Preprint typeset in JHEP style - HYPER VERSION
IFT-UAM/CSIC-09-14

Hydrodynamics of Holographic Superconductors

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Abstract: We study the poles of the retarded Green functions of a holographic superconductor. The model shows a second order phase transition where a charged scalar operator condenses and a $U(1)$ symmetry is spontaneously broken. The poles of the holographic Green functions are the quasinormal modes in an AdS black hole background. We study the spectrum of quasinormal frequencies in the broken phase, where we establish the appearance of a massless or hydrodynamic mode at the critical temperature as expected for a second order phase transition. In the broken phase we find the pole representing second sound. We compute the speed of second sound and its attenuation length as function of the temperature. In addition we find a pseudo diffusion mode, whose frequencies are purely imaginary but with a non-zero gap at zero momentum. This gap goes to zero at the critical temperature. As a technical side result we explain how to calculate holographic Green functions and their quasinormal modes for a set of operators that mix under the RG flow.

Keywords: Holography, Superconductivity.

Contents

1. Introduction

The AdS/CFT correspondence is a tool for studying field theories in the strong coupling regime [1–3]. The range of physical phenomena to which it can be applied is constantly increasing. One of the latest additions is the realization of spontaneous symmetry breaking and the appearance of a superfluid (often described as superconducting) phase at low temperature. A model with a charged scalar condensing in the background of a charged AdS black hole has first been introduced in [4]. Shortly afterwards it was realized that a charged scalar condenses as well when the black hole is neutral and it was shown explicitly that the DC conductivity is infinite in the broken phase [5].

By now there is a large variety of holographic models of superfluidity/superconductivity [6–21]. Specifically hydrodynamical behavior in these models has been addressed before in [22,23] where the speed of sound has been calculated from derivatives of thermodynamic quantities. The hydrodynamic poles of retarded Green functions have been studied in an analytical approximation for infinitesimal condensate in a p-wave model in [24]

In this work we are interested in the hydrodynamics of the holographic superconductor introduced in [5]. In general, hydrodynamic behavior is connected either to the presence of a conserved charge, a spontaneously broken symmetry or a second order phase transition. As we will see in our model all three possibilities are realized. In holographic models the hydrodynamic modes appear as quasinormal modes in the AdS black hole background whose frequency vanishes in the zero momentum limit [25,26]. Quasinormal modes play an important role in the physics of black holes and a very good review of their properties in asymptotically flat black holes is [27]. In the asymptotically flat situation the quasinormal modes are defined as solutions of linearized wave equations with purely infalling boundary conditions on the horizon outgoing ones at the boundary. In asymptotically AdS spaces the situation is somewhat different. On the horizon one still imposes infalling boundary conditions but on the conformal boundary of anti de Sitter space several possibilities of choosing boundary conditions arise: Dirichlet or Neumann or Robin (mixed) ones. The holographic interpretation fixes this degeneracy of boundary conditions by defining the quasinormal modes as the poles of the holographic Green functions [28–30]. This of course implies to know the holographic Green functions, which are computed using the prescription given in [31]. In situations in which there are several fields whose linearized wave equations form a coupled system of differential equations possibly subject to a constraint due to a gauge symmetry the construction of the holographic Green functions is a bit more complicated. We solve this problem in full generality and show that the quasinormal modes defined are the zeroes of the determinant spanned by the values at the boundary of a maximal set of linearly independent solutions to the field equations.

Having solved the problem of defining the holographic Green functions we concentrate on finding the lowest quasinormal modes and in particular the ones representing the hydrodynamic behavior of the system. Hydrodynamic modes can be understood as massless modes in the sense that $\lim_{k\to 0}\omega(k) = 0$. Such modes arise in the presence of a conserved charge. In this case a local charge distribution can not simply dissipate away but has to spread slowly over the medium according to a diffusion process. Other situations in which hydrodynamic modes appear are at a second order phase transition, characterized by the appearance of a new massless mode and spontaneous breaking of a global symmetry where a massless Goldstone boson appears. A discussion of hydrodynamics in systems with spontaneous breaking of global symmetries can be found in [32] and in the relativistic context in [33].

We will consider the abelian gauge model of [5] without backreaction, i.e. assuming that the metric is a simple AdS black hole with flat horizon topology. In [5] it has been established that this model undergoes a second order phase transition towards forming a condensate of the charged scalar field thus spontaneously breaking the $U(1)$ gauge symmetry. The conductivity in the broken phase has a delta function peak at zero frequency and a gap typical of superconductors. We add here that although this model is referred to as holographic superconductor it is more proper to speak of a holographic charged superfluid as in [22] since the $U(1)$ symmetry in the boundary field theory is global and there is no clear holographic description of how to add gauge fields.

An important particularity of the model is the missing backreaction. Since the metric fluctuations are set to zero this means that effectively there is no energy momentum tensor in the field theory dual. In particular the generators of translations and rotations are missing in the operator algebra. This does of course not mean that the model does not have these symmetries, they are however not realized as inner automorphisms of the operator algebra (they are still outer automorphisms). Besides the space time symmetries being realized as outer automorphisms there is a direct consequence of this in what concerns the hydrodynamics of the model: all hydrodynamic modes related to them are missing. There is no shear mode for the momentum diffusion and no sound mode for the energy transport. The hydrodynamic modes we find in the model are therefore only due to the presence of the $U(1)$ symmetry and its spontaneous breakdown.

Hydrodynamics can be understood as an effective field theory defined by the continuity equations of the conserved currents and so-called constitutive relations which encode the dissipative behavior of the system. The constitutive relations tell us how fast a current is built up due to gradients in the charge density or due to external fields. The constitutive relations depend on transport coefficients such as viscosity or conductivity. Transport coefficients can be divided into absorptive and reactive ones depending on whether they are odd or even under time reversal [32]. A typical example for an absorptive transport coefficient is the diffusion constant, an example for a reactive one is the speed of sound. Reactive transport coefficients such as the speed of sound (the static susceptibility is another example) can often be computed from purely thermodynamic considerations. This has been done for the speed of second sound in this model in [22] and for a variant of fourth sound in [23][1]. As we have argued before, the hydrodynamic modes of energy and momentum transport, shear and sound modes are missing. In the broken phase one expects however the appearance of a hydrodynamic mode with approximately linear dispersion relation for small momenta which represents the second sound present in superfluids. Indeed such a mode is bound to appear for each spontaneously broken continuous symmetry [32]. Our aim is to find the second sound mode directly as a pole in the holographic Green functions in the broken phase and to read off the speed of sound from its dispersion relation. We have found this mode numerically and our results for the speed of second sound agree (with numerical uncertainties) with the results in [22].

Below the critical temperature one expects actually only a part of the medium to be in the superfluid state whereas another part stays in the normal fluid phase. The fluid is a two component fluid and naively one might expect that this is reflected in the pole structure of the Green functions as the presence of the diffusive pole of the normal fluid component. As we will see, the hydrodynamic character of this diffusive pole is lost however below the critical temperature. We find a pole with purely imaginary frequencies obeying a dispersion relation roughly of the form $\omega = -i\gamma - iDk^2$. The gap γ goes to zero at the T_c such that at the critical temperature this mode goes over into the usual diffusive mode of the normal fluid.

Our goal in the following is to establish the presence of the second sound pole in the holographic superconductor model [5]. In section two we introduce the model and describe its properties. We compute the condensate as a function of temperature. This is basically a review of the results of [5] except the fact that we choose to work in the grand canonical

[1]Note that in [22] the mode has been called second sound whereas in [23] it was argued that it should be rather called fourth sound. There it has been argued that it is the fourth sound that survives the probe limit. In any case working directly in the probe limit we only have one sound mode. For convenience we will refer to it as second sound. Disentangling first, second or fourth sound would need to take into account the backreaction which is beyond the scope of this paper.

ensemble where we hold fixed the value of the chemical potential instead of the value of the charge density.

In section three we compute the quasinormal modes of the complex scalar field in the unbroken phase. As expected we find that at the critical temperature a quasinormal frequency crosses over into the upper half of the complex frequency plane. Since a pole in the upper half plane is interpreted as an instability this is an indication that the scalar field condenses. The quasinormal modes of gauge fields in the four dimensional AdS black hole have been studied before [34]. In particular the longitudinal gauge field channel shows a diffusion pole with diffusion constant $D = 3/(4\pi T)$. The holographic Green functions for gauge fields are often calculated in a formalism that employs gauge invariant variables, i.e. the electric field strengths. For reasons explained in section four we prefer however to work directly with the gauge fields. The longitudinal components obey two coupled differential equations subject to a constraint and we show in full generality how to compute the holographic Green function in such a situation. The quasinormal mode condition boils down to setting a determinant of field values at the boundary of AdS to zero. We find that this determinant is proportional to the electric field strength exemplifying that the poles of holographic Green functions are gauge invariant as expected on general grounds.

Section four is the core of our paper. Here we study the lowest modes of the quasinormal mode spectrum in the superfluid phase. We find that the longitudinal gauge fields at non zero momentum couple to real and imaginary part of the scalar field fluctuations, building up a system of four coupled differential equations subject to one constraint. We solve this system numerically and compute the quasinormal modes from our determinant condition. We find hydrodynamic modes with approximately linear real part of the dispersion relation. We compute the speed of second sound from it and find our results to be in good numerical agreement with what was found in [22] from thermodynamic considerations. The second sound pole has however also an imaginary part and we can fit the dispersion relation (for small momenta) to $\omega = \pm v_s k - i\Gamma_s k^2$ which allows us to read off the attenuation constant Γ_s of second sound. We also find a purely imaginary mode with a dispersion relation of the form $\omega = -i\gamma - iDk^2$ with D the diffusion constant. It is a sort of gapped diffusion mode. The gap γ goes to zero for $T \to T_c$. A simple two fluid model suggests that there is still a normal fluid component and in it charges should diffuse in the usual way right below T_c. Therefore we expect a diffusive pole to show up in the two-point function. The diffusive behavior is modified however at long wavelength by the presence of the gap γ. This mode is therefore not really a hydrodynamic mode.

We close this work with section five where we summarize and discuss our results.

2. The Model

As in [5] we consider a four dimensional planar AdS black hole with line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2)\,. \tag{2.1}$$

the blackening factor is $f(r) = \frac{r^2}{L^2} - \frac{M}{r}$. This metric has a horizon at $r_H = M^{1/3}L^{2/3}$, the Hawking temperature is $T = \frac{3}{4\pi}\frac{r_H}{L^2}$. In the following we will rescale coordinates according to

$$\begin{pmatrix} r \\ t \\ x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} r_H \rho \\ L^2/r_H \, \bar{t} \\ L^2/r_H \, \bar{x} \\ L^2/r_H \, \bar{y} \end{pmatrix} \tag{2.2}$$

In the new dimensionless coordinates the metric takes the form (2.1) with $M = 1$ times the overall AdS scale L^2.

We take an abelian gauge model with a massive charged scalar field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m^2\Psi\bar{\Psi} - (\partial_\mu\Psi - iA_\mu\Psi)(\partial^\mu\bar{\Psi} + iA^\mu\bar{\Psi}) \tag{2.3}$$

and a tachyonic mass $m^2 = -2/L^2$ above the Breitenlohner-Friedmann bound. As in [5] we ignore the backreaction of these fields onto the metric. We seek solutions for which the time component of the gauge field vanishes at the horizon and takes a non-zero value μ on the boundary. This value can be interpreted as the chemical potential. The boundary condition on the horizon is usually justified by demanding that the gauge field has finite norm there. Here this can be seen as follows: as in [22] we can chose a gauge that removes the phase of the scalar field Ψ from the equations of motion. In this gauge the scalar field current becomes $J_\mu = \psi^2 A_\mu$. Therefore the value of A_μ is directly related to a physical quantity, the current, and it is a well defined physical condition to demand the current to have finite norm at the horizon. This is achieved by taking the scalar field to be regular and the gauge field to vanish at the horizon.

In addition to the gauge field the scalar field might be non trivial as well. In fact for high chemical potential (low temperature) the scalar field needs to be switched on in order to have a stable solution. In our study of the quasinormal modes in the next section we will indeed see that a quasinormal mode crosses into the upper half plane at the critical temperature. We denote the temporal component of the gauge field in the dimensionless coordinates by Φ. The field equations for the background fields are

$$\Psi'' + (\frac{f'}{f} + \frac{2}{\rho})\Psi' + \frac{\Phi^2}{f^2}\Psi + \frac{2}{L^2 f}\Psi = 0\,, \tag{2.4}$$

$$\Phi'' + \frac{2}{\rho}\Phi' - \frac{2\Psi^2}{f}\Phi = 0\,. \tag{2.5}$$

The equations can be solved numerically by integration from the horizon out to the boundary. As we just argued for the current to have finite norm at the horizon we have to chose $\Phi(1) = 0$ and demand the scalar field to be regular at the horizon. These conditions leave two integration constants undetermined. The behavior of the fields at the conformal boundary is

$$\Phi = \bar{\mu} - \frac{\bar{n}}{\rho} + O(\frac{1}{\rho^2})\,, \tag{2.6}$$

$$\Psi = \frac{\psi_1}{\rho} + \frac{\psi_2}{\rho^2} + O(\frac{1}{\rho^2})\,. \tag{2.7}$$

The value of the mass of the scalar field chosen allows to define two different theories due to the fact that both terms in the expansion above are normalizable in AdS. The canonical choice of what one considers to be the normalizable mode gives a theory in which ψ_1 is interpreted as a coupling and ψ_2 as expectation value of an operator with mass dimension two. On the other hand one might consider ψ_2 as the coupling and ψ_1 as the expectation value of an operator of dimension one.

All our numerical calculations are done using the dimensionless coordinates. In order to relate the asymptotic values (2.6), (2.7) to the physical quantities we note that

$$\bar{\mu} = \frac{3L}{4\pi T}\mu\,, \tag{2.8}$$

$$\bar{n} = \frac{9L}{16\pi^2 T^2}n\,, \tag{2.9}$$

$$\psi_1 = \frac{3}{4\pi T L^2}\langle O_1\rangle\,, \tag{2.10}$$

$$\psi_2 = \frac{9}{16\pi^2 T^2 L^4}\langle O_2\rangle\,, \tag{2.11}$$

where μ is the chemical potential, n the charge density and $\langle O_i\rangle$ are the vacuum expectation values of the operators sourced by the scalar field. From now on we will set $L = 1$ and work in the grand canonical ensemble by fixing $\mu = 1$. Different values for $\bar{\mu}$ can now be interpreted as varying the temperature T. For high temperatures the scalar field is trivial and the gauge field equation is solved by $\Phi = \bar{\mu} - \frac{\bar{\mu}}{\rho}$. Spontaneous symmetry breaking means that an operator has a non trivial expectation value even when no source for the operator is switched on. We therefore look for nontrivial solutions of the scalar field with either $\psi_1 = 0$ or $\psi_2 = 0$. Numerically we find that a non-trivial scalar field is switched on at $\bar{\mu} = 1.1204$ corresponding to a critical temperature $T_c = 0.213\mu$ for the operator O_1 and at $\bar{\mu} = 4.0637$ corresponding to a critical temperature of $T_c = 0.0587\mu$ for the operator O_2. We chose to plot the squares of the condensates as a function of reduced temperature. It

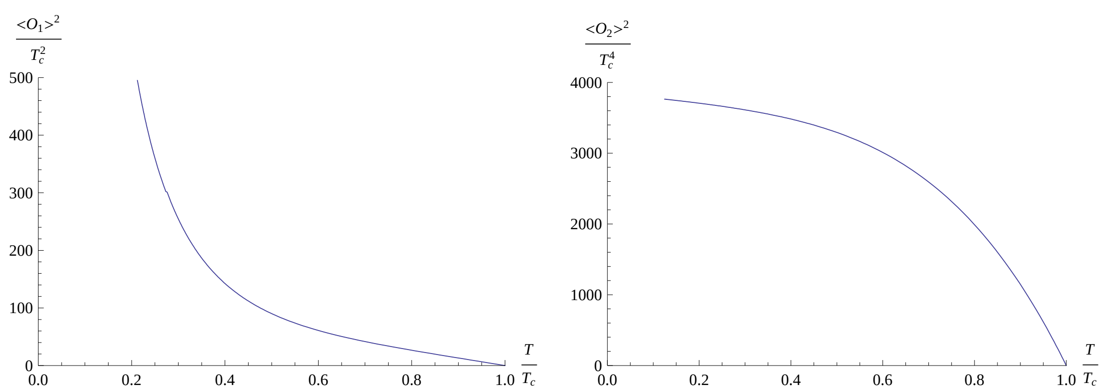

Figure 1: The condensates as function of the temperature in the two possible theories.

makes the linear behavior for temperatures just below the critical one manifest.

$$\langle O_i\rangle^2 \propto \left(1 - \frac{T}{T_c}\right)\,. \tag{2.12}$$

Since we want to compute the holographic two point functions we will have to expand the action to second order in field fluctuations around the background. We divide the fields into background plus fluctuations in the following way

$$\Psi = \psi(\rho) + \sigma(\rho,t,x) + i\eta(\rho,t,x)\,, \tag{2.13}$$

$$A_\mu = \mathcal{A}_m(\rho) + a_\mu(\rho,t,x)\,. \tag{2.14}$$

The gauge transformations act only on the fluctuations

$$\delta a_\mu = \partial_\mu \lambda\,, \tag{2.15}$$

$$\delta\sigma = -\lambda\eta\,, \tag{2.16}$$

$$\delta\eta = \lambda\sigma + \lambda\psi\,. \tag{2.17}$$

The action expanded out to second order is $S = S^{(0)} + S^{(1)} + S^{(2)}$

$$\begin{aligned}
S^{(0)} &= \int \sqrt{-g}\left(-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{2}{L^2}\psi^2 - (\partial_\mu\psi)(\partial^\mu\psi) - \mathcal{A}_\mu\mathcal{A}^\mu\psi^2\right)\,,\\
S^{(1)} &= \int \sqrt{-g}\left(-\frac{1}{2}\mathcal{F}_{\mu\nu}f^{\mu\nu} + \frac{4}{L}^2\psi\sigma - 2\partial_\mu\psi\partial^\mu\sigma - 2\partial_\psi\mathcal{A}^\mu\eta - 2\mathcal{A}^2\psi\sigma + 2\mathcal{A}_\mu\partial^\mu\eta - 2\mathcal{A}_\mu a^\mu\psi^2\right)\,,\\
S^{(2)} &= \int \sqrt{-g}\left(-\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - (\partial\sigma)^2 - (\partial\eta)^2 - \mathcal{A}^2\sigma^2 - \mathcal{A}^2\eta^2 + \frac{2}{L}^2\sigma^2 + \frac{2}{L}^2\eta^2 - 2\partial_\mu\psi a^\mu\eta\right.\\
&\quad\left. -2\mathcal{A}^\mu a_\mu\psi\sigma - 2\partial_\mu\sigma\mathcal{A}^\mu\eta + 2\mathcal{A}_\mu\sigma\partial\mu\eta + 2\partial_\mu\eta a^\mu\psi - a^2\psi^2\right)\,.
\end{aligned} \tag{2.18}$$

Up to the equations of motion these can be written as boundary terms

$$\begin{aligned}
S_B^{(1)} &= \int_B \sqrt{-g}\left(g^{\rho\rho}\mathcal{F}_{\rho\mu}a^\mu - 2g^{\rho\rho}\partial_\rho\psi\sigma + 2A^\rho\psi\eta\right)\,,\\
S_B^{(2)} &= -\int_B \sqrt{-g}g^{\rho\rho}\left(\frac{1}{2}g^{\nu\lambda}f_{\rho\nu}a_\lambda + \eta\partial_\rho\eta + \sigma\partial_\rho\sigma + a_\rho\psi\eta\right)\,.
\end{aligned} \tag{2.19}$$

Note that $S^{(1)}$ is not trivial since according to the holographic dictionary it has to encode the non-vanishing expectation values of the field theory operators.

3. Quasinormal Frequencies in the Unbroken Phase

We assume now that the scalar field Ψ has vanishing background value and take the field to depend on t, x, ρ. Frequency ω and momentum k in the dimensionless coordinates are related to the physical ones ω_{ph}, k_{ph} as

$$\omega = \frac{3\omega_{ph}}{4\pi T} \quad , \quad k = \frac{3k_{ph}}{4\pi T}\,. \tag{3.1}$$

The equations of motion in the unbroken phase are

$$
\begin{aligned}
0 &= \Psi'' + (\frac{f'}{f} + \frac{2}{\rho})\Psi' + (\frac{(\Phi+\omega)^2}{f^2} + \frac{2}{f} - \frac{k^2}{f\rho^2})\Psi\,, \\
0 &= a_t'' + \frac{2}{\rho}a_t' - \frac{k^2}{\rho^2}a_t - \frac{\omega k}{f\rho^2}a_x\,, \\
0 &= a_x'' + \frac{f'}{f}a_x' + \frac{\omega^2}{f^2}a_x + \frac{\omega k}{f\rho^2}a_t\,, \\
0 &= \frac{\omega}{f}a_t' + \frac{k}{\rho^2}a_x'\,.
\end{aligned}
\tag{3.2}
$$

The equation of motion for the complex conjugate scalar $\bar{\Psi}$ can be obtained by changing the sign of the gauge field background Φ in (3.2).

3.1 Green Functions

In order to calculate the quasinormal frequencies we impose ingoing boundary conditions at the horizon. Since the coefficients of the differential equations (3.2) are known analytically and are such that they are of Fuchsian type we can use the Frobenius method to approximate the solutions at the horizon and at the boundary by series expansions. As is well-known the holographic Green functions are proportional to the ratio of the connection coefficients. More precisely we demand

$$
\Psi_H = (\rho - 1)^{-i\omega/3}(1 + O(\rho - 1))\,, \tag{3.3}
$$

on the horizon and write the local solution at the AdS boundary as

$$
\Psi_B = \frac{A}{\rho} + \frac{B}{\rho^2} + \mathrm{O}\left(\frac{1}{\rho^3}\right)\,. \tag{3.4}
$$

In the theory with the dimension two operator we take A as the coefficient of the non-normalizable mode and B as the coefficient of the normalizable mode. Writing the local solution on the horizon as a linear combination of normalizable and non-normalizable modes on the boundary fixes the connection coefficient A and B. We have written the boundary action as a functional of real and imaginary part of the scalar field. We will rewrite the boundary action now in terms of the complex scalar Ψ and its conjugate. In addition we introduce a local boundary counterterm to regularize the action.

$$
S_\Psi^B = \int \left[-\frac{1}{2} f\rho^2(\bar{\Psi}\Psi' + \Psi\bar{\Psi}') - \rho^3\bar{\Psi}\Psi \right]_{\rho=\Lambda}\,. \tag{3.5}
$$

This allows to compute the Green functions $G_{\bar{O}O}(q) = \langle \bar{O}(-q)O(q)\rangle$ and $G_{O\bar{O}}(q) = \langle O(-q)\bar{O}(q)\rangle$ fulfilling $G_{O\bar{O}}(-q) = G_{\bar{O}O}(q)$, with q being the four momentum (ω, k). We write $\Psi(q,\rho) = \Psi_0(q)f_q(\rho)$, where we interpret $\Psi_0(q)$ as the source that inserts the operator $O(q)$ in the dual field theory. We introduce a cutoff and normalize the profile function f_q to $1/\Lambda$ at the cutoff. In terms of an arbitrary solution the normalized one is

$f_k(\rho) = \Psi_k(\rho)/(\Lambda\Psi(\Lambda))$ where $\Psi_k(\rho)$ has the boundary expansion (3.4). The boundary action is now

$$
\begin{aligned}
S^B &= -\frac{1}{2}\int \left[\Psi_0(-q)(\rho^2 f f_{-q}\bar{f}'_q + \rho^3 f_q \bar{f}_q)\bar{\Psi}_0(q) + \bar{\Psi}_0(-q)(\rho^2 f \bar{f}_{-q} f'_q + \rho^3 \bar{f}_{-q} f_q)\Psi_0(q)\right]_{\rho=\Lambda} \\
&= \int \Psi_0(-q)\mathcal{F}_{\Psi\bar{\Psi}}(\Lambda)\bar{\Psi}_0(q) + \bar{\Psi}_0(-q)\mathcal{F}_{\bar{\Psi}\Psi}(\Lambda)\Psi_0(q)\,. \qquad (3.6)
\end{aligned}
$$

According to the holographic dictionary the renormalized retarded Green functions are given by the limit $\lim_{\Lambda\to\infty} -2\mathcal{F}(\Lambda)$:

$$
G_{\bar{O}_2 O_2} = \frac{B}{A} \quad , \quad G_{O_2 \bar{O}_2} = \frac{\bar{B}}{\bar{A}}\,. \qquad (3.7)
$$

We denote the connection coefficients for the complex conjugate scalar as $\bar{A}$, $\bar{B}$[2]. According to (3.2) they can be obtained by switching the sign of the chemical potential in the expressions for A, B.

The theory with the operator of dimension one can be obtained through a Legendre transform. We note that the expectation value of the dimension two operator is $\langle O_2\rangle = -\left.\rho^2(\rho\Psi(\rho))'\right|_{\rho=\Lambda}$, whereas the source is given by $\Lambda\Psi(\Lambda)$. We therefore add the following terms to the boundary action

$$
S^B \to S^B + \int d^4k \left.\left[\rho^3(\bar{\Psi}(\rho\Psi(\rho))' + \Psi(\rho\bar{\Psi}(\rho))'\right]\right|_{\rho=\Lambda}\,. \qquad (3.8)
$$

Now we can evaluate the Green functions as before, the only difference being the normalization of the profile function $f_k(\rho) = \Psi_k(\rho)/[\Lambda\Psi'(\Lambda) + \Psi(\Lambda)]$. This normalization takes care that the term of order $1/\rho^2$ in the boundary expansion couples with unit strength to the source $\Psi_0(q)$. We find finally for the Green functions of the Legendre transformed theory

$$
G_{O_1\bar{O}_1} = \frac{A}{B} \quad , \quad G_{\bar{O}_1 O_1} = \frac{\bar{A}}{\bar{B}}\,, \qquad (3.9)
$$

as expected. The quasinormal modes in the scalar sector are given by the zeroes of the connection coefficients A and $\bar{A}$ in the theory with operator of dimension two and by the zeroes of B and $\bar{B}$ in the theory with the dimension one operator. In terms of the unnormalized solutions to the field equations we can write the Green functions as

$$
G_{\bar{O}_2 O_2} = -\lim_{\Lambda\to\infty}\left(\Lambda^2\frac{\Psi'_q(\Lambda)}{\Psi_q(\Lambda)} + \Lambda\right)\,, \qquad (3.10)
$$

$$
G_{O_1\bar{O}_1} = \lim_{\Lambda\to\infty}\frac{\Psi_q(\Lambda)}{\Lambda(\Lambda\Psi'_q(\Lambda) + \Psi_q(\Lambda))}\,. \qquad (3.11)
$$

3.2 Quasinormal Modes from Determinants

Before presenting the results for the quasinormal modes of the scalar field we would like to outline a method of how to calculate the holographic Green functions for the gauge fields without using gauge invariant variables such as the electric field strength $E = -i(ka_t + \omega a_x)$.

[2] Note that the infalling boundary condition for the conjugate scalar is $\bar{\Psi} \sim (\rho - 1)^{-i\omega/3}$.

The complicated structure of the gauge symmetry in the broken phase makes it rather difficult to express the boundary action in terms of gauge invariant field combinations. As a warm up for the problem of how to calculate the holographic Green functions in this situation we will consider how we can calculate them in the unbroken phase directly in terms of the gauge fields. We necessarily have to solve a system of coupled differential equations whose solutions are restricted by a constraint.

The correct boundary conditions for the gauge fields on the horizon are

$$a_t \propto (\rho - 1)^{1-i\omega/3}(a_t^0 + \dots)\,, \tag{3.12}$$

$$a_x \propto (\rho - 1)^{-i\omega/3}(a_x^0 + \dots)\,. \tag{3.13}$$

The two coefficients a_x^0 and a_t^0 are not independent but related by the constraint. At this point we have fixed the incoming wave boundary conditions and there seems to be now a unique solution to the field equations. We would expect however two linearly independent solutions with infalling boundary conditions on the horizon. The constraint reduces this to only one solution. The problem is now that in order to compute the Green function for the charge density and longitudinal current component separately we need solutions that asymptote to $(a_t, a_x) = (1, 0)$ and $(a_t, a_x) = (0, 1)$ respectively. This is of course not possible with only one available solution at the horizon. Because of the gauge symmetry the gauge field system (3.2) allows for an algebraic solution

$$a_t = -\omega\lambda\,, \tag{3.14}$$

$$a_t = k\lambda\,, \tag{3.15}$$

with $\lambda' = 0$, i.e. λ being independent of ρ. This is of course nothing but a gauge transformation of the trivial solution. Remember that even after fixing the radial gauge $a_\rho = 0$, gauge transformations with gauge parameters independent of ρ are still possible. These gauge transformations appear as algebraic solutions to the field equations. We also stress that the infalling boundary conditions really have to be imposed only on physical fields, i.e. the electric field strength. Having therefore an arbitrary non trivial gauge field solution corresponding to an electric field with infalling boundary conditions we can add to it the gauge mode (3.14).

We can use this to construct a basis of solutions that allows the calculation of the holographic Green functions. Let us now assume that there is a solution that takes the values $(a_t, a_x) = (1, 0)$ at the boundary. We will call this solution from now on α_i^t for $i \in t, x$. Analogously we define the solution α_i^x. According to the holographic dictionary the solution α_i^t couples to the boundary value $\lim_{\rho\to\infty} a_t(q, \rho) = \mathcal{A}_t(q)$, i.e. the source of the field theory operator J_t (the time component of the conserved current J_μ). In parallel α_i^x couples to the boundary value $\mathcal{A}_x(q)$. A generic solution of the gauge field equations can now be written in terms of the boundary fields as

$$a_i(q, \rho) = \mathcal{A}_x(q)\alpha_i^x(\rho) + \mathcal{A}_t(q)\alpha_i^t(\rho)\,. \tag{3.16}$$

Using this expansion the boundary action can be written as

$$S_B = \frac{1}{2}\int_B \mathcal{A}_i(-q)\left[\left(\rho^2\alpha^i_t(-q,\rho)\frac{d}{d\rho}(\alpha^j_t(q,\rho)) - f(\rho)\alpha^i_x(-q,\rho)\frac{d}{d\rho}(\alpha^j_x(q,\rho))\right)\right]_{\rho=\Lambda}\mathcal{A}_j(q)\,, \tag{3.17}$$

where again we have introduced a cutoff at $\rho=\Lambda$. From this it follows that the holographic Green functions are given by

$$2\mathcal{F}^{ij}(\rho) = \rho^2\alpha^i_t(-k,\rho)\frac{d}{d\rho}(\alpha^j_t(k,\rho)) - f(\rho)\alpha^i_x(-k,\rho)\frac{d}{d\rho}(\alpha^j_x(k,\rho))\,, \tag{3.18}$$

in the limit

$$G^{ij} = \lim_{\Lambda\to\infty} -2\mathcal{F}^{ij}(\Lambda)\,. \tag{3.19}$$

Notice also that $\frac{d}{d\rho}[\mathcal{F}^{ij}(\rho) - \mathcal{F}^{*ji}(\rho)] = 0$ by the field equations.

Although there are no terms of the form $a'_t a_x$ in the boundary action this formalism gives automatically expressions for the mixed Green functions G_{tx} and G_{xt}! But we still have to construct the solutions $\alpha^i_j(\rho)$. This can be done in the following way: suppose we have an arbitrary solution $(a_t(\rho), a_x(\rho))$ obeying the infalling boundary conditions (3.12). We can add to this now an appropriate gauge mode, such that at the cutoff the solution takes the form $(1,0)$ or $(0,1)$ in terms of $a_t(\Lambda), a_x(\Lambda)$. This is easily achieved by solving

$$\begin{pmatrix} c^t_t & c^t_x \\ c^x_t & c^x_x \end{pmatrix}\begin{pmatrix} a_t(\Lambda) & a_z(\Lambda) \\ -\omega\lambda & k\lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{3.20}$$

The linear combinations formed with the coefficients c^i_j give now new solutions $(\alpha^t_t(\rho), \alpha^t_x(\rho))$ and $(\alpha^x_t(\rho), \alpha^x_x(\rho)))$ obeying the correct boundary conditions on the AdS boundary. Using the general expression for the Green function (3.19) we get explicitly in terms of solutions obeying the infalling boundary conditions

$$G_{tt} = \lim_{\Lambda\to\infty}\Lambda^2\frac{k a'_t(\Lambda)}{k a_t(\Lambda) + \omega a_x(\Lambda)}\,, \tag{3.21}$$

$$G_{tx} = \lim_{\Lambda\to\infty}\Lambda^2\frac{\omega a'_t(\Lambda)}{k a_t(\Lambda) + \omega a_x(\Lambda)}\,, \tag{3.22}$$

$$G_{xt} = -\lim_{\Lambda\to\infty}\Lambda^2\frac{k a'_x(\Lambda)}{k a_t(\Lambda) + \omega a_x(\Lambda)}\,, \tag{3.23}$$

$$G_{xx} = -\lim_{\Lambda\to\infty}\Lambda^2\frac{\omega a'_x(\Lambda)}{k a_t(\Lambda) + \omega a_x(\Lambda)}\,. \tag{3.24}$$

Note that the denominator for all is given by $k a_t(\Lambda) + \omega a_x(\Lambda)$ which is up to an irrelevant constant nothing but the gauge invariant electric field E_x. Therefore we see immediately that the poles of these Green function are gauge invariant and coincide of course with the poles of the Green function in the gauge invariant formalism where $G \propto \frac{E'_x}{E_x}$. Indeed using the constraint on the boundary we find the well known expressions [35]

$$G_{tt} = \frac{k^2}{k^2-\omega^2}\lim_{\Lambda\to\infty}\Lambda^2\frac{E'_x}{E_x}\quad,\quad G_{tx} = \frac{k\omega}{k^2-\omega^2}\lim_{\Lambda\to\infty}\Lambda^2\frac{E'_x}{E_x}\,,$$
$$G_{xx} = \frac{\omega^2}{k^2-\omega^2}\lim_{\Lambda\to\infty}\Lambda^2\frac{E'_x}{E_x}\,. \tag{3.25}$$

On general grounds one expects indeed that the poles of the holographic Green functions for gauge fields are gauge independent.

If we are interested only in the location of quasinormal frequencies we do not even have to construct the holographic Green functions explicitly. From the linear system in (3.20) we infer that the quasinormal frequencies coincide with the zeroes of the determinant of the field values at the boundary. Indeed vanishing determinant means that there is a nontrivial zero mode solution to (3.20) such that the boundary values of the fields are $(0, 0)$ which in turn means that the coefficient in the solution of the non-normalizable mode vanishes. The determinant is $\lambda(ka_t + \omega a_x)$ and again given by the electric field strength. In fact these remarks apply to systems of coupled differential equations in AdS black hole metric in general: the quasinormal frequencies corresponding to the poles of the holographic Green functions are the zeroes of the determinant of the field values on the boundary for a maximal set of linearly independent solutions obeying infalling boundary conditions on the horizon. The fact that the differential equations are coupled is the holographic manifestation of mixing of operators under the RG flow. Therefore one has to specify at which scale one is defining the operators. The scheme outlined above is dual to define the operators at the cutoff Λ.

3.3 Hydrodynamic and higher QNMs

We have numerically computed the quasinormal frequencies for the fluctuations satisfying the equations of motion (3.2) for both the O_2 and the O_1 theories. The quasinormal modes of the scalar field correspond to zeroes of A in the theory of dimension two operator and to zeroes of B in the dimension one operator theory, where A and B are the connection coefficients of the boundary solution (3.4). Results for the lowest three poles of the scalar field at zero momentum are shown in Figure 2.

The poles with positive real part correspond to the quasinormal modes of the complex scalar Ψ, while those with negative real part are the quasinormal modes of $\bar{\Psi}$, obtained by changing the overall sign of the gauge field background Φ. As the temperature is decreased, the poles get closer to the real axis, until at the critical temperature T_c the lowest mode crosses into the upper half of the complex frequency plane. It happens at $T_c = 4.0637$ for the theory of dimension two operator and at $T_c = 1.1204$ in the case of the dimension one operator theory. For $T < T_c$ the mode would become tachyonic, i. e. unstable. This instability indicates that the scalar field condenses and the system undergoes a phase transition at $T = T_c$. At the critical point, the lowest scalar quasinormal mode is a hydrodynamic mode in the sense that it is massless, $\lim_{k\to 0}\omega(k) = 0$. This mode is identified with the Goldstone boson that appears after the spontaneous breaking of the global $U(1)$ symmetry and in the next section we will see that it evolves into the second sound mode characteristic of superfluid models.

The quasinormal modes correspond to simple poles of the retarded Green function, so close to the nth pole the Green function can be approximated by $G(\omega, k, T) \sim \frac{R_n(\omega,k,T)}{\omega-\omega_n(k,T)}$. Knowing the connection coefficients we can compute the Green functions and therefore the residue for each quasinormal mode as explained in [36, 37]. For the lowest quasinormal mode at $k = 0$ and at the critical temperature, the residue takes the value $R_2(T_c) =$

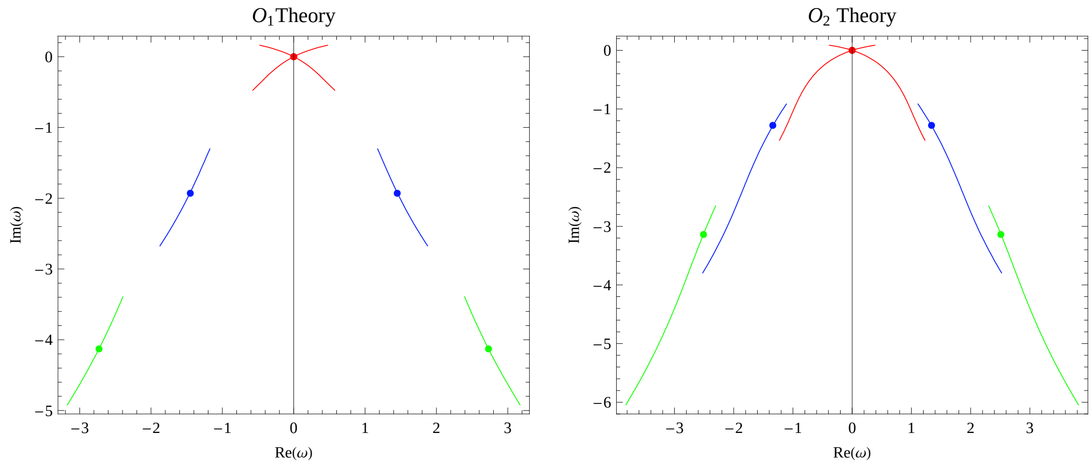

Figure 2: Lowest scalar quasinormal frequencies as a function of the temperature and at momentum $k = 0$, from $T/T_c = \infty$ to $T/T_c = 0.81$ in the O_2 theory (right) and to $T/T_c = 0.56$ in the O_1 theory (left). The dots correspond to the critical point $T/T_c = 1$ where the phase transition takes place. Red, blue and green correspond to first, second and third mode respectively.

$-2.545 + 0.825i$ in the O_2 theory and $R_1(T_c) = 0.686 - 0.348i$ in the O_1 theory. In general, one expects the residues of hydrodynamic modes that correspond to conserved quantities of the system to vanish in the limit of zero momentum, since its susceptibility remains constant. Consider for instance the diffusion mode associated to conserved density. The susceptibility is defined through the two point correlation function as

$$\chi = \lim_{k,\omega\to 0} \langle \rho\rho \rangle = \lim_{k,\omega\to 0} \frac{i\sigma k^2}{\omega + iDk^2} = \frac{\sigma}{D}\,, \tag{3.26}$$

where D is the diffusion constant and σ is the conductivity. The residue, $i\sigma k^2$, vanishes and one recovers the well-known Einstein relation $\sigma = D\chi$. However, for hydrodynamic modes appearing at second order phase transitions the order parameter susceptibility should diverge at the critical point. This order parameter susceptibility is given in our case by the correlator of the boundary operator sourced by the scalar field. At the critical temperature it is

$$\chi_{\bar{O}_i O_i} = \lim_{k,\omega\to 0} \langle \bar{O}_i O_i \rangle = \lim_{k,\omega\to 0} \frac{R_i(k, T_c)}{\omega - \omega_H(k, T_c)} \to \infty \tag{3.27}$$

since $\omega_H(0, T_c) = 0$ while the residue remains finite. This result allows us to identify the lowest scalar quasinormal mode in the unbroken phase with the Goldstone boson appearing at the critical point.

In the model under consideration one can also compute the gauge field fluctuations in the normal phase. Nevertheless, as the model does not include the backreaction of the metric, the computation is not sensitive to temperature anymore. This can be seen from the equations of motion (3.2) of the gauge fluctuations, that do not depend on the background solutions thus are independent of the temperature. Hence we recover the results for the quasinormal modes of vector field perturbations in the AdS_4 black hole background computed by [34]. For our purposes the most important fact is the presence of a hydrodynamic mode corresponding to diffusion. For small momenta that mode has dispersion relation $\omega = -iDk^2$ with $D = 1$ (which is $D = 3/(4\pi T)$ in physical units). In order to study the behavior of the diffusion pole in the unbroken phase as a function of the temperature one has to consider the backreacted model described in [38].

4. Quasinormal Frequencies in the Broken Phase

In this section we will apply our determinant method for finding quasinormal modes of a coupled system of field equations. With this technique we follow the model analyzed in the previous section into its broken phase. Figure 3 schematically summarizes our analysis. The two formerly separate sets of scalar (grey dots) and longitudinal vector poles (black dots) present in the unbroken phase where the scalars and vectors decouple, are now unified into one inseparable pole structure in the coupled system. This is in analogy to a coupled system of two harmonic oscillators in which it makes no sense to ask for the eigenfrequencies of the single oscillators. One could of course try to diagonalize the system of differential equations, in our case however this looks rather complicated and we prefer to work directly with the coupled system and with the gauge fields instead of gauge invariant variables. The lowest modes (see figure 3) are two hydrodynamic second sound modes originating from the two lowest scalar quasinormal modes in the unbroken phase. In addition we will find a non-hydrodynamic pseudodiffusion mode staying on the imaginary axis in the range of momenta we consider. This mode can be thought of as the prolongation of the diffusion mode into the unbroken phase.

4.1 Application of the determinant method

The equations of motion in the broken phase couple the scalar fluctuations η, σ to the longitudinal vector components a_t, a_x

$$0 = f\eta'' + \left(f' + \frac{2f}{\rho}\right)\eta' + \left(\frac{\phi^2}{f} + \frac{2}{L^2} + \frac{\omega^2}{f} - \frac{k^2}{\rho^2}\right)\eta - \frac{2i\omega\phi}{f}\sigma - \frac{i\omega\psi}{f}a_t - \frac{ik\psi}{r^2}a_x\,, \tag{4.1}$$

$$0 = f\sigma'' + \left(f' + \frac{2f}{\rho}\right)\sigma' + \left(\frac{\phi^2}{f} + \frac{2}{L^2} + \frac{\omega^2}{f} - \frac{k^2}{\rho^2}\right)\sigma + \frac{2\phi\psi}{f}a_t + \frac{2i\omega\phi}{f}\eta\,, \tag{4.2}$$

$$0 = f{a_t}'' + \frac{2f}{\rho}{a_t}' - \left(\frac{k^2}{\rho^2} + 2\psi^2\right)a_t - \frac{\omega k}{\rho^2}a_x - 2i\omega\psi\,\eta - 4\psi\phi\,\sigma\,, \tag{4.3}$$

$$0 = f{a_x}'' + f'{a_x}' + \left(\frac{\omega^2}{f} - 2\psi^2\right)a_x + \frac{\omega k}{f}a_t + 2ik\psi\,\eta\,. \tag{4.4}$$

This system of four coupled equations is subject to the constraint

$$\frac{\omega}{f}{a_t}' + \frac{k}{\rho^2}{a_x}' = 2i\left(\psi'\,\eta - \psi\,\eta'\right)\,, \tag{4.5}$$

where the left hand side known from the unbroken phase is amended by the condensate terms on the right. Note that the real part σ of the scalar fluctuation is not involved in the constraint.

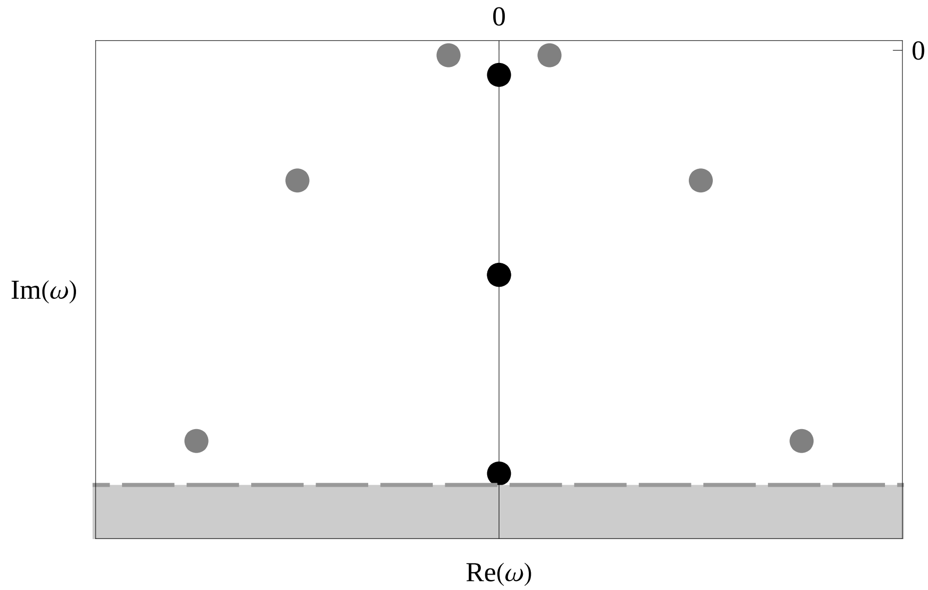

Figure 3: Schematic plot of the poles in the coupled system, i.e. in the broken phase at small finite momentum right below T_c. These poles are present in each retarded correlation function for the coupled fields η, σ, A_t, A_x, while their residues might vanish for specific fields. Close to the origin we find the (pseudo)diffusion mode and two hydrodynamic second sound modes. In addition two sets of higher (non-hydrodynamic) quasinormal modes are shown. In the unbroken phase these poles originate from the scalar (grey dots). We also expect a tower of purely imaginary poles stemming from the longitudinal vector channel (black dots). The grey area indicates where our present numerical methods break down. In this paper we will be concerned primarily with the hydrodynamic modes and will touch upon the higher quasinormal modes only briefly.

The constraint can be interpreted as the Ward identity of current conservation in the presence of the condensate. We expand the gauge fields near the boundary and note that $a_0' = \langle n\rangle/\rho^2$ and $a_x' = -\langle j_x\rangle/\rho^2$, where j_x is the x-component of the current. Expanding also the r.h.s. and comparing the leading orders in ρ we find

$$\partial_\mu \langle j^\mu\rangle = 2\langle O_i\rangle \eta_0^i \tag{4.6}$$

where η_0^i is the source for the insertion of the imaginary part of the operator O_i, i.e. the Goldstone field in the dual field theory. This equation is to be understood as the local Ward identity encoding current conservation in the presence of the condensate $\langle O_i\rangle$. It follows for example that the two point function of the divergence of the current with the operator $O_i^{(\eta)}$ is zero only up to a contact term $\langle \partial_\mu j^\mu(x) O_i^{(\eta)}(y)\rangle = \langle O_i\rangle \delta(x-y)$.

The gauge field component a_y being transverse to the momentum decouples from the above system and assumes the form

$$0 = f\, a_y{}'' + f' a_y{}' + \left(\frac{\omega^2}{f} - \frac{k^2}{\rho^2} - 2\psi^2\right) a_y\,. \tag{4.7}$$

Since we do not expect any hydrodynamic modes in the transverse vector channel we will not study this equation further.

Applying the indicial procedure to the system (4.1) to find the exponents for the singular and the coefficients for the regular parts of the fields, we obtain the following behavior at the horizon

$$\eta = (\rho-1)^{\zeta}\left(\eta^{(0)} + \eta^{(1)}(\rho-1) + \dots\right), \tag{4.8}$$

$$\sigma = (\rho-1)^{\zeta}\left(\sigma^{(0)} + \sigma^{(1)}(\rho-1) + \dots\right), \tag{4.9}$$

$$a_t = (\rho-1)^{\zeta+1}\left(a_t^{(0)} + a_t^{(1)}(\rho-1) + \dots\right), \tag{4.10}$$

$$a_x = (\rho-1)^{\zeta}\left(a_t^{(0)} + a_t^{(1)}(\rho-1) + \dots\right), \tag{4.11}$$

with the exponent $\zeta = -i\omega/3$ obeying the incoming wave boundary condition.

Due to the constraint we can choose only three of the four parameters at the horizon. Using the constraint and without loss of generality we can eliminate the time component a_t^0 and parametrize the solutions by $(\eta^{(0)}, \sigma^{(0)}, a_x^{(0)})$. We choose three linearly independent combinations I, II, III. A fourth solution can be found from the gauge transformations

$$\eta^{IV} = i\lambda\psi\,, \quad \sigma^{IV} = 0\,, \quad a_t^{IV} = \lambda\omega\,, \quad a_x^{IV} = -\lambda k\,. \tag{4.12}$$

with λ being an arbitrary constant with respect to ρ. It is not an algebraic solution to the equations of motion since η has non trivial dependence on the bulk variable ρ. The gauge solution solves the equations (4.1) not exactly but only up to terms proportional to the background equations (2.4).

Our goal is to find the poles in the retarded correlation functions of the four fields appearing in the coupled system of equations of motion (4.1). A convenient way of imposing the appropriate boundary conditions is given by redefining the scalar fields as

$$\tilde\eta(\rho) = \rho\eta(\rho) \quad , \quad \tilde\sigma(\rho) = \rho\sigma(\rho)\,. \tag{4.13}$$

Then the most general solution for each field $\varphi_i \in \{\tilde\eta, \tilde\sigma, a_t, a_x\}$ including gauge degrees of freedom can be written

$$\varphi_i = \alpha_1 {\varphi_i}^I + \alpha_2 {\varphi_i}^{II} + \alpha_3 {\varphi_i}^{III} + \alpha_4 {\varphi_i}^{IV}\,. \tag{4.14}$$

In the theory with the dimension two operator the sources for the various gauge invariant operators are given by $\varphi_i(\Lambda)$. We are interested in the quasinormal modes of the system (4.1) and as we have argued in the previous section these are the special values of the frequency where the determinant spanned by the values $\varphi_i^{I,II,III,IV}$ vanishes. Expanding this determinant we get

$$\begin{aligned} 0 &= \frac{1}{\lambda}\det\begin{pmatrix} {\varphi_\eta}^I & {\varphi_\eta}^{II} & {\varphi_\eta}^{III} & {\varphi_\eta}^{IV} \\ {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} & {\varphi_\sigma}^{IV} \\ \varphi_t^I & \varphi_t^{II} & \varphi_t^{III} & \varphi_t^{IV} \\ \varphi_x^I & \varphi_x^{II} & \varphi_x^{III} & \varphi_x^{IV} \end{pmatrix} \\ &= i\varphi_\eta^{IV}\det\begin{pmatrix} {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} \\ \varphi_t^I & \varphi_t^{II} & \varphi_t^{III} \\ \varphi_x^I & \varphi_x^{II} & \varphi_x^{III} \end{pmatrix} + \omega\det\begin{pmatrix} {\varphi_\eta}^I & {\varphi_\eta}^{II} & {\varphi_\eta}^{III} \\ {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} \\ \varphi_x^I & \varphi_x^{II} & \varphi_x^{III} \end{pmatrix} + k\det\begin{pmatrix} {\varphi_\eta}^I & {\varphi_\eta}^{II} & {\varphi_\eta}^{III} \\ {\varphi_\sigma}^I & {\varphi_\sigma}^{II} & {\varphi_\sigma}^{III} \\ \varphi_t^I & \varphi_t^{II} & \varphi_t^{III} \end{pmatrix}, \end{aligned} \tag{4.15}$$

which needs to be evaluated at the cutoff $\rho = \Lambda$. The first term in (4.15) vanishes at the cutoff since $\varphi_4^{IV} = \Lambda\psi = 0$ is just the condition that the operator O_2 is not sourced by the background.

We first find three linearly independent numerical solutions and then solve condition (4.15) numerically. Explicit checks confirm that the choice of a solution basis $\varphi^{I,II,III,IV}$ is completely arbitrary and does not change the results. Note also that in our present case all the remaining determinants can not be factorized. But if the momentum is set to zero, the only remaining term is the one with ω and the determinant factorizes into a scalar part and a vector part since the system of equations decouples.

In the theory with the dimension one operator the sources are given by $-\Lambda^2\tilde{\eta}$ and $-\Lambda^2\tilde{\sigma}$ for the scalar fields. Let us call $\varphi_1 = -\rho^2\tilde{\eta}'$ and $\varphi_2 = -\rho^2\tilde{\sigma}'$ in this case. The determinant has therefore the same form and again the first term vanishes due to the absence of sources for O_1 in the background solution. The quasinormal modes can again be found by integrating three arbitrary solutions with infalling boundary conditions from the horizon to the cutoff and finding numerically the zeroes of the determinant (4.15).

4.2 Hydrodynamic and Goldstone modes

Sound mode The scalar modes originally destabilizing the unbroken phase turn into Goldstone modes at T_c instead of becoming tachyonic. Below T_c they evolve into the two second sound modes. Figure 4 shows their movement when momentum is changed at different temperatures. Note that we focus on the positive real frequency axis because of the mirror symmetry sketched in figure 3. From the dispersion relation at small frequencies and long wavelengths we extract the speed of second sound v_s and the second sound attenuation Γ_s using the hydrodynamic equation

$$\omega = v_s k - i\Gamma_s k^2 \,. \tag{4.16}$$

It turns out that the hydrodynamic regime, i.e. that range of momenta in which the dispersion relation is well approximated by (4.16), is very narrow for temperatures just below the critical one since the speed of sound vanishes at T_c. Fits to the hydrodynamic form at a high temperature $T \approx 0.9999T_c$ are plotted in figure 5 for the O_2-theory, the results for the O_1 theory are qualitatively similar.

The speed of second sound is shown in figure 6. We have a good numerical agreement with the thermodynamic value of the second sound velocity given in [22]. This nicely confirms validity of our method. In particular, within our numerical precision we find that the value of the square of the speed of sound tends to $v_s^2 \approx 1/3$ in the O_1 theory and to $v_s^2 \approx 1/2$ in the O_2 theory[3] . However, the numerics becomes rather unstable for low temperatures, especially for the O_1 theory. Near but below the critical temperature we

[3]In [23] it was argued that conformal symmetry implies $v_s^2 = 1/2$ at zero temperature. Due to the divergence in the order parameter for the O_1 theory conformal symmetry could be broken and allow thus for a different value.

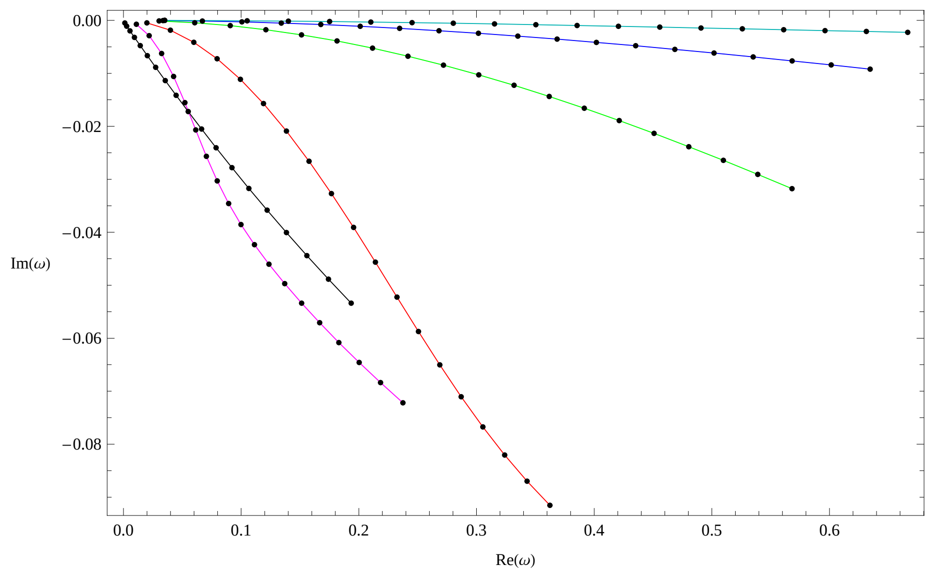

Figure 4: Movement of the positive frequency sound pole away from $\omega = 0$ with increasing spatial momentum. Distinct curves correspond to temperatures below the phase transition $T/T_c = 0.999$ (black), 0.97 (pink), 0.91 (red), 0.71 (green), 0.52 (blue), 0.26 (light blue). Dots on one curve are separated by $\Delta k = 0.05$. All curves start at $k = 0.05$ and end at $k = 1.00$.

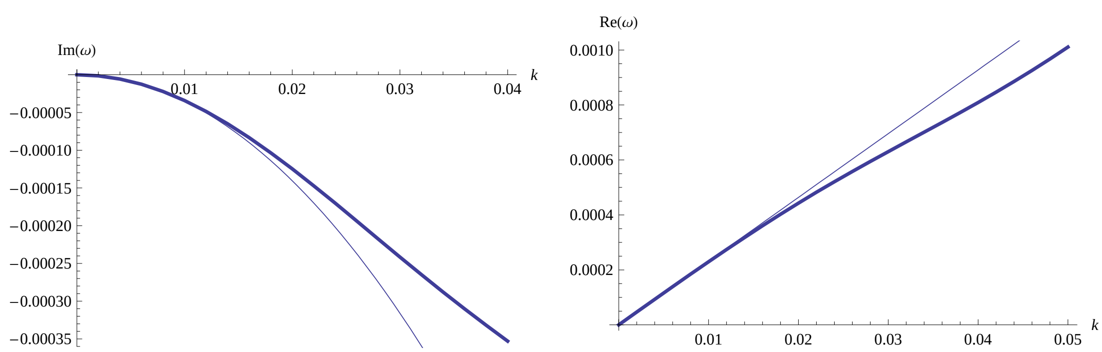

Figure 5: Fits of the real and imaginary part of the hydrodynamic modes in the broken phase to the lowest order approximation $\omega = v_s k - i\Gamma_s k^2$. The left figure shows the real part and the right one the imaginary part. The thick lines are the numerical results and the thin lines are the linear and quadratic fits. The fit is done for a temperature just below the critical one where the range of the approximation is rather small.

find

$$v_s^2 \approx 1.9 \left(1 - \frac{T}{T_c}\right) \qquad \mathrm{O}_1 - \text{Theory}\,, \tag{4.17}$$

$$v_s^2 \approx 2.8 \left(1 - \frac{T}{T_c}\right) \qquad \mathrm{O}_2 - \text{Theory}\,. \tag{4.18}$$

Moreover, as a benefit of our effort considering the fluctuations, we are also able to extract non-thermodynamic quantities in this channel. Specifically we examine the attenuation of the second sound mode as shown in figure 7. The curve shows how attenuation

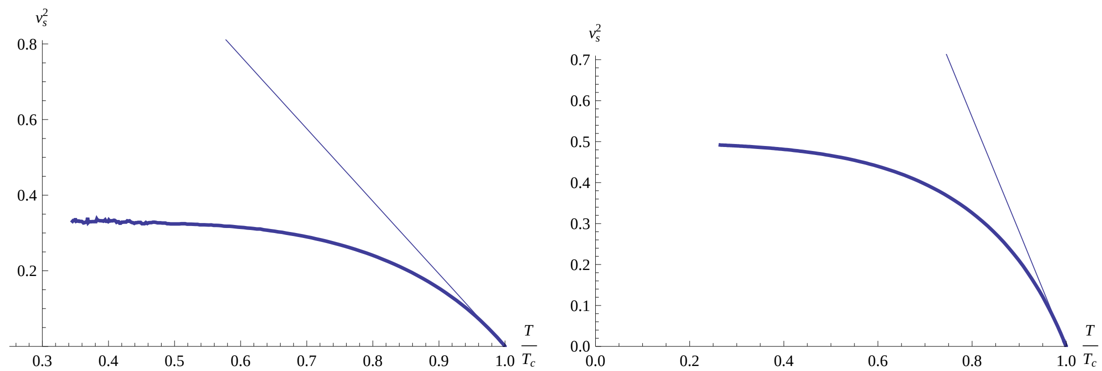

Figure 6: The plots show the squares of the speed of sound as extracted from the location of the lowest quasinormal mode in the broken phase. The left figure is for the O_1 theory and the right one for the O_2 theory. We also indicate the linear behavior close to T_c. As can be seen the numerics for the O_1 theory becomes somewhat unstable for low temperatures.

smoothly asymptotes to zero as the superfluid becomes more and more ideal at low temperatures. This effect is however much stronger in the O_2 theory. Near T_c the attenuation is growing. Within our numerical precision it seems however that the attenuation constant is taking a finite value at the critical temperature. Numerically we find

$$\Gamma_s = 1.87 T_c \quad \text{at} \quad T = 0.9991 T_c \qquad \mathrm{O}_1 - \mathrm{Theory}\,, \tag{4.19}$$

$$\Gamma_s = 1.48 T_c \quad \text{at} \quad T = 0.9998 T_c \qquad \mathrm{O}_2 - \mathrm{Theory}\,. \tag{4.20}$$

A similar behavior has been observed in [39], where the attenuation of the normal sound mode asymptotes to a finite value near a phase transition.

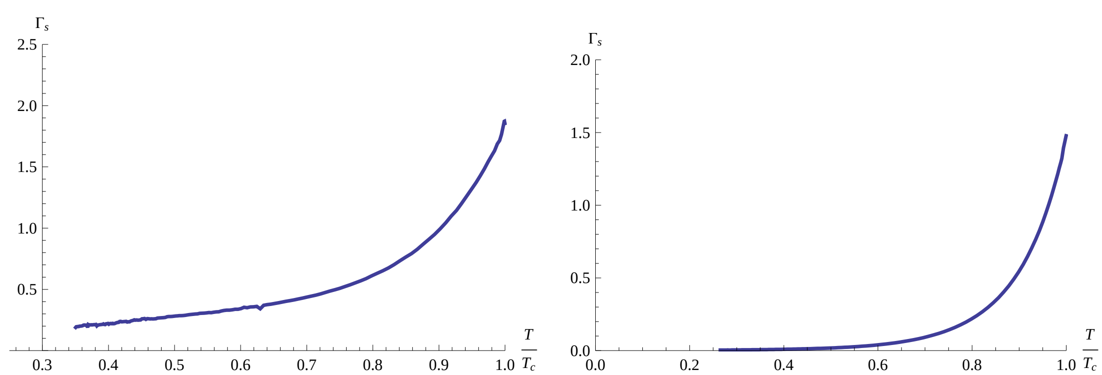

Figure 7: The plots show the attenuation constants of second sound as extracted from the location of the lowest quasinormal mode in the broken phase. The left figure is for the O_1 theory and the right one for the O_2 theory. Again it can be noticed that the O_1 theory is numerically more challenging at low temperatures.

(Pseudo) diffusion mode The vector diffusion mode from the unbroken phase turns into a (pseudo) diffusion mode below T_c[4]. For not too low temperatures and not too large

[4] An analytical result obtained for second sound in a non-abelian model [24] also shows the appearance of a pseudo diffusion mode with a gap that vanishes as the condensate goes to zero.

momenta the dispersion relation for this mode is well approximated by

$$\omega = -iDk^2 - i\gamma(T)\,, \tag{4.21}$$

with a gap $\gamma \in \mathbb{R}$ in imaginary frequency direction. Thus the pole is shifted from its unbroken phase position such that it does not approach zero at vanishing momentum any more, i.e. it is not anymore a hydrodynamic mode.

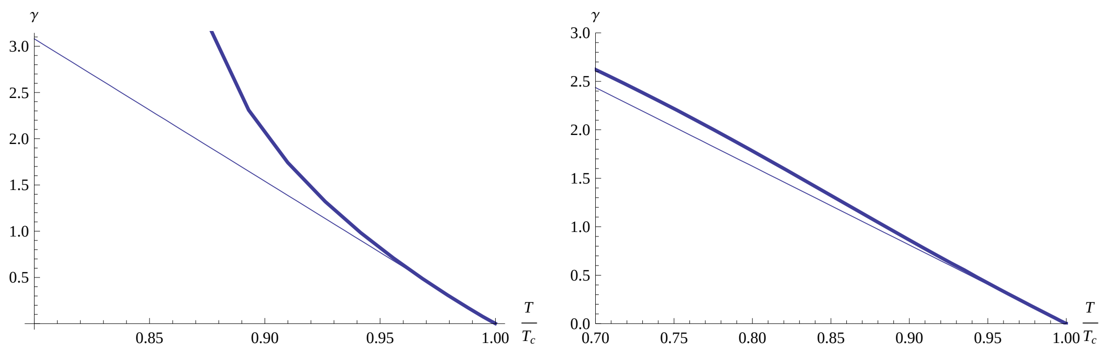

Figure 8: The plots show the gap of the pseudo diffusion mode as a function of reduced temperature. On the left the O_1 theory and on the right the O_2 theory.

In figure 8 we have plotted the gap γ as a function of the reduced temperature and we can see that it vanishes linearly near T_c.

$$\gamma \approx 15.4T_c\left(1-\frac{T}{T_c}\right) \qquad \mathrm{O}_1-\text{Theory}\,, \tag{4.22}$$

$$\gamma \approx 8.1T_c\left(1-\frac{T}{T_c}\right) \qquad \mathrm{O}_2-\text{Theory}\,. \tag{4.23}$$

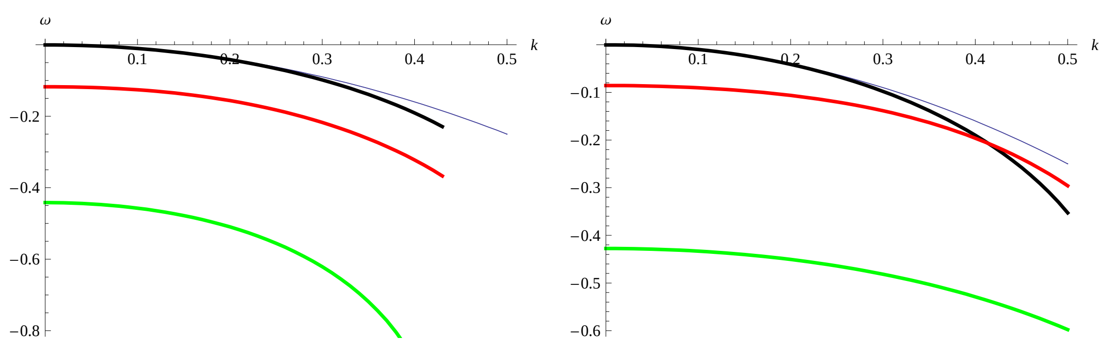

Figure 9: The plots show the dispersion relations for the pseudo diffusion mode at different temperatures and the diffusion dispersion relation with $D = 3/(4\pi T)$ (note that even in the unbroken phase the latter approximates the diffusive quasinormal mode only for small momenta). On the left we have the O_1 theory at temperatures $T = 0.999T_c$ (black), $T = 0.97Tc$ (red) and $T = 0.91T_c$ (green). On the right the same for the O_2 theory at temperatures $T = 0.999T_c$ (black), $T = 0.97T_c$ (red) and $T = 0.87T_c$ (green).

Figure 9 shows the dispersion relation for the diffusion pole at different temperatures. The offset at $k = 0$ is the gap size γ depending linearly on T only near T_c. This implies that the relation (4.21) asymptotes to the ordinary diffusion equation near the critical temperature. As expected the highest temperature curve $T = 0.999T_c$ (black) matches the hydrodynamic approximation (thin line) very well at small momenta. That agreement becomes worse around $k \sim 0.25$. Also as the condensate grows below T_c the behavior of this (pseudo) diffusion mode becomes less hydrodynamic.

4.3 Higher quasinormal modes

In addition to the hydrodynamic sound modes and the pseudo diffusion mode there are higher quasinormal modes. They are not the main focus of this paper and we have not studied them in detail. We have however traced the prolongation of the second and third quasinormal modes in the scalar sector from the unbroken phase into the broken phase.

The former scalar modes evolve continuously into higher modes of the coupled system through the phase transition as seen from the two kinked dashed (unbroken phase) and solid (broken phase) lines in figure 10. The kink indicates that the poles move continuously but change direction at the critical temperature. We show only the plot for the theory with the dimension two operator, similar results hold for the O_1 theory. At the critical temperature the locations of the quasinormal frequency calculated with the Frobenius method in the unbroken phase and by the method of finding the zeroes of the determinant spanned by the solutions match with impressively high precision. We might take this as a highly non trivial test of the accuracy of the numerical integration method.

We expect that all quasinormal frequencies are shifted *continuously* in the complex frequency plane across the phase transition. This means that there are no jumps in any of the dispersion relations. There is simply an infinite set of poles corresponding to the degrees of freedom of the system which are continuously shifted when parameters are changed.

Our numerical invesigations also reveal poles developing an increasing real part with decreasing temperature which most likely originate from the longitudinal vector modes in the unbroken phase. Note that we have not shown these modes for simplicity.

The two higher modes shown in figure 10 move parallel to the real axis and to each other with decreasing temperature. [5] At low temperatures their real parts are almost the same. This is also true for the longitudinal vectors as far as we can tell within our numerical uncertainties. We suspect that this *alignment of higher poles* has to do with the appearence of the conductivity gap showing up at low temperatures [5].

5. Summary and Discussion

We have studied the hydrodynamics of a holographic model of a superfluid. The model is an Abelian gauge field model with a charged scalar field in a four dimensional AdS black hole background.

As is well known by now the holographic dictionary allows to interpret the quasinormal frequencies as the poles of the retarded Green functions of the dual field theory. By

[5] Note that this behavior is very different from the behavior usually found in five-dimensional holographic setups when temperature is decreased [40–42].

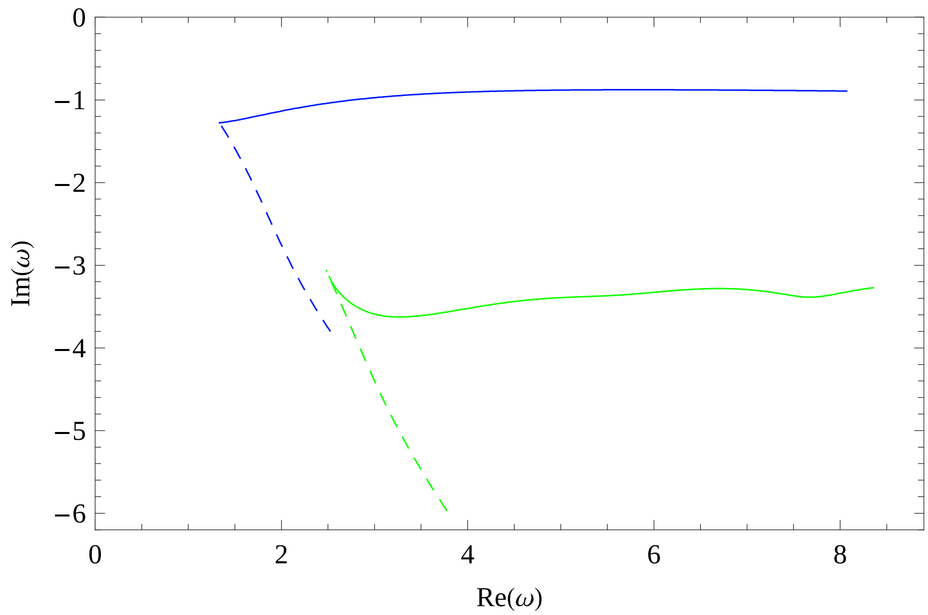

Figure 10: Movement of the higher poles in the complex frequency plane at vanishing momentum $k = 0$. Scalar modes 2 (blue) and 3 (green) in the unbroken phase (dashed) evolve continuously into the higher poles of the broken phase (solid). The right end points are evaluated for $T/T_c = 0.25$.

calculating the low lying quasinormal frequencies numerically in the broken and unbroken phases we were able to identify the hydrodynamics. We found that at high temperatures, $T > T_c$ there is only one hydrodynamic mode representing the diffusion of the conserved $U(1)$ charge. As one lowers the temperature reaching the critical temperature T_c two of the quasinormal frequencies of the charged scalar field approach the origin of the complex frequency plane. Precisely at the critical temperature at the onset of the phase transition these modes become massless, giving rise to new hydrodynamic variables. We also have calculated the residue of these modes and found that it stays finite at $T = T_c$ resulting in a divergence of the order parameter susceptibility, as expected.

Below the critical temperature these modes stay massless and show a dispersion relation with a linear real part and a quadratic imaginary one, allowing an interpretation as the modes of second sound in the superfluid. On the other hand the diffusion mode starts to develop a gap and stops to be hydrodynamic. The counting is therefore one hydrodynamic mode at high temperatures, three at the critical temperature and two at low temperatures.

In the low temperature phase we were able to calculate the speed of sound as well as its attenuation constant as a function of temperature. Also the gap in the pseudo diffusion mode has been determined.

We have also been able to follow some of the higher quasinormal modes through the phase transition and found that they evolve continuously albeit non-smoothly with temperature in the complex frequency plane, showing a sharp kink at the critical temperature.

On a technical side we have developed a method to determine the quasinormal frequencies and the holographic Green functions for systems of coupled differential equations and without using gauge invariant variables. The quasinormal frequencies correspond simply to the zeroes of the determinant spanned by a maximal set of linearly independent solutions. We have furthermore seen that the poles of the Green functions stemming from bulk gauge fields are gauge invariant as expected.

There are now several interesting questions that should be investigated in the future. The most obvious one is to apply the methods developed here to the model where the backreaction of the matter and gauge field onto the metric are properly taken into account. Due to the presence of the metric fluctuations the hydrodynamics of such a model is certainly much richer. In addition to the diffusion and second sound modes found here one expects also shear and sound modes stemming from bulk metric fluctuations. The corresponding system of differential equations promises to be rather involved. However there should be no principal obstacle to apply our methods also in these cases. Such an investigation is currently underway [43].

Another interesting direction of research should be to reinterpret the results obtained here and analogous ones for related models with different scalar mass and living in different dimensions from the point of view of dynamical critical phenomena [44]. We have seen already that the speed of sound scales with exponent one half, whereas the gap in the pseudo diffusion mode scales with exponent one. The situation for the sound attenuation is unfortunately less clear. As far as our numerics indicates the sound attenuation reaches a finite value at $T = T_c$. An extensive study of related models possibly with enhanced numerical efforts might give new insights here.

Of course an effort should also be undertaken to extend the results at hand to models of p-wave superconductors [45]. For infinitesimal condensates and analytic study of the hydrodynamics has already been done in [24]. It might be of interest to supplement these analytical result with a numeric study that allows to go further away from the phase transition point deep into the broken phases. Another very interesting class of holographic p-wave superconductors are the ones realized on $D7$ brane embeddings [8, 13, 21]. Due to the presence of fundamental matter these should be especially interesting to study.

We hope to come back to all or at least some of these and other questions in future publications.

Acknowledgments

We thank S. Hartnoll and S. Sachdev for discussions.
This work has been supported partially by the Plan Nacional de Altas Energías FPA-2006-05485, FPA-2006-05423, EC Commission under grant MRTN-CT-2004-005104 and Comunidad de Madrid HEPHACOS P-ESP-00346. I. A. is supported by grant BES-2007-16830.

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