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Hydrodynamic Calculations of Spherical Gravitational Collapse in the Scalar-Tensor Theory of Gravity

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Hydrodynamic equations for spherical gravitational collapse in the scalar-tensor theory of gravity are approximated by finite-difference equations. The dynamical motion of a gaseous sphere is calculated numerically on the assumption that the sphere consists of a perfect gas without energy flow and, therefore, its total mass is conserved. In order to avoid the difficulty of matching of the metric and scalar fields at the surface of the gaseous sphere, the sphere is divided into two parts, i.e., a central core and an extended tenuous atmosphere. In the collapsing core, scalar waves are generated around its central region at the final stage, but their effect is not so large as to deviate various physical quantities appreciably from those to be obtained in the general relativistic treatment, except in the inner-most region of self-closure.

§ 1. Introduction and summary

Brans and Dicke[1] have proposed a theory of gravitation called the scalar-tensor theory, according to which gravitational phenomena should be described by a combination of the metric tensor and a scalar field ϕ whose reciprocal is in proportion to the Newtonian gravitation constant G, in contrast with Einstein's general theory of relativity. Relevance of a theory of gravity should be decided by its experimental or observational test, but not by our taste or philosophy. It is therefore important to investigate how a prediction of the Brans-Dicke theory would depart from that in general relativity. As regards the solar gravitational field, their predictions for the deflection of a light ray and the peripherion advance of the Mercury are somewhat different from Einstein's. There are, however, many unknown factors to settle the problem, such as an inner rotation of the sun and the nature of inter-planetary plasma. Accordingly it is difficult to compare directly their respective predictions with observational data.

The departure of the two theories will become important only when the general relativistic effect itself becomes dominant. Salmona[2] has examined the structure of a neutron star in the Brans-Dicke theory and shown that the effect of the inertial scalar field is small. Matsuda[3] has reexamined the problem, but his conclusion has been similar to Salmona's except for a slight difference in numerical results. As the gravitational field of a collapsing object is stronger than that of a neutron star, the phenomenon of gravitational collapse will be suitable to find out some characteristic feature of the Brans-Dicke theory of gravity to be compared with Einstein's.

As regards the gravitational collapse occurring in the process of galaxy formation, there is a numerical analysis by Nariai and Fujimoto[4] for the dynamical behavior of a rotating gaseous ellipsoid with uniform density in the Friedmann universe, according to which the ellipsoid becomes free from the cosmic expansion and starts to collapse when the density contrast relative to the cosmological background attains the values of 5.4 to 5.6, provided that its angular momentum is moderate. Dynamical equations for such an ellipsoid in the Brans-Dicke universe have been derived by Nariai,[5] but their numerical analysis is not yet performed.

On the other hand, Thorne and Dykla[6] have analyzed qualitatively the problem of gravitational collapse in the scalar-tensor theory and concluded that the gravitational collapse produces black holes identical with those in general relativity. Their argument is mainly based on the points: a) in the spherical gravitational collapse of an uncharged star, the scalar field ϕ approaches a constant value when $t\rightarrow\infty$, b) as a result, the field equations and their solutions must agree with those in general relativity. However, Matsuda[7] has pointed out that their argument is not always true if the scalar field is taken to be asymptotically constant. Another reason is that the Brans type-I solution for a spherically symmetric vacuum region does not approach the usual Schwarzschild solution, but the truncated Schwarzschild solution having a naked singularity.[7] This means that the gravitational collapse beyond a non-singular event horizon may not occur in the scalar-tensor theory, while the optical appearance of a collapsing star is not so different from its general relativistic counterpart because of the fact that the metrics in the region $R\geq R_g=2GM/c^2$ (M is the total mass) are similar in both theories.[8]

The aim of this paper is to study the gravitational collapse of a gaseous sphere by solving numerically the full field equations in the scalar-tensor theory, in contrast to Thorne and Dykla's[6] method relying on a power series expansion of the field equations with respect to $1/\omega$, where ω is the Dicke coupling constant.

The gravitational collapse of a gaseous sphere has been investigated by many authors in terms of general relativity. The hydrodynamic equations have been formulated by Misner and Sharp,[9] and numerical calculations of the gravitational collapse of various astronomical objects have been performed by May and White,[10a),10b)] Voropinov et al.,[11] Schwarz,[12] Matsuda and Sato.[13] It has been shown that gaseous spheres with masses larger than a certain critical value cannot settle into hydrostatic equilibrium states but become black holes.

The hydrodynamic equations requisite for our numerical analysis have recently been formulated by Nariai,[14] where we must solve an inhomogeneous wave equation for the scalar field as well as the field equations for the metric tensor. We have encountered with many difficulties in numerical calculations. a) As a scalar wave generated in the interior region propagates with the light velocity, finer time meshes for dealing with the finite-difference scheme are needed than in the general relativistic treatment, and therefore much computer time is needed. b) Unless we employ a suitable difference scheme, which will be described in the Appendix, the wave equation for ϕ leads to the occurrence of oscillations of two zones in wave-length and large amplitudes. c) As the Birkoff theorem does not hold in the scalar-tensor theory of gravity, we cannot settle the time-dependent vacuum metric surrounding the collapsing object even in the case of spherical symmetry. Therefore we cannot settle the boundary conditions of the metric and scalar fields at the surface of the object. [In Ref. 14), the Brans type-I metric was proposed to be adopted as the vacuum metric, but this was shown to be insufficient in the course of our numerical analysis.] To avoid this difficulty, we must introduce a tenuous envelope surrounding the main bulk of the object such that any signal caused at the surface of the envelope cannot reach the bulk within the time considered. Then the evolution of bulk is not affected by the manner specifying the boundary conditions.

In § 2 the hydrodynamic equations in the scalar-tensor theory are summarized. Our main concern is to clarify how the characters of gravitational collapse in that theory deviate from those in general relativity. Accordingly we assume a simple equation of state for gas and a simple configuration for gaseous sphere, so that the model considered does not necessarily correspond to a real astronomical object.

In § 3 the numerical results are presented, which shows that the deviation is small if the relativistic effect itself is small and the bounce phenomenon occurs in the central region. In the case of a continued collapse, oscillations of the scalar field develop in the central region, but they cannot halt the collapsing. Therefore, characteristic nature of the collapse resembles qualitatively that in general relativity. The present method is, for numerical difficulties, incompetent to see the asymptotic ($t\rightarrow\infty$) state of the metric and scalar fields, so that we cannot say anything about the final state of the collapsing object, e.g., the validity of Thorne and Dykla's conjecture.

§ 2. Hydrodynamic equations

In this section, we briefly summarize the hydrodynamic equations in terms of the scalar-tensor theory of gravity. We consider an ideal fluid and neglect heat transfer, pair creations and magnetic fields. Assuming a spherical symmetry leads to the metric

$$dS^2=e^{\nu}c^2dt^2-e^{\lambda}dr^2-R^2(t,r)\,d\Omega^2, \tag{1}$$

where ν and λ are functions of t and r; r is a radial coordinate defined by the rest mass of matter included in a sphere with radius r, i.e.,

$$4\pi e^{\lambda/2}R^2\rho=1\,, \tag{2}$$

where ρ is the rest-mass density. The coordinate system which moves with the fluid leads to the energy-momentum tensor:

$$T_0^{\ 0}=-c^2\varepsilon\,,\quad T_0^{\ \nu}=0\,,\quad T_\mu^{\ \nu}=P\delta_\mu^{\ \nu},\quad T=3P-c^2\varepsilon\,, \tag{3}$$

where P is the pressure, ε is the total density defined by $\varepsilon=\rho(1+e/c^2)$ and e is the internal energy per gram. We introduce comoving derivatives

$$D_t\equiv e^{-\nu/2}\partial_t\,,\quad D_r\equiv e^{-\lambda/2}\partial_r\,, \tag{4}$$

and define the following quantities similar to those defined originally by Misner and Sharp:[7)]

$$U\equiv D_tR\,,\quad u\equiv\tfrac{1}{2}D_t\psi\,, \tag{5}$$

$$\Gamma\equiv D_rR\,,\quad \gamma\equiv\tfrac{1}{2}D_r\psi \tag{6}$$

and

$$\tilde{U}\equiv U+uR\,,\quad \tilde{\Gamma}\equiv\Gamma+\gamma R\,, \tag{7}$$

$$G\tilde{m}\equiv\tfrac{1}{2}c^2\,\mathrm{R}e^{\psi}\{1+(\tilde{U}/c)^2-\tilde{\Gamma}^2\}; \tag{8}$$

where $\psi\equiv\ln(G\phi)$ (ϕ is the inertial scalar field).

On inserting above relations into the Brans-Dicke field equations, we obtain the following set of equations:[14)]

$$D_t\tilde{U}=-\tilde{\Gamma}\left(\frac{1}{\rho w}D_rP-c^2\gamma\right)-\frac{G\tilde{m}}{R^2}e^{-\psi}-4\pi GPc^{-2}e^{-\psi}R-\frac{1}{2}(3+2\omega)\,(u^2+c^2\gamma^2)R\,, \tag{9}$$

$$D_tu=-\frac{\gamma}{\rho w}D_rP+\frac{4\pi Ge^{-\psi}}{3+2\omega}\left(\varepsilon-\frac{3P}{c^2}\right)+c^2\left(D_r+\frac{2\tilde{\Gamma}}{R}\right)\gamma$$
$$-u\left(\frac{D_r\tilde{U}}{\tilde{\Gamma}}+\frac{2\tilde{U}}{R}\right)+u^2\left\{1-(3+2\omega)\,\frac{\gamma R}{\tilde{\Gamma}}\right\}, \tag{10}$$

$$D_t\ln(\rho R^2)=-\frac{D_r\tilde{U}}{\tilde{\Gamma}}+u\left\{1-(3+2\omega)\,\frac{\gamma R}{\tilde{\Gamma}}\right\}, \tag{11}$$

$$D_te=-PD_t\left(\frac{1}{\rho}\right), \tag{12}$$

$$D_r(\nu/2)=-\frac{1}{c^2\rho w}D_rP\,, \tag{13}$$

$$D_tm=-4\pi P\left(\frac{R}{c}\right)^2e^{-\psi/2}\tilde{U}\,, \tag{14}$$

$$D_t\sigma=-\{(u^2+c^2\gamma^2)\tilde{U}-2c^2u\gamma\tilde{\Gamma}\}R^2e^{\psi/2}, \tag{15}$$

where m and σ are defined by

$$G\tilde{m}=e^{\psi/2}\{Gm+\tfrac{1}{2}(3+2\omega)\sigma\}, \tag{16}$$

and w is the relativistic enthalpy defined by

$$w=\frac{\varepsilon+Pc^{-2}}{\rho}=1+\frac{e}{c^2}+\frac{P}{\rho c^2}\,. \tag{17}$$

Note that the mass function m is conserved at the surface of a star, but not $\widetilde{m}$.

As we are concerned only with the dynamical behaviour of a collapsing object in the scalar-tensor theory of gravity, in comparison with that in general relativity, we assume the following simple form as the equation of state:

$$P=\tfrac{2}{3}\rho e\,. \tag{18}$$

These equations can be solved numerically, using a finite-difference method similar to the one used in the case of a general relativistic collapse.[13] Shocks are treated by means of an artificial viscosity similar to that of Richtmyer and von Neumann.

The boundary conditions used are

$$R=U=\widetilde{U}=m=\sigma=\gamma=0 \quad \text{at} \quad r=0\,, \tag{19}$$

$$P=\nu=\gamma=0\,,\quad \psi=\ln[(4+2\omega)/(3+2\omega)] \quad \text{at} \quad r=r_b\,, \tag{20}$$

where r_b $(=\text{const})$ is the radial coordinate specifying the surface of the collapsing object. It should be noted that true boundary conditions for γ and ψ at $r=r_b$ are not known to us, unless we solve the time-dependent vacuum metric surrounding the object. As was discussed in § 1, we avoid this difficulty by assuming a tenuous extended envelope surrounding the main bulk of the object, which we call a core, and removing the surface r_b so far that any signal caused at r_b cannot reach the core surface r_c within the time considered. Therefore any choices of boundary conditions at r_b do not affect the evolution of the core. We assume $R_b=R_c+1.2c\cdot t_f$, where t_f is a free-fall time of the core; $t_f=(G\rho_0)^{-1/2}$, where ρ_0 is the central density and r_b is determined by R_b.

Initial conditions are assumed to be

$$U=\widetilde{U}=u=\gamma=\sigma=0\,,\quad \psi=\ln[(4+2\omega)/(3+2\omega)]. \tag{21}$$

The internal energy e is assumed to be constant, e_0, in the core and decreases as $e=e_0(R_c/R)^2$ in the envelope, where R_c is the radius of the core. As the density distribution, we adopt the Emden function of index 0.1 in the core and $\rho=\rho_c(R_c/R)^5$ in the envelope, where ρ_c is the density of the outermost shell of the core. The choice of that index is made, in part, because for a core with higher index there appears a central region of extreme curvature necessitating a finer zoning; and, in part, because the assumption of uniform density (index 0) leads to an unfavourable gap of physical quantities at the core surface. The mass function $\widetilde{m}$ is determined by Eqs. (6), (7), (8), (19) and

$$D_r(G\widetilde{m})=4\pi G\rho(1+e/c^2)R^2\widetilde{\Gamma}+\gamma G\widetilde{m}+\frac{3+2\omega}{2}\{(u^2+c^2\gamma^2)\widetilde{\Gamma}-2u\gamma\widetilde{U}\}R^2e^{\psi}, \tag{22}$$

and m is determined by Eqs. (16) and (21). If we put $\psi=u=\gamma=0$ identically, the above equations are reduced to the hydrodynamic equations in general relativity.[9]

§ 3. Numerical results

Numerical calculations have been performed for various initial configurations. When the mass of a gaseous sphere is so small that its contracting motion is halted, it has been found that there is no noticeable difference between the numerical results obtained by using the Brans-Dicke theory and the general relativity. Therefore, in this section, we present an example of continued collapse, in which we may expect noticeable differences.

Initial model of the gaseous sphere is assumed as $R_c=3\cdot 10^9$ cm, $\rho_0=10^8$ g/cm^3, $e_0=10^{15}$ erg/g. As was noted before, this model does not correspond to a real astronomical object. The Dicke coupling constant ω is assumed to be 10, because a larger ω does not make noticeable difference and a smallar ω has been rejected by recent observations.[15]

In Figs. 1 through 7, spatial distributions of ρ, R, m, U, Γ, γR, ψ and $e^{\nu/2}$ are represented in the case of the Brans-Dicke hydrodynamics from the initial rest state to the final state where the calculation is stopped by numerical difficulty. The abscissa is the zone number of a space mesh. In these calculations, rather coarse zoning such as 36 zones for the core and 10 zones for the atmosphere is employed, partly because the hydrodynamic calculations in the Brans-Dicke theory consume computer time much longer than that in general relativity and partly because we are concerned only with the qualitative nature of gravitational collapse but not precise numerical results.

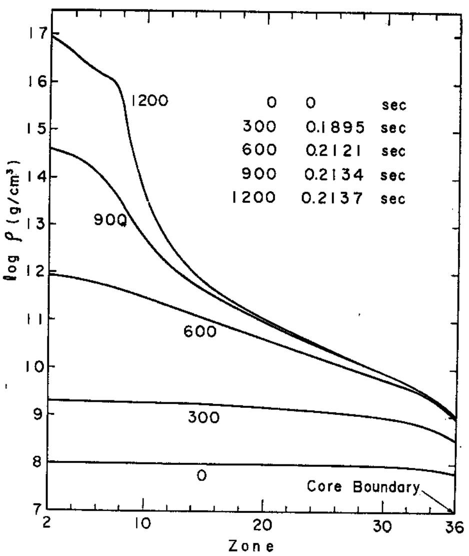

Fig. 1. log ρ versus zone number J at various time during the collapse from rest of a spherical gaseous object. The center of the object is $J=1$ which is not expressed in the figure, the surface of the core $J=36$, and the surface of the atmosphere $J=46$. Initially, density distribution is expressed by the Emden function of index 0.1 in the core and by $\rho\sim R^{-5}$ in the atmosphere, the central density is $\rho_0=10^8$ g/cm^3, and internal energy is assumed to be homogeneous in the core, $e_0=10^{15}$ erg/g, and $e\sim R^{-2}$ in the envelope. Initial core radius is R_c $(=3\cdot 10^9$ cm) and atmospheric radius R_b is determined $R_b=R_c+1.2c(G\rho_0)^{-1/2}$. The core baryon mass is r_c $(=6.97\cdot 10^{36}$ g) and the atmospheric baryon mass is r_b $(=9.06\cdot 10^{36}$ g). The numbers denoted in the figure are numbers N of time mesh; $N=300$ corresponds to $t=0.1895$ sec, $N=600$ to $t=0.2121$ sec, $N=900$ to $t=0.2134$ sec, $N=1000$ to $t=0.2135$ sec, $N=1100$ to $t=0.2136$ sec, $N=1200$ to $t=0.2137$ sec.

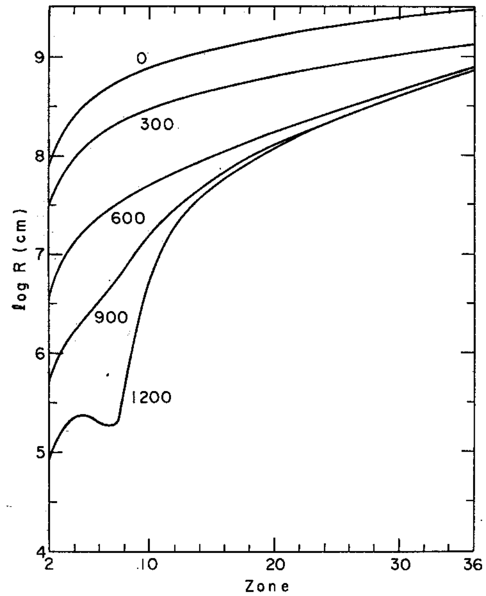

Fig. 2. log R versus zone number at various time for the same model as in Fig. 1.

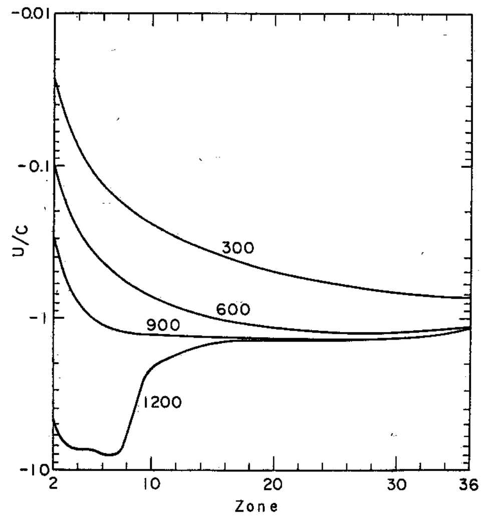

Fig. 4.

Fig. 4. U versus zone number for the same model as in Fig. 1.

Fig. 5. Γ versus zone number (upper figure) and $10^2\gamma R$ versus zone number (lower figure). Note that the departure of the Brans-Dicke hydrodynamics from the general relativistic one can be estimated by the ratio $|\gamma R/\Gamma|$; the larger $|\gamma R/\Gamma|$, the larger departure. Therefore, in the region of $\Gamma\simeq 0$, the largest departure exists.

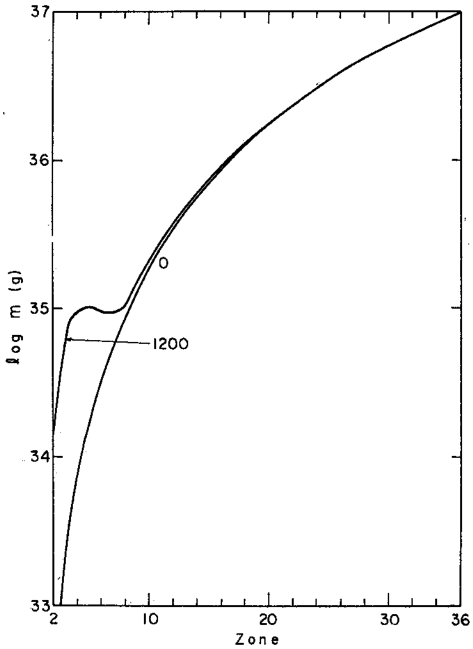

Fig. 3. m versus zone number for the same model as in Fig. 1.

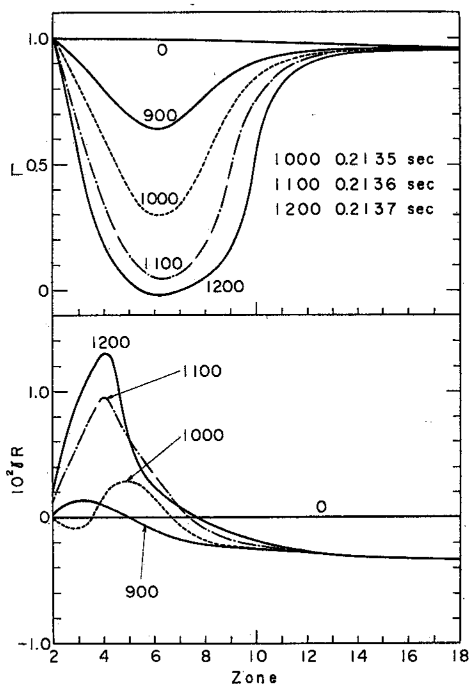

Fig. 5.

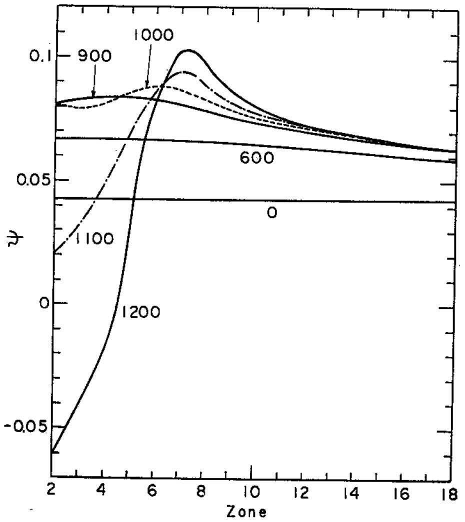

Fig. 6. $\psi(\equiv \ln(G\phi))$ versus zone number. Note that scalar waves are generated in the inner region of the object at the later stage of collapse.

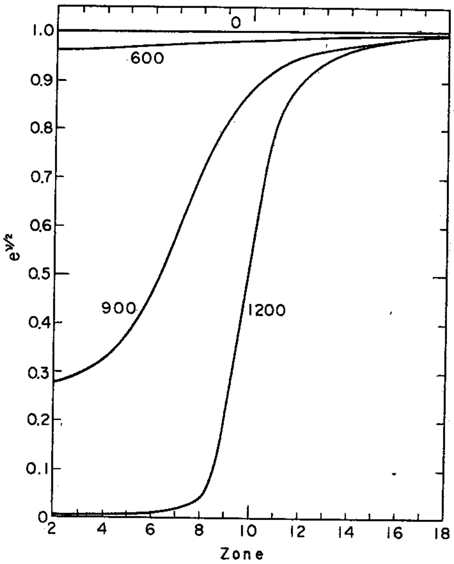

Fig. 7. $e^{\nu/2}$ versus zone number.

These figures show the characteristic aspects of relativistic collapse with pressure, such as the proceeding of a central condensation, Fig. 1, the increase of mass in the central region, Fig. 3, and the occurrence of $\Gamma\to 0$ (or self-closure) at some intermediate shell, Figs. 2 and 5; compare these figures with those of general relativistic collapse.[10a),10b),13)] Our calculation was stopped by a difficulty of increasing numerical errors around inner zones.

The growing of scalar waves in the central region is shown in the lower figure of Fig. 5 and in Fig. 6. The departure of the Brans-Dicke hydrodynamics from the Einstein's can be estimated by the values of $|uR/U|$ and $|\gamma R/\Gamma|$. If these values are small, the departure can be neglected, except for the fact that the gravitational coupling constant ϕ^{-1} become smaller than G for positive ω. In our case, the value of $|uR/U|$ is found to be smaller than 0.1 throughout the interior region, although it is not shown in these figures. As can be seen in Fig. 5, $|\gamma R/\Gamma|$ is small except in the region of $\Gamma\sim 0$. This means that the effect of the scalar field on the dynamics can be neglected in the gaseous sphere except for the region of $\Gamma\sim 0$.

In the region of $\Gamma\sim 0$, however, the departure of the Brans-Dicke hydrodynamics from the Einstein's is so large that the approximation[6)] to the Brans-Dicke theory, constructed by the method of a power series expansion with respect to $1/\omega$, breaks down. In spite of this, the present method is, for numerical difficulty, incompetent to see the asymptotic ($t\to\infty$) state of the metric and scalar fields, so that we cannot say any conclusive statement about the final state of the collapsing object, e.g., the validity of Thorne and Dykla's conjecture or not. Only some conjectures can be given. The asymptotic metric may have a naked singularity: (a) The Brans type-I solution for a static spherically symmetric vacuum region approaches the truncated Schwarzshild solution having a naked singularity when ϕ approaches a constant value;[7] (b) True singularity will appear first in the neighborhood of the region of $\Gamma=0$, which will become an event horizon, and there is no other event horizon. (We conjecture that even in general relativity metrics with a singularity covered by a non-singular event horizon, such as Schwarzshild metric and Kerr metric, are exceptional ones and general metric may have a naked singularity. Though this idea is not currently accepted, the discovery of the rotating Weyl metric by Tomimatsu and Sato,[16] which has a naked singularity outside an event horizon, has strengthen the idea.) If we define a black hole by a mass covered by non-singular event horizons and without naked singularity, it is conjectured that there are scarcely black holes in the universe.

The optical appearance of a collapsing star for a distant observer in the Brans-Dicke theory is not so different from that in general relativity, because the metrics outside the event horizon are not so different. The only remarkable difference for a distant observer is the emission of scalar waves, while gravitational waves are not emitted from a spherical collapsing object in general relativity.

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Appendix

——*Technique of solving a wave equation numerically*——

The technique of solving hydrodynamic equations by a difference scheme is discussed by many authors (see, for example, Refs. 10b), 12) and 13)). Here we discuss the method of solving Eq. (10), which governs the behavior of scalar field ϕ and is originated from

$$\Box\phi=\frac{8\pi T}{(3+2\omega)c^4}\,.$$

Although hydrodynamic equations are of the hyperbolic type, too, they are rather stable when pressure is negligible. The collapsing problem is this case. However, Eq. (10) is unstable and scalar waves with two zones in wavelength develop unless some suitable procedure to damp out waves with short wavelength is employed. We are not interested in scalar waves with short wavelength but those with wavelength comparable with the radius of the collapsing object.

Let us consider the following equation:

$$\frac{\partial y}{\partial t}=f(y, y', y''),$$

where prime denotes spatial derivative. A natural difference scheme is

$$\tilde{y}^{n+1/2}=y^n+\tfrac{1}{2}f^n\cdot\Delta t\,,$$

$$y^{n+1}=y^n+\tilde{f}^{n+1/2}(\tilde{y}^{n+1/2}\cdots)\Delta t\,.$$

This scheme is known to be unstable. To get stable solutions, we use the following scheme

$$\tilde{y}^{n+1}=y^n+f^n\cdot\Delta t\,,$$

$$y^{n+1}=y^n+[(1-\alpha)f^n+\alpha\tilde{f}^{n+1}(\tilde{y}^{n+1}\cdots)]\Delta t\,,\qquad \alpha>\tfrac{1}{2}\,.$$

In the present calculations, we adopt $\alpha=1$. This difference scheme is a kind of low path filter, which damps out oscillations of high frequencies. This scheme is used recently in the numerical weather prediction.

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