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HYBRIDIZED GLOBALLY DIVERGENCE-FREE LDG METHODS. PART I: THE STOKES PROBLEM

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ABSTRACT. We devise and analyze a new local discontinuous Galerkin (LDG) method for the Stokes equations of incompressible fluid flow. This optimally convergent method is obtained by using an LDG method to discretize a vorticity-velocity formulation of the Stokes equations and by applying a new *hybridization* to the resulting discretization. One of the main features of the hybridized method is that it provides a globally divergence-free approximate velocity without having to construct globally divergence-free finite-dimensional spaces; only elementwise divergence-free basis functions are used. Another important feature is that it has significantly less degrees of freedom than all other LDG methods in the current literature; in particular, the approximation to the pressure is only defined on the faces of the elements. On the other hand, we show that, as expected, the condition number of the Schur-complement matrix for this approximate pressure is of order h^{-2} in the mesh size h. Finally, we present numerical experiments that confirm the sharpness of our theoretical a priori error estimates.

1. INTRODUCTION

This is the first in a series of papers in which we propose and analyze hybridized, globally divergence-free local discontinuous Galerkin (LDG) methods. In this paper, we present the main ideas in a simple setting and consider the Stokes equations

$$-\Delta\vec{u} + \operatorname{grad} p = \vec{f}, \qquad \operatorname{div}\vec{u} = 0 \qquad \text{in } \Omega; \qquad \vec{u} = \vec{u}_D \qquad \text{on } \Gamma = \partial\Omega;$$

which model the flow of a viscous, incompressible fluid. Here Ω is taken to be a bounded polygonal domain in $\mathbb{R}^2$, $\vec{f} \in L^2(\Omega)^2$ a given source term, and $\vec{u}_D \in H^{1/2}(\Gamma)^2$ is a prescribed Dirichlet datum satisfying the usual compatibility condition

$$\int_\Gamma \vec{u}_D \cdot \vec{n}\, ds = 0, \tag{1.1}$$

with $\vec{n}$ denoting the outward normal unit vector on the boundary Γ of the domain Ω.

We devise an LDG method that is locally conservative, optimally convergent and provides an approximate velocity which is globally divergence-free, that is, which

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belongs to the space

$$H(\mathrm{div}^0;\Omega) = \{\, \vec{v} \in L^2(\Omega)^2 : \mathrm{div}\, \vec{v} = 0 \text{ in } \Omega\}.$$

One of the advantages of working with finite-dimensional subspaces of $H(\mathrm{div}^0;\Omega)$ is that the approximate velocity has a smaller number of degrees of freedom in comparison with standard finite element spaces with similar approximation properties. Another advantage is that the pressure can be completely eliminated from the equations, and the resulting problem for the velocity becomes a simple second-order elliptic problem. An LDG method for the velocity can be devised in a most straightforward way; the pressure can be easily computed once the velocity is obtained. On the other hand, finite-dimensional subspaces of $H(\mathrm{div}^0;\Omega)$ are extremely difficult to deal with because their basis functions are complicated to construct and do not have local support. The main contribution of this paper is to show that this difficulty can be overcome by a simple *hybridization* of the LDG method.

This hybridization is similar to that of mixed methods for second-order elliptic problems; see [1], [8], [9], and [12]. It is achieved as follows. First, we relax the continuity of the normal components of the approximate velocity across interelement boundaries. This allows us to use velocity spaces whose elements are completely discontinuous and divergence-free inside each element. In this procedure, however, the pressure has to be reintroduced in the equations as the corresponding Lagrange multiplier. Furthermore, new equations have to be added to the system to render it solvable. These equations enforce the continuity of the normal components of the new approximate velocity. Thus, although the approximate velocity is divergence-free only inside each element, the additional equations automatically ensure that it is also globally divergence-free. In this way, the proposed hybridization allows one to base the approximate velocity on the space $H(\mathrm{div}^0;\Omega)$ without having to work with globally divergence-free finite-dimensional spaces. Another advantage of this approach is that the resulting hybridized LDG method produces a remarkably *small* Schur-complement matrix for the pressure, given that this variable is only defined on the edges of the triangulation; the values of the pressure inside the elements can be computed in an element-by-element fashion.

The price to pay for these advantages is the fact that the condition number of the Schur-complement matrix for the pressure is of order h^{-2}, instead of being of order one, as is typical of classical mixed methods. Fortunately, good preconditioners can be obtained by using the techniques developed in [5]. We also refer the reader to [24] for Schwarz preconditioners for hybridized discretizations of second-order problems.

We carry out the numerical analysis of the resulting hybridized LDG method. We show that it is well defined and derive optimal a priori error bounds in the classical norms. Thus, for approximations using polynomials of degree $k \geq 1$, the error in the velocity, measured in an H^1-like norm, is proven to converge with the optimal order k with respect to the mesh size. Similarly, we show that the L^2-errors in the vorticity and the pressure converge with the optimal order k when these variables are taken to be piecewise polynomials of degree $k-1$. The same rates are obtained if all the unknowns are approximated by polynomials of degree k. We display numerical results showing that these results are sharp.

Finally, let us point out that, although here we use the LDG approach to discretize the Stokes problem, our theoretical results are equally valid for any other stable and consistent DG discretization as described in the unifying analysis for second-order problems proposed in [2].

This paper is organized as follows. In Section 2, we present a brief overview of the development of divergence-free methods for the Stokes problem, in order to place our work into perspective. The hybridized LDG method is then described in Section 3. In Section 4, we present our main theoretical results, namely, optimal error estimates for the velocity and the vorticity (Theorem 4.1) and optimal error estimates for the pressure (Theorem 4.2). We also outline the main steps of the proof of these estimates while the details of some of the auxiliary results are given in Section 6. In Section 5, we discuss a modification of the matrix equations that allows us to use minimization algorithms to actually solve the resulting linear systems. We further state an upper bound on the condition number of the Schur-complement matrix for the pressure (Theorem 5.2). This bound follows from an inf-sup condition whose proof is contained in Section 7. Then in Section 8, we present numerical experiments that confirm that the theoretical orders of convergence are sharp. Finally, we end in Section 9 by discussing some extensions of the proposed method.

2. A historical overview of the main ideas for enforcing incompressibility

To properly motivate the devising of hybridized LDG methods, we briefly discuss the evolution of some of the main ideas and techniques used to deal with the incompressibility condition $\operatorname{div} \vec{u} = 0$. To simplify the exposition, we take $\vec{u}_D = \vec{0}$.

2.1. **Enforcing exact incompressibility.** By considering finite-dimensional subspaces $\vec{V}_h \subset H_0^1(\Omega)^2 \cap H(\operatorname{div}^0; \Omega)$, exactly incompressible velocity approximations $\vec{u}_h \in \vec{V}_h$ can be readily defined by requiring that

$$(\operatorname{grad} \vec{u}_h, \operatorname{grad} \vec{v}) = (\vec{f}, \vec{v}), \qquad \vec{v} \in \vec{V}_h.$$

Unfortunately, since the early beginnings of the development of finite element methods for incompressible flow, it was clear that the construction of such finite-dimensional spaces $\vec{V}_h$ was an extremely difficult goal to achieve. Indeed, in his pioneering work of 1972, Fortin [20] was able to construct spaces of that type, but they turned out to be “complex elements of limited applicability”, as Crouzeix and Raviart said in their seminal paper of 1973, [18]. In an effort to be able to use simpler methods, these authors proposed an alternative approach.

2.2. **Enforcing weak incompressibility.** Crouzeix and Raviart [18] sacrificed the exact verification of the incompressibility condition and opted instead for enforcing it only weakly; the pressure must then be considered simultaneously with the velocity. In the case of conforming methods, for example, we take $(\vec{u}_h, p_h)$ in a finite-dimensional space $\vec{V}_h \times Q_h \subset H_0^1(\Omega)^2 \times L^2(\Omega)/\mathbb{R}$, and determine it by requiring that

$$(\operatorname{grad} \vec{u}_h, \operatorname{grad} \vec{v}) - (p_h, \operatorname{div} \vec{v}) = (\vec{f}, \vec{v}), \quad (q, \operatorname{div} \vec{u}_h) = 0, \qquad (\vec{v}, q) \in \vec{V}_h \times Q_h.$$

Of course, we can still try to solve only for the velocity in the above method. Indeed, since the approximate velocity $\vec{u}_h$ belongs to the finite-dimensional space

$$\vec{Z}_h = \left\{ \vec{v} \in \vec{V}_h : (q, \operatorname{div} \vec{v}) = 0, \quad q \in Q_h \right\},$$

it can be characterized as the only element of $\vec{Z}_h$ such that

$$(\operatorname{grad}\vec{u}_h, \operatorname{grad}\vec{v}) = (\vec{f}, \vec{v}), \qquad \vec{v} \in \vec{Z}_h.$$

The pressure can thus be eliminated from the equations and recovered once the velocity $\vec{u}_h$ is computed (see [10] and [11]) by solving

$$-(p_h, \operatorname{div}\vec{v}) = (\vec{f}, \vec{v}) - (\operatorname{grad}\vec{u}_h, \operatorname{grad}\vec{v}), \qquad \vec{v} \in \vec{V}_h.$$

The solvability of this equation is guaranteed by a discrete inf-sup condition for the method; see the books [9] and [22], and the recent article [27].

Unfortunately, bases for the space $\vec{Z}_h$ of weak incompressibility are also very difficult to construct. Bases for spaces of this type were constructed by Griffiths in 1979 [25], by Thomasset in 1981 [38], both for the two-dimensional case, and by Hecht in 1981 [28] for the three-dimensional case and the nonconforming method using piecewise-linear functions. However, they have a support that is not necessarily local; this has a negative impact on the sparsity of the stiffness matrix. Yet another problem with this approach was discovered in 1990 by Dörfler [19] who showed that the condition number of the stiffness matrix for the velocity is of order h^{-4}. Because of all these difficulties, this approach was never very popular.

2.3. **Weak incompressibility can ensure exact incompressibility.** Of course, exact incompressibility follows from weakly imposed incompressibility if $\operatorname{div}\vec{V}_h \subset Q_h$. This condition is satisfied, for example, if we take $\vec{V}_h$ to be the space of continuous vector fields which are polynomials of degree k on each triangle and W_h the space of discontinuous functions which are piecewise polynomials of degree $k-1$. In 1985, Scott and Vogelius [37] showed that the constant of the inf-sup condition for this conforming method is independent of the mesh size, provided $k \geq 4$ (and a minor condition on the triangulation holds). This implies that the approximate velocity $\vec{u}_h$ is globally divergence-free and that the solution $(\vec{u}_h, p_h)$ is optimally convergent; see other cases in [6]. Lower order polynomial spaces with this property do not exist, as was proven in 1975 by Fortin [21]. Extensions of these results to the three-dimensional case seem to be quite challenging and remain an interesting open problem. Other examples of how weak incompressibility implies exact incompressibility can be found in the book by Gunzburger [26].

2.4. **Locally incompressible approximations.** In 1990, Baker, Jureidini and Karakashian considered locally incompressible velocities, that is, they took

$$\vec{V}_h \subset \{\vec{v} \in L^2(\Omega)^2 : \vec{v}|_K \in H(\operatorname{div}^0; K),\ K \in \mathcal{T}_h\},$$

with $\mathcal{T}_h$ being a triangulation of Ω. Since the use of this space rendered *impossible* the satisfaction of the continuity constraints typical of conforming and nonconforming methods, the authors were led to use *completely discontinuous* approximations for the velocity; the approximate pressures where, however, continuous. This optimally convergent method was extended in the late 90's to the Navier–Stokes equations by Karakashian and Katsaounis [30] and by Karakashian and Jureidini [29] with excellent results.

Recently, several DG methods for incompressible flow have been proposed in the literature, all of which impose the incompressibility condition weakly; see [4], [17], [16], [14], [15], [23] and [36]. General DG methods face an important difficulty when applied to the Navier–Stokes equations. In this case, the appearance of the nonlinear convection introduces a phenomenon that is not present in the Stokes or Oseen equations, namely, that energy-stable DG methods, like the methods proposed in [29] and in [23], *cease* to be locally conservative *because* the incompressibility condition is enforced only weakly.

2.5. **Exactly incompressible, locally conservative LDG methods.** To recover the highly valued property of local conservation for DG methods for the Navier–Stokes equations, a new way to deal with the incompressibility condition was introduced by Cockburn, Kanschat and Schötzau in [15]. It is based on the observation that, for mixed DG discretizations of the incompressibility condition, a globally divergence-free velocity can be easily computed by using an element-by-element post-processing of the approximate velocity $\vec{u}_h$. Since the velocity $\vec{u}_h$ is only weakly incompressible, this gives another way of enforcing the incompressibility condition strongly by only enforcing it weakly.

For this locally conservative, optimally convergent DG method, the velocity is taken to be a piecewise polynomial of degree $k \geq 1$ and the pressure a piecewise polynomial of degree $k-1$. This is the first method to produce globally divergence-free velocities by using polynomials of degree equal or bigger than *one*, in both two and three space dimensions.

2.6. **Hybridized, exactly incompressible LDG methods.** The LDG method we propose in this paper is devised in an effort to *reduce* the number of unknowns of the LDG method in [15] while maintaining the exact incompressibility of its approximate velocity. This is achieved in two steps. First, we devise an LDG method which uses approximate velocities in spaces $\vec{\mathcal{V}}_h$ such that $\vec{\mathcal{V}}_h \subset H(\mathrm{div}^0;\Omega)$. Then, we hybridize the method. That is, we base its velocity approximation on spaces of the form

$$\vec{V}_h \subset \{\vec{v} \in L^2(\Omega)^2 : \vec{v}|_K \in H(\mathrm{div}^0;K),\ K \in \mathcal{T}_h\},$$

and still get the approximate velocity in $\vec{\mathcal{V}}_h$. The spaces $\vec{V}_h$ are remarkably simpler to deal with from the implementational point of view.

It must be emphasized that it is not known how to carry out a similar hybridization for classical conforming and nonconforming methods for the Stokes problem. Let us briefly discuss the nature of this difficulty. When dealing with the LDG method proposed in this paper, the original velocity spaces are such that $\vec{\mathcal{V}}_h \subset H(\mathrm{div}^0;\Omega)$, and hence to perform an hybridization, we only have to deal with the continuity of the normal components across interelement boundaries. On the other hand, for conforming methods, the velocity spaces satisfy the inclusion $\vec{V}_h \subset H^1_0(\Omega)^2$, and hence their hybridization must involve the more difficult issue of how to handle the continuity of the *whole* function and not only that of its normal component. For classical nonconforming methods, a similar situation arises.

3. The hybridized LDG method

In this section, we introduce the hybridized LDG method. After introducing the notation we are going to use, we first describe the LDG method with globally divergence-free velocities and then present its hybridization. Finally, we show how to compute the pressure in an element-by-element fashion in terms of the approximate solution of the hybridized LDG method.

3.1. **Preliminaries.** To define the method, we need to introduce some notation. Let us begin with the notation related to the triangulation of the domain. We denote by $\mathcal{T}_h$ a regular and shape-regular triangulation of Ω into triangles and set $h = h_{\max} = \max_{K\in\mathcal{T}_h}\{h_K\}$, where h_K is the diameter of the triangle K. We write $(p,\psi)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h}(p,\psi)_K$, where $(p,\psi)_K = \int_K p\,\psi\,dx$. Further, we denote by $\mathcal{E}_h^{\mathcal{I}}$ the set of interior edges of $\mathcal{T}_h$ and by $\mathcal{E}_h^{\mathcal{B}}$ the set of boundary edges. We define $\mathcal{E}_h = \mathcal{E}_h^{\mathcal{B}}\cup\mathcal{E}_h^{\mathcal{I}}$. If $\mathcal{F}_h$ is a subset of $\mathcal{E}_h$, we use the notation $\langle f,\phi\rangle_{\mathcal{F}_h} = \sum_{e\in\mathcal{F}_h}\langle f,\phi\rangle_e$, where $\langle f,\phi\rangle_e = \int_e f\,\phi\,ds$. For an element $K\in\mathcal{T}_h$, the boundary ∂K will be understood as a subset of $\mathcal{E}_h$.

Next, let us deal with the notation associated with weak formulations. Thus, if φ satisfies the equation $\mathsf{O}\,\varphi = \Phi$, where O is a first-order differential operator and φ is a scalar- or vector-valued function, we can write, for any element $K\in\mathcal{T}_h$, that

$$(\mathsf{O}\varphi,\psi)_K = (\varphi,\mathsf{O}^\star\,\psi)_K + \langle\varphi,\psi\odot\vec{n}_K\rangle_{\partial K} = (\Phi,\psi)_K,$$

where ψ is any smooth function, $\mathsf{O}^\star$ is the formal adjoint operator to O and the symbol $\odot$ stands for the corresponding multiplication operator on the boundary (with respect to the outward unit normal vector $\vec{n}_K$ on ∂K). Adding over the triangles of the triangulation, we obtain

$$(\varphi,\mathsf{O}^\star\,\psi)_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h}\langle\varphi,\psi\odot\vec{n}_K\rangle_{\partial K}, = (\Phi,\psi)_{\mathcal{T}_h}.$$

Next, we define the jump operator $[\![\cdot]\!]$ by

$$[\![\varphi\,(\psi\odot\vec{n})]\!] = \begin{cases}\varphi\,(\psi\odot\vec{n}) & \text{on boundary edges in } \mathcal{E}_h^{\mathcal{B}},\\ \varphi^+\,(\psi^+\odot\vec{n}_{K^+}) + \varphi^-\,(\psi^-\odot\vec{n}_{K^-}) & \text{on interior edges in } \mathcal{E}_h^{\mathcal{I}}.\end{cases}$$

Here $\varphi^\pm$ and $\psi^\pm$ denote the traces of φ and ψ on the edge $e = \partial K^+\cap\partial K^-$ taken from within the interior of the triangles $K^\pm$. We can thus rewrite the above equation as

$$(\varphi,\mathsf{O}^\star\,\psi)_{\mathcal{T}_h} + \langle 1,[\![\varphi\,(\psi\odot\vec{n})]\!]\rangle_{\mathcal{E}_h} = (\Phi,\psi)_{\mathcal{T}_h}.$$

Finally, to be able to define the discrete traces of the LDG method, it only remains to introduce the average $\{\!\{\cdot\}\!\}$ of the traces, namely,

$$\{\!\{\varphi\}\!\} = \begin{cases}\varphi & \text{on boundary edges in } \mathcal{E}_h^{\mathcal{B}},\\ \frac{1}{2}(\varphi^+ + \varphi^-) & \text{on interior edges in } \mathcal{E}_h^{\mathcal{I}}.\end{cases}$$

We are now ready to define the LDG method.

3.2. **An LDG method for the vorticity-velocity formulation.** We first introduce the LDG method that is based on velocities in $H(\mathrm{div}^0;\Omega)$. To introduce its weak formulation, we rewrite the Stokes system as

$$\omega - \mathrm{curl}\,\vec{u} = 0 \qquad \text{in } \Omega, \tag{3.1}$$

$$\vec{\mathrm{curl}}\,\omega + \mathrm{grad}\,p = \vec{f} \qquad \text{in } \Omega, \tag{3.2}$$

$$\mathrm{div}\,\vec{u} = 0 \qquad \text{in } \Omega, \tag{3.3}$$

$$\vec{u} = \vec{u}_D \qquad \text{on } \Gamma, \tag{3.4}$$

where ω is the scalar vorticity, $\omega = \mathrm{curl}\,\vec{u} = \partial_1 u_2 - \partial_2 u_1$, and $\vec{\mathrm{curl}}\,\omega$ is the vector-valued curl given by $\vec{\mathrm{curl}}\,\omega = (\partial_2\omega, -\partial_1\omega)$.

Next, we multiply equation (3.1) by a test function $\sigma \in L^2(\Omega)$, equation (3.2) by $\vec{v} \in H(\mathrm{div}^0;\Omega)$, such that $\vec{v}\cdot\vec{n} = 0$ on Γ, and equation (3.4) by $\vec{n}\,q$ with $q \in L^2(\Gamma)$. We assume that the test functions $\vec{v}$ and σ are smooth inside the triangles K but might be discontinuous in Ω. Then, after integrating by parts and making use of the identities $\vec{a}\times\vec{b} = a_1b_2 - a_2b_1$, $c\times\vec{a} = c\,(-a_2, a_1)$, and $\vec{a}\times c = -c\times\vec{a}$, we obtain

$$\begin{aligned}
&(\omega,\sigma)_{\mathcal{T}_h} - (\vec{u}, \vec{\mathrm{curl}}\,\sigma)_{\mathcal{T}_h} - \langle \vec{u}, [\![\sigma\times\vec{n}]\!]\rangle_{\mathcal{E}_h} = 0,\\
&(\omega, \mathrm{curl}\,\vec{v})_{\mathcal{T}_h} + \langle \omega, [\![\vec{v}\times\vec{n}]\!]\rangle_{\mathcal{E}_h} = (\vec{f},\vec{v})_{\mathcal{T}_h},\\
&\langle \vec{u}\cdot\vec{n}, q\rangle_{\mathcal{E}_h^{\mathcal{B}}} = \langle \vec{u}_D\cdot\vec{n}, q\rangle_{\mathcal{E}_h^{\mathcal{B}}}.
\end{aligned}$$

Here we have also used the continuity of the tangential components of $\vec{u}$ and ω.

Note that the pressure p does not appear in the equations; this is due to the fact that $\vec{v} \in H(\mathrm{div}^0;\Omega)$. Moreover, only the information about the *tangential* component of the Dirichlet boundary condition $\vec{u}_D$ appears in the first equation, whereas the information about its *normal* component is contained in the third equation. Finally, if q is a constant, we have

$$\langle \vec{u}_D\cdot\vec{n}, q\rangle_{\mathcal{E}_h^{\mathcal{B}}} = q\,\langle \vec{u}_D\cdot\vec{n}, 1\rangle_{\mathcal{E}_h^{\mathcal{B}}} = 0,$$

by the compatibility condition (1.1) on $\vec{u}_D$. Thus, in order to ensure that the formulation is well defined, we must take q modulo constants.

Motivated by the fact that the exact solution satisfies the above equations, we define the LDG approximation $(\omega_h, \vec{u}_h)$ in the finite-dimensional space $\Sigma_h^k \times \vec{\mathcal{V}}_h^k$ by requiring that

$$(\omega_h,\sigma)_{\mathcal{T}_h} - (\vec{u}_h, \vec{\mathrm{curl}}\,\sigma)_{\mathcal{T}_h} - \left\langle \widehat{\vec{u}}_h, [\![\sigma\times\vec{n}]\!]\right\rangle_{\mathcal{E}_h} = 0, \tag{3.5}$$

$$(\omega_h, \mathrm{curl}\,\vec{v})_{\mathcal{T}_h} + \langle \widehat{\omega}_h, [\![\vec{v}\times\vec{n}]\!]\rangle_{\mathcal{E}_h} = (\vec{f},\vec{v})_{\mathcal{T}_h}, \tag{3.6}$$

$$\langle \vec{u}_h\cdot\vec{n}, q\rangle_{\mathcal{E}_h^{\mathcal{B}}} = \langle \vec{u}_D\cdot\vec{n}, q\rangle_{\mathcal{E}_h^{\mathcal{B}}} \tag{3.7}$$

for all $(\sigma,\vec{v},q) \in \Sigma_h^k \times \vec{\mathcal{V}}_h^k \times \mathcal{Q}_h^k/\mathbb{R}$ with $\vec{v}\cdot\vec{n} = 0$ on Γ, where

$$\begin{aligned}
\Sigma_h^k &= \{\,\sigma \in L^2(\Omega) :\, \sigma|_K \in \mathbb{P}_{k-1}(K),\ K \in \mathcal{T}_h\,\},\\
\vec{\mathcal{V}}_h^k &= \{\,\vec{v} \in H(\mathrm{div}^0;\Omega) :\, \vec{v}|_K \in \vec{\mathbb{P}}_k(K),\ K \in \mathcal{T}_h\,\},\\
\mathcal{Q}_h^k &= \{\,q \in L^2(\Gamma) :\ q|_e \in \mathbb{P}_k(e),\ e \in \mathcal{E}_h^{\mathcal{B}}\,\}.
\end{aligned}$$

Here $\mathbb{P}_\ell(D)$ denotes the space of polynomials of degree $\leq \ell$ on D. The quantities $\widehat{\vec{u}}_h$ and $\widehat{\omega}_h$ are approximations to the traces of $\vec{u}_h$ and ω_h; they must be suitably defined to render the method stable and optimally convergent. On interior edges in $\mathcal{E}_h^{\mathcal{I}}$, these discrete traces are chosen as

$$\widehat{\omega}_h = \{\!\{\omega_h\}\!\} + \vec{\mathsf{E}}\cdot[\![\omega_h\times\vec{n}]\!] + \mathsf{D}\,[\![\vec{u}_h\times\vec{n}]\!], \qquad \widehat{\vec{u}}_h = \{\!\{\vec{u}_h\}\!\} + \vec{\mathsf{E}}\,[\![\vec{u}_h\times\vec{n}]\!]. \tag{3.8}$$

Similarly, on boundary edges in $\mathcal{E}_h^{\mathcal{B}}$, we take

$$\widehat{\omega}_h = \omega_h + \mathsf{D}\,(\vec{u}_h - \vec{u}_D)\times\vec{n}, \qquad \widehat{\vec{u}}_h = \vec{u}_D. \tag{3.9}$$

Here D and $\vec{\mathsf{E}}$ are functions defined on $\mathcal{E}_h$ and $\mathcal{E}_h^{\mathcal{I}}$, respectively. The function D has a stabilization role, whereas the proper definition of the function E can reduce the sparsity of the resulting matrices and might even have a positive impact on the accuracy of the approximation in some cases; see [13]. This completes the definition of the LDG method with globally divergence-free velocity spaces.

With arguments similar to those in [17, Proposition 2.1], it can be readily seen that LDG method in (3.5)–(3.9) has a unique solution $(\omega_h, \vec{u}_h) \in \Sigma_h^k \times \vec{\mathcal{V}}_h^k$, provided that $\mathsf{D} > 0$.

3.3. **The hybridized LDG method.** Next, we hybridize the LDG method described in (3.5)–(3.9). The purpose of the hybridization is to base the approximation of the velocities on the space of locally divergence-free functions given by

$$\vec{V}_h^k = \{\, \vec{v} \in L^2(\Omega)^2 : \vec{v}|_K \in \vec{\mathbb{J}}_k(K),\ K \in \mathcal{T}_h \,\},$$

where

$$\vec{\mathbb{J}}_k(K) = \{\, \vec{v} \in \vec{\mathbb{P}}_k(K) : \operatorname{div} \vec{v} = 0 \text{ on } K \,\}.$$

This space is noticeably bigger than the space $\vec{\mathcal{V}}_h^k$ but has the advantage of being a set of functions which are totally discontinuous across interelement boundaries. As a consequence, only local basis functions are needed for its implementation. On the other hand, the price we pay for this is that we need to reintroduce the pressure in the equations. To this end, we define the space

$$Q_h^k = \{\, q \in L^2(\mathcal{E}_h) : q|_e \in \mathbb{P}_k(e),\ e \in \mathcal{E}_h \,\}.$$

Thus, we define the LDG approximation $(\omega_h, \vec{u}_h, p_h) \in \Sigma_h^k \times \vec{V}_h^k \times Q_h^k/\mathbb{R}$ by requiring that

$$(\omega_h, \sigma)_{\mathcal{T}_h} - (\vec{u}_h, \vec{\operatorname{curl}}\, \sigma)_{\mathcal{T}_h} - \langle \widehat{\vec{u}}_h, [\![\sigma \times \vec{n}]\!] \rangle_{\mathcal{E}_h} = 0, \tag{3.10}$$

$$(\omega_h, \operatorname{curl} \vec{v})_{\mathcal{T}_h} + \langle \widehat{\omega}_h, [\![\vec{v} \times \vec{n}]\!] \rangle_{\mathcal{E}_h} + \langle [\![\vec{v} \cdot \vec{n}]\!], p_h \rangle_{\mathcal{E}_h} = (\vec{f}, \vec{v})_{\mathcal{T}_h}, \tag{3.11}$$

$$\langle [\![\vec{u}_h \cdot \vec{n}]\!], q \rangle_{\mathcal{E}_h} = \langle \vec{u}_D \cdot \vec{n}, q \rangle_{\mathcal{E}_h^{\mathcal{B}}} \tag{3.12}$$

for all $(\sigma, \vec{v}, q) \in \Sigma_h^k \times \vec{V}_h^k \times Q_h^k/\mathbb{R}$.

Note that the pressure p_h now appears in (3.11); this is because $[\![\vec{v} \cdot \vec{n}]\!]$ is not necessarily equal to zero. Note also that if q is a constant on $\mathcal{E}_h$, we have that, for $\vec{v} \in \vec{V}_h^k$,

$$\langle [\![\vec{v} \cdot \vec{n}]\!], q \rangle_{\mathcal{E}_h} = \sum_{K \in \mathcal{T}_h} \langle \vec{v} \cdot \vec{n}_K, q \rangle_{\partial K} = q \sum_{K \in \mathcal{T}_h} (\operatorname{div} \vec{v}, 1)_K = 0,$$

and that, in view of (1.1), $\langle \vec{u}_D \cdot \vec{n}, q \rangle_{\mathcal{E}_h^{\mathcal{B}}} = q \, \langle \vec{u}_D \cdot \vec{n}, 1 \rangle_{\mathcal{E}_h^{\mathcal{B}}} = 0$. Thus, to ensure that the above weak formulation is well defined, we must take q in the space $Q_h^k/\mathbb{R}$.

Next, we establish that the LDG method in (3.10)–(3.12), completed with the definition of the discrete traces (3.8)–(3.9), is well defined. Moreover, we show that its approximate vorticity and velocity are *also* the solution of the original LDG method in (3.5)–(3.9). In other words, even though the hybridized LDG method works with locally divergence-free, but totally discontinuous, approximate velocities, it provides a globally divergence-free velocity field $\vec{u}_h \in H(\operatorname{div}^0; \Omega)$.

Proposition 3.1. *If $\mathsf{D} > 0$ and $k \geq 1$, there is a unique solution $(\omega_h, \vec{u}_h, p_h) \in \Sigma_h^k \times \vec{V}_h^k \times Q_h^k/\mathbb{R}$ of (3.10)–(3.12) with the numerical fluxes in (3.8)–(3.9). Moreover, $(\omega_h, \vec{u}_h)$ is also the solution of (3.5)–(3.9). In particular, $\vec{u}_h \in \vec{\mathcal{V}}_h^k \subset H(\operatorname{div}^0; \Omega)$.*

To prove this result, we use the following stability result whose detailed proof is presented in Section 7 below; cf. Proposition 7.1.

Proposition 3.2. *Let $q \in Q_h^k$, where $k \geq 1$. Then for any $K \in \mathcal{T}_h$ there is a velocity field $\vec{v} \in \vec{\mathbb{J}}_k(K)$ such that*

$$\langle \vec{v} \cdot \vec{n}_K, q \rangle_{\partial K} \geq \| q - m_{\partial K}(q) \|_{0,\partial K}^2,$$

with $m_{\partial K}(q)$ denoting the mean value $m_{\partial K}(q) = \frac{1}{|\partial K|} \langle 1, q \rangle_{\partial K}$. Here $|\partial K|$ is the one-dimensional measure of ∂K.

Proof of Proposition 3.1. To prove the existence and uniqueness of the approximate solution in (3.10)–(3.12), it is sufficient to show that the only LDG approximation to the homogeneous Stokes problem with $\vec{u}_D = \vec{0}$ and $\vec{f} = \vec{0}$ is the trivial one. In this case, by using arguments similar to those in [17, Proposition 2.1], it can be readily seen that $(\omega_h, \vec{u}_h) = (0, \vec{0})$, provided that $\mathsf{D} > 0$. Equation (3.11) then becomes

$$0 = \langle [\![\vec{v} \cdot \vec{n}]\!], p_h \rangle_{\mathcal{E}_h} = \sum_{K \in \mathcal{T}_h} \langle \vec{v} \cdot \vec{n}_K, p_h \rangle_{\partial K}, \qquad \vec{v} \in \vec{V}_h^k.$$

From the stability result in Proposition 3.2, we can find a field $\vec{v} \in \vec{V}_h^k$ such that

$$0 = \sum_{K \in \mathcal{T}_h} \langle \vec{v} \cdot \vec{n}_K, p_h \rangle_{\partial K} \geq \sum_{K \in \mathcal{T}_h} \| p_h - m_{\partial K}(p_h) \|_{0,\partial K}^2.$$

This immediately implies that p_h must be constant on ∂K for all $K \in \mathcal{T}_h$. Hence, the multiplier p_h is constant on $\mathcal{E}_h$, which shows the first claim.

To show the second claim, note that (3.12) implies that the normal component of $\vec{u}_h$ is continuous across interelement boundaries and that

$$\langle \vec{u}_h \cdot \vec{n}, q \rangle_{\mathcal{E}_h^{\mathcal{B}}} = \langle \vec{u}_D \cdot \vec{n}, q \rangle_{\mathcal{E}_h^{\mathcal{B}}}, \qquad q \in \mathcal{Q}_h^k.$$

We conclude that $\vec{u}_h$ is in $H(\mathrm{div}^0; \Omega)$ and satisfies (3.7). Furthermore, in view of the inclusion $\vec{\mathcal{V}}_h^k \subset \vec{V}_h^k$, we obtain that $(\omega_h, \vec{u}_h)$ is also the solution of (3.5)–(3.7). □

3.4. **Recovering the pressure.** Next, we show that once the solution of the hybridized LDG method is computed, it is possible to recover the pressure on the whole domain Ω in an element-by-element fashion.

To see this, note that if we multiply the equation (3.2) by a test function $\vec{\mathsf{v}}$ and integrate over the triangle K, we get

$$-(p, \mathrm{div}\, \vec{\mathsf{v}})_K = (\vec{f}, \vec{\mathsf{v}})_K - (\omega, \mathrm{curl}\, \vec{\mathsf{v}})_K - \langle \vec{\mathsf{v}} \cdot \vec{n}_K, p \rangle_{\partial K} - \langle \omega, \vec{\mathsf{v}} \times \vec{n}_K \rangle_{\partial K}.$$

We then set

$$\mathsf{Q}_h^k = \{ \mathsf{q} \in L^2(\Omega) \,:\, \mathsf{q}|_K \in \mathbb{P}_{k-1}(K),\ K \in \mathcal{T}_h \},$$

and take the pressure $\mathsf{p}_h \in \mathsf{Q}_h^k$ on each triangle K as the element of $\mathbb{P}_{k-1}(K)$ such that

$$(3.13)\quad -(\mathsf{p}_h, \mathrm{div}\, \vec{\mathsf{v}})_K = (\vec{f}, \vec{\mathsf{v}})_K - (\omega_h, \mathrm{curl}\, \vec{\mathsf{v}})_K - \langle \vec{\mathsf{v}} \cdot \vec{n}_K, p_h \rangle_{\partial K} - \langle \widehat{\omega}_h, \vec{\mathsf{v}} \times \vec{n}_K \rangle_{\partial K}$$

for all $\vec{\mathsf{v}}$ in $\vec{\mathbb{P}}_k(K)$.

Proposition 3.3. *The pressure $\mathsf{p}_h \in \mathsf{Q}_h^k$ given by* (3.13) *is well defined for $k \geq 1$.*

Proof. Since $\mathbb{P}_{k-1}(K) = \mathrm{div}\, \vec{\mathbb{P}}_k(K)$, we only have to prove that if $\mathrm{div}\, \vec{\mathsf{v}}|_K = 0$, then the right-hand side of (3.13) is also equal to zero. But in that case, $\vec{\mathsf{v}} \in \vec{\mathbb{J}}_k(K)$, and the right-hand side of (3.13) is zero by (3.11) defining the hybridized LDG method. □

This concludes the presentation of the method.

Let us point out that this method has a considerable smaller number of degrees of freedom than the other LDG methods for incompressible fluid flow, a typical example of which is the one considered in [15]. To see this, let us recall that the method in [15] provides approximations for $(\vec{\sigma}_1, \vec{\sigma}_2, \vec{u}, p)$, since to define it, we need to rewrite the Stokes system as follows:

$$\vec{\sigma}_i - \operatorname{grad} \vec{u}_i = 0, \qquad \operatorname{div} \vec{\sigma}_i + \partial_i p = \vec{f}_i \qquad \text{i=1,2}, \qquad \operatorname{div} \vec{u} = 0 \qquad \text{in } \Omega. \tag{3.14}$$

Instead, the LDG method proposed in this paper only provides approximations for $(\omega, \vec{u}, p)$. Thus, we see that the LDG method under consideration approximates a considerably smaller number of variables.

Of course, an extension of the LDG method in [15] to the system of equations (3.1)–(3.4) we are using to define our LDG method is not difficult to devise and analyze. Conversely, the hybridized LDG method proposed in this paper can be straightforwardly extended to the system in (3.14).

4. Error analysis of the method

In this section, we present two a priori error estimates for the hybridized LDG method. The first, Theorem 4.1, states that the vorticity ω_h and velocity $\vec{u}_h$ converge at an optimal rate. The second, Theorem 4.2, states that the pressure at the interior of the elements, p_h, is also optimally convergent. These are our main theoretical results.

4.1. The error estimates.

The error estimates we obtain are for the hybridized LDG method for which we take

$$\mathsf{D}|_e = \mathsf{d} h_e^{-1} \qquad \forall e \in \mathcal{E}_h, \tag{4.1}$$

with h_e denoting the length of the edge e and d being a positive parameter that is independent of the mesh size, and

$$|\vec{\mathsf{E}}|_e \le C \qquad \forall e \in \mathcal{E}_h^{\mathcal{I}}, \tag{4.2}$$

with a constant C independent of the mesh size. We use the norm

$$\|\vec{v}\|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} \| \operatorname{grad} \vec{v}\|_{0,K}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [\![\vec{v} \otimes \vec{n}]\!] \|_{0,e}^2 .$$

Here, for $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, we denote by $\vec{a} \otimes \vec{b}$ the matrix given by $[\vec{a} \otimes \vec{b}]_{ij} = a_i b_j$.

Theorem 4.1. *Let $(\omega, \vec{u}, p)$ be the exact solution of the Stokes system, and let $(\omega_h, \vec{u}_h, p_h) \in \Sigma_h^k \times \vec{V}_h^k \times Q_h^k/\mathbb{R}$ be the approximation given by the hybridized* LDG *method with $k \ge 1$. If $\vec{u} \in H^{s+1}(\Omega)^2$ for $s \ge 1$, then*

$$\|\omega - \omega_h\|_0 + \|\vec{u} - \vec{u}_h\|_{1,h} \le C\, h^{\min\{k,s\}} \, \|\vec{u}\|_{s+1},$$

where the constant C is independent of the mesh size and the exact solution.

Theorem 4.2. *Let p be the exact pressure of the Stokes system and let $\mathsf{p}_h \in \mathsf{Q}_h^k$ be its approximation given by the post-processing method* (3.13) *with $k \ge 1$. If $\vec{u} \in H^{s+1}(\Omega)^2$ and $p \in H^s(\Omega)$ for $s \ge 1$, then*

$$\|p - \mathsf{p}_h\|_{L^2(\Omega)/\mathbb{R}} \le C\, h^{\min\{k,s\}} \left[\|\vec{u}\|_{s+1} + \|p\|_s \right],$$

where the constant C is independent of the mesh size and the exact solution.

Remark 4.3. The convergence rates in the above results are optimal with respect to the approximation properties of the underlying finite element spaces. The same convergence rates are obtained if the approximate vorticity ω_h is taken to be a piecewise polynomial of degree k, that is, if it belongs to the space Σ_h^{k+1}.

Remark 4.4. Note that, in the first result in Theorem 4.1, nothing is said about the error of the approximation provided by the pressure on the mesh edges, p_h. This reflects the fact that the original LDG method in (3.5)–(3.7) does not involve any pressure variable in its definition. While it can be readily proven, using the inf-sup condition in Theorem 5.1 below, that, on quasi-uniform meshes, p_h converges to p with order $k-1$ in an L^2-like norm, the numerical experiments of Section 8 actually indicate that p_h converges to p with order k. However, since p_h is a piecewise polynomial of degree k, this observed rate of convergence is suboptimal with respect to the approximation properties of Q_h^k. A complete theoretical understanding of this result, certainly linked with the fact that the vorticity and the velocity converge with order k, still remains to be achieved.

4.2. **The compact form of the method.** To facilitate the analysis we rewrite the hybridized LDG method in a compact from. We do this by fully exploiting the fact that the vorticity can be easily eliminated from the equations; we follow [2]. For $\vec{v} \in \vec{V}(h)$, where

$$\vec{V}(h) = \vec{V}_h^k + \left[H^1(\Omega)^2 \cap H(\text{div}^0;\Omega)\right],$$

we define the lifted element $L_{\vec{\mathsf{E}}}(\vec{u}) \in \Sigma_h^k$ by

$$(L_{\vec{\mathsf{E}}}(\vec{u}), \sigma)_{\mathcal{T}_h} = \left\langle \vec{\mathsf{E}}[\![\vec{u} \times \vec{n}]\!], [\![\sigma \times \vec{n}]\!] \right\rangle_{\mathcal{E}_h^{\mathcal{I}}} + \langle \{\!\{\sigma\}\!\}, [\![\vec{u} \times \vec{n}]\!] \rangle_{\mathcal{E}_h} \qquad \forall \sigma \in \Sigma_h^k. \tag{4.3}$$

Similarly, the lifting $U_D \in \Sigma_h^k$ of the boundary datum $\vec{u}_D$ is given by

$$(U_D, \sigma)_{\mathcal{T}_h} = \langle \vec{u}_D, \sigma \times \vec{n} \rangle_{\mathcal{E}_h^{\mathcal{B}}} \qquad \forall \sigma \in \Sigma_h^k.$$

We note that, for the exact velocity $\vec{u}$, we have

$$L_{\vec{\mathsf{E}}}(\vec{u}) = -U_D. \tag{4.4}$$

This can easily be seen using the fact that the jump of $\vec{u}$ vanishes over interior edges and that $\vec{u} = \vec{u}_D$ on boundary edges.

By integration by parts, it is easy to see that the first equation (3.10) in the definition of the hybridized LDG method can be rewritten as

$$\omega_h = \text{curl}\, \vec{u}_h + L_{\vec{\mathsf{E}}}(\vec{u}_h) + U_D \qquad \text{on each } K \in \mathcal{T}_h. \tag{4.5}$$

Next, we use this expression to eliminate the vorticity from the equations. Thus, from the definition of the numerical fluxes and the lifting $L_{\vec{\mathsf{E}}}$, the second equation (3.11) can be expressed as

$$\begin{aligned}(\omega_h, \text{curl}\, \vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + \langle \mathsf{D}[\![\vec{u}_h \times \vec{n}]\!], [\![\vec{v} \times \vec{n}]\!] \rangle_{\mathcal{E}_h} + \langle [\![\vec{v} \cdot \vec{n}]\!], p_h \rangle_{\mathcal{E}_h} \\ = (\vec{f}, \vec{v})_{\mathcal{T}_h} + \langle \mathsf{D}(\vec{u}_D \times \vec{n}), \vec{v} \times \vec{n} \rangle_{\mathcal{E}_h^{\mathcal{B}}},\end{aligned}$$

and, using (4.5), becomes

$$\begin{aligned}(\text{curl}\, \vec{u}_h + L_{\vec{\mathsf{E}}}(\vec{u}_h), \text{curl}\, \vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + \langle \mathsf{D}[\![\vec{u}_h \times \vec{n}]\!], [\![\vec{v} \times \vec{n}]\!] \rangle_{\mathcal{E}_h} + \langle [\![\vec{v} \cdot \vec{n}]\!], p_h \rangle_{\mathcal{E}_h} \\ = (\vec{f}, \vec{v})_{\mathcal{T}_h} - (U_D, \text{curl}\, \vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + \langle \mathsf{D}(\vec{u}_D \times \vec{n}), \vec{v} \times \vec{n} \rangle_{\mathcal{E}_h^{\mathcal{B}}}.\end{aligned} \tag{4.6}$$

Finally, by introducing the bilinear forms

$$A_h(\vec{u},\vec{v}) = (\operatorname{curl}\vec{u} + L_{\vec{\mathsf{E}}}(\vec{u}), \operatorname{curl}\vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + \langle \mathsf{D}[\![\vec{u}\times\vec{n}]\!], [\![\vec{v}\times\vec{n}]\!]\rangle_{\mathcal{E}_h}, \tag{4.7}$$

$$B_h(\vec{v},q) = \langle [\![\vec{v}\cdot\vec{n}]\!], q\rangle_{\mathcal{E}_h}, \tag{4.8}$$

as well as the functionals

$$F_h(\vec{v}) = (\vec{f},\vec{v})_{\mathcal{T}_h} - (U_D, \operatorname{curl}\vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + \langle \mathsf{D}(\vec{u}_D\times\vec{n}), \vec{v}\times\vec{n}\rangle_{\mathcal{E}_h^{\mathcal{B}}}, \tag{4.9}$$

$$G_h(q) = \langle \vec{u}_D\cdot\vec{n}, q\rangle_{\mathcal{E}_h^{\mathcal{B}}}, \tag{4.10}$$

we are led to consider the following mixed formulation:

Find $(\vec{u}_h, p_h) \in \vec{V}_h^k \times Q_h^k/\mathbb{R}$ such that

$$\left\{ \begin{array}{rcll} A_h(\vec{u}_h,\vec{v}) + B_h(\vec{v},p_h) &=& F_h(\vec{v}) & \forall \vec{v}\in\vec{V}_h^k, \\ B_h(\vec{u}_h,q) &=& G_h(q) & \forall q\in Q_h^k/\mathbb{R}. \end{array} \right. \tag{4.11}$$

The formulation in (4.11) is the compact form of the LDG method that we are going to use in our error analysis.

Furthermore, in order to write the post-processing procedure (3.13) in a compact form, we introduce the space

$$\vec{\mathsf{V}}_h^k = \{\vec{\mathsf{v}} \in L^2(\Omega)^2 \, : \, \vec{\mathsf{v}}|_K \in \vec{\mathbb{P}}_k(K), \ K\in\mathcal{T}_h\}.$$

Now, if we set

$$\mathsf{B}_h(\vec{\mathsf{v}},\mathsf{p}) = -(\mathsf{p}, \operatorname{div}\vec{\mathsf{v}})_{\mathcal{T}_h},$$

then the recovered pressure p_h is the element of the space Q_h^k given by

$$\mathsf{B}_h(\vec{\mathsf{v}},\mathsf{p}_h) = F_h(\vec{\mathsf{v}}) - A_h(\vec{u}_h,\vec{\mathsf{v}}) - B_h(\vec{\mathsf{v}},p_h)$$

for all $\vec{\mathsf{v}}\in\vec{\mathsf{V}}_h^k$.

Here we note that the definition of the form A_h can be straightforwardly extended to the the space $\vec{\mathsf{V}}(h) = \vec{\mathsf{V}}_h^k + \big[H^1(\Omega)^2 \cap H(\operatorname{div}^0;\Omega)\big]$ by extending the lifting operator $L_{\vec{\mathsf{E}}}$ to an operator $\vec{\mathsf{V}}(h) \to \Sigma_h^k$, using the same definition.

This is the framework we are going to use to carry out the analysis of the method.

4.3. **Stability properties.** Next, we state the main stability properties of the forms A_h, B_h and B_h. The proofs of the most difficult properties are presented in full detail in Section 6.

4.3.1. *Continuity.* We start by noting the following continuity properties:

$$|A_h(\vec{\mathsf{u}},\vec{\mathsf{v}})| \le C_{A,\mathrm{cont}} \|\vec{\mathsf{u}}\|_{1,h} \|\vec{\mathsf{v}}\|_{1,h}, \qquad \vec{\mathsf{u}},\vec{\mathsf{v}}\in\vec{\mathsf{V}}(h), \tag{4.12}$$

$$|B_h(\vec{\mathsf{v}},q)| \le C_{B,\mathrm{cont}} \|\vec{\mathsf{v}}\|_{1,h} \|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}, \qquad \vec{\mathsf{v}}\in\vec{\mathsf{V}}(h), \ q\in Q_h^k, \tag{4.13}$$

$$|\mathsf{B}_h(\vec{\mathsf{v}},\mathsf{q})| \le C_{\mathsf{B},\mathrm{cont}} \|\vec{\mathsf{v}}\|_{1,h} \|\mathsf{q}\|_0, \qquad \vec{\mathsf{v}}\in\vec{\mathsf{V}}(h), \ \mathsf{q}\in\mathsf{Q}_h^k, \tag{4.14}$$

with continuity constants $C_{A,\mathrm{cont}}$, $C_{B,\mathrm{cont}}$ and $C_{\mathsf{B},\mathrm{cont}}$ that are independent of the mesh size. Here we use the norms

$$\|q\|_{L^2(\mathcal{E}_h;h)}^2 = \sum_{e\in\mathcal{E}_h} h_e \|q\|_{0,e}^2, \qquad \|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|q-c\|_{L^2(\mathcal{E}_h;h)}. \tag{4.15}$$

To see the continuity of A_h, we first note that there holds

$$\|L_{\vec{\mathsf{E}}}(\vec{\mathsf{v}})\|_0^2 \le C_{\mathrm{lift}}^2 \sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\![\vec{\mathsf{v}}\times\vec{n}]\!]\|_{0,e}^2, \qquad \vec{\mathsf{v}}\in\vec{\mathsf{V}}(h), \tag{4.16}$$

with a constant C_{lift} independent of the mesh size; see [33, Section 3] or [34, Proposition 4.2] for details. Property (4.12) then follows from the above estimate and an application of the Cauchy–Schwarz inequality. The continuity properties in (4.13) and (4.14) are readily obtained from the weighted Cauchy–Schwarz inequality.

4.3.2. *Coercivity of A_h.* Next, we discuss the coercivity properties of the form A_h. The following trivial stability result holds:

$$A_h(\vec{v},\vec{v}) \geq C\,\Big[\sum_{K\in\mathcal{T}_h}\big(\|\operatorname{curl}\vec{v}\|_{0,K}^2 + \|\operatorname{div}\vec{v}\|_{0,K}^2\big) + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\vec{v}\times\vec{n}]\!]\|_{0,e}^2\Big], \qquad \vec{v}\in\vec{V}_h^k; \tag{4.17}$$

see [2] or [34, Lemma 4.5]. Here we have used the fact that functions in $\vec{V}_h^k$ are locally incompressible. Then, if we introduce the norm (see [22, Lemma I.2.5 and Remark I.2.7])

$$\|\vec{v}\|_\star^2 = \sum_{K\in\mathcal{T}_h}\big(\|\operatorname{curl}\vec{v}\|_{0,K}^2+\|\operatorname{div}\vec{v}\|_{0,K}^2\big)+\sum_{e\in\mathcal{E}_h} h_e^{-1}\big(\|[\![\vec{v}\times\vec{n}]\!]\|_{0,e}^2+\|[\![\vec{v}\cdot\vec{n}]\!]\|_{0,e}^2\big), \tag{4.18}$$

we immediately get that

$$A_h(\vec{v},\vec{v}) \geq C\|\vec{v}\|_\star^2, \qquad \vec{v}\in\vec{V}_h^k\cap H_0(\operatorname{div};\Omega), \tag{4.19}$$

where we have set

$$H_0(\operatorname{div};\Omega) = \{\,\vec{v}\in L^2(\Omega)\,:\,\operatorname{div}\vec{v}\in L^2(\Omega),\ \vec{v}\cdot\vec{n}=0 \text{ on } \Gamma\,\}.$$

The following result shows that the norm $\|\cdot\|_\star$ is actually equivalent to the norm $\|\cdot\|_{1,h}$.

Proposition 4.5. *On $\vec{\mathsf{V}}_h^k$, the norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_\star$ are equivalent uniformly in the mesh size. That is, there are constants C_1 and C_2 independent of the mesh size such that $C_1\|\vec{\mathsf{v}}\|_{1,h} \leq \|\vec{\mathsf{v}}\|_\star \leq C_2\|\vec{\mathsf{v}}\|_{1,h}$ for all $\vec{\mathsf{v}}\in\vec{\mathsf{V}}_h^k$.*

The proof of Proposition 4.5 is carried out in Section 6.1. Combining Proposition 4.5 and (4.19), we obtain the following coercivity property:

$$A_h(\vec{v},\vec{v}) \geq C_{A,\text{coer}}\|\vec{v}\|_{1,h}^2, \qquad \vec{v}\in\vec{V}_h^k\cap H_0(\operatorname{div};\Omega), \tag{4.20}$$

with a coercivity constant $C_{A,\text{coer}}$ that is independent of the mesh size.

4.3.3. *The* inf-sup *condition for B_h.* To derive the error estimates in the pressure p_h, we will make use of the following inf-sup condition. There is a constant $\mathsf{C}_{\mathsf{B},\text{is}}$ independent of the mesh size such that

$$\sup_{\vec{\mathsf{v}}\in\vec{\mathsf{V}}_h^k\cap H_0(\operatorname{div};\Omega)} \frac{\mathsf{B}_h(\vec{\mathsf{v}},q)}{\|\vec{\mathsf{v}}\|_{1,h}} \geq \mathsf{C}_{\mathsf{B},\text{is}}\|\,P_{\mathsf{Q}_h^k}\,q\,\|_{L^2(\Omega)/\mathbb{R}}, \qquad q\in L^2(\Omega), \tag{4.21}$$

where $P_{\mathsf{Q}_h^k}$ is the L^2-projection onto Q_h^k.

The proof of this result is carried out in Section 6.2.

4.4. **Sketch of the proofs of the error estimates.** Next, we outline the main steps of the proofs of our error estimates. We begin by addressing the fact that, after elimination of the vorticity, the method under consideration does not have the so-called Galerkin orthogonality property.

4.4.1. *Approximate Galerkin orthogonality.* It is well known that the property of Galerkin orthogonality is crucial in the analysis of classical finite element methods. However, due to the use of the lifting operators in the definition of the form A_h, such a property does not hold; see [33] and [34]. Instead, we have what we refer to as approximate Galerkin orthogonality. This is stated in the following result.

Lemma 4.6. *The exact solution $(\omega, \vec{u}, p)$ of the Stokes problem satisfies*

$$\left\{ \begin{array}{lllll} A_h(\vec{u},\vec{v}) & + & B_h(\vec{v},p) & = & F_h(\vec{v}) + R_h(\omega,\vec{v}) \qquad \forall \vec{v} \in \vec{V}_h^k, \\ & & B_h(\vec{u},q) & = & G_h(q) \qquad \forall q \in Q_h^k/\mathbb{R}, \end{array} \right. \tag{4.22}$$

as well as

$$\mathsf{B}_h(\vec{\mathsf{v}},p) = F_h(\vec{\mathsf{v}}) - B_h(\vec{\mathsf{v}},p) - A_h(\vec{u},\vec{\mathsf{v}}) + R_h(\omega,\vec{\mathsf{v}}) \qquad \forall \vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k. \tag{4.23}$$

Here $R_h(\omega,\vec{\mathsf{v}})$ is the expression

$$R_h(\omega,\vec{\mathsf{v}}) = \langle \vec{\mathsf{E}}[\![\vec{\mathsf{v}} \times \vec{n}]\!], [\![P_{\Sigma_h^k}\omega \times \vec{n}]\!]\rangle_{\mathcal{E}_h^{\mathcal{I}}} + \langle \{\!\{P_{\Sigma_h^k}\omega - \omega\}\!\}, [\![\vec{\mathsf{v}} \times \vec{n}]\!]\rangle_{\mathcal{E}_h},$$

with $P_{\Sigma_h^k}$ denoting the L^2-projection onto Σ_h^k.

Proof. We first note that the second equation in (4.22) is trivially satisfied since $\vec{u}$ is continuous across interelement boundaries and satisfies $\vec{u} = \vec{u}_D$ on Γ. To prove the first equation in (4.22), consider the expression

$$\Theta_h = A_h(\vec{u},\vec{v}) + B_h(\vec{v},p) - F_h(\vec{v}),$$

for $\vec{v} \in \vec{V}_h^k$. As a direct consequence of the definitions of the forms A_h, (4.7), B_h, (4.8), and F_h, (4.9), of the fact that $L_{\vec{\mathsf{E}}}(\vec{u}) = -U_D$, (4.4), of the continuity of $\vec{u}$ across interelement boundaries, and of the fact that on the boundary $\vec{u} = \vec{u}_D$, we immediately get

$$\begin{aligned} \Theta_h =& (\omega, \operatorname{curl}\vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + \langle [\![\vec{v}\cdot\vec{n}]\!], p\rangle_{\mathcal{E}_h} - (\vec{f},\vec{v})_{\mathcal{T}_h} \\ =& (\omega, \operatorname{curl}\vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + (\operatorname{grad} p - \vec{f},\vec{v})_{\mathcal{T}_h} && \text{since } \vec{v} \in \vec{V}_h^k, \\ =& (\omega, \operatorname{curl}\vec{v} + L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} + (-\vec{\operatorname{curl}}\,\omega,\vec{v})_{\mathcal{T}_h} && \text{by (3.2)}, \\ =& (\omega, L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} - \langle \omega, [\![\vec{v}\times\vec{n}]\!]\rangle_{\mathcal{E}_h} && \text{by integration by parts}, \\ =& (P_{\Sigma_h^k}\omega, L_{\vec{\mathsf{E}}}(\vec{v}))_{\mathcal{T}_h} - \langle \omega, [\![\vec{v}\times\vec{n}]\!]\rangle_{\mathcal{E}_h} \\ =& \langle \vec{\mathsf{E}}[\![\vec{v}\times\vec{n}]\!], [\![P_{\Sigma_h^k}\omega\times\vec{n}]\!]\rangle_{\mathcal{E}_h^{\mathcal{I}}} + \langle (\{\!\{P_{\Sigma_h^k}\omega\}\!\} - \omega), [\![\vec{v}\times\vec{n}]\!]\rangle_{\mathcal{E}_h}, \end{aligned}$$

by the definition of the lifting $L_{\vec{\mathsf{E}}}$, (4.3). This means that $\Theta_h = R_h(\omega,\vec{v})$ and so the first equation in (4.22) holds true.

Equation (4.23) follows in a completely analogous fashion. This completes the proof. □

4.4.2. *Error analysis for the velocity and vorticity.* To obtain the error estimates for the velocity and the vorticity, we introduce the set

$$\vec{Z}_h(\vec{w}) = \{\vec{v} \in \vec{V}_h^k \,:\, B_h(\vec{v},q) = \langle \vec{w}\cdot\vec{n}, q\rangle_{\mathcal{E}_h^{\mathcal{B}}},\ q \in Q_h^k\}, \tag{4.24}$$

where $\vec{w}$ is any given function with a well-defined normal component in the boundary of Ω. From the equations of the mixed formulation (4.11), we can see that the approximate velocity $\vec{u}_h$ can be characterized as the element of the set $\vec{Z}_h(\vec{u}_D)$ such that

$$A_h(\vec{u}_h,\vec{v}) = F_h(\vec{v}), \tag{4.25}$$

for all $\vec{v} \in \vec{Z}_h(\vec{0})$. Note that this is nothing but a compact form of the formulation of the original unhybridized LDG method after the elimination of the vorticity. This follows from the definition of the method and from the fact that

$$\vec{Z}_h(\vec{w}) = \{\vec{v} \in \vec{\mathcal{V}}_h^k \, : \, \langle \vec{v} \cdot \vec{n}, q \rangle_{\mathcal{E}_h^{\mathcal{B}}} = \langle \vec{w} \cdot \vec{n}, q \rangle_{\mathcal{E}_h^{\mathcal{B}}} \, , \; q \in \mathcal{Q}_h^k\}.$$

Since, by Lemma 4.6, we have that

$$A_h(\vec{u}, \vec{v}) = F_h(\vec{v}) + R_h(\omega, \vec{v}),$$

for all $\vec{v} \in Z_h(\vec{0})$, we proceed as in [33] and [34] and apply the technique proposed by Strang to analyze nonconforming methods to obtain that

$$\| \, \vec{u} - \vec{u}_h \, \|_{1,h} \leq \left(1 + \frac{C_{A,\text{cont}}}{C_{A,\text{coer}}}\right) \inf_{\vec{v} \in \vec{Z}_h(\vec{u}_D)} \| \, \vec{u} - \vec{v} \, \|_{1,h} + \frac{1}{C_{A,\text{coer}}} \sup_{\vec{v} \in \vec{Z}_h(\vec{0})} \frac{|R_h(\omega, \vec{v})|}{\| \, \vec{v} \, \|_{1,h}}.$$

Here we have used the continuity of A_h in (4.12) and the coercivity property (4.20) on $\vec{Z}_h(\vec{0}) \subset \left(\vec{V}_h^k \cap H_0(\text{div}; \Omega)\right)$.

The error estimate for the velocity then follows immediately if we show that

$$\inf_{\vec{v} \in \vec{Z}_h(\vec{u}_D)} \| \, \vec{u} - \vec{v} \, \|_{1,h} \leq C \, h^{\min\{k,s\}} \, \| \, \vec{u} \, \|_{s+1}, \tag{4.26}$$

$$\sup_{\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k} \frac{|R_h(\omega, \vec{\mathsf{v}})|}{\| \, \vec{\mathsf{v}} \, \|_{1,h}} \leq C \, h^{\min\{k,s\}} \, \| \, \vec{u} \, \|_{s+1}. \tag{4.27}$$

The properties in (4.26) and (4.27) are proven in Section 6.3.

Now, it only remains to obtain the estimate for the vorticity from that of the velocity. Indeed, in view of the definition of the approximate vorticity (4.5) and (4.4), we have that, on each element $K \in \mathcal{T}_h$,

$$\omega - \omega_h = \text{curl}(\vec{u} - \vec{u}_h) + L_{\vec{\mathsf{E}}}(\vec{u} - \vec{u}_h),$$

and hence

$$\|\omega - \omega_h\|_0 \leq \left(\sqrt{2} + C_{\text{lift}}\right) \, \|\vec{u} - \vec{u}_h\|_{1,h},$$

where we used the stability property (4.16) for the lifting operator $L_{\vec{\mathsf{E}}}$.

4.4.3. *Error analysis for the pressure.* To obtain the estimate of the error $p - \mathsf{p}_h$ in the approximation of the pressure in the interior of the elements, we proceed as follows. Note that, by the definition of the approximate pressure p_h and Lemma 4.6, we have

$$\mathsf{B}_h(\vec{\mathsf{v}}, p - \mathsf{p}_h) = -A_h(\vec{u} - \vec{u}_h, \mathsf{v}) + R_h(\omega, \vec{\mathsf{v}}),$$

for all $\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k \cap H_0(\text{div}; \Omega)$. Here we have used that $B_h(\vec{\mathsf{v}}, p - p_h) = 0$ for $\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k \cap H_0(\text{div}; \Omega)$. Then, from the inf-sup condition in (4.21) we obtain

$$\sup_{\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k \cap H_0(\text{div};\Omega)} \frac{\mathsf{B}_h(\vec{\mathsf{v}}, p - \mathsf{p}_h)}{\|\vec{\mathsf{v}}\|_{1,h}} \geq \mathsf{C}_{\mathsf{B},\text{is}} \| \, P_{\mathsf{Q}_h^k} p - \mathsf{p}_h \, \|_{L^2(\Omega)/\mathbb{R}},$$

where we recall that $P_{\mathsf{Q}_h^k}$ is the L^2-projection onto Q_h^k. With the continuity property (4.12), we immediately conclude that

$$\| \, P_{\mathsf{Q}_h^k} \, p - \mathsf{p}_h \, \|_{L^2(\Omega)/\mathbb{R}} \leq \frac{C_{A,\text{cont}}}{C_{\mathsf{B},\text{is}}} \, \| \, \vec{u} - \vec{u}_h \, \|_{1,h} + \frac{1}{\mathsf{C}_{\mathsf{B},\text{is}}} \sup_{\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k \cap H_0(\text{div};\Omega)} \frac{|R_h(\omega, \vec{\mathsf{v}})|}{\| \, \vec{\mathsf{v}} \, \|_{1,h}}.$$

The error estimate for $p - \mathsf{p}_h$ now follows from the estimate of the error of the velocity, the estimate (4.27) of R_h, and the well-known approximation result

$$\| p - P_{\mathsf{Q}_h^k} p \|_{L^2(\Omega)/\mathbb{R}} \leq C\, h^{\min\{k,s\}} \| p \|_s.$$

This concludes the description of the main steps in the error analysis. In Section 6 we prove in full detail the auxiliary results that remain to be proven.

5. The Schur-complement matrix for the pressure

In this section, we discuss a subtle but important issue related to the actual solution of the matrix equation associated to the hybridized LDG method. This issue does not appear in the classical mixed methods for the Stokes system.

From the weak formulation (4.11), it is easy to see that the matrix equation of the hybridized LDG method is of the form

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{U} \\ \mathbb{P} \end{pmatrix} = \begin{pmatrix} \mathbb{F} \\ \mathbb{G} \end{pmatrix},$$

where $\mathbb{U}$ and $\mathbb{P}$ are the vectors of coefficients of the velocity $\vec{u}_h$ and the pressure p_h with respect to their corresponding finite element basis, respectively. A popular method used to solve this system of equations is the Uzawa method, which is nothing but a simplified version of the steepest descent method applied to the Schur-complement matrix for the pressure.

Unfortunately, we cannot use such a method since the Schur-complement matrix for the pressure is *not* defined, given that the matrix $\mathbb{A}$ is *not* invertible. Indeed, we can see in the proof of Theorem 4.1 that its inverse is only defined in the kernel of the matrix $\mathbb{B}$. This suggests the following remedy. Let us define the mapping $M_{h,\varepsilon} : \vec{V}_h^k \mapsto Q_h^k$ by

$$M_{h,\varepsilon}(\vec{v})|_e = \varepsilon_e^{-1} [\![\vec{v} \cdot \vec{n}]\!]|_e \qquad \text{for } e \in \mathcal{E}_h,$$

for some strictly positive function piecewise-constant function $\varepsilon \in L^2(\mathcal{E}_h)$. It is then clear that the approximate solution $(\vec{u}_h, p_h) \in \vec{V}_h^k \times Q_h^k/\mathbb{R}$ also satisfies the weak formulation

$$\begin{cases} A_h(\vec{u}_h, \vec{v}) + B_h(\vec{u}_h, M_{h,\varepsilon}(\vec{v})) \; + \; B_h(\vec{v}, p_h) & = \; F_h(\vec{v}) + G_h(M_{h,\varepsilon}(\vec{v})), \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad B_h(\vec{u}_h, q) & = \; G_h(q), \end{cases}$$

for all $(\vec{v}, q) \in \vec{V}_h^k \times Q_h^k/\mathbb{R}$. The matrix equation of this new formulation is of the form

$$\begin{pmatrix} \mathfrak{A}_\varepsilon & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{U} \\ \mathbb{P} \end{pmatrix} = \begin{pmatrix} \mathfrak{F}_\varepsilon \\ \mathbb{G} \end{pmatrix}, \tag{5.1}$$

where $\mathfrak{A}_\varepsilon$ can now be proven to be symmetric and positive definite (and thus invertible). Indeed, since

$$B_h(\vec{u}, M_{h,\varepsilon}(\vec{v})) = \sum_{e \in \mathcal{E}_h} \varepsilon_e^{-1} \langle [\![\vec{u} \cdot \vec{n}]\!], [\![\vec{v} \cdot \vec{n}]\!] \rangle_e, \tag{5.2}$$

the matrix $\mathfrak{A}_\varepsilon$ is readily seen to be symmetric. Furthermore, we easily see that, for any $\vec{v} \in \vec{V}_h^k$,

$$A_h(\vec{v}, \vec{v}) + B_h(\vec{v}, M_{h,\varepsilon}(\vec{v})) \geq 0,$$

and that we have equality if and only if $A_h(\vec{v}, \vec{v}) = 0$ and $B_h(\vec{v}, M_{h,\varepsilon}(\vec{v})) = 0$. The second equation implies that $\vec{v}$ is in the kernel of the matrix $\mathbb{B}$. Since $\mathbb{A}$ is positive definite on this subspace according to (4.20), the first equation implies that $\vec{v} = \vec{0}$. This shows that the matrix $\mathfrak{A}_\varepsilon$ is positive definite.

Consequently, the Schur-complement matrix for the pressure, $\mathbb{S}_\varepsilon = \mathbb{B}\, \mathfrak{A}_\varepsilon^{-1} \mathbb{B}^t$, can be formed and used to efficiently solve for the unknowns. To bound the condition number of $\mathbb{S}_\varepsilon$, we will use the following inf-sup stability result.

Theorem 5.1. *There is a constant $C_{B,\mathrm{is}}$ independent of the mesh size such that*

$$\sup_{\vec{v} \in \vec{V}_h^k} \frac{B_h(\vec{v}, q)}{\| \vec{v} \|_{1,h}} \geq C_{B,\mathrm{is}} h_{\min} \| q \|_{L^2(\mathcal{E}_h; h)/\mathbb{R}}, \qquad q \in Q_h^k,$$

where $h_{\min} = \min_{K \in \mathcal{T}_h} h_K$.

The detailed proof of Theorem 5.1 can be found in Section 7. Here we show how these results allow us to bound the condition numbers of $\mathfrak{A}_\varepsilon$ and $\mathbb{S}_\varepsilon$.

Theorem 5.2. *The augmented stiffness matrix for the velocity, $\mathfrak{A}_\varepsilon$, as well as the Schur-complement matrix for the pressure, $\mathbb{S}_\varepsilon = \mathbb{B}\, \mathfrak{A}_\varepsilon^{-1} \mathbb{B}^t$, are symmetric, positive definite matrices. Moreover, their condition number is of order h^{-2} provided*

$$\varepsilon_e = \mathsf{f}_e\, h_e \quad \text{for } e \in \mathcal{E}_h,$$

and the mesh is quasi-uniform. Here the positive piecewise-constant functions f and f^{-1} are uniformly bounded.

Thus, we can apply a minimization algorithm to the Schur-complement matrix for the pressure of the new system. Note also that new system of equations is associated with the application of the classical augmented Lagrangian method to the original equations.

For classical mixed methods, the condition number of the Schur-complement matrix for the pressure is of order one. The fact that in our case it is of order h^{-2} is directly associated with the fact that our approximate velocity is divergence-free. A similar result was obtained by Dörfler [19] for a nonconforming method whose approximate velocity was weakly divergence-free. He found that the condition number of the stiffness matrix for the velocity was of order h^{-4} instead of order h^{-2}, as is the case for classical weakly divergence-free methods.

Proof of Theorem 5.2. In what follows, we drop the subindex ε, since in this result, this piecewise-constant function has been chosen. As already discussed, the matrix $\mathfrak{A}$ is symmetric and positive definite. By construction, the Schur-complement matrix $\mathbb{S}$ is symmetric. Furthermore, in view of the inf-sup condition in Theorem 5.1, it is positive definite as well; see [9] for details.

Next, let us prove the claimed upper bounds on the condition numbers of $\mathfrak{A}$ and $\mathbb{S}$, denoted by $\kappa_{\mathfrak{A}}$ and $\kappa_{\mathbb{S}}$, respectively. We start by estimating $\kappa_{\mathfrak{A}}$, which can be expressed as follows:

$$\kappa_{\mathfrak{A}} = \frac{\max_{\mathbb{V} \neq 0} \frac{\mathbb{V}^t \mathfrak{A} \mathbb{V}}{\mathbb{V}^t \mathbb{V}}}{\min_{\mathbb{V} \neq 0} \frac{\mathbb{V}^t \mathfrak{A} \mathbb{V}}{\mathbb{V}^t \mathbb{V}}}.$$

Since $\mathbb{V}^t \mathfrak{A} \mathbb{V} = A_h(\vec{v}, \vec{v}) + B_h(\vec{v}, M_h(\vec{v}))$, by the continuity properties of A_h and B_h in (4.12) and (4.13) and the definition of M_h, we immediately have that

$$\mathbb{V}^t \mathfrak{A} \mathbb{V} \leq C_{\mathfrak{A},\mathrm{cont}} \|\vec{u}\|_{1,h} \|\vec{v}\|_{1,h}, \qquad \vec{u}, \vec{v} \in \vec{V}_h^k,$$

for a continuity constant $C_{\mathfrak{A},\mathrm{cont}}$ that is independent of the mesh size. Furthermore, the coercivity result (4.17), the definition of the operator M_h, and the equivalence result in Proposition 4.5 yield

$$\mathbb{V}^t \mathfrak{A} \mathbb{V} \geq C_{\mathfrak{A},\mathrm{coer}} \|\vec{v}\|_{1,h}^2, \qquad \vec{v} \in \vec{V}_h^k, \tag{5.3}$$

for a coercivity constant $C_{\mathfrak{A},\mathrm{coer}}$ that is independent of the mesh size. We thus can conclude that

$$C_{\mathfrak{A},\mathrm{coer}} \|\vec{v}\|_{1,h}^2 \leq \mathbb{V}^t \mathfrak{A} \mathbb{V} \leq C_{\mathfrak{A},\mathrm{cont}} \|\vec{v}\|_{1,h}^2.$$

Furthermore, from the Poincaré inequality in [7] and standard inverse estimates, we obtain

$$\frac{\| \vec{v} \|_{1,h}^2}{\| \vec{v} \|_0^2} \in \left[\frac{1}{C_{\mathrm{P}}^2}, C_{\mathrm{inv}}^2 \, h_{\min}^{-2} \right],$$

with constants C_{P} and C_{inv} that are independent of the mesh size. Similarly, by the regularity of the mesh,

$$\frac{\| \vec{v} \|_0^2}{\mathbb{V}^t \mathbb{V}} \in \left[\frac{h_{\min}^2}{C_{\mathrm{reg}}}, C_{\mathrm{reg}} \, h_{\max}^2 \right],$$

with a constant C_{reg} that is independent of the mesh size. Combining these estimates yields

$$\kappa_{\mathfrak{A}} \leq \left(\frac{C_{\mathfrak{A},\mathrm{cont}} \, C_{\mathrm{P}}^2 \, C_{\mathrm{inv}}^2 \, C_{\mathrm{reg}}^2}{C_{\mathfrak{A},\mathrm{coer}}} \right) \left(\frac{h_{\max}^2}{h_{\min}^4} \right) = \mathcal{O}(h^{-2}),$$

provided the mesh is quasi-uniform.

Next, we let us bound $\kappa_{\mathbb{S}} = \frac{\max_{\mathbb{Q} \neq 0} \frac{\mathbb{Q}^t \mathbb{S} \mathbb{Q}}{\mathbb{Q}^t \mathbb{Q}}}{\min_{\mathbb{Q} \neq 0} \frac{\mathbb{Q}^t \mathbb{S} \mathbb{Q}}{\mathbb{Q}^t \mathbb{Q}}}$. Let us begin by noting that we have

$$\frac{\mathbb{Q}^t \mathbb{S} \mathbb{Q}}{\mathbb{Q}^t \mathbb{Q}} = \frac{\mathbb{Q}^t \mathbb{B} \mathbb{V}}{\mathbb{Q}^t \mathbb{Q}} = \left(\frac{B_h(\vec{v}, q)}{\| \vec{v} \|_{1,h} \, \| q \|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}} \right) \left(\frac{\| \vec{v} \|_{1,h}}{\| q \|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}} \right) \left(\frac{\| q \|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}^2}{\mathbb{Q}^t \mathbb{Q}} \right),$$

where $\mathfrak{A} \mathbb{V} = \mathbb{B}^t \, \mathbb{Q}$, that is,

$$A_h(\vec{v}, \vec{w}) + B_h(\vec{v}, M_h(\vec{w})) = B_h(\vec{w}, q) \qquad \forall \vec{w} \in \vec{V}_h^k. \tag{5.4}$$

Since, by the regularity of the mesh, we have

$$\frac{\| q \|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}^2}{\mathbb{Q}^t \mathbb{Q}} \in \left[\frac{h_{\min}^2}{\mathsf{C}_{\mathrm{reg}}}, \mathsf{C}_{\mathrm{reg}} \, h_{\max}^2 \right],$$

with a constant $\mathsf{C}_{\mathrm{reg}}$ independent of the mesh size. Furthermore, from the continuity and inf-sup stability for B_h in (4.13) and Theorem 5.1, respectively, we conclude that

$$\frac{B_h(\vec{v}, q)}{\| \vec{v} \|_{1,h} \, \| q \|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}} \in \left[C_{B,\mathrm{is}} \, h_{\min}, C_{B,\mathrm{cont}} \right].$$

Hence, we obtain

$$\kappa_{\mathbb{S}} \leq \left(\frac{C_{B,\mathrm{cont}}^2 \, C_{\mathfrak{A},\mathrm{coer}} \, \mathsf{C}_{\mathrm{reg}}^2}{C_{B,\mathrm{is}}^2 \, C_{\mathfrak{A},\mathrm{cont}}} \right) \left(\frac{h_{\max}^2}{h_{\min}^4} \right) = \mathcal{O}(h^{-2}),$$

provided the mesh is quasi-uniform and provided we show that

$$\frac{\| \vec{v} \|_{1,h}}{\| q \|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}} \in \left[\frac{C_{B,\mathrm{is}}}{C_{\mathfrak{A},\mathrm{cont}}} \, h_{\min}, \frac{C_{B,\mathrm{cont}}}{C_{\mathfrak{A},\mathrm{coer}}} \right]. \tag{5.5}$$

To prove property (5.5), we first take $\vec{w} = \vec{v}$ in (5.4), use (5.3) and (4.13), and get

$$\|\,\vec{v}\,\|_{1,h}^2 \leq C_{\mathfrak{A},\mathrm{coer}}^{-1} B_h(\vec{v},q) \leq C_{\mathfrak{A},\mathrm{coer}}^{-1} C_{B,\mathrm{cont}} \|\vec{v}\|_{1,h} \|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}.$$

Furthermore, from the inf-sup condition in Theorem 5.1 and the weak form of the problem in (5.4), we conclude that

$$C_{B,\mathrm{is}} h_{\min} \|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}} \leq \sup_{\vec{w}\in\vec{V}_h^k} \frac{B_h(\vec{w},q)}{\|\vec{w}\|_{1,h}} = \sup_{\vec{w}\in\vec{V}_h^k} \frac{A_h(\vec{v},\vec{w}) + B_h(\vec{v},M_h(\vec{w}))}{\|\vec{w}\|_{1,h}}.$$

Hence, we obtain that

$$C_{B,\mathrm{is}} h_{\min} \|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}} \leq C_{\mathfrak{A},\mathrm{cont}} \|\vec{v}\|_{1,h}.$$

These estimates show property (5.5) and complete the proof of Theorem 5.2. □

6. Auxiliary results in the proofs of Theorems 4.1 and 4.2

In this section, we complete the proofs of the auxiliary results that we used in Section 4.4 to derive the error estimates in Theorems 4.1 and 4.2.

6.1. **The norm equivalence result in Proposition 4.5.**

To prove the result in Proposition 4.5, we are going to use the following result shown in [31, Theorem 2.2].

Lemma 6.1. *For each* $\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k$ *there is a function* $\vec{\mathsf{A}}\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k \cap H_0^1(\Omega)^2$ *such that*

$$\|\vec{\mathsf{v}} - \vec{\mathsf{A}}\vec{\mathsf{v}}\|_{1,h}^2 \leq C \sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\![\vec{\mathsf{v}} \otimes \vec{n}]\!]\|_{0,e}^2,$$

with a constant C *independent of the mesh size.*

Proof of Proposition 4.5. We first note that that the inequality on the right-hand side is trivial and that we only need to establish the one on the left-hand side.

To do so, we set

$$\vec{\mathsf{V}}_h^{k,c} = \vec{\mathsf{V}}_h^k \cap \big[H_0(\mathrm{div};\Omega) \cap H_0(\mathrm{curl};\Omega)\big],$$

where

$$H_0(\mathrm{curl};\Omega) = \{\, \vec{v} \in L^2(\Omega)^2 \;:\; \mathrm{curl}\,\vec{v} \in L^2(\Omega),\ \vec{v}\times\vec{n} = 0 \text{ on } \Gamma \,\}.$$

Clearly, $\big[\vec{\mathsf{V}}_h^k \cap H_0^1(\Omega)^2\big] \subset \vec{\mathsf{V}}_h^{k,c}$. Moreover, the result in [22, Lemma I.2.5 and Remark I.2.7] ensures the algebraic and topological equality of $H_0(\mathrm{div};\Omega) \cap H_0(\mathrm{curl};\Omega)$ and $H_0^1(\Omega)^2$. This implies that there exist constants c_1 and c_2 such that

$$c_1 \|\vec{\mathsf{v}}\|_1 \leq \|\vec{\mathsf{v}}\|_\star \leq c_2 \|\vec{\mathsf{v}}\|_1 \qquad \forall \vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^{k,c}. \tag{6.1}$$

Let $\vec{\mathsf{V}}_h^{k,\perp}$ be the orthogonal complement of $\vec{\mathsf{V}}_h^{k,c}$ in $\vec{\mathsf{V}}_h^k$ with respect to the norm $\|\cdot\|_\star$. Hence,

$$\vec{\mathsf{V}}_h^k = \vec{\mathsf{V}}_h^{k,c} \oplus \vec{\mathsf{V}}_h^{k,\perp}. \tag{6.2}$$

Now, fix $\vec{\mathsf{v}}$ in $\vec{\mathsf{V}}_h^k$ arbitrary. We decompose $\vec{\mathsf{v}}$ into $\vec{\mathsf{v}} = \vec{\mathsf{v}}^c + \vec{\mathsf{v}}^\perp$, according to (6.2). We have

$$\begin{aligned}
\|\vec{\mathsf{v}}\|_{1,h}^2 &\leq C\big[\|\vec{\mathsf{v}} - \vec{\mathsf{A}}\vec{\mathsf{v}}\|_{1,h}^2 + \|\vec{\mathsf{A}}\vec{\mathsf{v}} - \vec{\mathsf{v}}^c\|_1^2 + \|\vec{\mathsf{v}}^c\|_1^2\big] \\
&\leq C\big[\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\![\vec{\mathsf{v}} \otimes \vec{n}]\!]\|_{0,e}^2 + \|\vec{\mathsf{A}}\vec{\mathsf{v}} - \vec{\mathsf{v}}^c\|_\star^2 + \|\vec{\mathsf{v}}^c\|_\star^2\big].
\end{aligned}$$

Here we have used the triangle inequality, the approximation result in Lemma 6.1, and the norm equivalence property in (6.1). Since, on each edge $e \in \mathcal{E}_h$,

$$\|[\![\vec{\mathsf{v}} \otimes \vec{n}]\!]\|_{0,e}^2 = \|[\![\vec{\mathsf{v}} \times \vec{n}]\!]\|_{0,e}^2 + \|[\![\vec{\mathsf{v}} \cdot \vec{n}]\!]\|_{0,e}^2 = \|[\![\vec{\mathsf{v}}^\perp \times \vec{n}]\!]\|_{0,e}^2 + \|[\![\vec{\mathsf{v}}^\perp \cdot \vec{n}]\!]\|_{0,e}^2,$$

we have

$$\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\![\vec{\mathsf{v}} \otimes \vec{n}]\!]\|_{0,e}^2 \leq \|\vec{\mathsf{v}}^\perp\|_\star^2.$$

Then, by the triangle inequality, the trivial bound $\|\cdot\|_\star \leq C_2\|\cdot\|_{1,h}$, the approximation property in Lemma 6.1, and the previous estimate,

$$\begin{aligned} \|\vec{\mathsf{A}}\vec{\mathsf{v}} - \vec{\mathsf{v}}^c\|_\star^2 &\leq C\big[\|\vec{\mathsf{v}} - \vec{\mathsf{A}}\vec{\mathsf{v}}\|_\star^2 + \|\vec{\mathsf{v}} - \vec{\mathsf{v}}^c\|_\star^2\big] \leq C\big[\|\vec{\mathsf{v}} - \vec{\mathsf{A}}\vec{\mathsf{v}}\|_{1,h}^2 + \|\vec{\mathsf{v}}^\perp\|_\star^2\big] \\ &\leq C\big[\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\![\vec{\mathsf{v}} \otimes \vec{n}]\!]\|_{0,e}^2 + \|\vec{\mathsf{v}}^\perp\|_\star^2\big] \leq C\|\vec{\mathsf{v}}^\perp\|_\star^2. \end{aligned}$$

Gathering these bounds and using the orthogonality of the decomposition (6.2) with respect to $\|\cdot\|_\star$, we obtain $\|\vec{\mathsf{v}}\|_{1,h}^2 \leq C\big[\|\vec{\mathsf{v}}^\perp\|_\star^2 + \|\vec{\mathsf{v}}^c\|_\star^2\big] \leq C\|\vec{\mathsf{v}}\|_\star^2$. This completes the proof. ☐

6.2. **The** inf-sup **condition for** B_h**.** To complete the proof of the error estimate for the pressure, it only remains to prove the inf-sup condition for B_h in (4.21). To do that, we first note that

$$\mathsf{B}_h(\vec{\mathsf{w}}, q) = \mathsf{B}_h(\vec{\mathsf{w}}, P_{\mathsf{Q}_h^k} q),$$

for any $\vec{\mathsf{w}} \in \vec{\mathsf{V}}_h^k \cap H_0(\mathrm{div};\Omega)$ and $q \in L^2(\Omega)$. Hence, to prove (4.21) it is enough to construct a velocity field $\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k \cap H_0(\mathrm{div};\Omega)$ such that

$$\|P_{\mathsf{Q}_h^k}\mathsf{q}\|_{L^2(\Omega)/\mathbb{R}} \leq \mathsf{B}_h(\vec{\mathsf{v}}, P_{\mathsf{Q}_h^k}\mathsf{q}), \qquad \|\vec{\mathsf{v}}\|_{1,h} \leq C. \tag{6.3}$$

To do that, we first use the continuous inf-sup condition to find a field $\vec{v} \in H_0^1(\Omega)^2$ such that

$$\|P_{\mathsf{Q}_h^k}\mathsf{q}\|_{L^2(\Omega)/\mathbb{R}} \leq \mathsf{B}_h(\vec{v}, P_{\mathsf{Q}_h^k}\mathsf{q}), \qquad \|\vec{v}\|_1 \leq C;$$

see, e.g., [9] or [22]. Then let $\vec{\mathsf{v}} = \vec{I}_k\vec{v}$ be the BDM projection of $\vec{v}$ of degree k; see [9, Section III.3.3] for details. Clearly, $\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k \cap H_0(\mathrm{div};\Omega)$, and, by the definition of $\vec{I}_k$,

$$\begin{aligned} \Big(P_{\mathsf{Q}_h^k}\mathsf{q}, \mathrm{div}\,\vec{\mathsf{v}}\Big)_{\mathcal{T}_h} &= -\Big(\mathrm{grad}(P_{\mathsf{Q}_h^k}\mathsf{q}), \vec{\mathsf{v}}\Big)_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h}\Big\langle \vec{\mathsf{v}}\cdot\vec{n}_K, P_{\mathsf{Q}_h^k}\mathsf{q}\Big\rangle_{\partial K} \\ &= -\Big(\mathrm{grad}(P_{\mathsf{Q}_h^k}\mathsf{q}), \vec{v}\Big)_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h}\Big\langle \vec{v}\cdot\vec{n}_K, P_{\mathsf{Q}_h^k}\mathsf{q}\Big\rangle_{\partial K} \\ &= \Big(P_{\mathsf{Q}_h^k}\mathsf{q}, \mathrm{div}\,\vec{v}\Big)_{\mathcal{T}_h}, \end{aligned}$$

which shows that $\mathsf{B}_h(\vec{v}, P_{\mathsf{Q}_h^k}\mathsf{q}) = \mathsf{B}_h(\vec{\mathsf{v}}, P_{\mathsf{Q}_h^k}\mathsf{q})$. Furthermore, it can be readily seen that $\|\vec{\mathsf{v}}\|_{1,h} \leq C\|\vec{v}\|_1$, which yields the result in (6.3).

6.3. **Approximation results.** Next, we prove the approximation results in (4.27) and (4.26).

6.3.1. *The estimate of R_h in* (4.27). For functions $\vec{\mathsf{v}} \in \vec{\mathsf{V}}_h^k$, the estimate

$$R_h(\omega, \vec{\mathsf{v}}) \leq C\, h^{\min\{k,s\}} \|\vec{\mathsf{v}}\|_{1,h} \|\omega\|_s \leq C\, h^{\min\{k,s\}} \|\vec{\mathsf{v}}\|_{1,h} \|\vec{u}\|_{s+1},$$

with C independent of the mesh size, is readily obtained from the weighted Cauchy–Schwarz inequality and the following standard approximation property for the L^2-projection $P_{\Sigma_h^k}$ onto Σ_h^k:

$$\|\omega - P_{\Sigma_h^k}\omega\|_{0,e}^2 \leq C h_K^{2\min\{k,s\}-1} \|\omega\|_{s,K}^2, \qquad e \in \partial K,\ K \in \mathcal{T}_h.$$

This shows (4.27).

6.3.2. *The approximation inequality in* (4.26). To show (4.26), we proceed as follows. Let $\vec{I}_k \vec{u}$ denote the BDM projection of $\vec{u}$ of degree k; see [9, Section III.3.3]. By construction of this projection, we have $\vec{I}_k \vec{u} \in H(\mathrm{div}; \Omega)$. Furthermore, the commuting diagram property in [9, Proposition III.3.7] gives

$$\mathrm{div}(\vec{I}_k \vec{u}) = P_{k-1} \,\mathrm{div}\, \vec{u} = 0 \qquad \text{on each } K \in \mathcal{T}_h,$$

with P_{k-1} denoting the L^2-projection onto the space of piecewise polynomials of degree $k-1$. This shows that $\vec{I}_k \vec{u} \in \vec{V}_h^k \cap H(\mathrm{div}; \Omega)$. Finally, for a boundary edge e, the definition of $\vec{I}_k$ ensures that

$$\left\langle \vec{I}_k \vec{u} \cdot \vec{n}, q \right\rangle_e = \langle \vec{u}_D \cdot \vec{n}, q \rangle_e, \qquad q \in \mathbb{P}_k(e).$$

Hence, we have $\vec{I}_k \vec{u} \in \vec{Z}_h(\vec{u}_D)$ and can use $\vec{v} = \vec{I}_k \vec{u}$ to bound the infimum in (4.26). The approximation properties of $\vec{I}_k$ can be found in [9, Proposition III.3.7]: there holds

$$\|\vec{u} - \vec{I}_k \vec{u}\|_{m,K} \leq C h_K^{\min\{k,s\}+1-m} \|\vec{u}\|_{s+1,K}, \qquad m = 0, 1,\ K \in \mathcal{T}_h, \tag{6.4}$$

with a constant independent of the mesh size. Using the multiplicative trace inequality and the above approximation result, we conclude that, for an edge e of an element $K \in \mathcal{T}_h$,

$$\begin{aligned} h_e^{-1} \|\vec{u} - \vec{I}_k \vec{u}\|_{0,e}^2 &\leq C h_K^{-2} \|\vec{u} - \vec{I}_k \vec{u}\|_{0,K}^2 + C h_K^{-1} \|\vec{u} - \vec{I}_k \vec{u}\|_{0,K} \|\vec{u} - \vec{I}_k \vec{u}\|_{1,K} \\ &\leq C h_K^{2\min\{k,s\}} \|\vec{u}\|_{s+1,K}^2. \end{aligned} \tag{6.5}$$

Using (6.4)–(6.5), we immediately obtain

$$\inf_{\vec{v} \in \vec{Z}_h(\vec{u}_D)} \|\vec{u} - \vec{v}\|_{1,h} \leq \|\vec{u} - \vec{I}_k \vec{u}\|_{1,h} \leq C h^{\min\{k,s\}} \|\vec{u}\|_{s+1}.$$

This proves (4.26).

7. The inf-sup condition for B_h

In this section, we first prove a slightly stronger version of the stability result in Proposition 3.2 and then show the inf-sup condition in Theorem 5.1.

7.1. **The stability result in Proposition 3.2.** To prove Proposition 3.2, we introduce, for each element $K \in \mathcal{T}_h$, the following local forms: $B_K(\vec{v}, q) = \langle \vec{v} \cdot \vec{n}_K, q \rangle_{\partial K}$, and the local spaces $Q_h^k(K) = \{q \in L^2(\partial K) : q|_e \in \mathbb{P}_k(e),\ e \in \partial K\}$. We show the following result from which Proposition 3.2 immediately follows after suitably scaling the local field $\vec{v}$.

Proposition 7.1. *There is a constant C independent of the mesh size such that for any $K \in \mathcal{T}_h$ and $q \in Q_h^k(K)$ there is a velocity field $\vec{v} \in \vec{\mathbb{J}}_k(K)$ with*

$$B_K(\vec{v}, q) \geq C \frac{1}{|\partial K|} \|q - m_{\partial K}(q)\|_{0,\partial K}^2$$

and

$$\|\vec{v}\|_{0,K}^2 + h_K^2 \|\operatorname{grad} \vec{v}\|_{0,K}^2 \leq C \frac{1}{|\partial K|} \|q - m_{\partial K}(q)\|_{0,\partial K}^2.$$

Here we recall that $m_{\partial K}(q) = \frac{1}{|\partial K|} \langle 1, q \rangle_{\partial K}$.

Proof. We proceed in several steps.

Step 1 : We begin by introducing an operator that will be crucial in our proof. To this end, let $\widehat{K} = \{x = (x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1 - x_1\}$ denote the reference triangle. We denote the vertices of $\widehat{K}$ by P_i, $i = 1, 2, 3$; we further set $P_4 = P_1$. The edge that connects P_i and P_{i+1} is denoted by e_i.

Given real numbers $\{\Lambda_i\}_{i=1}^3$ and a function $\lambda \in L^2(\partial \widehat{K})$, we define the polynomial $\Psi \in \mathbb{P}_{k+1}(\widehat{K})$ by

$$\Psi(P_i) = \Lambda_i, \qquad i = 1, 2, 3, \tag{7.1}$$

$$\langle \Psi, r \rangle_{e_i} = \langle \lambda, r \rangle_{e_i}, \qquad r \in \mathbb{P}_{k-1}(e_i),\ i = 1, 2, 3, \tag{7.2}$$

$$(\Psi, w)_{\widehat{K}} = 0, \qquad w \in \mathbb{P}_{k-2}(\widehat{K}). \tag{7.3}$$

In order to see that the conditions (7.1)–(7.3) uniquely define Ψ, it is enough to show that $\Lambda_i = 0$, $i = 1, 2, 3$, and $\lambda = 0$ imply $\Psi = 0$. In that case, however, conditions (7.1) and (7.2) imply that Ψ vanishes on $\partial \widehat{K}$. Hence, Ψ is of the form $\Psi(x) = b(x)\Psi_I(x)$, where b is the cubic bubble function $b(x) = x_1 x_2 (1 - x_1 - x_2)$ and Ψ_I is a polynomial of degree $k - 2$. Condition (7.3) then yields

$$(b\, \Psi_I, w)_{\widehat{K}} = 0, \qquad w \in \mathbb{P}_{k-2}(\widehat{K}).$$

Since $b > 0$ on $\widehat{K}$, this implies that $\Psi_I = 0$. Consequently, we have $\Psi = 0$.

Next, we claim that

$$\|\Psi\|_{2,\widehat{K}}^2 \leq \widehat{C} \left[\sum_{i=1}^3 \Lambda_i^2 + \sum_{i=1}^3 \|\lambda\|_{0,e_i}^2 \right], \tag{7.4}$$

with a constant $\widehat{C}$ only depending on the polynomial degree k.

Using similar arguments as before, it can be seen that the expression $\varphi \mapsto \|\varphi\|_E$, given by

$$\|\varphi\|_E^2 = \sum_{i=1}^3 \varphi(P_i)^2 + \sum_{i=1}^3 \|P_{k-1}^{e_i} \varphi\|_{0,e_i}^2$$

is a norm on $\mathbb{P}_k^E(\widehat{K}) = \{\varphi \in \mathbb{P}_k(\widehat{K}) \,|\, (\varphi, w)_{\widehat{K}} = 0\ \forall w \in \mathbb{P}_{k-2}(\widehat{K})\}$. Here $P_{k-1}^{e_i}$ is the L^2-projection on the edge e_i onto the polynomial space $\mathbb{P}_{k-1}(e_i)$. Since all norms are equivalent on the finite-dimensional space $\mathbb{P}_k^E(\widehat{K})$, we conclude that

$$\|\Psi\|^2_{2,\widehat{K}} \leq \widehat{C}\|\Psi\|^2_E,$$

with a constant $\widehat{C}$ only depending on the polynomial degree k. Since $\Psi(P_i) = \Lambda_i$ by (7.1) and $P^{e_i}_{k-1}\Psi = P^{e_i}_{k-1}\lambda$ on e_i by (7.2), we obtain

$$\|\Psi\|^2_{2,\widehat{K}} \leq \widehat{C}\Big[\sum_{i=1}^3 \Lambda_i^2 + \sum_{i=1}^3 \|P^{e_i}_{k-1}\lambda\|^2_{0,e_i}\Big].$$

The bound (7.4) follows by noting that $\|P^{e_i}_{k-1}\lambda\|_{0,e_i} \leq \|\lambda\|_{0,e_i}$.

Step 2 : We are now ready to show the result of Proposition 7.1 on the reference element $\widehat{K}$. To that end, let $q \in Q_h^k(\widehat{K})$. Using the results of Step 1, we define the polynomial $\Psi \in \mathbb{P}_{k+1}(\widehat{K})$ by

$$\begin{aligned} \Psi(P_i) &= -[q]_{P_i}, && i = 1,2,3,\\ \langle \Psi, r\rangle_{e_i} &= -\langle q_{t_{e_i}}, r\rangle_{e_i}, && r \in \mathbb{P}_{k-1}(e_i),\ i = 1,2,3,\\ (\Psi, w)_{\widehat{K}} &= 0, && w \in \mathbb{P}_{k-2}(\widehat{K}). \end{aligned}$$

Here $[q]_{P_i}$ denotes the jump of q at the vertex P_i given by

$$[q]_{P_i} = q|_{e_i}(P_i) - q|_{e_{i-1}}(P_i), \qquad i = 1,2,3,$$

where we have set $e_0 = e_3$. Furthermore, $q_{t_{e_i}}$ denotes the derivative of q in the direction of the unit vector t_{e_i} tangential to e_i.

We then set $\vec{v} = \vec{\text{curl}}\,\Psi$. Evidently, $\vec{v} \in \vec{\mathbb{J}}_k(\widehat{K})$. Integration by parts and the defining properties of Ψ yield

$$\begin{aligned} \langle \vec{v}\cdot\vec{n}_{\widehat{K}}, q\rangle_{\partial\widehat{K}} &= \Big\langle \vec{\text{curl}}\,\Psi\cdot\vec{n}_{\widehat{K}}, q\Big\rangle_{\partial\widehat{K}} = \sum_{i=1}^3 \langle \text{grad}\,\Psi\cdot t_{e_i}, q\rangle_{e_i} = \sum_{i=1}^3 \big\langle \Psi_{t_{e_i}}, q\big\rangle_{e_i}\\ &= \sum_{i=1}^3 \Big(-\big\langle \Psi, q_{t_{e_i}}\big\rangle_{e_i} + \Psi q\big|_{P_i}^{P_{i+1}}\Big) = \sum_{i=1}^3 \big(\|q_{t_{e_i}}\|^2_{0,e_i} + [q]^2_{P_i}\big). \end{aligned}$$

From (7.4), we further have

$$\|\vec{v}\|^2_{0,\widehat{K}} + \|\,\text{grad}\,\vec{v}\|^2_{0,\widehat{K}} \leq \widehat{C}\|\Psi\|^2_{2,\widehat{K}} \leq \widehat{C}\sum_{i=1}^3 \big[\|q_{t_{e_i}}\|^2_{0,e_i} + [q]^2_{P_i}\big].$$

The expressions $q \mapsto \big(\sum_{i=1}^3 \|q_{t_{e_i}}\|^2_{0,e_i} + [q]^2_{P_i}\big)^{\frac{1}{2}}$ and $q \mapsto \|q - m_{\partial\widehat{K}}(q)\|_{0,\partial\widehat{K}}$ are both norms on the finite-dimensional space $Q_h^k(\widehat{K})/\mathbb{R}$. Hence, they are equivalent and we obtain

$$B_{\widehat{K}}(\vec{v}, q) \geq \widehat{C}\|q - m_{\partial\widehat{K}}(q)\|^2_{0,\partial\widehat{K}}, \tag{7.5}$$

as well as

$$\|\vec{v}\|^2_{0,\widehat{K}} + \|\,\text{grad}\,\vec{v}\|^2_{0,\widehat{K}} \leq \widehat{C}\|q - m_{\partial\widehat{K}}(q)\|^2_{0,\partial\widehat{K}}, \tag{7.6}$$

with a constant $\widehat{C}$ only depending on the polynomial degree k. The results in (7.5)–(7.6) show the desired estimates on the reference element $\widehat{K}$.

Step 3 : Let us now show the result of Proposition 7.1 for an arbitrary element K in $\mathcal{T}_h$. To this end, fix $q \in Q_h^k(K)$. We denote by F_K the affine transformation that maps $\widehat{K}$ onto K and by $\widehat{q}$ the function in $L^2(\partial\widehat{K})$ given by $\widehat{q} = q \circ F_K$. Since F_K

is affine, we have $\widehat{q} \in Q_h^k(\widehat{K})$. By Step 1, there is a velocity field $\widehat{\vec{v}}$ in $\vec{\mathbb{J}}_k(\widehat{K})$ such that (7.5) and (7.6) are satisfied on $\widehat{K}$. We then use the Piola transform to define the velocity field $\vec{v}$ on K by

$$\vec{v} = \frac{1}{|\det DF_K|} DF_K \widehat{\vec{v}} \circ F_K^{-1}.$$

Here DF_K is the Jacobian matrix of the transformation F_K. Since we have $\operatorname{div} \vec{v} = \frac{1}{|\det DF_K|}\widehat{\operatorname{div}} \widehat{\vec{v}}$ (see [9, Section III.1.3]), the field $\vec{v}$ belongs to the space $\vec{\mathbb{J}}_k(K)$. Using the properties of the Piola transform in [9, Lemma III.1.5] and a density argument as in equation (3.4.25) of [35], it can be seen that

$$B_K(\vec{v}, q) = B_{\widehat{K}}(\widehat{\vec{v}}, \widehat{q}).$$

Hence, using a standard scaling argument we obtain

$$\begin{aligned} B_K(\vec{v}, q) = B_{\widehat{K}}(\widehat{\vec{v}}, \widehat{q}) &\geq C\|\widehat{q} - m_{\partial\widehat{K}}(\widehat{q})\|_{0,\partial\widehat{K}}^2 = C \inf_{\kappa\in\mathbb{R}} \|\widehat{q} - \kappa\|_{0,\partial\widehat{K}}^2 \\ &\geq C|\partial K|^{-1} \inf_{\kappa\in\mathbb{R}} \|q - \kappa\|_{0,\partial K}^2 = C|\partial K|^{-1}\|q - m_{\partial K}(q)\|_{0,\partial K}^2. \end{aligned} \tag{7.7}$$

Similarly, we have from [9, Lemma III.1.7]

$$\|\vec{v}\|_{0,K}^2 + h_K^2\|\operatorname{grad} \vec{v}\|_{0,K}^2 \leq C\big[\|\widehat{\vec{v}}\|_{0,\widehat{K}}^2 + \|\operatorname{grad} \widehat{\vec{v}}\|_{0,\widehat{K}}^2\big].$$

Therefore,

$$\begin{aligned} \|\vec{v}\|_{0,K}^2 + h_K^2\|\operatorname{grad} \vec{v}\|_{0,K}^2 &\leq C\|\widehat{q} - m_{\partial\widehat{K}}(\widehat{q})\|_{0,\partial\widehat{K}}^2 \leq C \inf_{\kappa\in\mathbb{R}} \|\widehat{q} - \kappa\|_{0,\partial\widehat{K}}^2 \\ &\leq C|\partial K|^{-1} \inf_{\kappa\in\mathbb{R}} \|q - \kappa\|_{0,\partial K}^2 \\ &= C|\partial K|^{-1}\|q - m_{\partial K}(q)\|_{0,\partial K}^2 \leq B_K(\vec{v}, q), \end{aligned}$$

by (7.7). This completes the proof of Proposition 7.1. □

7.2. **Proof of Theorem 5.1.** Let us now prove the inf-sup condition for B_h in Theorem 5.1. We proceed in several steps.

Step 1 : Motivated by the result in Proposition 7.1, on $Q_h^k/\mathbb{R}$, we define the norm $|\!|\!| \cdot |\!|\!|_{Q_h^k/\mathbb{R}}$ by

$$|\!|\!| \cdot |\!|\!|_{Q_h^k/\mathbb{R}}^2 = \sum_{K\in\mathcal{T}_h} \frac{1}{|\partial K|}\|q - m_{\partial K}(q)\|_{0,\partial K}^2.$$

Proposition 7.2. *There is a constant C independent of the mesh size such that*

$$\sup_{\vec{v}\in\vec{V}_h^k} \frac{B_h(\vec{v}, q)}{\|\vec{v}\|_{1,h}} \geq Ch_{\min}|\!|\!| q |\!|\!|_{Q_h^k/\mathbb{R}}, \qquad q \in Q_h^k/\mathbb{R},$$

with $h_{\min} = \min_{K\in\mathcal{T}_h} h_K$.

Proof. Let $q \in Q_h^k/\mathbb{R}$. Proposition 7.1 ensures the existence of a velocity field $\vec{v} \in \vec{V}_h^k$ such that

$$B(\vec{v}, q) = \sum_{K\in\mathcal{T}_h} B_K(\vec{v}, q) \geq C|\!|\!| q |\!|\!|_{Q_h^k/\mathbb{R}}^2 \tag{7.8}$$

and

$$\sum_{K\in\mathcal{T}_h} \big[h_K^{-2}\|\vec{v}\|_{0,K}^2 + \|\operatorname{grad} \vec{v}\|_{0,K}^2\big] \leq Ch_{\min}^{-2}|\!|\!| q |\!|\!|_{Q_h^k/\mathbb{R}}^2.$$

From the multiplicative trace inequality, we conclude that for each edge e of an element $K \in \mathcal{T}_h$ there holds

$$h_e^{-1}\|\vec{v}\|_{0,e}^2 \leq C\|\vec{v}\|_{0,K}\big[h_K^{-2}\|\vec{v}\|_{0,K}+h_K^{-1}\|\operatorname{grad}\vec{v}\|_{0,K}\big] \leq Ch_K^{-2}\|\vec{v}\|_{0,K}^2+C\|\operatorname{grad}\vec{v}\|_{0,K}^2.$$

This readily implies that

$$\|\vec{v}\|_{1,h}^2 \leq \sum_{K\in\mathcal{T}_h}\big[h_K^{-2}\|\vec{v}\|_{0,K}^2 + \|\operatorname{grad}\vec{v}\|_{0,K}^2\big] \leq Ch_{\min}^{-2}|\!|\!| q |\!|\!|_{Q_h^k/\mathbb{R}}^2. \tag{7.9}$$

The assertion now follows from the inequalities (7.8) and (7.9). □

Step 2 : The following result was inspired by a similar result proven in [24].

Lemma 7.3. *There is a constant independent of the mesh size such that*

$$|\!|\!| q |\!|\!|_{Q_h^k/\mathbb{R}} \geq C\|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}$$

for any $q \in Q_h^k/\mathbb{R}$, with the norm $\|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}$ given in (4.15).

Proof. We proceed as in the proof of Theorem 2.3 in [24] and define for $q \in Q_h^k$ the lifted element v_q as follows: for any $K \in \mathcal{T}_h$ let $v_q|_K$ be the unique polynomial in $\mathbb{P}_k(K)$ that satisfies

$$(\operatorname{grad} v_q, \operatorname{grad} w)_K + \frac{1}{|\partial K|}\,\langle v_q, w\rangle_{\partial K} = \frac{1}{|\partial K|}\,\langle q, w\rangle_{\partial K} \qquad \forall w \in \mathbb{P}_k(K).$$

The assertion follows from the following three inequalities: for $q \in Q_h^k$, there holds

$$\|q\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}} \leq C\inf_{\kappa\in\mathbb{R}}\|v_q-\kappa\|_0, \tag{7.10}$$

$$\inf_{\kappa\in\mathbb{R}}\|v_q-\kappa\|_0 \leq C\big[\sum_{K\in\mathcal{T}_h}\|\operatorname{grad} v_q\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h^{\mathcal{I}}}h_e^{-1}\|[\![v_q\vec{n}]\!]\|_{0,e}^2\big]^{\frac{1}{2}}, \tag{7.11}$$

$$\big[\sum_{K\in\mathcal{T}_h}\|\operatorname{grad} v_q\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h^{\mathcal{I}}}h_e^{-1}\|[\![v_q\vec{n}]\!]\|_{0,e}^2\big]^{\frac{1}{2}} \leq C|\!|\!| q |\!|\!|_{Q_h^k/\mathbb{R}}. \tag{7.12}$$

It remains to show (7.10)–(7.12).

The bound (7.10)*:* As in the proof of Theorem 2.3 in [24], it follows from a local scaling argument that $\|q\|_{L^2(\mathcal{E}_h;h)} \leq C\|v_q\|_0$. Furthermore, if $q = \kappa$ on $\mathcal{E}_h$ for a constant $\kappa \in \mathbb{R}$, we have $v_q = \kappa$. Hence, we obtain from the previous estimate

$$\|q-\kappa\|_{L^2(\mathcal{E}_h;h)} \leq C\|v_{q-\kappa}\|_0 = C\|v_q-\kappa\|_0.$$

Taking the infimum over $\mathbb{R}$ shows the bound (7.10).

The bound (7.11)*:* This bound follows from the fact that

$$\inf_{\kappa\in\mathbb{R}}\|v_q-\kappa\|_0 = \|v_q-\kappa_{v_q}\|_0,$$

where $\kappa_{v_q} = \frac{1}{|\Omega|}(1, v_q)$, and the corresponding discrete Poincaré inequality, namely,

$$\|v_q-\kappa_{v_q}\|_0^2 \leq C\big[\sum_{K\in\mathcal{T}_h}\|\operatorname{grad} v_q\|_{0,K}^2 + \sum_{e\in\mathcal{E}_h^{\mathcal{I}}}h_e^{-1}\|[\![v_q\,\vec{n}]\!]\|_{0,e}^2\big]; \tag{7.13}$$

see [7] for details.

The bound (7.12)*:* We recall that if q is constant on ∂K, then v_q is constant on K. Hence, a local scaling argument yields

$$\|\operatorname{grad} v_q\|_{0,K}^2 \leq C\frac{1}{|\partial K|}\|q-m_{\partial K}(q)\|_{0,\partial K}^2. \tag{7.14}$$

Further, it can be shown as in equation (2.13) of [24] that

$$\frac{1}{|\partial K|}\|v_q - q\|_{0,\partial K}^2 \le C\frac{1}{|\partial K|}\|q - m_{\partial K}(q)\|_{0,\partial K}^2.$$

For an interior edge e shared by two elements K and K', this inequality allows us to conclude that

$$\begin{aligned}\frac{1}{h_e}\|[\![v_q\vec{n}]\!]\|_{0,e}^2 &\le \frac{1}{h_e}\|(v_q|_K - q) - (v_q|_{K'} - q)\|_{0,e}^2 \\ &\le C\frac{1}{|\partial K|}\|q - m_{\partial K}(q)\|_{0,\partial K}^2 + C\frac{1}{|\partial K'|}\|q - m_{\partial K'}(q)\|_{0,\partial K'}^2.\end{aligned} \tag{7.15}$$

Adding the local bounds in (7.14) and (7.15) over all elements and edges, respectively, yields the bound (7.12).

This completes the proof. □

Step 3 : Combining Proposition 7.2 and Lemma 7.3 results in the inf-sup in Theorem 5.1.

8. Numerical experiments

In this section, we present numerical experiments that show that the theoretical orders of convergence are sharp. We also display the orders of convergence of the L^2-norm of the error in the velocity and the error in the approximation of the pressure on the edges.

The exact solution we take is the two-dimensional analytical solution of the incompressible Navier–Stokes equations obtained by Kovasznay in [32], namely,

$$\vec{u}(x_1, x_2) = (1 - e^{\lambda x_1}\cos 2\pi x_2, \tfrac{\lambda}{2\pi}e^{\lambda x_1}\sin 2\pi x_2), \qquad p(x_1, x_2) = \tfrac{1}{2}e^{2\lambda x_1} + C,$$

where $\lambda = Re/2 - \sqrt{Re^2/4 + 4\pi^2}$ and Re is the Reynolds number. The Kovasznay solution is also a solution of the Stokes problem

$$-\frac{1}{Re}\Delta\vec{u} + \operatorname{grad} p = \vec{f}, \qquad \operatorname{div}\vec{u} = 0 \qquad \text{in } \Omega,$$

with $\vec{f} = -(\vec{u}\cdot\nabla)\vec{u}$. Of course, we take as Dirichlet boundary conditions for the velocity the restriction of the Kovasznay velocity to $\partial\Omega$.

In our numerical experiments, we consider the domain $\Omega = (-1/2, 3/2) \times (0, 2)$. The meshes we take are refinements of a uniform mesh of 32 congruent triangles. Each refinement is obtained by dividing each triangle into four congruent triangles.

Table 1. History of convergence for Kovasznay flow with $Re = 10$.

	mesh	$\|e_\omega\|_0$		$\|e_{\vec{u}}\|_{1,h}$		$\|e_{\mathsf{p}}\|_{L^2(\Omega)/\mathbb{R}}$		$\|e_{\vec{u}}\|_0$		$\|e_p\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}$	
k	ℓ	error	order	error	order	error	order	error	order	error	order
1	0	7.7e+00	–	1.4e+01	–	2.2e+00	–	7.3e-01	–	2.5e+01	–
	1	3.4e+00	1.19	8.8e+00	0.71	1.3e+00	0.70	3.9e-01	0.93	8.1e+00	1.63
	2	2.2e+00	0.60	4.4e+00	1.00	7.3e-01	0.90	6.6e-02	2.50	5.7e+00	0.50
	3	1.1e+00	1.07	2.1e+00	1.05	3.7e-01	0.97	1.4e-02	2.25	2.6e+00	1.13
	4	5.2e-01	1.03	1.0e+00	1.02	1.8e-01	0.99	3.1e-03	2.10	1.2e+00	1.06
	5	2.5e-01	1.01	5.0e-01	1.01	9.4e-02	0.99	7.5e-04	2.06	6.1e-01	1.02
2	0	3.7e+00	–	8.0e+00	–	7.4e-01	–	2.9e-01	–	1.0e+01	–
	1	9.7e-01	1.94	3.1e+00	1.34	2.4e-01	1.59	7.0e-02	2.00	2.5e+00	2.06
	2	2.6e-01	1.91	8.8e-01	1.85	6.7e-02	1.86	1.2e-02	2.61	6.2e-01	2.00
	3	7.6e-02	1.76	2.4e-01	1.86	1.7e-02	1.96	1.7e-03	2.71	1.9e-01	1.74
	4	2.0e-02	1.87	6.2e-02	1.93	4.3e-03	1.99	2.4e-04	2.87	5.0e-02	1.88
3	0	1.3e+00	–	3.2e+00	–	1.7e-01	–	7.8e-02	-	6.2e+00	–
	1	1.2e-01	3.42	4.5e-01	2.82	3.0e-02	2.54	6.0e-03	3.70	3.1e-01	4.33
	2	1.6e-02	2.92	5.4e-02	3.04	4.2e-03	2.85	3.4e-04	4.15	4.4e-02	2.80
	3	1.9e-03	3.04	5.9e-03	3.20	5.4e-04	2.96	1.8e-05	4.24	4.7e-03	3.24

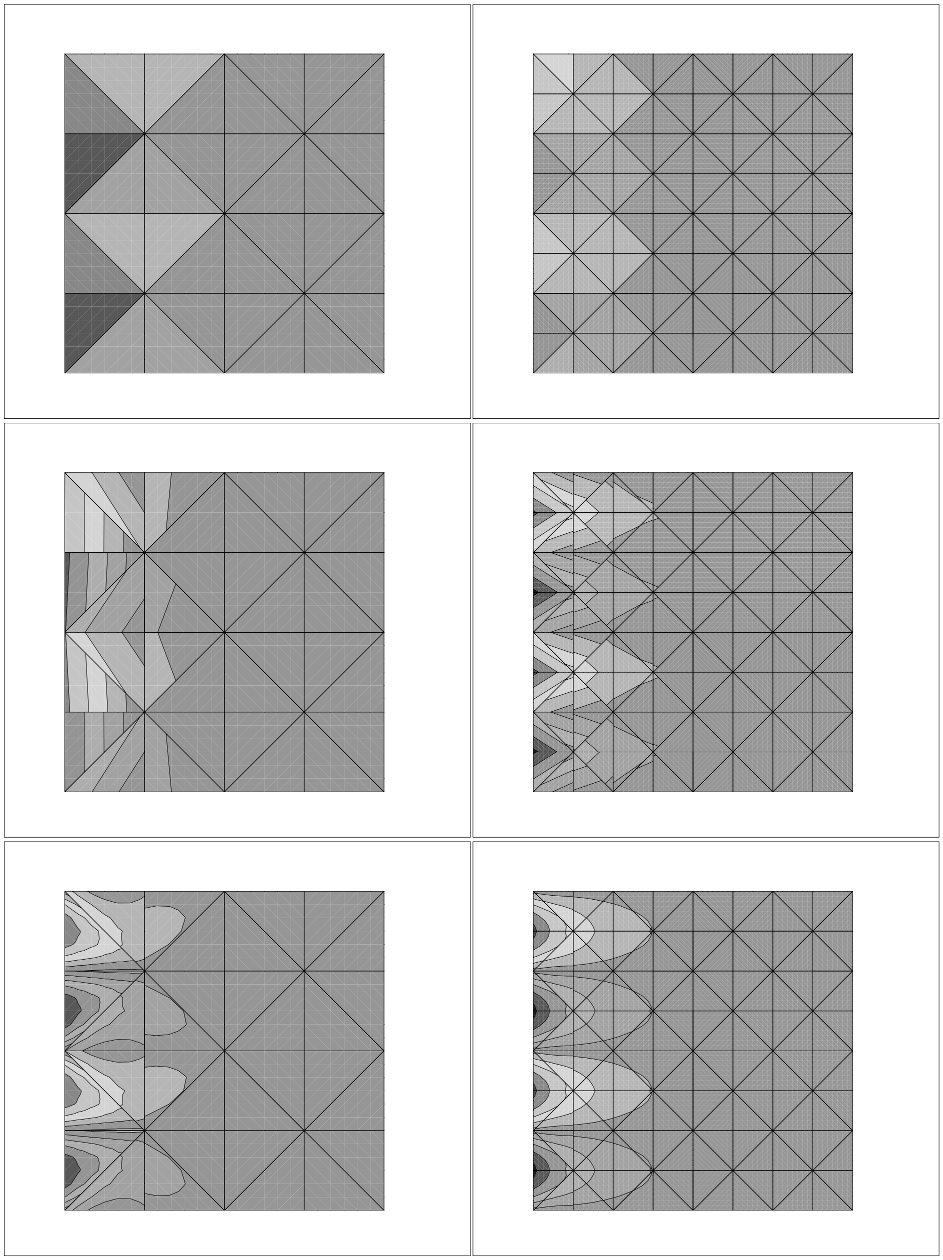

FIGURE 1. Approximation of the vorticity for Kovasznay flow with $Re = 10$: Original mesh (left) and mesh of level 1 (right); polynomial degree of the vorticity: $k = 0$ (top), $k = 1$ (middle), and $k = 2$ (bottom). Ten isolines are shown ranging from $\omega = -20.74$ to $\omega = +20.74$.

We say that the mesh has level ℓ if it is obtained from the original mesh by ℓ of these refinements. In all our tests, the stabilization parameter D has been chosen

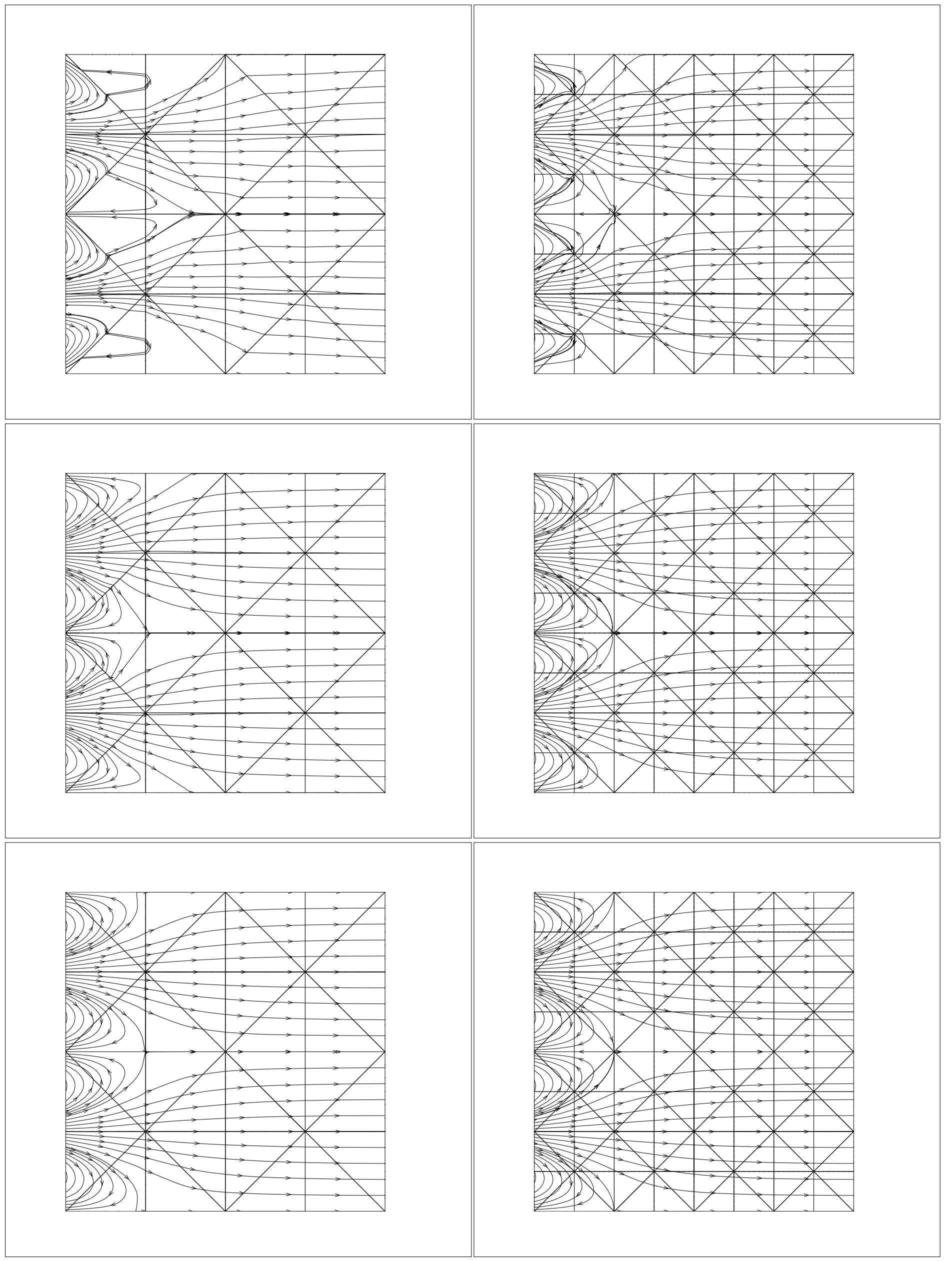

FIGURE 2. Streamlines of the approximate velocity for Kovasznay flow with $Re = 10$: Original mesh (left) and mesh of level 1 (right); polynomial degree of the velocity: $k = 1$ (top), $k = 2$ (middle), and $k = 3$ (bottom).

as in (4.1) with $\mathsf{d} = 1$ and the parameter $\vec{E}$ has been set to $\vec{0}$; the inequality (4.2) is thus trivially satisfied. Finally, the resulting linear system of equations has been solved iteratively by applying the method of conjugate gradients to the Schur complement matrix of the pressure for the augmented system (5.1) (see Section 5) with a suitably chosen parameter ε.

In Table 1, we display the errors and convergence history of the method for Kovasznay flow with $Re = 10$. For fixed polynomial degree k, we see that the orders of convergence of the L^2-norm of the error in the vorticity, $\|e_\omega\|_0$, of the H^1-like norm of the error in the velocity, $\|e_{\vec{u}}\|_{1,h}$, and the L^2-norm of the error in the pressure in the interior, $\|e_{\mathsf{p}}\|_{L^2(\Omega)/\mathbb{R}}$, are of the optimal order of k. These rates clearly confirm our theoretical results in Theorems 4.1 and 4.2. We also see that the L^2-error in the velocity, $\|e_{\vec{u}}\|_0$, converges with the optimal order $k+1$. Furthermore, the L^2-like norm of the error in the pressure on the edges, $\|e_p\|_{L^2(\mathcal{E}_h;h)/\mathbb{R}}$, converges with order k, which is suboptimal with respect to the approximation properties of Q_h^k; see the discussion in Remark 4.4. Note that to obtain the errors for the pressures we make sure to take the proper constants out.

In Figure 1, we show the approximate vorticity on the meshes of level 0 (left) and 1 (right), respectively. We have plotted the discontinuous approximations as they are computed without smoothing them out or making them continuous; all of them with the very same Tecplot layout (URL http:\\www.tecplot.com). Similarly, in Figure 2, we plot the streamlines of the approximate velocity. (The differences in the velocities can be better seen in their streamlines than in vector plots.) In all these figures, but especially in Figure 2, we see a clear improvement of the approximations as we go from $k = 1$ to $k = 3$.

9. Extensions

Note that although we chose meshes made of triangles, we could have easily considered squares or rectangular elements. Note also that the hybridization technique we have presented for the Stokes problem can be applied to other problems with the same structure. Maybe the simplest example is the computation of an approximation $\vec{u}_h$ to the L^2-projection of a vector field $\vec{f}$ into $H(\mathrm{div}^0;\Omega)$, $\vec{u}$. To see this, we only have to realize that we can write

$$\vec{u} + \mathrm{grad}\, p = \vec{f}, \qquad \mathrm{div}\,\vec{u} = 0 \qquad \text{in } \Omega; \qquad p = 0 \qquad \text{on } \Gamma;$$

and so, if $\vec{v} \in H(\mathrm{div}^0;\Omega)$, we have that $(\vec{u},\vec{v}) = (\vec{f},\vec{v})$. We then approximate $\vec{u}$ by $\vec{u}_h \in \vec{\mathcal{V}}_h \subset H(\mathrm{div}^0;\Omega)$ by using the above weak formulation. We immediately see that $\vec{u}_h$ converges to $\vec{u}$ with an optimal rate. Moreover, the hybridization procedure allows us to avoid the construction of the space $\vec{\mathcal{V}}_h$ and solve instead for the pressure on the edges. The Schur-complement matrix for the pressure is well defined, can be computed directly, and has a condition number of order h^{-2} provided that the mesh is quasi-uniform.

The extension of this work to the three-dimensional case is almost straightforward, except for a few nontrivial technical results. It is going to be considered in a forthcoming paper. Also, the extension of this approach to the Maxwell equations and to the Navier–Stokes equations constitutes the subject of ongoing work.

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