Hydrodynamic reductions and solutions of a universal hierarchy

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Abstract

The diagonal hydrodynamic reductions of a hierarchy of integrable hydrodynamic chains are explicitly characterized. Their compatibility with previously introduced reductions of differential type is analyzed and their associated class of hodograph solutions is discussed.

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1 Introduction

In a series of papers [1]-[4] we have considered an infinite hierarchy of integrable systems [5] which admits many interesting $(1+1)$-dimensional reductions like the Burgers, KdV and NLS hierarchies as well as different classes of *energy-dependent* hierarchies [6]-[9] including the Camassa-Holm model also [10] . It motivated the term *universal hierarchy* we proposed in [1].

This universal hierarchy can be defined in terms of a generating function $G = G(\lambda, \boldsymbol{x})$ depending on an spectral parameter λ and an infinite set of variables $\boldsymbol{x} := (\ldots, x_{-1}, x_0, x_1, \ldots)$, which admits expansions

$$G = 1 + \frac{g_1(\boldsymbol{x})}{\lambda} + \frac{g_2(\boldsymbol{x})}{\lambda^2} + \cdots, \quad \lambda \to \infty, \tag{1}$$

$$G = b_0(\boldsymbol{x}) + b_1(\boldsymbol{x})\lambda + b_2(\boldsymbol{x})\lambda^2 + \cdots, \quad \lambda \to 0. \tag{2}$$

The hierarchy is provided by the system of flows

$$\partial_n G = \langle A_n, G \rangle, \quad n \in \mathbb{Z}, \tag{3}$$

where

$$\langle U, V \rangle := U(\partial_x V) - (\partial_x U)V,$$

and

$$A_n := \lambda^n + g_1(\boldsymbol{x})\lambda^{n-1} + \cdots + g_{n-1}(\boldsymbol{x})\lambda + g_n(\boldsymbol{x}), \quad n \geq 0 \tag{4}$$

$$A_{-n} := \frac{b_0(\boldsymbol{x})}{\lambda^n} + \frac{b_1(\boldsymbol{x})}{\lambda^{n-1}} + \cdots + \frac{b_{n-1}(\boldsymbol{x})}{\lambda}, \quad n > 0. \tag{5}$$

In terms of the coefficients $\{g_n(\boldsymbol{x})\}_{n\geq 1} \bigcup \{b_n(\boldsymbol{x})\}_{n\geq 0}$ the system (3) becomes a hierarchy of hydrodynamic chains.

An alternative and useful formulation of (3) is obtained by introducing the generating function

$$H := \frac{1}{G}. \tag{6}$$

Thus Eq.(3) is equivalent to

$$\partial_n H = \partial_x \Big(A_n H \Big), \quad n \in \mathbb{Z}, \tag{7}$$

which means, in particular, that the coefficients $\{h_n(\boldsymbol{x})\}_{n\geq 1}$ supply an infinite set of conservation laws for (3).

The hierarchy (3) forms a compatible system. Indeed, as a consequence of (3) one derives the consistency conditions [1]-[4]

$$\partial_n A_m - \partial_m A_n = \langle A_n, A_m\rangle, \quad n, m \in \mathbb{Z}, \tag{8}$$

It is important to notice that (8) implies that the pencil of differential forms

$$\omega(\lambda) := \sum_{n\in\mathbb{Z}} (H\, A_n)\,\mathrm{d}\,x_n, \tag{9}$$

is closed with respect to the variables $\boldsymbol{x}$ since it satisfies

$$\begin{aligned}\partial_m(H\,A_n) - \partial_n(H\,A_m) &= \partial_m H\,A_n - \partial_n H\,A_m + H(\partial_m A_n - \partial_n A_m)\\ &= \partial_x(HA_m)A_n - \partial_x(HA_n)A_m + H\langle A_n, A_m\rangle = 0.\end{aligned}$$

The potential function $Q = Q(\lambda, \boldsymbol{x})$ corresponding to ω

$$\mathrm{d}\,Q = \omega, \tag{10}$$

leads to another useful formulation of our hierarchy. Indeed, according to (9)

$$\partial_n Q = A_n\,\partial_x Q, \quad n \in \mathbb{Z}. \tag{11}$$

Moreover, (11) is completely determined by Q as

$$G = \frac{1}{\partial_x Q}. \tag{12}$$

In Section 2 of this paper the formulation (11) is applied to prove that (3) includes multidimensional models such as

$$u_{tx} = u_x u_{yz} - u_{yx} u_z, \tag{13}$$

$$u_{yy} = u_y u_{xz} - u_{xy} u_z, \tag{14}$$

$$u_{zx} = u_x u_{yy} - u_{xy} u_y, \tag{15}$$

$$\left(\frac{u_t}{u_x}\right)_t = \left(\frac{u_y}{u_x}\right)_z, \tag{16}$$

$$\left(\frac{u_t}{u_x}\right)_t = \left(\frac{u_y}{u_x}\right)_x. \tag{17}$$

Section 3 deals with the main aim of the present paper: to characterize the *hydrodynamic reductions* of (3). These reductions are given by the solutions of (3) of the form $G = G(\lambda, \boldsymbol{R})$, where $\boldsymbol{R} = (R^1, \dots, R^N)$ denotes a finite set of functions (*Riemann invariants*) satisfying a system of hydrodynamic equations of diagonal form

$$\partial_n R^i = \Lambda_n^i(\boldsymbol{R})\partial_x R^i, \quad n \in \mathbb{Z}. \tag{18}$$

This type of reductions appeared in the context of the dispersionless KP hierarchy [11]-[14] and has been also used in [15] to characterize the integrability of (2 + 1)-dimensional quasilinear systems. In the present paper we study these reductions for the whole set of flows of the hierarchy(3) and we obtain their explicit form . Our results are summarized in Theorem 1. It should be noticed that a similar analysis for the first member (t_1-flow) of (3) has been recently performed in [16]. The compatibility between hydrodynamic and differential reductions considered in part 2 of Section 3. The paper finishes with a discussion of the solutions supplied by the generalized hodograph method [17].

The following notation conventions are henceforth used. Firstly, G_∞ and G_0 stand for the expansions (1) (2) of G as $\lambda \to \infty$ and $\lambda \to 0$, respectively. Furthermore, let $\mathbb{V}$ be the space of formal Laurent series

$$V = \sum_{n=-\infty}^{\infty} a_n \lambda^n.$$

We will denote by $\mathbb{V}_{r,s}$ $(r \leq s)$ the subspaces of elements

$$V = \sum_{n=r}^{s} a_n \lambda^n.$$

and by $P_{r,s} : \mathbb{V} \mapsto \mathbb{V}_{r,s}$ the corresponding projectors. Given $V \in \mathbb{V}$ we will also denote

$$(V)_{r,s} := P_{r,s}(V).$$

In particular, notice that we can write

$$A_n = \Big(\lambda^n G_\infty\Big)_{0,+\infty}, \quad A_{-n} = \Big(\lambda^{-n} G_0\Big)_{-\infty,-1}, \quad n \geq 1.$$

2 Integrable models arising in the hierarchy

2.1 Multidimensional models

If we use the potential function $Q(\lambda,\boldsymbol{x})$ for the differential form (9)

$$\mathrm{d}\,Q=\sum_{n\in\mathbb{Z}}(H\,A_n)\,\mathrm{d}\,x_n, \tag{19}$$

then from (1) one readily deduces that Q admits expansions of the form

$$Q=\sum_{n\geq 0}\lambda^n x_n+\frac{q_1(\boldsymbol{x})}{\lambda}+\frac{q_2(\boldsymbol{x})}{\lambda^2}+\cdots, \quad \lambda\to\infty, \tag{20}$$

$$Q=\sum_{n\geq 1}\frac{x_n}{\lambda^n}+p_0(\boldsymbol{x})+p_1(\boldsymbol{x})\lambda+p_2(\boldsymbol{x})\lambda^2+\cdots, \quad \lambda\to 0. \tag{21}$$

By substituting (20) into (19) and by identifying coefficients of equal powers of λ one obtains formulas for the differentials of the functions $q_n(\boldsymbol{x})$ and $p_n(\boldsymbol{x})$ in terms of the coefficients of the expansions (1) and (2). For example, the simplest ones are

$$\mathrm{d}\,q_1=b_0\,\mathrm{d}\,x_{-1}+\sum_{n\geq 1}(b_n\;\mathrm{d}\,x_{-n-1}-g_n\;\mathrm{d}\,x_{n-1}), \tag{22}$$

$$\mathrm{d}\,p_0=\frac{1}{b_0}\Big(\mathrm{d}\,x_0+\sum_{n\geq 1}(g_n\;\mathrm{d}\,x_n-b_n\;\mathrm{d}\,x_{-n})\Big). \tag{23}$$

They imply

$$\mathrm{d}\,q_1=\frac{1}{\partial_x p_0}\Big(\mathrm{d}\,x_{-1}-\sum_{n\neq -1}\partial_{n+1}p_0\;\mathrm{d}\,x_n\Big) \tag{24}$$

$$\mathrm{d}\,p_0=\frac{1}{\partial_{-1}q_1}\Big(\mathrm{d}\,x_0-\sum_{n\neq 0}\partial_{n-1}q_1\;\mathrm{d}\,x_n\Big). \tag{25}$$

Permutability of crossing derivatives of q_1 and p_0 in these identities lead at once to multidimensional nonlinear equations for the functions g_n and b_n. For example, starting from

$$\partial_m\partial_{-1}q_1=\partial_{-1}\partial_m q_1, \quad m\neq -1,$$

and using (24), the following nonlinear equation results

$$\partial_n\partial_0 p_0 = \partial_x p_0\,(\partial_{-1}\partial_{n+1}p_0) - (\partial_{-1}\partial_x p_0)\,D_{n+1}p_0, \quad n \neq -1. \tag{26}$$

In the same way, from the crossing relation

$$\partial_m\partial_n q_1 = \partial_n\partial_m q_1, \quad m, n \neq -1,$$

and (24) we get the following nonlinear equation

$$\partial_m\Big(\frac{\partial_{n+1}p_0}{\partial_x p_0}\Big) = \partial_n\Big(\frac{\partial_{m+1}p_0}{\partial_x p_0}\Big), \quad m, n \neq -1. \tag{27}$$

The same type of equations can be derived for q_1. The different choices available for n, m in the equations (26) and (27) give rise to the models (13)-(17).

2.2 2-dimensional integrable models

In [1]-[3] we developed a theory of differential reductions of our hierarchy based on imposing differential constraints on $G \approx G_\infty$ of the form

$$\Big(\mathcal{F}(\lambda, G, G_x, G_{xx}, \ldots)\Big)_{-\infty,-1} = 0, \quad x := x_0. \tag{28}$$

In particular the following three classes of reductions associated to arbitrary polynomials $a = a(\lambda)$ in λ were characterized:

Zero-order reductions

$$a(\lambda)G = U(\lambda, \boldsymbol{x}), \quad U := \Big(a(\lambda)G\Big)_{0,+\infty}, \tag{29}$$

First-order reductions

$$G_x + a(\lambda) = U(\lambda, \boldsymbol{x})G, \quad U := \Big(\frac{a}{G}\Big)_{0,+\infty}, \tag{30}$$

Second-order reductions

$$\frac{1}{2}GG_{xx}-\frac{1}{4}G_x^2+a(\lambda)=U(\lambda,\boldsymbol{x})G^2,\quad U:=\Big(\frac{a}{G^2}\Big)_{0,+\infty}. \tag{31}$$

The first-order reduction for a linear function $a(\lambda)$ determines the Burgers hierarchy. On the other hand, under the differential constraints (31) the hierarchy (18) describes the KdV hierarchy and its generalizations associated to energy-dependent Schrödinger spectral problems. In particular, the linear and quadratic choices for $a(\lambda)$ lead to the KdV (Korteweg-deVries) and NLS (Nonlinear-Schrödinger) hierarchies, respectively. Indeed, if we define the functions $\psi(\lambda,\boldsymbol{x})$ by

$$\psi(\lambda,\boldsymbol{x}):=\exp(D_x^{-1}\phi),\quad \phi:=-\frac{1}{2}\frac{H_x}{H}\pm\sqrt{a(\lambda)}\,H, \tag{32}$$

then from (31) it is straightforward to deduce that

$$\begin{aligned}\partial_n\psi&=-\frac{1}{2}(\partial_n\log H)\psi\pm\sqrt{a}A_nH\psi\\&=A_n(-\frac{1}{2}D_x\log H\pm\sqrt{a}H)\psi-\frac{1}{2}A_{n,x}\psi\\&=A_n\psi_x-\frac{1}{2}A_{n,x}\psi,\\ \psi_{xx}&=(\phi_x+\phi^2)\psi=(\{D_x,H\}+aH^2)\psi=U\psi.\end{aligned}$$

In other words, the functions ψ are wave functions for the integrable hierarchies associated to energy-dependent Schrödinger problems. The evolution law of the potential function U under the flows (18) can be determined from the equation

$$\partial_nU=-\frac{1}{2}A_{n,xxx}+2UA_{n,x}+U_xA_n, \tag{33}$$

which arises as an straightforward consequence of (18) and (31).

Additional reduced hierarchies including nonlinear integrable models such as the Camassa-Holm equation can be also deduced (see [4]).

3 Hydrodynamic reductions and solutions

3.1 Hydrodynamic reductions

Let us consider now the hydrodynamic reductions of (3). We look for classes a solutions

$$G = G(\lambda, \boldsymbol{R}), \tag{34}$$

of (3) where $\boldsymbol{R} = (R^1, \dots, R^N)$ satisfies a infinite system of hydrodynamic equations of diagonal form (18). Our aim is to characterize both the form of $G = G(\lambda, \boldsymbol{R})$ and the *characteristic speeds* Λ^i_n defining the system (18). By substituting (35) into (3) then by using (18) the identification of coefficients of the derivatives $\partial_x R^i,\ i = 1, \dots, N$ implies

$$(D_i G)\Lambda^i_n = A_n(D_i G) - (D_i A_n)G, \quad 1 \leq i \leq N, \ \ n \in \mathbb{Z} \tag{35}$$

where

$$D_i := \frac{\partial}{\partial R^i}.$$

In addition to these equations we impose the requirement of the commutativity of the flows (18), which is equivalent to the following restrictions on the characteristic speeds

$$\frac{D_j \Lambda^i_n}{\Lambda^j_n - \Lambda^i_n} = \frac{D_j \Lambda^i_m}{\Lambda^j_m - \Lambda^i_m}, \quad i \neq j, \quad m \neq n. \tag{36}$$

We start our analysis by considering the *positive flows* $n \geq 1$. From (35) we get the following system for $G \approx G_\infty$

$$(D_i G)\Lambda^i_n = (\lambda^n G)_{-\infty,0}(D_i G) - (D_i(\lambda^n G)_{-\infty,0})G, \quad n \geq 1. \tag{37}$$

By substituting in these equations the expansion $G \approx G_\infty$ and identifying the coefficients in $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$ we get

$$D_i g_{n+1} = \Lambda^i_n D_i g_1, \tag{38}$$
$$D_i g_{n+2} = \Lambda^i_n D_i g_2 + g_{n+1}(D_i g_1) - (D_i g_{n+1}) g_1. \tag{39}$$

As an immediate consequence it follows

$$\Lambda^i_{n+1} = g_{n+1} + \Lambda^i_n(\Lambda^i_1 - g_1), \tag{40}$$

which implies

$$\Lambda_n^i = A_n\Big(\lambda = \Lambda_1^i - g_1\Big), \quad n \geq 1. \tag{41}$$

We now look for a system characterizing Λ_1^i and g_1. To this end we differentiate (38) and find

$$\begin{aligned} D_j D_i g_{n+1} &= D_j D_i g_1 \Lambda_n^i + D_i g_1 D_j \Lambda_n^i \\ &= D_i D_j g_1 \Lambda_n^j + D_j g_1 D_i \Lambda_n^i, \end{aligned}$$

so that

$$D_{ij} g_1 = \frac{D_j \Lambda_n^i}{\Lambda_n^j - \Lambda_n^i} D_i g_1 + \frac{D_i \Lambda_n^j}{\Lambda_n^i - \Lambda_n^j} D_j g_1,$$

and from (36) we may write

$$D_{ij} g_1 = \frac{D_j \Lambda_1^i}{\Lambda_1^j - \Lambda_1^i} D_i g_1 + \frac{D_i \Lambda_1^j}{\Lambda_1^i - \Lambda_1^j} D_j g_1, \quad i \neq j. \tag{42}$$

On the other hand according to (40)

$$\Lambda_2^i = g_2 + \Lambda_1^i(\Lambda_1^i - g_1),$$

so that from (36) and with the help of (38) we find

$$\frac{D_j \Lambda_1^i}{\Lambda_1^j - \Lambda_1^i} = \frac{D_j \Lambda_2^i}{\Lambda_2^j - \Lambda_2^i} = \frac{D_j \Lambda_1^i (2\Lambda_1^i - g_1) + (\Lambda_1^j - \Lambda_1^i) D_j g_1}{(\Lambda_1^j - \Lambda_1^i)(\Lambda_1^j + \Lambda_1^i - g_1)}, \quad i \neq j,$$

which reduces to

$$D_j \Lambda_1^i = D_j g_1, \quad i \neq j.$$

In this way

$$\Lambda_1^i = g_1 + f^i(R^i), \tag{43}$$

where the functions f^i are arbitrary. By using this result in (42) it follows

$$D_{ij} g_1 = 0, \quad i \neq j,$$

so that

$$g_1 = \sum_{k=1}^{N} h^k(R^k), \tag{44}$$

where the functions h^k are arbitrary.

By using these results we may determine $G(\lambda, \boldsymbol{R})$ since from the equation (35) with $n = 1$

$$D_i \ln G = \frac{D_i A_1}{A_1 - \Lambda_1^i} = \frac{\dot{h}^i(R^i)}{\lambda - f^i(R^i)},$$

where $\dot{h}^i := D_i h^i$. Thus we get

$$G(\lambda, \boldsymbol{R}) = \exp\Big(\sum_{i=1}^N \int^{R^i} \frac{\dot{h}^i(R^i)}{\lambda - f^i(R^i)} \, \mathrm{d}\, R^i \Big), \tag{45}$$

where the undefinite integrations are determined up to a function of λ decaying at $\lambda \to \infty$. The expression (45) coincides with the generating function of the conservation laws densities found in [16].

Let us now determine the characteristic speeds Λ_n^i for the *negative flows* $n \leq -1$. The equations (35) imply the following system for $G \approx G_0$

$$(D_i G)\Lambda_{-n}^i = (\lambda^{-n} G)_{0,\infty}(D_i G) - (D_i(\lambda^{-n} G)_{0,\infty})G, \quad n \geq 1. \tag{46}$$

Then by inserting the expansion $G \approx G_0$ of (2) and identifying the coefficients in λ^0 and λ we get

$$D_i b_n = (b_n + \Lambda_{-n}^i) D_i \ln b_0, \tag{47}$$

$$b_0 D_i b_{n+1} = (b_n + \Lambda_n^i) D_i b_1 + b_{n+1}(D_i b_0) - (D_i b_n) b_1. \tag{48}$$

It implies the following recurrence relation for the characteristic speeds

$$\Lambda_{-n-1}^i = \frac{\Lambda_{-1}^i}{b_0}(b_n + \Lambda_{-n}^i), \tag{49}$$

which implies

$$\Lambda_{-n}^i = A_{-n}\Big(\lambda = \frac{b_0}{\Lambda_{-1}^i}\Big), \quad n \geq 1. \tag{50}$$

The only unknown now is Λ_{-1}^i since according to (45)

$$b_0 = \exp\Big(-\sum_{i=1}^N \int^{R^i} \frac{\dot{h}^i(R^i)}{f^i(R^i)} \, \mathrm{d}\, R^i \Big).$$

To find Λ_{-1}^i we use (49) for $n = 1$

$$\Lambda_{-2}^i = \frac{\Lambda_{-1}^i}{b_0}(b_1 + \Lambda_{-1}^i),$$

and the commutativity condition for the $n = -1$ and $n = -2$ flows

$$\frac{D_j \Lambda^i_{-1}}{\Lambda^j_{-1} - \Lambda^i_{-1}} = \frac{D_j \Lambda^i_{-2}}{\Lambda^j_{-2} - \Lambda^i_{-2}}.$$

Thus by eliminating λ^i_{-2} one finds at once

$$D_j \ln \Lambda^i_{-1} = D_j \ln b_0, \quad i \neq j.$$

Therefore,

$$\Lambda^i_{-1} = b_0\, g^i(R^i), \tag{51}$$

with g^i being arbitrary functions.

At this point we have determined all the unknowns of our problem. However, in our calculation we only used a subset of the equations required, so we must prove that our solution satisfies the full system of equations (35)-(36).

Let us begin with (35) for $n \geq 1$

$$D_i \ln G = \frac{D_i A_n}{A_n - \Lambda^i_n}, \quad n \geq 1,$$

which according to (41), (43)-(45) reads

$$D_i A_n(\lambda) = \frac{\dot{h}^i}{\lambda - f^i}\Big(A_n(\lambda) - A_n(\lambda = f^i)\Big). \tag{52}$$

To prove this identity we express $A_n(\lambda)$ as

$$A_n(\lambda) = (\lambda^n G)_{0,\infty} = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{(\widetilde{\lambda}^n G(\widetilde{\lambda}))_{0,\infty}}{\widetilde{\lambda} - \lambda}\,\mathrm{d}\,\widetilde{\lambda} = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{\widetilde{\lambda}^n G(\widetilde{\lambda})}{\widetilde{\lambda} - \lambda}\,\mathrm{d}\,\widetilde{\lambda}$$

$$= \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{\widetilde{\lambda}^n}{\widetilde{\lambda} - \lambda} \exp\Big(\sum_{i=1}^{N} \int^{R^i} \frac{\dot{h}^i}{\widetilde{\lambda} - f^i}\,\mathrm{d}\,R^i\Big)\,\mathrm{d}\,\widetilde{\lambda},$$

where γ_∞ is a positively oriented closed loop around ∞ in the complex plane of $\widetilde{\lambda}$ (λ and f^i are assumed to lie inside the loop). Now by differentiating this expression with respect to R^i one finds

$$D_i A_n(\lambda) = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{\dot{h}^i \widetilde{\lambda}^n G(\widetilde{\lambda})}{(\widetilde{\lambda} - \lambda)(\widetilde{\lambda} - f^i)}\,\mathrm{d}\,\widetilde{\lambda} = \frac{1}{2\pi i}\int_{\gamma_\infty} \frac{\dot{h}^i A_n(\widetilde{\lambda})}{(\widetilde{\lambda} - \lambda)(\widetilde{\lambda} - f^i)}\,\mathrm{d}\,\widetilde{\lambda}$$

$$= \frac{\dot{h}^i}{\lambda - f^i}\Big(A_n(\lambda) - A_n(\lambda = f^i)\Big),$$

which proves (52). This identity leads also at once to (36) since it implies

$$\frac{D_j A_n(\lambda = f^i)}{A_n(\lambda = f^j) - A_n(\lambda = f^i)} = \frac{\dot{h}^j}{f^j - f^i}, \quad i \neq j,$$

which means that

$$\frac{D_j \Lambda_n^i}{\Lambda_n^j - \Lambda_n^i} = \frac{D_j \Lambda_1^i}{\Lambda_1^j - \Lambda_1^i}, \quad n > 1.$$

Let us consider now (35) for $n \leq -1$

$$D_i \ln G = \frac{D_i A_{-n}}{A_{-n} - \Lambda_{-n}^i}, \quad n \geq 1.$$

From (45), (50) and (51) it takes the form

$$D_i A_{-n}(\lambda) = \frac{\dot{h}^i}{\lambda - f^i}\Big(A_{-n}(\lambda) - A_{-n}(\lambda = \frac{1}{g^i})\Big). \tag{53}$$

In order to proof (53) we express $A_{-n}(\lambda)$ in the form

$$A_{-n}(\lambda) = (\lambda^{-n} G)_{-\infty,-1} = \frac{1}{2\pi i}\int_{\gamma_0} \frac{(\widetilde{\lambda}^{-n} G(\widetilde{\lambda}))_{-\infty,-1}}{\widetilde{\lambda} - \lambda}\, \mathrm{d}\,\widetilde{\lambda} = \frac{1}{2\pi i}\int_{\gamma_0} \frac{\widetilde{\lambda}^{-n} G(\widetilde{\lambda})}{\widetilde{\lambda} - \lambda}\, \mathrm{d}\,\widetilde{\lambda}$$

$$= \frac{1}{2\pi i}\int_{\gamma_0} \frac{\widetilde{\lambda}^{-n}}{\widetilde{\lambda} - \lambda} \exp\Big(\sum_{i=1}^{N} \int^{R^i} \frac{\dot{h}^i}{\widetilde{\lambda} - f^i}\, \mathrm{d}\, R^i\Big)\, \mathrm{d}\,\widetilde{\lambda},$$

where γ_0 is a closed small loop with negative orientation around $\widetilde{\lambda} = 0$ (λ and f^i are assumed to lie outside the loop). By differentiating with respect to R^i it yields

$$D_i A_{-n}(\lambda) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{\dot{h}^i \widetilde{\lambda}^{-n} G(\widetilde{\lambda})}{(\widetilde{\lambda} - \lambda)(\widetilde{\lambda} - f^i)}\, \mathrm{d}\,\widetilde{\lambda}$$

$$= \frac{1}{2\pi i}\int_{\gamma_0} \frac{\dot{h}^i A_{-n}(\widetilde{\lambda})}{(\widetilde{\lambda} - \lambda)(\widetilde{\lambda} - f^i)}\, \mathrm{d}\,\widetilde{\lambda} = \frac{\dot{h}^i}{\lambda - f^i}\Big(A_{-n}(\lambda) - A_{-n}(\lambda = f^i)\Big),$$

Hence (53) holds if we set

$$g^i(R^i) = \frac{1}{f^i(R^i)}.$$

The remaining commutativity conditions (36) follow at once.

Therefore we may summarize our analysis in the next theorem

Theorem 1. *The hydrodynamic reductions of the hierarchy* (3) *are determined by*

$$G(\lambda,\boldsymbol{R})=\exp\Big(\sum_{i=1}^{N}\int^{R^i}\frac{\dot{h}^i(R^i)}{\lambda-f^i(R^i)}\,\mathrm{d}\,R^i\Big), \tag{54}$$

$$\partial_n R^i=\Lambda^i_n(\boldsymbol{R})\,\partial_x R^i,\quad \Lambda^i_n(\boldsymbol{R}):=A_n(\lambda=f^i(R^i)), \tag{55}$$

where the functions h^i and f^i are arbitrary.

Since the systems (18) are invariant under local transformations of the form $R^i\to\widetilde{R^i}(R^i)$, without loss of generality we will henceforth set

$$h^i(R^i)=R^i, \tag{56}$$

so that the form of the reduced generating function is

$$G(\lambda,\boldsymbol{R})=\exp\Big(\sum_{i=1}^{N}\int^{R^i}\frac{\mathrm{d}\,R^i}{\lambda-f^i(R^i)}\Big), \tag{57}$$

In this way some of the simplest hydrodynamic reductions (54) are given by

$$f^i(R^i)=c_i,\quad G=\exp\Big(\sum_{i=1}^{N}\frac{R^i}{\lambda-c_i}\Big), \tag{58}$$

$$f^i(R^i)=-\frac{R^i}{\epsilon_i},\quad G=\prod_{i=1}^{N}\Big(\frac{\epsilon_i\lambda+R^i}{\epsilon_i\lambda+\lambda_i}\Big)^{\epsilon_i}. \tag{59}$$

We also notice that the characteristic velocities of the hydrodynamic systems (55) can be written in terms of the Schur polynomials

$$\exp(\sum_{n\geq 1}k^n x_n)=\sum_{n\geq 0}k^n S_n(x_1,\ldots,x_n),$$

as

$$\Lambda^i_n(\boldsymbol{R}) = \sum_{j=0}^{n} S_j(I_1, \dots, I_j)\Big(f^i(R^i)\Big)^{n-j}, \quad n \geq 0,$$

$$\Lambda^i_{-n}(\boldsymbol{R}) = \exp(-I_0)\sum_{j=0}^{n-1} S_j(I_{-1}, \dots, I_{-j})\Big(f^i(R^i)\Big)^{-n+j}, \quad n > 0,$$

where

$$I_n := sgn(n)\sum_{i=1}^{N}\int^{R^i}(f^i(R^i))^{n-1}\,\mathrm{d}\,R^i.$$

3.2 Compatibility with differential reductions

A natural question is to find the hydrodynamic reductions compatible with the differential reductions (30) and (31). Let us prove the following result:

Theorem 2. *The only hydrodynamic reductions* (57) *compatible with either first or second order differential constraints are characterized by*

$$f^i(R^i) = -R^i + c_i, \quad i = 1, \dots, N,$$

which correspond to generating functions of the form

$$G(\lambda, \boldsymbol{R}) = \alpha(\lambda)\prod_{i=1}^{N}(\lambda + R^i - c_i).$$

Proof. If we substitute (57) into the differential constraint (31) for second-order differential reductions we get

$$\frac{a(\lambda)}{G^2} = U - \frac{1}{2}\sum_i\Big(\dot{f}^i\frac{(\partial_x R^i)^2}{(\lambda - f^i)^2} + \frac{\partial_{xx}R^i}{\lambda - f^i}\Big) - \frac{1}{4}\Big(\sum_i\frac{\partial_x R^i}{\lambda - f^i}\Big)^2.$$

This means that G has a simple zero at each $\lambda = f^i(R^i)$ so that from (57) we have

$$\exp\Big(\int^{R^i}\frac{\mathrm{d}\,R^i}{\lambda - f^i(R^i)}\Big) = (\lambda - f^i(R^i))H(\lambda, R^i),$$

where H is different from zero at $\lambda = f^i(R^i)$. If we now differentiate with respect to R^i we deduce that

$$\frac{1}{\lambda - f^i} = -\frac{\dot{f}^i}{\lambda - f^i} + \mathcal{O}(1), \quad \lambda \to \infty.$$

Therefore the statement of the theorem for second-order differential constraints follows. The corresponding proof for first-order constraints is similar. □

3.3 Hodograph solutions

The general solution of the infinite system (55) is provided by the implicit *generalized hodograph* formula [17]

$$x + \sum_{n\in\mathbb{Z}-\{0\}} \Lambda^i_n(\boldsymbol{R})\, x_n = \Gamma^i(\boldsymbol{R}), \quad i = 1, \ldots, N, \tag{60}$$

where the functions Γ^i are the general solution of the linear system

$$\frac{D_j\Gamma^i}{\Gamma^j - \Gamma^i} = \frac{1}{f^j - f^i}, \quad i \neq j. \tag{61}$$

By introducing the potential function $\Phi(\boldsymbol{R})$

$$\Gamma^i = D_i\Phi, \quad i = 1, \ldots, N,$$

the system (61) reduces to the Laplace type form

$$(f^i - f^j)D_iD_j\Phi = D_i\Phi - D_j\Phi, \quad i \neq j, \tag{62}$$

the general solution of which depends on N arbitrary functions of one variable.

In particular it is immediate to deduce [16] that the generating function (57) provides a one-parameter family of solutions of (62). Thus we can produce important solutions of (61) from linear superpositions of $G(\lambda, \boldsymbol{R})$. For example

$$\begin{aligned}
\Lambda^i_n = D_i\Phi, \;\; \Phi := \frac{1}{2\pi i}\int_{\gamma_\infty} \lambda^n G(\lambda, \boldsymbol{R})\,\mathrm{d}\,\lambda, \quad n \geq 0,\\
\Lambda^i_{-n} = D_i\Phi, \;\; \Phi := \frac{1}{2\pi i}\int_{\gamma_0} \lambda^{-n} G(\lambda, \boldsymbol{R})\,\mathrm{d}\,\lambda, \quad n > 0.
\end{aligned}$$

Furthermore, in several important cases the general solution of (62) can be written in terms of $G(\lambda, \boldsymbol{R})$.

Examples

If we set

$$f^i(R^i) = \frac{R^i}{n}, \quad n = 1, 2, \dots,$$

then the generating function (non normalized as $\lambda \to \infty$) is

$$G(\lambda, \boldsymbol{R}) = \prod_{i=1}^{N} \Big(\lambda - \frac{R^i}{n}\Big)^{-n}.$$

Hence, by taking for each $i = 1, \dots, N$ a closed loop γ_i in the complex λ-plane with positive orientation around $\lambda_i = \frac{R^i}{n}$, the general solution of (62) can be expressed as

$$\Phi := \frac{1}{2\pi i} \sum_{i=1}^{N} \int_{\gamma_i} \phi_i(\lambda) G(\lambda, \boldsymbol{R}) \,\mathrm{d}\,\lambda,$$

where the functions $\phi_i(\lambda)$ are arbitrary. For example, if $n = 1$ it takes the form

$$\Phi = \sum_{i=1}^{N} \phi_i(R^i) \prod_{k \neq i} \frac{1}{R^i - R^k}.$$

Similarly we may deal with the case

$$f^i(R^i) = -\frac{R^i}{n}, \quad n = 1, 2, \dots,$$

which leads to

$$G(\lambda, \boldsymbol{R}) = \prod_{i=1}^{N} \Big(\lambda + \frac{R^i}{n}\Big)^{n}.$$

Now to generate the general solution of (62) we take for each $i = 1, \dots, N$ a path $\gamma_i(R^i)$ in the complex λ-plane ending at $\lambda_i = -R^i/n$ so that we can write

$$\Phi := \frac{1}{2\pi i} \sum_{i=1}^{N} \int_{\gamma_i(R^i)} \phi_i(\lambda) G(\lambda, \boldsymbol{R}) \,\mathrm{d}\,\lambda,$$

where the functions $\phi_i(\lambda)$ are arbitrary.

Let us consider in detail the case

$$f^i(R^i) = -R^i, \quad i = 1, 2.$$

One finds

$$\Phi = \Big(\theta_1(-R^1) - \theta_2(-R^2)\Big)(R^1 - R^2) + 2\int^{-R^1} \theta_1(\lambda)\,\mathrm{d}\,\lambda + 2\int^{-R^2} \theta_2(\lambda)\,\mathrm{d}\,\lambda,$$

with $\theta_i(\lambda)$ being arbitrary functions. A normalized generating function is given by

$$G(\lambda, \boldsymbol{R}) = \frac{(\lambda + R^1)(\lambda + R^2)}{(\lambda + \lambda_0)^2}, \quad \lambda_0 \neq 0. \tag{63}$$

We may characterize the general solution of the hydrodynamic flows (55) corresponding to $n = -1$ and $n = -2$ by means of the hodograph formula

$$x + A_{-1}(\lambda = -R^i)\, y + A_{-2}(\lambda = -R^i)\, z = D_i\Phi, \quad i = 1, 2, \tag{64}$$

where $y := x_{-1},\ z := x_{-2}$. From (63) one calculates

$$b_0 = c^2 R^1 R^2, \quad b_1 = c^2(R^1 + R^2) - 2c^3 R^1 R^2, \quad c := \frac{1}{\lambda_0},$$

and gets that the system (64) reads

$$\begin{aligned}(x - c^2 z) - c^2(y - 2cz)R^2 &= \dot{\theta}_1(-R^1)(R^2 - R^1) - \theta_1(-R^1) - \theta_2(-R^2),\\ (x - c^2 z) - c^2(y - 2cz)R^1 &= \dot{\theta}_2(-R^2)(R^1 - R^2) - \theta_1(-R^1) - \theta_2(-R^2).\end{aligned} \tag{65}$$

In particular, it implies

$$\boldsymbol{R} = \boldsymbol{R}(x - c^2 z, y - 2cz),$$

so that $\boldsymbol{R}$ is constant on the straight lines

$$\vec{x} := (x, y, z) = (x_0, y_0, 0) + (c^2 + 2c, 1)s.$$

We notice that starting from these solutions we may generate solutions

$$u(x, y, z) := \int^y b_0(\boldsymbol{R})\,\mathrm{d}\,y + \int^z b_1(\boldsymbol{R})\,\mathrm{d}\,z,$$

of the nonlinear equation

$$u_{yy} = u_u u_{zx} - u_{yx} u_z.$$

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References

[1] L. Martinez Alonso and A. B. Shabat, Phys.Lett. **A 300**, 58 (2002).

[2] L. Martinez Alonso and A. B. Shabat, J. Non. Math. Phys. **10** No 2, 1 (2003).

[3] L. Martinez Alonso and A. B. Shabat, *" On the prolongation of a hierarchy of hydrodynamic chains"* to appear in *New Trends in Integrability and Partial Solvability* proceedings NATO ARW.978791, NATO Science Series , Kluwer Academic Publishers.

[4] A. Shabat, Theor. and Math. Phys. **136** No 2, 1066 (2003).

[5] V. G. Mikhalev, Funct. Anal. Appl. **26** No 2, 140 (1992).

[6] M. Jaulent and I. Miodek, Lett. Math. Phys. **1**, 243 (1976); Lett. Nuovo Cimento **20**, 655 (1977).

[7] L. Martínez Alonso, J. Math. Phys. **21**, 2342 (1980).

[8] M. Antonowicz and A. P. Fordy, Physica **28 D**, 345 (1987).

[9] A. N. W. Hone, Phys. Lett. **A 249**, 46 (1998).

[10] R. Camassa and D. Holm, Phys. Rev. Letters **71**, 1661 (1993).

[11] J. Gibbons and Y. Kodama. Phys. Lett. **A 135**, 167 (1989).

[12] J. Gibbons and S. P. Tsarev. Phys. Lett. **A 211** , 19 (1996).

[13] J. Gibbons and S. P. Tsarev. Phys. Lett. **A 258** , 263 (1999).

[14] M. Mañas, L. Martinez Alonso and E. Medina. J. Phys. A: Math. Gen. **35**, 401 (2002)

[15] E. V. Ferapontov and K. R. Khusnutdinova. *On integrability of* $(2+1)$-*dimensional quasilinear systems* nlin.SI/0305044, (2003).

[16] M. V. Pavlov. *Integrable hydrodynamic chains* nlin.SI/0301010, (2003).

[17] S. P. Tsarev. Izvestija AN USSR Math. **54**, N5, 1048 (1990).