The hydrodynamic gradient expansion in linear response theory

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One of the foundational questions in relativistic fluid mechanics concerns the properties of the hydrodynamic gradient expansion at large orders. Studies of expanding systems arising in heavy-ion collisions and cosmology show that the expansion in real space gradients is divergent. On the other hand, expansions of dispersion relations of hydrodynamic modes in powers of momenta have a non-vanishing radius of convergence. We resolve this apparent tension finding a beautifully simple and universal result: the real space hydrodynamic gradient expansion diverges if initial data have support in momentum space exceeding a critical value, and converges otherwise. This critical value is an intrinsic property of the microscopic theory, and corresponds to a branch point of the spectrum where hydrodynamic and nonhydrodynamic modes first collide.

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Introduction– The goal of relativistic hydrodynamics is to provide an effective description of long-lived, long wavelength degrees of freedom – hydrodynamic modes – which are generally expected to dominate nonequilibrium dynamics of collective states of quantum field theories at macroscopic scales and sufficiently late times [1]. Understanding what exact scales and times these are has been a very active field of research of the past decade in connection with studies of collective phases of strong interactions in relativistic heavy-ion collisions at RHIC and LHC [2, 3]. In these settings, relativistic hydrodynamics is the framework translating between the spectrum of low-energy particles observed in detectors and microscopic features such as information about initial state, equation of state and interaction strength [4, 5]. Related recent developments in relativistic hydrodynamics go well beyond the realm of nuclear physics and extend also to astrophysics [6–8], as well as to studies of strong gravity [9, 10].

Much progress on the emergence of relativistic hydrodynamics has occurred recently thanks to, one one hand, viewing hydrodynamics as an effective field theory formulated in a spacetime derivative expansion [11] and, on the other, using insights from linear response theory [12].

The effective field theory approach expresses expectation values of conserved currents in terms of derivatives of local classical fields. For the energy-momentum tensor $\langle T^{\mu\nu}\rangle$ these can be chosen as the energy density $\mathcal{E}$ and a normalized fluid velocity U^{μ}. The energy momentum tensor is represented as a sum of all possible terms graded by the number of derivatives, starting with the perfect fluid contribution. The foundational importance of this expansion is that at a formal level it is unique and well defined in any system which is known to equilibrate. By comparing this formal series to the analogous gradient expansion calculated in a microscopic theory one can express the parameters appearing in the hydrodynamic series – transport coefficients – in terms of microscopic quantities. Interestingly, the gradient series evaluated on a solution of the evolution equations can have a vanishing radius of convergence at least in the case of highly-symmetric flows describing rapidly expanding matter, as was discovered in AdS/CFT calculations [13–15], hydrodynamic models [16–18] and kinetic theory [19, 20].

In linear response theory [21], the response of the system is governed by sums of harmonic contributions with complex frequencies which encode Fourier space singularities of retarded correlators [22]. Imaginary parts of these frequencies capture effects of dissipation. Terms associated with frequencies which vanish at small momentum correspond to shear and sound mode hydrodynamic excitations, while the rest represents transient phenomena [2]. The gradient expansion of the hydrodynamic constitutive relations translates here into series in spatial momentum for shear and sound mode frequencies. In Ref. [23] and later in Refs. [24, 25] it was observed that such a series has a finite non-zero radius of convergence, which is governed by the presence of nonhydrodynamic modes. This parallels the fact that the Borel transform of the gradient expansion in an expanding plasma similarly reveals information about the nonhydrodynamic sectors. These transient excitations are present in all relativistic models which do not violate causality.

The present Letter combines these two lines of research [26] in a novel way, which allows us to make for the first time rather generic statements about the convergence of the hydrodynamic gradient expansion across microscopic theories and models. In particular, we show that the convergence of the real space gradient expansion of the constitutive relations in the linearized regime is governed by the same mechanism that yields a finite radius of convergence of series expansions of hydrodynamic mode frequencies at small momentum.

Hydrodynamics – The expectation value of the conserved energy-momentum tensor can be expressed as the perfect-fluid part plus corrections $\Pi^{\mu\nu}$

$$\langle T^{\mu\nu}\rangle = (\mathcal{E}+\mathcal{P})\,U^\mu U^\nu + \mathcal{P}\,g^{\mu\nu} + \Pi^{\mu\nu}. \tag{1}$$

In hydrodynamics, $\Pi^{\mu\nu}$ is represented in terms of derivatives of the hydrodynamic fields which we take as the energy density $\mathcal{E}$ and flow velocity U^μ with $U\cdot U=-1$. The pressure $\mathcal{P}$ is related to $\mathcal{E}$ via an equation of state [2, 3].

We consider flat d-dimensional spacetime and use the Landau frame where $U_\mu \Pi^{\mu\nu}=0$. We focus on conformal and parity-invariant theories. Conformal symmetry forces $\Pi^\mu_{\ \mu}=0$ and $\mathcal{P}=\mathcal{E}/(d-1)$. Under these conditions, the most general hydrodynamic $\Pi^{\mu\nu}$ takes the form [27, 28]

$$\Pi^{\mu\nu} = -\eta\,\sigma^{\mu\nu} + \tau_\pi \eta\,\mathcal{D}\sigma^{\mu\nu} - \\ -\frac{1}{2}\theta_1\,\mathcal{D}_\alpha \mathcal{D}^\alpha \sigma^{\mu\nu} - \theta_2\,\mathcal{D}^{\langle\mu}\mathcal{D}^{\nu\rangle}\mathcal{D}_\alpha U^\alpha + \ldots, \tag{2}$$

where the ellipsis denotes terms higher than third order in derivatives and we display only terms which contribute at the linearized level. The angle-brackets in Eq. (2) denote the tensors made symmetric, transverse and traceless, $\mathcal{D}=U^\mu\partial_\mu$ and $\mathcal{D}^\mu=(g^{\mu\nu}+U^\mu U^\nu)\,\partial_\nu$ are respectively a comoving and a transverse derivative, $\sigma^{\mu\nu}=2\mathcal{D}^{\langle\mu}U^{\nu\rangle}$ denotes the shear tensor and η is the shear viscosity, τ_π the Israel-Stewart relaxation time and θ_1, θ_2 are third order transport coefficients.

We focus on small perturbations away from thermal equilibrium, i.e., we consider

$$U^\mu = (1,\mathbf{u})^\mu \quad \text{and} \quad \mathcal{E}=\mathcal{E}_0+\epsilon \tag{3}$$

with $|\epsilon/\mathcal{E}_0|, |u_l u^l| \ll 1$. We denote spatial indices with Latin letters and spatial vectors with bold font. It is useful to work in Fourier space with a plane-wave Ansatz

$$u^i(t,\mathbf{x}) = \hat{u}^i(\mathbf{k})\,e^{-i\,\omega\,t+i\,\mathbf{k}\cdot\mathbf{x}},\ \ \epsilon(t,\mathbf{x})=\hat{\epsilon}(\mathbf{k})\,e^{-i\,\omega\,t+i\,\mathbf{k}\cdot\mathbf{x}}. \tag{4}$$

The perturbations can be decomposed into shear and sound channel components [1], labelled here by $\perp$ and $\|$ subscripts. They are given by

$$\hat{\mathbf{u}}_\| = \frac{\mathbf{k}\cdot\hat{\mathbf{u}}}{\mathbf{k}^2}\,\mathbf{k}, \qquad \hat{\mathbf{u}}_\perp = \hat{\mathbf{u}}-\hat{\mathbf{u}}_\|. \tag{5}$$

with $\hat{\epsilon}=0$ vanishing in the shear channel. With no loss of generality, due to rotational invariance, we take

$$\mathbf{k}=(0,\ldots,0,k). \tag{6}$$

Conservation of the energy-momentum tensor together with the hydrodynamic constitutive relation (2) determines the frequencies ω appearing in Eq. (4) as functions of k. The dispersion relations take the form [27, 28]

$$\tilde{\omega}_\perp = -i\frac{\eta}{s\,T}k^2 - i\left(\frac{\eta^2\,\tau_\pi}{s^2T^2}-\frac{\theta_1}{2\,s\,T}\right)k^4+\ldots, \\ \tilde{\omega}_\|^\pm = \pm c_s\,k - i\Gamma k^2 \mp \frac{\Gamma}{2\,c_s}\left(\Gamma-2\,c_s^2\,\tau_\pi\right)k^3 - \\ i\left(2\,\Gamma^2\,\tau_\pi - \frac{(d-2)\,(\theta_1+\theta_2)}{2\,(d-1)\,s\,T}\right)k^4+\ldots, \tag{7}$$

where the tilde means that these are frequencies in the hydrodynamic theory rather than in a microscopic theory. In the above equation, T and s are the temperature and entropy density associated with $\mathcal{E}_0$, $c_s=1/\sqrt{(d-1)}$ is the speed of sound, and $\Gamma=(d-2)/(d-1)\eta/(sT)$. As is clear from these expressions, hydrodynamic excitations of arbitrarily small momentum are arbitrarily long-lived.

Calculations in holography [23–25] reveal that the series (7) have a finite and non-zero radius of convergence, with evidence that goes to back to the studies of causal second order hydrodynamics in Ref. [11]. In physically interesting cases, linear response theory shows that apart from the hydrodynamic modes, there are additional excitations that are short-lived, i.e. whose complex frequency $\omega(k)$ has a non-vanishing imaginary part even as $k\to 0$ [11, 12, 29, 30]. Explicit calculations in several representative cases show that the radius of convergence of hydrodynamic dispersion relations is set by the magnitude k_* of a (possibly complex) momentum for which the frequency of a hydrodynamic mode coincides with that of a nonhydrodynamic one at a branch point of $\omega(k)$ [23–25].

Constitutive relations – Our goal is to understand the properties of the gradient expansion (2) in linearized hydrodynamics in real space that would facilitate comparison with earlier studies of nonlinear evolution in expanding plasma systems. To this end, we propose a novel way of parametrizing $\Pi^{\mu\nu}$, involving only spatial derivatives. We find that the most general form of $\Pi^{\mu\nu}$ in this setting can be constructed from three elementary tensorial structures that are first, second and third order in gradients and linear in the hydrodynamic fluctuations. These are respectively

$$\sigma_{jl} = \left(\partial_j u_l + \partial_l u_j - \frac{2}{d-1}\delta_{jl}\partial_r u^r\right), \tag{8}$$

$$\pi^\epsilon_{jl} = \left(\partial_j\partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\epsilon, \tag{9}$$

$$\pi^u_{jl} = \left(\partial_j\partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\partial_r u^r. \tag{10}$$

With no loss of generality we write the constitutive relations in the form

$$\Pi_{jl} = -A(\partial^2)\,\sigma_{jl} - B(\partial^2)\,\pi^u_{jl} - C(\partial^2)\,\pi^\epsilon_{jl}, \tag{11}$$

where A, B and C are infinite series in spatial Laplacians,

$$A = \sum_{n=0}^{\infty} a_n\left(-\partial^2\right)^n, \tag{12}$$

and the a_n are transport coefficients, with similar expressions for B and C involving transport coefficients b_n and c_n. The remaining components are $\Pi_{tt}=\Pi_{ti}=0$ by the Landau frame condition. In principle, A, B and C could also depend on ∂_t, but in the hydrodynamic gradient expansion one can use the conservation equations to replace temporal derivatives by spatial ones in a systematic way [31].

It follows from Eq. (11) that each even order in gradients introduces one new transport coefficient, while each odd order higher than one introduces two. We find it remarkable that such a simple argument implies that the number of independent transport coefficients at a given order in the gradient expansion of linearized hydrodynamics does not grow with the order, but is limited.

An analogous situation occurs in the series expansions of $\omega_\perp$, $\omega_\parallel^\pm$ around $k=0$. Since $\omega_\parallel^+$, $\omega_\parallel^-$ obey the relation $\omega_\parallel^+(k) = -\omega_\parallel^-(k)^*$, their series coefficients are not independent. These coefficients are real for odd powers of k, and purely imaginary for even powers of k. $\omega_\perp$ is given by a series expansion in k^2 with purely imaginary coefficients. Therefore, each even order in Eq. (7) introduces two new real parameters, while each odd order introduces just one. This counting matches the number of independent transport coefficients in Eq. (11), and suggests that it is possible to express a_n, b_n and c_n, see Eq. (12), in terms of the hydrodynamic dispersion relations (7).

Matching– We now show explicitly that there is a direct relation between A, B and C defined in Eq. (11) and the hydrodynamic dispersion relations (7). Any observable can be used to perform matching, and here we choose to match to the microscopic shear and sound mode dispersion relations, in turn.

For the shear mode, with the wave vector choice we made in (6), the only non-zero components of σ_{jl} are

$$\sigma_{1,d-1} = \sigma_{d-1,1} = i\,k\,u_1 \tag{13}$$

where we have taken $\mathbf{u} = (u_1, 0, \ldots, 0)$ with no loss of generality due to rotational invariance. π^u_{jl} and π^ϵ_{jl} vanish identically for this mode since $\partial_i u^i = \epsilon = 0$.

The conservation of the energy-momentum tensor (1), in combination with the hydrodynamic constitutive relation (11), predicts the following dispersion relation

$$\tilde{\omega}_\perp(k) = -i\frac{1}{s\,T}\sum_{n=0}^{\infty} a_n k^{2n+2}. \tag{14}$$

Demanding that $\tilde{\omega}_\perp(k)$ agrees with the microscopic shear hydrodynamic mode $\omega_\perp$ at every order in an expansion around $k^2 = 0$ fixes the a_n coefficients to be

$$a_n = [k^{2n+2}]\,(i\,s\,T\,\omega_\perp)\,, \tag{15}$$

where the notation $[k^p]\,(f)$ denotes the coefficient of k^p in the series expansion of f around $k=0$.

With $A(\partial^2)$ fixed, we determine $B(\partial^2)$ and $C(\partial^2)$ by considering the sound mode. Now $\mathbf{u} = (0, \ldots, 0, u_{d-1})$, $\epsilon \neq 0$ and

$$\pi^u_{jl} = -\frac{1}{2}k^2\sigma_{jl}. \tag{16}$$

Furthermore, the only non-zero components of σ_{jl} and π^ϵ_{jl} are

$$\sigma_{jj} = -\frac{2}{d-1}\,i\,k\,u_{d-1}, \;\; j = 1 \ldots d-2, \tag{17a}$$

$$\sigma_{d-1,d-1} = \frac{2\,(d-2)}{d-1}\,i\,k\,u_{d-1} \tag{17b}$$

$$\pi^\epsilon_{jj} = \frac{1}{d-1}\,k^2\,\epsilon, \;\; j = 1 \ldots d-2, \tag{17c}$$

$$\pi^\epsilon_{d-1,d-1} = -\frac{d-2}{d-1}\,k^2\,\epsilon. \tag{17d}$$

In the end, the conservation equations reduce to

$$-\,i\,\omega\,\epsilon + i\,k\,s\,T\,u_{d-1} = 0, \tag{18a}$$

$$\begin{aligned} &-\,i\,\omega\,s\,T\,u_{d-1} + \frac{1}{d-1}\,i\,k\,\epsilon + \\ &+ \frac{d-2}{d-1}\sum_{n=0}^{\infty}(2\,a_n - b_{n-1})\,k^{2n+2}\,u_{d-1} + \\ &+ \frac{d-2}{d-1}\sum_{n=0}^{\infty} i\,c_n\,k^{2n+3}\,\epsilon = 0, \end{aligned} \tag{18b}$$

where we have introduced $b_{-1} \equiv 0$ for brevity. Note that the conservation equation (18a) does not depend on transport coefficients as a result of our frame choice. Eqs. (18) has two solutions, $\tilde{\omega}^+_\parallel(k)$ and $\tilde{\omega}^-_\parallel(k)$, given as series expansions around $k=0$, whose coefficients depend on a_n, b_n and c_n. Demanding that these quantities agree with the microscopic sound modes $\omega^+_\parallel(k)$ and $\omega^-_\parallel(k)$, the matching conditions for b_n and c_n are

$$b_n = [k^{2n+4}]\left(-i\,\frac{d-1}{d-2}\,s\,T\left(\omega^+_\parallel + \omega^-_\parallel\right) + 2\,i\,s\,T\omega_\perp\right), \tag{19a}$$

$$c_n = [k^{2n+4}]\left(-\frac{k^2}{d-2} - \frac{d-1}{d-2}\,\omega^+_\parallel\,\omega^-_\parallel\right). \tag{19b}$$

The coefficients a_n, b_n and c_n are directly related to the transport coefficients defined in the standard way. Up to third order in gradients one has

$$\begin{aligned} &a_0 = \eta, \;\; a_1 = \frac{\eta^2\,\tau_\pi}{s\,T} - \frac{1}{2}\theta_1, \quad c_0 = \frac{2\,\eta\,\tau_\pi}{(d-1)\,s\,T}, \\ &b_0 = \theta_2 - \frac{2\,(d-3)\,\eta^2\,\tau_\pi}{(d-1)\,s\,T}. \end{aligned} \tag{20}$$

The explicit relation between hydrodynamic dispersion relations (7) and hydrodynamic constitutive relations as encapsulated by Eqs. (15) and (19) is our main result. Its importance stems from the fact that it connects well-studied hydrodynamic dispersion relations as series in small k with real space hydrodynamic constitutive relations which previously have only been tested at large orders for expanding plasma systems. In the rest of the paper, we explore the implications of this relation on the radius of convergence of the hydrodynamic gradient expansion in real space.

Large order behaviour– The analytic properties of the dispersion relations can be used to constrain the growth of transport coefficients. We expect that in a microscopic theory which respects relativistic causality, the hydrodynamic dispersion relations $\omega_\perp(k)$ and $\omega_\parallel^\pm(k)$ have at least one branch-point singularity in the complex k-plane. One justification for this expectation is of empirical nature, as it is realized in theories of causal hydrodynamics and holography. In the Supplemental Material we provide an additional argument in favour of it. Importantly, it implies that $\omega_\perp(k)$ and $\omega_\parallel(k)$ cannot be polynomials in k, so the hydrodynamic gradient expansion (11) following from the matching conditions (15) and (19) must contain an infinite number of terms. Moreover, the transport coefficients a_n, b_n and c_n grow geometrically in a manner controlled by the position of the branch points closest to $k = 0$ [32]

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = |k_*^{(A)}|^{-2}, \tag{21}$$

where $|k_*^{(A)}|$ denotes the modulus of the branch point location, and analogous expressions hold for b_n and c_n. Note that $|k_*^{(A)}|, |k_*^{(B)}|, |k_*^{(C)}|$ correspond to the closest branch point between $\omega_\perp$ and $\omega_\parallel$ as dictated by Eqs. (15) and (19). The power appearing on the right hand side of Eq. (21) is due to the fact that the transport coefficients are coefficients of a Taylor series in k^2.

Convergence– The convergence properties of the series (11) depend on the behavior of the transport coefficients a_n, b_n and c_n as well as on the particular solution ϵ and $\mathbf{u}$. In this Section we show that the support in momentum space of the latter plays a crucial role in determining the radius of convergence of the gradient expansion. We will focus on square-integrable functions, thus excluding trivial cases for which the gradient expansion truncates at a finite order.

We start by assuming that the flow is homogeneous in the $x^1, ..., x^{d-2}$ directions, and define $x \equiv x^{d-1}$. Furthermore, we take the Fourier transforms of $\epsilon(t,x)$ and $u^i(t,x)$, $\hat{\epsilon}(t,k)$ and $\hat{u}^i(t,k)$, to vanish for $|k| > |k_{max}|$. In the linearized regime, the support is time-independent and thus this condition is a restriction on the support of the initial data.

According to the Paley-Wiener theorem [33], the Fourier transform of a square-integrable function $\hat{f}(k)$ supported in $|k| \le |k_{max}|$ is an entire function of exponential type $|k_{max}|$ [34]. In particular, it follows that

$$\limsup_{n\to\infty} |f^{(n)}(x)|^{\frac{1}{n}} = |k_{max}|. \tag{22}$$

Let us consider now the A-contribution to (11). For a compactly-supported $\hat{u}^i$, $\sigma_{jl}(t,x)$ will be of exponential type $|k_{max}|$ for all times. Hence,

$$\limsup_{n\to\infty} |\partial_x^{2n} \sigma_{jl}(t,x)|^{\frac{1}{n}} = |k_{max}|^2. \tag{23}$$

Applying the root test results in the following convergence criterium for the A-contribution to (11)

$$\limsup_{n\to\infty} |a_n \partial_x^{2n} \sigma_{jl}(t,x)|^{\frac{1}{n}} = \frac{|k_{max}|^2}{|k_*^{(A)}|^2} < 1. \tag{24}$$

Analogous arguments apply to the remaining pieces of (11), with the conclusion that the gradient expansion of the constitutive relations will be a convergent series if and only if the support of the hydrodynamic perturbations and their time-derivatives at $t = 0$ does not exceed the smallest of $|k_*^{(A)}|, |k_*^{(B)}|$ and $|k_*^{(C)}|$.

The condition for the convergence of the gradient expansion spelled out above applies to arbitrary longitudinal fluid flows. Note that previous real space statements about the convergence or divergence of the hydrodynamic expansion were based on case studies of comoving flows in simple expanding spacetimes. Our analysis here covers a large class of models and does not make any simplifying symmetry assumptions about the longitudinal spacetime dependence.

Even if divergent, the partial sums of the gradient expansion only grow geometrically as long as the support of the initial data in k-space does not extend to infinity. If it does, this geometric divergence is enhanced to the factorial one known from the studies of expanding geometries [13, 16–20, 35]. The ambiguity of the sum is then related to the multi-sheeted structure of the dispersion relations $\omega(k)$ for $|k| > |k_*|$ [23–25] and requires contributions from nonhydrodynamic modes to be resolved.

For a flow without any symmetry restrictions, we can argue heuristically that the same convergence conditions hold. Let us focus again on the A-contribution to (11). Truncating the series to N-th order results in

$$-\int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{k} \left[\sum_{n=0}^{N} a_n \left(\mathbf{k}^2\right)^n\right] \hat{\sigma}_{ij}(t,\mathbf{k})\, e^{i\,\mathbf{k}\cdot\mathbf{x}}, \tag{25}$$

where we have interchanged the order of summation and integration. According to Eq. (21), the partial sums appearing in Eq. (25) are convergent as $N \to \infty$, provided that they are evaluated at $|\mathbf{k}| < |k_*^{(A)}|$. Outside this $(d-1)$-dimensional sphere we get a non-convergent series. Hence, it seems natural to assume that the condition for Eq. (25) to converge as $N \to \infty$ is that the hydrodynamic variable $\hat{\mathbf{u}}$ does not have support past $|k_*^{(A)}|$. Analogous arguments would hold also for the B- and C-pieces, supporting the fact that the convergence criterion spelled out before is fully general.

An illustrative example– For illustration, we now consider a shear channel perturbation in the Müller-Israel-Stewart (MIS) theory of hydrodynamics [36–38],

$$\epsilon = 0, \;\; \mathbf{u} = (u_1(t,x), 0, ..., 0). \tag{26}$$

The only tensor structure contributing to Eq. (11) is the shear tensor and the only nontrivial independent component of the constitutive relations is

$$\Pi_{1,d-1}(t,x) = -\sum_{n=0}^{\infty} a_n (-1)^n \partial_x^{2n+1} u_1(t,x). \qquad (27)$$

The a_n transport coefficients can be computed in closed form, since the shear hydrodynamic mode is known exactly [11],

$$\omega_\perp(k) = i\frac{-1+\sqrt{1-4\,D\,\tau_\pi\,k^2}}{2\,\tau_\pi}, \qquad (28)$$

where $D \equiv \eta/(s\,T) = (d-1)/(d-2)\,\Gamma$ is the diffusion constant. MIS contains also a single nonhydrodynamic shear mode which differs from Eq. (28) by the sign of the square root. The final result for the a_n coefficients is

$$a_n = s\,T\,\mathcal{C}_n D^{n+1} \tau_\pi^n, \qquad (29)$$

where $\mathcal{C}_n$ are the Catalan numbers. Therefore,

$$|k_*^{(A)}| = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1/2} = 1/\sqrt{4\,D\,\tau_\pi}, \qquad (30)$$

which is also the location of the branch points of Eq. (28), where the hydrodynamic and the nonhydrodynamic mode collide.

The initial state of the system is fully specified by $u_1(0,x)$ and $\partial_t u_1(0,x)$. We take $u_1(0,x) = 0$ and

$$\partial_t \hat{u}_1(0,k) = \frac{1}{2\pi} e^{-\frac{1}{2}\gamma^2 k^2} \Theta(k_{max}^2 - k^2), \qquad (31)$$

where Θ is the Heaviside step function. As seen in Fig. 1, the real space gradient expansion is convergent for $k_{max}^2 < 1/(4\,D\,\tau_\pi)$, geometrically divergent for $1/(4\,D\,\tau_\pi) \leq k_{max}^2 < \infty$, and factorially divergent for $k_{max} \to \infty$. This is exactly what is expected on the basis of our general analysis.

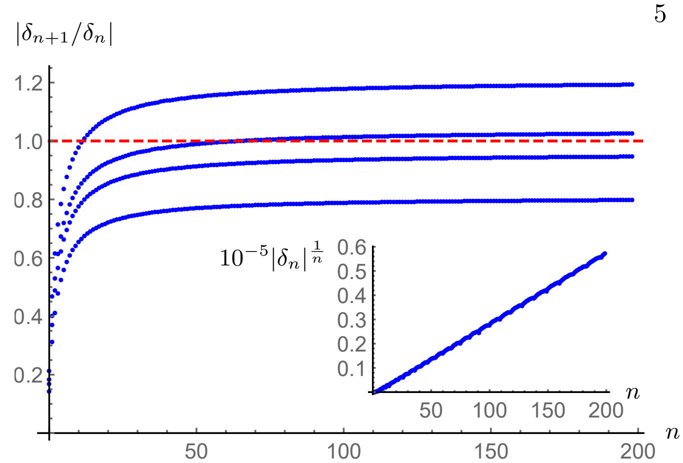

FIG. 1. Main plot: Ratio test applied to the gradient expansion (27), where δ_n denotes the n-th contribution. We use $\gamma = 0.1$ and consider $t = 1$, $x = 0.5$ with $s = T = \eta = \tau_\pi = 1$, in such a way that $|k_*^{(A)}| = 0.5$. From top to bottom, $k_{max} = 0.55, 0.51, 0.49, 0.45$. The gradient expansion is convergent for $k_{max} < |k_*^{(A)}|$ and geometrically divergent otherwise, as expected. Inset: root test applied to δ_n when $k_{max} \to \infty$. The geometric divergence of the gradient expansion is enhanced to a factorial one, manifest here in the asymptotic linear growth of $|\delta_n|^{\frac{1}{n}}$ with n.

Discussion and outlook– We have shown that the radius of convergence of the real space hydrodynamic gradient expansion evaluated on a solution of the evolution equations is determined by the momentum space support of the initial data. This represents a major step forward beyond earlier studies of expanding systems. Any statement about the convergence of the derivative series should thus be viewed as pertaining to the asymptotics of specific solutions and does not impact the definition of hydrodynamics which rests on its ability to match these asymptotics to those of underlying microscopic theories.

The applicability of hydrodynamics is connected with the radius of convergence of the gradient expansion only in the sense that both issues reflect the presence of a regulator sector consisting of transient, nonhydrodynamic modes required by causality. The regime of applicability of hydrodynamics is determined by the scale where specific calculations begin to be sensitive to the nonhydrodynamic mode spectrum [39].

It is very important that complete information about the nonhydrodynamic sector is encoded in the gradient series itself. In the case of an expanding plasma this is very beautifully expressed by the phenomenon of resurgence [40], which makes it possible to extract the form of the full solution from the asymptotic series [16, 18, 41]. The integration constants necessary to describe any complete solution enter that procedure as transseries parameters. An analogous encoding of nonhydrodynamic data in the hydrodynamic sector is seen in the analytic continuation of dispersion relations [23]. Generalizations of these ideas based on developments reported in this Letter are the subject of ongoing research [42].

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Supplementary material

Our main objective in this appendix is to provide additional arguments in favor of the two hypothesis regarding the behavior of the hydrodynamic dispersion relations put forward in the main text:

1. $\omega(k)$ has at least one singularity in the complex k-plane.

2. This singularity is a branch point.

We start by recalling that, under a metric fluctuation $\eta^{\mu\nu} \to \eta^{\mu\nu} + h^{\mu\nu}$, the response of the energy-momentum tensor expectation value in the thermal state is controlled by the retarded two-point function

$$G^{\mu\nu,\alpha\beta}(t,\mathbf{x}) = -i\Theta(t)\langle\left[T^{\mu,\nu}(t,\mathbf{x}),T^{\alpha\beta}(0,0)\right]\rangle \quad \text{(A.32)}$$

as

$$\delta\langle T^{\mu\nu}(t,\mathbf{x})\rangle = \\ = -\frac{1}{2}\int_{\mathbb{R}^{1,d-1}} dt'\, d^d\mathbf{x}'\, G^{\mu\nu,\alpha\beta}(t-t',\mathbf{x}-\mathbf{x}')h_{\alpha\beta}(t',\mathbf{x}').$$

The expectation values are taken in the background thermal state. Defining

$$G^{\mu\nu,\alpha\beta}(t,\mathbf{x}) = \int_{\mathbb{R}^{1,d-1}} d\omega\, d^{d-1}\mathbf{k}\, e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\hat{G}^{\mu\nu,\alpha\beta}(\omega,\mathbf{k}), \quad \text{(A.33)}$$

and similarly for $h^{\mu\nu}$, Eq. (A.33) can be written as

$$2(2\pi)^{-d}\delta\langle T^{\mu\nu}(t,\mathbf{x})\rangle = \\ = -\int_{\mathbb{R}^{1,d-1}} d\omega\, d^{d-1}\mathbf{k}\, e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\hat{G}^{\mu\nu,\alpha\beta}(\omega,\mathbf{k})\hat{h}_{\alpha\beta}(\omega,\mathbf{k}).$$

Hydrodynamic and nonhydrodynamic frequencies appear as poles of $\hat{G}^{\mu\nu,\alpha\beta}(\omega,\mathbf{k})$ which, due to rotational invariance, only depend on $\mathbf{k}^2$ [48]. To discuss the interplay between relativistic causality and the analyticity properties of these frequencies, we consider the following setup: we imagine that our metric fluctuation is only active at $t=0$ and, furthermore, we also assume that it only depends on $x^{d-1} \equiv x$,

$$h^{\mu\nu}(t,\mathbf{x}) = \delta(t)f^{\mu\nu}(x). \quad \text{(A.34)}$$

In momentum space,

$$\hat{h}^{\mu\nu}(\omega,\mathbf{k}) = \frac{1}{2\pi}\delta(k_1)...\delta(k_{d-2})\hat{f}^{\mu\nu}(k), \quad \text{(A.35)}$$

where we have also defined $k \equiv k_{d-1}$. Hence,

$$2(2\pi)^{1-d}\delta\langle T^{\mu\nu}(t,x)\rangle = \\ = -\int_{\mathbb{R}^{1,1}} d\omega\, dk\, e^{-i\omega t+ikx}\hat{G}^{\mu\nu,\alpha\beta}(\omega,0,...,0,k)\hat{f}_{\alpha\beta}(k).$$

Perfoming the integral with respect to ω, we obtain

$$\delta\langle\hat{T}^{\mu\nu}(t,k)\rangle = \sum_{q=0}^{N_H}\xi_q^{\mu\nu}(k)e^{-i\omega_q(k)t} + \\ + \sum_{q=0}^{N_{NH}}\Xi_q^{\mu\nu}(k)e^{-i\Omega_q(k)t} + \text{b.c.} \quad \text{(A.36)}$$

In writing the spectral decomposition (A.36), we have deformed our original integration contour along the real ω-axis to isolate the contributions coming from the singularities of $\hat{G}^{\mu\nu,\alpha\beta}(\omega,k)$ in the lower half of the complex ω-plane. N_H, N_{NH} refer respectively to the number of hydrodynamic ω_q and nonhydrodynamic Ω_q modes excited by the metric fluctuation, while the excitation coefficients $\xi_q^{\mu\nu}$ and $\Xi_q^{\mu\nu}$ are determined by the residues of the retarded correlator at its poles and the initial data. Finally, b.c. denotes the continuous contributions coming from the branch cuts that might be present. These contributions are absent in theories of causal relativistic hydrodynamics and AdS/CFT in the semiclassical limit, but do appear in kinetic theory.

As a final comment about Eq. (A.36), note that we have also assumed that any remaining contribution coming from an integral around infinity can be neglected. This is justified in the case in which our microscopic theory is a CFT and $t>0$: for $|\omega| \to \infty$ the retarded correlator should reduce to the vacuum result, which does not grow exponentially fast in the same limit.

Imagine now that $f^{\mu\nu}(x)$ is a square-integrable function supported only for $|x| \leq R$. Relativistic causality demands that, at $t>0$, the support of $\delta\langle T^{\mu\nu}(t,x)\rangle$ is at most $R+t$. Let us assume that $\delta\langle T^{\mu\nu}(t,x)\rangle$ is also square-integrable at all times. Then, the Paley-Wiener theorem [33] tells us that the spatial Fourier transform of $\delta\langle T^{\mu\nu}(t,x)\rangle$, $\delta\langle T^{\mu\nu}(t,k)\rangle$, is an entire function of exponential type at most $R+t$, also square-integrable along the real k-axis. We remind the reader that an entire function $f(z)$ is a function analytic everywhere in the complex z-plane, and that an entire function of exponential type σ is an entire function obeying the bound

$$|f(z)| \leq Ce^{\sigma|z|}, \;\; \forall\, z\in\mathbb{C}, \;\; C\in\mathbb{R}^+. \quad \text{(A.37)}$$

In the light of the Paley-Wiener theorem, and when the spectral decomposition (A.36) holds, property 1 follows by contradiction: if the frequency $\omega(k)$ were entire, its Laurent series expansion

$$\omega(k) = \sum_{n=1}^{\infty} w_n k^n \quad \text{(A.38)}$$

would be convergent $\forall\, k\in\mathbb{C}$, and the bound (A.37), as applied to $\delta\langle T^{\mu\nu}(t,k)\rangle$, would be violated. This result is in line with the conclusions of Ref. [49]. Since $\omega_\perp$ is given by a Taylor series in k^2, while $\omega_\parallel^\pm$ are series in k, the only possible exception to this behavior would be the case in which $\omega_\perp = 0$, $|\omega_\parallel^\pm| \propto |k|$, which corresponds precisely to ideal hydrodynamics.

On the other hand, property 2 can be justified as follows: if $\omega(k)$ had a pole, $\delta\langle T^{\mu\nu}(t,k)\rangle$ would develop an essential singularity at the pole location, thus failing to be entire. Furthermore, as argued in Ref. [49], for systems with a finite number of modes a pole in some dispersion relation entails that the initial value problem does not have a unique solution.

A final consequence of property 2 is that nonhydrodynamic modes must exist in a theory that respects relativistic causality. These modes, which in principle could be absent if the singularities in the hydrodynamic dispersion relations were poles, appear naturally when analytically continuing these functions past the branch cuts that are actually present.