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Hybrid modelling techniques using acoustic boundary and finite element methods

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1. Introduction

Acoustic analysis is becoming more commonplace due to the evolution of suitable analysis techniques and the increase in computational power of affordable computers. For small acoustic problems, i.e., where the region to be modelled is not large numbers of acoustic wavelengths in extent, the two most commonly used methods are finite elements (FE) and boundary elements (BE). These two techniques both have their advantages and disadvantages, depending on the problem to be solved. FE are definitely better if the fluid medium has inhomogeneous properties. BE do not deal as well with natural frequency calculations. For steady state harmonic response the BE method is better at exterior problems. For interior problems, either technique could be used - the author's opinion is that FE has the edge. The relative computational advantage of the FE will increase with the ratio of boundary to volume, as the geometry of the problem changes. Ease of mesh creation is another factor to be taken into account.

This paper considers a method of coupling the FE and BE for steady state acoustic problems. The intention is to enable an efficient solution for situations where neither method is ideally suited.

Fluid Equations

For small amplitude oscillations in an irrotational, inviscid, compressible fluid with no mean flow the pressure distribution satisfies the wave equation.

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \qquad (1)$$

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where c is the wavespeed. For steady state vibration at frequency ω this reduces to the Helmholtz equation

$$\nabla^2 p + k^2 p = 0 \tag{2}$$

where $k = \dfrac{\omega}{c}$ is the acoustic wavenumber. On a boundary with normal displacement u_n, the condition

$$\frac{\partial p}{\partial n} = -\rho \frac{\partial^2 u_n}{\partial t^2} \tag{3}$$

where ρ is the density, is required. For external problems, the total pressure p can be decomposed as

$$p = p_I + p_s \tag{4}$$

where p_I is the free field incident pressure and p_s is the scattered pressure, which must satisfy the Sommerfield radiation condition to ensure that it consists only of outgoing waves.

Acoustic Finite Elements

The acoustic FE equations can be derived using weighted residual methods, as described briefly below, or by applying variational principles. The weighted residuals statement is

$$\int_V W\left(\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}\right)dV + \int_\Gamma W\left(\frac{\partial p}{\partial n} + \rho\frac{\partial^2 u_n}{\partial t^2}\right)d\Gamma = 0 \tag{5}$$

where W is an arbitrary weighing function, Γ is the boundary of the fluid region and the normal n is taken positive into the fluid. Integrating by parts using divergence theorem, using an interpolation

$$p = [N]\{p\} \tag{6}$$

for pressure over a mesh of volume elements and using the Galerkin principle results in a set of equations which can be put in the form

$$[M_a]\{\ddot{p}\} + [S_a]\{p\} - \int_\Gamma [N]^T \ddot{u}_n d\Gamma = \{0\} \tag{7}$$

where $[M_a]$ and $[S_a]$ are square symmetric matrices which can be conveniently computed for each element and merged together for the whole system. These matrices are sparse.

More detailed derivation of the acoustic FE method together with some examples of its use is contained in Ref Petyt, Lea & Koopman[1].

Acoustic Boundary Elements

The direct formulation of the acoustic BE method is based on the Helmholtz formula,

$$\varepsilon p(\underline{x}) = \int_\Gamma \left(p(\underline{y}) \frac{\partial g(\underline{x},\underline{y})}{\partial n_y} - g(\underline{x},\underline{y}) \frac{\partial p(\underline{y})}{\partial n_y} \right) d\Gamma(\underline{y}) + p_I(\underline{x}) \tag{8}$$

where Γ is the boundary of the fluid region, $4\pi\varepsilon$ is the solid angle in the fluid domain for point $\underline{x}$ and

$$g(\underline{x},\underline{y}) = \frac{e^{-ikr}}{4\pi r} \tag{9}$$

is the free space Green's function and r is $|\underline{x}-\underline{y}|$. To obtain numerical solutions Γ is divided into patches R_i over which p and $\frac{\partial p}{\partial n}$ are interpolated using the same shape functions [N]. Equation (8) becomes

$$\varepsilon p(\underline{x}) - \sum_{i=1}^{m} \int_{R_i} \frac{\partial g}{\partial n_y} [N] d\Gamma \{p\} = -\sum_{i=1}^{m} \int_{R_i} g[N] d\Gamma \left\{ \frac{\partial p}{\partial n} \right\} + \{p_I\} \tag{10}$$

where m is the number of patches. Taking $\underline{x}$ to be at each nodal point in turn, a set of linear equations arise, which can be expressed in matrix form as

$$[H]\{p\} = [G]\left\{ \frac{\partial p}{\partial n} \right\} + \{p_I\} \tag{11}$$

For exterior problems, difficulties arise with the surface Helmholtz formulation at certain characteristic frequencies. The matrices $[H]$ and $[G]$ become singular and are ill-conditioned at nearby frequencies. More elaborate methods can be used to overcome these problems, such as the CHIEF method due to Schenck[2] or the Burton-Miller[3] formulation.

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Coupling To Structural Finite Elements

An acoustic FE or BE mesh can be coupled to a structural FE mesh. It is necessary to add a loading term in the structural equations due to the pressure distribution in the fluid. The usual structural FE equations become modified to

$$\left([S]+i\omega[C]-\omega^2[M]\right)\{u\}+[T]^T\{p\}=\{F\} \qquad (12)$$

where $[S]$, $[C]$ and $[M]$ are the structural stiffness, damping and mass matrices respectively $\{u\}$ is a vector of displacements on the structural mesh, $\{F\}$ is a vector of externally applied forces and $[T]$ is a coupling matrix defined by

$$[T]=\int_{\Gamma}[N_a]\underline{n}^T[N_s]^T d\Gamma \qquad (13)$$

where $[N_a]$ are the acoustic shape functions and $[N_s]$ are structural shape functions interpolating the displacements $\underline{u}$.

For coupling with acoustic FE it is necessary to express the loading term due to the boundary accelerations in terms of the structural motion. Equation (7) becomes modified to

$$\left([S_a]-\omega^2[M_a]\right)\{p\}+\omega^2[T]\{u\}=\{0\} \qquad (14)$$

Coupling between structural FE and an acoustic BE requires the construction of a matrix $[E]$ such that $[E]^T\{u\}$ gives the normal displacements at the degrees of freedom on the BE mesh. Equation (11) becomes

$$[H]\{p\}=\omega^2\rho[G][E]^T\{u\}+\{p_I\} \qquad (15)$$

An example of analyses using structural FE, acoustic FE and acoustic BE is contained in Ref Macey and Hardie[4].

Coupling Acoustic Finite Elements And Boundary Elements

Let the acoustic FE and BE pressure vectors be denoted by $\{p_f\}$ and $\{p_b\}$ respectively. The latter can be further split into $\{p_{b1}\}$ and $\{p_{b2}\}$ where $\{p_{b1}\}$ are the pressures on the interface with the acoustic FE mesh and $\{p_{b2}\}$ are on the interface with the structural mesh. The accelerations on the boundary Γ in equation (7) can be related in part to the surface motion of the structural FE mesh and in part to the normal pressure gradients on the BE mesh. This results in the equation

$$\left([S_a]-\omega^2[M_a]\right)\{p_f\}+\omega^2[T_f]\{u\}+[Q]\{p_b\}=-[R]\{p_{bI}\} \qquad (16)$$

where

$$[R]=\frac{1}{\rho}\int_{\Gamma_r}[N_a]^T[N_b]\; d\Gamma\; [G]^{-1},\; [Q]=-[R][H] \qquad (17)$$

and Γ_r is the interface between the acoustic FE and BE. The pressures on the two meshes can be related as:

$$\{p_{b1}\}=[W]\{p_f\} \qquad (18)$$

where $[W]$ just contains rows of shape functions. The full set of equations is

$$\begin{bmatrix} [S]+i\omega[C]-\omega^2[M] & [T_f]^T & [T_{b1}]^T & [T_{b2}]^T \\ \omega^2[T_f] & [S_a]-\omega^2[M_a] & [Q_1] & [Q_2] \\ [O] & [W] & -[I] & [O] \\ -\omega^2\rho[E]^T & [O] & [D_{21}] & [D_{22}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{p_f\} \\ \{p_{b1}\} \\ \{p_{b2}\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ -[R]\{p_{bI}\} \\ \{O\} \\ [K_{22}]\{p_{bI}\} \end{Bmatrix} \qquad (19)$$

where $[I]$ is the unit matrix and

$$[D]=[H][G]^{-1},\; [K]=[G]^{-1} \qquad (20)$$

Numerical Results

Three test problems have been used to check the new hybrid formulation, a piston vibrating in an infinite rigid baffle radiating into a half space, a radially poled piezoelectric cylinder transducer and three coaxial rigid disks scattering an axially incident plane wave. Results presented in this paper use axisymmetric meshes, although the PAFEC acoustics program also permits acoustic FE/BE coupling for 3D meshes. Quadratic elements are used throughout, so that the acoustic FE mesh is composed of 8 noded quadrilaterals and 6 noded triangles and the BE is composed of 3 noded line patches.

The piston problem has an analytical solution for comparison. The piston was taken to have radius 1m and vibrate with unit velocity. The fluid was taken to be water with ρ=1000kgm^{-3} and c=1500ms^{-1}. Acoustic FE were used to mesh the space immediately in front of the piston and the BE was used for the rest of the half space, see figure 1. Figure 2 shows a comparison of pressure computed at the piston centre with the analytical result.

The details of the cylinder together with the experimental results are given by Rogers[5], Figure 3 shows the FE/BE mesh used. Axisymmetric piezoelectric FE

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were used to model the cylinder. Figure 4 gives a comparison against the experimental results.

The coaxial disks were taken to have radius 0.75m, thickness 0.25m and separation of 0.075m. The surrounding fluid was taken to be water with properties as above. Figure 5 shows the mesh used. Figure 6 compares results with a pure BE solution at a point on axis, 0.25m before the first disk. The new hybrid model ran at least twice as fast as the pure BE solution, due to the reduction in the number of nodes on the BE.

In all cases, the comparison is good.

Conclusion

The hybrid FE/BE formulation has been shown to work correctly. It is able to reduce the solution time in many cases if the 'amount of boundary' can be reduced. Another speed enhancement from this method is to model the fluid around a vibrating structure with FE and use an approximate but faster BE on the remote boundary, as tried successfully by Hardie[6]. It will also find application where the near field fluid is not homogeneous.

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Figure 1

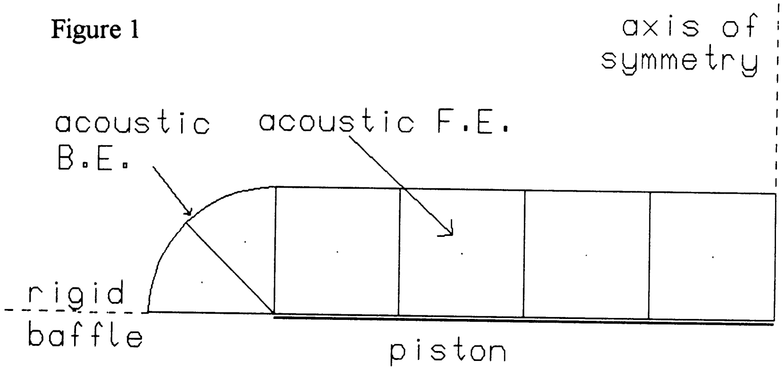

Figure 2

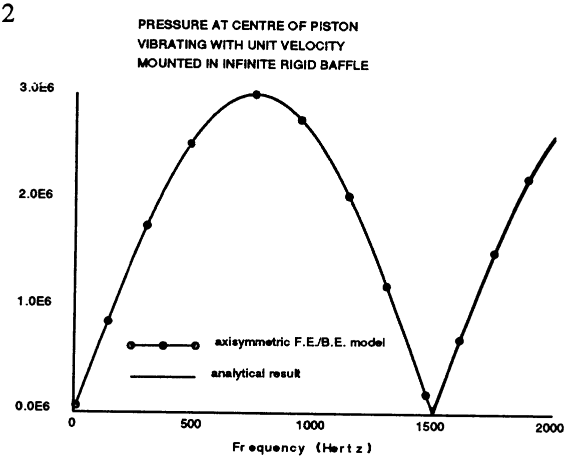

Figure 3

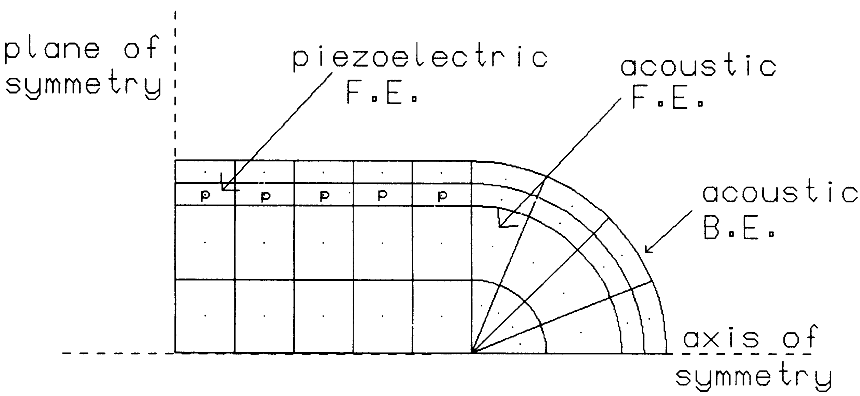

Figure 4

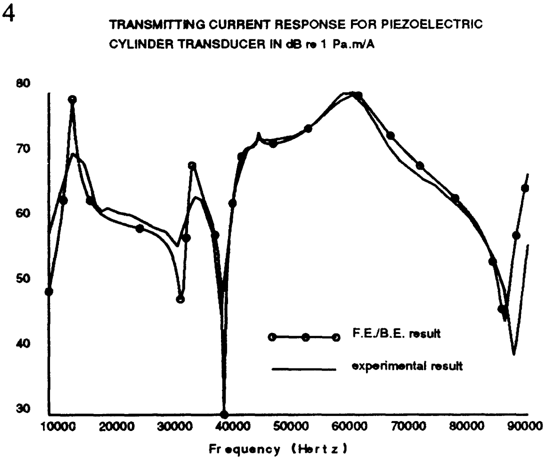

Figure 5

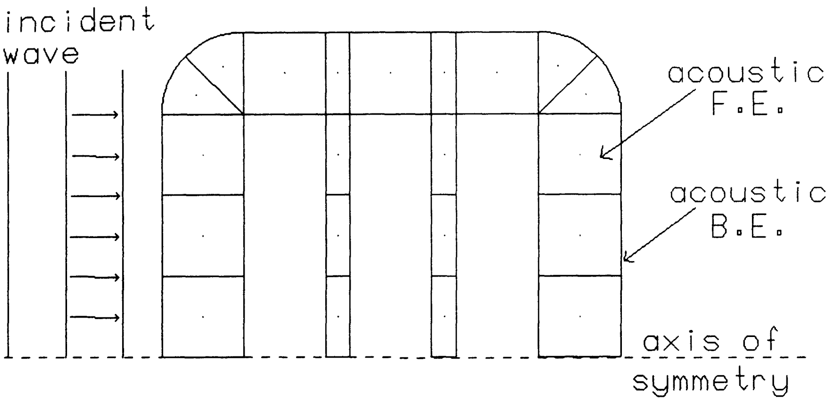

Figure 6