidly oscillating in the direction ρ_i. Nonetheless, on a compact domain such as X, then $|u_\rho|$ is uniformly bounded from below by a positive constant.

Let now $f_\rho = u_{\rho|\partial X}$ the trace of the CGO solution on ∂X. Then for f close to f_ρ and u the solution to, say, (79) or (90), we also obtain that $|u|$ is bounded from below by a positive constant. Such results were used in [Triki 2010].

Lower bound for vector fields.

Theorem 5.6 [Bal and Uhlmann 2010]. *Let u_{ρ_j} for $j = 1, 2$ be CGO solutions with q as above for both ρ_j and $k \geq 1$ and with $c_0^{-1}|\rho_1| \leq |\rho_2| \leq c_0|\rho_1|$ for some $c_0 > 0$. Then we have*

$$\hat{\beta} := \frac{1}{2|\rho_1|} e^{-(\rho_1+\rho_2)\cdot x}\Big(u_{\rho_1}\nabla u_{\rho_2} - u_{\rho_2}\nabla u_{\rho_1}\Big) = \frac{\rho_1 - \rho_2}{2|\rho_1|} + \hat{h}, \tag{92}$$

where the vector field $\hat{h}$ satisfies the constraint

$$\|\hat{h}\|_{C^k(\bar{X})} \leq \frac{C_0}{|\rho_1|}, \tag{93}$$

for some constant C_0 independent of ρ_j, $j = 1, 2$.

With $\rho_2 = \overline{\rho_1}$ so that $u_{\rho_2} = \overline{u_{\rho_1}}$, the imaginary part of (92) is a vector field that does not vanish on X for $|\rho_1|$ sufficiently large. Moreover, let $u_{\rho_1} = v + iw$

and $u_{\rho_2} = v - iw$ for v and w real-valued functions. Then the imaginary and real parts of (92) are given by

$$\Im\hat\beta = \frac{1}{|\rho_1|} e^{-2\Re\rho_1\cdot x}(w\nabla v - v\nabla w) = \frac{\Im\rho_1}{|\rho_1|} + \Im\hat h, \quad \Re\hat\beta = 0.$$

Let u_1 and u_2 be solutions of the elliptic problem (79) on X such that $u_1 + iu_2$ on ∂X is close to the trace of u_{ρ_1}. The above result shows that

$$|u_1\nabla u_2 - u_2\nabla u_1| \geq c_0 > 0 \quad \text{in } X.$$

This yields (24) and the result on unique and stable reconstructions in QPAT.

The above derivation may be generalized to the vector field β_α in (32) with applications in elastography. Indeed let us start from (81) with $\rho = \boldsymbol{k} + i\boldsymbol{l}$ such that $\boldsymbol{k}\cdot\boldsymbol{l} = 0$ and $k := |\boldsymbol{k}| = |\boldsymbol{l}|$. Then, using Corollary 5.4, we find that

$$\begin{aligned} \Re u_\rho &= e^{\boldsymbol{k}\cdot x}(\mathfrak{c} + \varphi_\rho^r), & \Im u_\rho &= e^{\boldsymbol{k}\cdot x}(\mathfrak{s} + \varphi_\rho^i), \\ \nabla\Re u_\rho &= k e^{\boldsymbol{k}\cdot x}(\mathfrak{c}\hat{\boldsymbol{k}} - \mathfrak{s}\hat{\boldsymbol{l}} + \chi_\rho^r), & \nabla\Im u_\rho &= k e^{\boldsymbol{k}\cdot x}(\mathfrak{s}\hat{\boldsymbol{k}} + \mathfrak{c}\hat{\boldsymbol{l}} + \chi_\rho^i), \end{aligned} \tag{94}$$

where $\mathfrak{c} = \cos(\boldsymbol{l}\cdot x)$, $\mathfrak{s} = \sin(\boldsymbol{l}\cdot x)$, $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$, $\hat{\boldsymbol{l}} = \boldsymbol{l}/|\boldsymbol{k}|$, and where $|\rho|\,|\zeta|$ is bounded as indicated in Corollary 5.4 for $\zeta \in \{\varphi_\rho^r, \varphi_\rho^i, \chi_\rho^r, \chi_\rho^i\}$.

Let u_1 on ∂X be close to $\Re u_\rho$. Then we find by continuity that $|\nabla u_1|$ is close to $|\nabla\Re u_\rho|$ so that for k sufficiently large, we find that

$$|\nabla u_1| \geq c_0 > 0 \quad \text{in } X. \tag{95}$$

This proves that $H(x) = \gamma|\nabla u|$ is bounded from below by a positive constant provided that the boundary condition f is in a well-chosen open set of illuminations.

For the application to elastography, define

$$\beta_\alpha = \Im u_\rho \nabla\Re u_\rho - \alpha\Re u_\rho\nabla\Im u_\rho, \quad \alpha > 0.$$

For $|\rho| > \rho_\alpha$ large enough that $|\zeta| < \dfrac{(\min(1,\alpha))^2}{4(1+\alpha)}$ for $\zeta \in \{\varphi_\rho^r, \varphi_\rho^i, \chi_\rho^r, \chi_\rho^i\}$, we verify using (94) that

$$|\beta_\alpha| \geq k e^{2\boldsymbol{k}\cdot x}\tfrac{1}{2}\big((\mathfrak{c}\mathfrak{s}(1-\alpha))^2 + (\mathfrak{s}^2 + \alpha\mathfrak{c}^2)\big) \geq k e^{2\boldsymbol{k}\cdot x}\tfrac{1}{2}(\min(1,\alpha))^2. \tag{96}$$

This provides a lower bound for β_α uniformly on compact sets. For an open set of illuminations (f_1, f_2) close to the traces of $(\Im u_\rho, \Re u_\rho)$ on ∂X, we find by continuity that the vector field $\beta_\alpha = u_1\nabla u_2 - \alpha u_2\nabla u_1$ in (32) also has a norm bounded from below uniformly on X.

Lower bound for determinants. The reconstruction in Theorem 4.7 requires that the determinants in (74) be bounded from below. In specific situations, for instance when the conductivity is close to a given constant, such a determinant is indeed bounded from below by a positive constant for a large class of boundary conditions. However, it has been shown in [Briane et al. 2004] that the determinant of the gradients of three solutions could change signs on a domain with conductivities with large gradient. Unlike what happens in two dimensions of space, it is therefore not possible in general to show that the determinant of gradients of solutions has a given sign. However, using CGO solutions, we can be assured that on given bounded domains, the larger of two determinants is indeed uniformly positive for well-chosen boundary conditions.

Let $u_\rho(x)$ be given by (91) solution of the elliptic problem (90). Upon treating the term ψ_ρ and its derivative as in (94) above and making them arbitrary small by choosing ρ sufficiently large, we find that $\sqrt{\gamma}u_\rho = e^{\rho\cdot x} + \text{l.o.t.}$, so that, to leading order,

$$\sqrt{\gamma}\nabla u_\rho = e^{\boldsymbol{k}\cdot x}(\boldsymbol{k}+i\boldsymbol{l})\big(\cos(\boldsymbol{l}\cdot x)+i\,\sin(\boldsymbol{l}\cdot x)\big)+\text{l.o.t.},\qquad \rho=\boldsymbol{k}+i\boldsymbol{l}.$$

Let $n=3$ and (e_1,e_2,e_3) a constant orthonormal frame of $\mathbb{R}^3$. It remains to take the real and imaginary parts of the above terms and choose $\hat{\boldsymbol{k}}=e_2$ or $\hat{\boldsymbol{k}}=e_3$ with $\hat{\boldsymbol{l}}=e_1$ to obtain, up to normalization and negligible contributions (for $k=|\boldsymbol{k}|$ sufficiently large), that for

$$\tilde{S}_1=e_2\cos kx_1-e_1\sin kx_1,\quad \tilde{S}_2=e_1\cos kx_1+e_2\sin kx_1,$$
$$\tilde{S}_3=e_3\cos kx_1-e_1\sin kx_1,\quad \tilde{S}_4=e_2\cos|k|x_1+e_3\sin|k|x_1,$$

we verify that $\det(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3)=-\cos kx_1$ and that $\det(\tilde{S}_1,\tilde{S}_2,\tilde{S}_4)=-\sin kx_1$. Upon changing the sign of S_3 or S_4 if necessary to make both determinants nonnegative, we find that the maximum of these two determinants is always bounded from below by a positive constant uniformly on X. This result is sufficient to prove Theorem 4.7; see [Bal et al. 2011a].

Hyperbolicity of a Lorentzian metric. As a final application of CGO solutions, we mention the proof that a given constant vector field remains a timelike vector of a Lorentzian metric. This finds applications in the proof of Theorem 4.1 in [Bal 2012].

Indeed, let $\hat{\boldsymbol{k}}$ be a given direction in $\mathbb{S}^{n-1}$ and $\rho=i\boldsymbol{k}+\boldsymbol{k}^\perp$ and $u_\rho=e^{\rho\cdot x}$, once again neglecting ψ_ρ. The real and imaginary parts of ∇u_ρ are such that

$$e^{-\boldsymbol{k}^\perp\cdot x}\Im\nabla e^{\rho\cdot x}=|\boldsymbol{k}|\theta(x),\;\; e^{-\boldsymbol{k}^\perp\cdot x}\Re\nabla e^{\rho\cdot x}=|\boldsymbol{k}|\theta^\perp(x),\tag{97}$$

where $\theta(x)=\hat{\boldsymbol{k}}\cos\boldsymbol{k}\cdot x+\hat{\boldsymbol{k}}^\perp\sin\boldsymbol{k}\cdot x$ and $\theta^\perp(x)=-\hat{\boldsymbol{k}}\sin\boldsymbol{k}\cdot x+\hat{\boldsymbol{k}}^\perp\cos\boldsymbol{k}\cdot x$. As usual, $\hat{\boldsymbol{k}}=\boldsymbol{k}/|\boldsymbol{k}|$.

Define the Lorentzian metrics

$$\mathfrak{h}_\theta = 2\theta \otimes \theta - I, \quad \mathfrak{h}_{\theta^\perp} = 2\theta^\perp \otimes \theta^\perp - I.$$

Note that $\theta(x)$ and $\theta^\perp(x)$ oscillate in the plane $(\boldsymbol{k}, \boldsymbol{k}^\perp)$. The given vector $\hat{\boldsymbol{k}}$ thus cannot be a time like vector for one of the Lorentzian metrics for all $x \in X$ (unless X is a domain included in a thin slab). However, in the vicinity of any point x_0, we can construct a linear combination $\psi(x) = \cos\alpha\, \theta(x) + \sin\alpha\, \theta^\perp(x)$ for $\alpha \in [0, 2\pi)$ such that

$$\hat{\boldsymbol{k}} \text{ is a timelike vector for } \mathfrak{h}_\psi = 2\psi \otimes \psi - I, \quad \text{i.e.,} \quad \mathfrak{h}_\psi(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}) = 2(\psi \cdot \hat{\boldsymbol{k}})^2 - 1 > 0,$$

uniformly for x close to x_0; see [Bal 2012] for more details. When $\theta(x)$ is constructed as $\widehat{\nabla u}$ for u solution to (79) or (90) for boundary conditions f close to the trace of the corresponding CGO solution u_ρ, then the Lorentzian metric $\mathfrak{h}_\psi$ constructed above still verifies that $\hat{\boldsymbol{k}}$ is a timelike vector with $\mathfrak{h}_\psi(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}})$ uniformly bounded from below by a positive constant locally.

6. Conclusions and perspectives

Research in hybrid inverse problems has been very active in recent years, primarily in the mathematical and medical imaging communities but also in geophysical imaging, see [White 2005] and references on the electrokinetic effect. This review focused on time-independent equations primarily with scalar-valued solutions. We did not consider the body of work done in the setting of time-dependent measurements, which involves different techniques than those presented here; see [McLaughlin et al. 2010] and references. We considered scalar equations with the exception of the system of Maxwell's equations as it appears in thermoacoustic tomography. Very few results exist for systems of equations. The diffusion and conductivity equations considered in this review involve a scalar coefficient γ. The reconstruction of more general tensors remains an open problem.

Compared to boundary value inverse problems, inverse problems with internal measurements enjoy better stability estimates precisely because local information is available. However, the derivation of such stability estimates often requires that specific, qualitative properties of solutions be satisfied, such as for instance the absence of critical points. This imposes constraints on the illuminations (boundary conditions) used to generate the internal data that forms one of the most difficult mathematical questions raised by the hybrid inverse problems.

What are the "optimal" illuminations (boundary conditions) for a given class of unknown parameters and how robust will the reconstructions be when such illuminations are modified are questions that are not fully answered. The theory of complex geometrical optics (CGO) solutions provides a useful tool to address these questions and construct suitable illuminations or prove their existence in several cases of interest. Numerical simulations will presumably be of great help to better understand whether such theoretical predictions are useful or reasonable in practice. Many numerical simulations performed in two dimensions of space confirm the good stability properties predicted by theory [Ammari et al. 2008; Bal and Ren 2011a; Bal et al. 2011b; Capdeboscq et al. 2009; Gebauer and Scherzer 2008; Kuchment and Kunyansky 2011; Nachman et al. 2009]. The two-dimensional setting is special as we saw in Section 5A. Very few simulations have been performed in the theoretically more challenging case of three (or more) dimensions of space. Simulations in [Kuchment and Kunyansky 2011] show very promising three-dimensional reconstructions in the setting of diffusion coefficients that are close to the constant case, which is also understood theoretically since $|\nabla u|$ then does not vanish for a large class of boundary conditions.

The main interest of hybrid inverse problems is that they combine high contrast with high resolution. This translates mathematically into good (Lipschitz or Hölder) stability estimates. Ideally, we would like to reconstruct highly oscillatory coefficients with a minimal influence of the noise in the measurements. Yet, all the results presented in this review paper and the cited references require that the coefficients satisfy some unwanted smoothness properties. To focus on one example for concreteness, the reconstructions in photoacoustic tomography involve the solution of the transport equation (26), which is well-posed provided that the vector field β is sufficiently smooth. Using theories of renormalization, the regularity of such vector fields can be decreased to $W^{1,1}$ or to the BV category [Ambrosio 2004; Bouchut and Crippa 2006; DiPerna and Lions 1989]. Yet, $u_1 \nabla u_2 - u_2 \nabla u_1$ is a priori only in L^2 when γ is arbitrary as a bounded coefficient [Hauray 2003]. The construction of CGO solutions presented in Section 5B also requires sufficient smoothness of the coefficients. How such reconstructions and stability estimates might degrade in the presence of nonsmooth coefficients is quite open. Many similar problems are also open for boundary-value inverse problems [Uhlmann 2009].

Finally, we have assumed in this review that the first step of the hybrid inverse problems had been done accurately. In practice, this may not quite always be so. PAT and TAT require that we solve an inverse source problem for a wave equation, which is a difficult problem in the presence of partial data and variable sound speed and is not entirely understood when realistic absorbing effects are accounted for [Kowar and Scherzer 2012; Stefanov and Uhlmann 2011]. In UMEIT and UMOT, we have assumed in the derivation in Section 2 that standing plane waves could be generated. This is practically difficult to achieve and different (equivalent) mechanisms have been proposed [Ammari et al. 2008; Kuchment and Kunyansky 2011]. In transient elastography, we have assumed that the full (scalar) displacement could be reconstructed as a function of time and space. This is also sometimes an idealized approximation of what can be achieved in practice [McLaughlin et al. 2010]. Finally, we have assumed knowledge of the current $\gamma|\nabla u|$ in CDII, which is also difficult to acquire in practical settings as typically only the z component of the magnetic field B_z can be constructed; see the recent review [Seo and Woo 2011]. The modeling of errors generated during the first step of the procedure and the influence that such errors may have on the reconstructions during the second step of the hybrid inverse problem remain active areas of research.

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