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Hydromagnetic Equilibrium of a Thin-Skin, Finite-Beta Toroidal Plasma Column

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HYDROMAGNETIC EQUILIBRIUM OF A THIN-SKIN, FINITE-BETA TOROIDAL PLASMA COLUMN

by

J. L. Johnson, R. L. Morse, W. B. Riesenfeld

ABSTRACT

This report presents an analytic approach to describing the toroidal equilibrium of a finite-beta plasma column in the hydromagnetic approximation. The thickness of the sheath region separating the plasma from the confining magnetic field is assumed to be negligibly small compared to all other dimensions (the "thin-skin" approximation). The deviations of this sharp interface from a toroidal cylinder shape are treated by perturbation theory, a procedure which is shown to be consistent with the usual series expansion in powers of reciprocal aspect ratio. This analytic formulation has the merit of providing engineering design estimates for the auxiliary external current strengths required in a toroidal theta-pinch device (Scyllac) to compensate for toroidal drifts and provide an equilibrium configuration for a finite-beta plasma. The auxiliary fields are superpositions of azimuthally modulated multipole fields applied in addition to an azimuthal modulation component of the main toroidally directed theta-pinch field. It is shown that the presence of this latter component is necessary for a convergent superposition.

I. INTRODUCTION

A major problem in the design of plasma experiments using toroidal configurations, such as the toroidal Scyllac device,[1] is to construct coils such that equilibrium can exist. Much work has been done on this problem using a simple hydromagnetic model for low-β systems with magnetic shear[2,3] (here β refers to the ratio of plasma particle pressure to the pressure, $B^2/8\pi$, of the confining magnetic field, B). The analysis of high-β systems, in which the plasma currents generate magnetic fields of the same order of magnitude as the externally applied fields, poses special difficulties with respect to both the equilibrium and the stability aspects of the problem. A major achievement in the analysis of toroidal high-β devices was the demonstration, by construction[4] and by calculation,[5] that there exists an equilibrium unity-β plasma with a sharp boundary separating the plasma from the field. Although this type of configuration can be successfully investigated by numerical techniques,[6] engineering design is greatly facilitated by an analytic formulation. This report provides such a study, incorporating simplifying approximations where demanded by analytical tractability without excessively compromising the engineering design implications.

We restrict the discussion to systems in which all magnetic field lines close upon themselves after azimuthally traversing the torus. The hydromagnetic equilibrium problem consists of finding solutions of the system of equations

$$\vec{\nabla} p = \vec{j} \times \vec{B} \tag{1}$$

and

$$\vec{j} = \frac{1}{4\pi} \operatorname{curl} \vec{B}, \tag{2}$$

with closed toroidal, constant-pressure surfaces, where p is the plasma pressure, $\vec{j}$ the electric current density, and $\vec{B}$ the magnetic field as a function of position. Equation (1) determines the component of current density, $j_\perp$, perpendicular to the magnetic field:

$$\vec{j}_\perp = \frac{\vec{B} \times \vec{\nabla} p}{B^2} . \tag{3}$$

Let us consider a flux tube (a bundle of neighboring field lines) whose nearly rectangular cross section is depicted in Fig. 1.

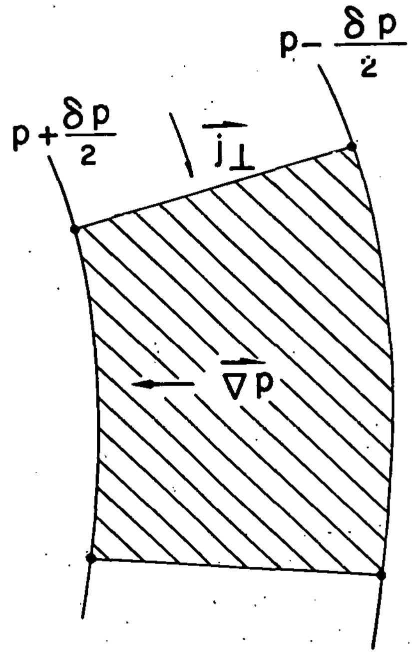

Fig. 1. Cross section of a flux tube with the magnetic field directed out of the plane of the diagram.

The magnetic field has an instantaneous direction perpendicular to the plane of the diagram and pointing toward the reader. The pressure gradient, $\vec{\nabla}p$, is a vector which, according to Eq. (1), must lie in the plane of the diagram; hence we may choose two opposite bounding surfaces of the flux tube to be neighboring constant-pressure surfaces corresponding to pressures p + (δp/2) and p - (δp/2), respectively. Since $\vec{j}_\perp$ is a vector perpendicular to both $\vec{\nabla}p$ and $\vec{B}$, we choose the remaining two boundaries of the flux tube to have their outward normals parallel and antiparallel to $\vec{j}_\perp$, respectively. Let $d\ell$ be the differential of length along a field line. Then the area, dS, of the upper boundary in Fig. 1 corresponding to a thickness, $d\ell$, along the field lines is given by

$$dS = \frac{\delta p}{|\vec{\nabla} p|}\, d\ell .$$

The current flowing into the flux tube across this boundary is, according to Eq. (3), given by

$$\begin{aligned} j_\perp\, dS &= \frac{|\vec{\nabla} p|}{B} \frac{\delta p}{|\vec{\nabla} p|}\, d\ell \\ &= \delta p \frac{d\ell}{B} . \end{aligned} \tag{4}$$

Hence the total current, δI, flowing into the flux tube is obtained by taking the line integral of Eq. (4) along the total length of the (closed) field lines:

$$\delta I = \delta p \oint \frac{d\ell}{B} . \tag{5}$$

The current flowing out of the flux tube has identical form except that the integral is evaluated along a field line at the bottom boundary of Fig. 1. But the steady-state equilibrium condition demands that there be no net charge buildup in any flux tube. This condition can be satisfied only by demanding that the quantity $\delta I/\delta p = \oint d\ell/B$ be constant on a constant-pressure surface:

$$\oint \frac{d\ell}{B} = \oint \frac{d\ell}{B} (p) . \tag{6}$$

Conversely, since a surface of constant $\oint d\ell/B$ is a a constant $\delta I/\delta p$ surface, and since the current density is always normal to the pressure gradient, it follows from the conservation of current that such a surface is also a constant-pressure surface:

$$p = p\left(\oint \frac{d\ell}{B}\right) . \qquad (7)$$

Thus, in equilibrium, the surfaces of constant pressure coincide with the surfaces of constant $\oint d\ell/B$.[7]

Substitution of Eq. (2) into Eq. (1) yields the condition

$$\vec{\nabla}\left(p + \frac{B^2}{8\pi}\right) = \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B}, \qquad (8)$$

i.e. the volume forces on the left-hand side of the equation, describing the compression of plasma and field, must balance the tension along the field lines.[8] The absolute value of the right-hand side of Eq. (8) may be simply expressed in terms of the local radius of curvature, ρ, of the field lines:

$$|(\vec{B}\cdot\vec{\nabla})\vec{B}| = \frac{B^2}{\rho} . \qquad (9)$$

When there is a transition layer of thickness Δ separating plasma and field, the magnetic contribution from the left-hand side of Eq. (8) is of order $B^2/8\pi\Delta$. The ratio of these terms is therefore given by

$$\frac{2|(\vec{B}\cdot\vec{\nabla})\vec{B}|}{|\vec{\nabla}(B^2)|} = \frac{2\Delta}{\rho} . \qquad (10)$$

Hence, in the limit where there is a sharp discontinuity in the plasma and field pressures, i.e., when there exists a sharp interface between plasma and field, the right-hand side of Eq. (8) does not contribute and the equation may be integrated to yield the jump condition across the constant-$\oint d\ell/B$ interface,

$$B^2_{out} - B^2_{in} = 8\pi p, \qquad (11)$$

where B_{out} and B_{in} are the field magnitudes just outside and inside of the interface.

The general hydromagnetic toroidal equilibrium problem consists of finding solutions of Eqs. (7) and (8) with toroidal constant-pressure surfaces. We restrict ourselves to the more tractable problem of constant plasma pressure except for a jump discontinuity across a mathematically thin boundary surface, so that Eq. (11) can be used. Since Meyer and Schmidt[4] have shown that such a "thin-skin" equilibrium exists, our purpose is to provide a guide for an explicit engineering design of such a configuration.

The major difficulty is that in a toroidal device the azimuthal field is inversely proportional to the radial distance from the toroidal major axis, so that the constant-$\oint d\ell/B$ surfaces would be cylinders concentric with the major axis and thus would not close on themselves within the machine. This problem is overcome in low-β Stellarators and pinches by providing a rotational transform[2,3] so that the magnetic field lines curve around a closed magnetic axis and thus form surfaces on which the pressure is constant. In systems with closed field lines, additional magnetic fields must be applied to ripple the field lines so that the inner lines with higher values of magnetic field are lengthened and the constant-$\oint d\ell/B$ surfaces are changed into toroids.

It is convenient to work in the coordinate system illustrated in Fig. 2, which is an orthogonal curvilinear system having a differential element of distance $d\ell$ given by

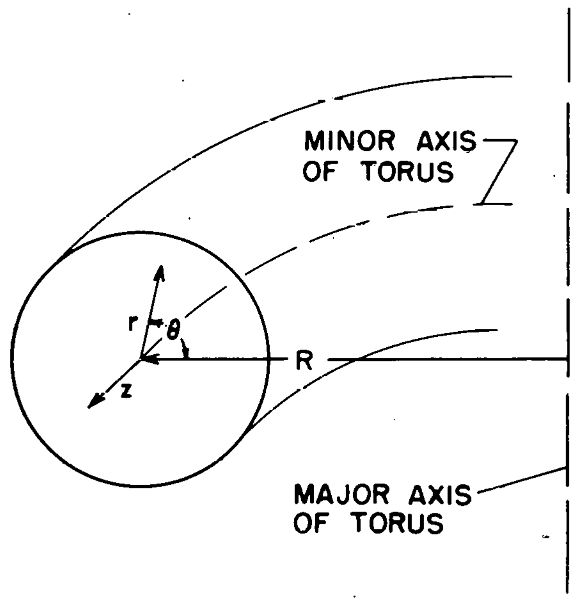

Fig. 2. Toroidal coordinate system.

$$d\ell^2 = dr^2 + r^2 d\theta^2 + (1 - \frac{r}{R}\cos\theta)^2 dz^2 . \qquad (12)$$

From the form of the metric it is evident that the z-coordinate is not the usual cylindrical coordinate; however, in the limit of large aspect ratio, $a/R \to 0$, the coordinates reduce to the cylindrical system (here R is the major radius of the torus and a is some relevant minor radius, such as the radius of the unperturbed toroidal plasma boundary). While this coordinate system does not yield separable solutions for the rigorous toroidal harmonics, it is nevertheless very suitable for carrying out perturbation calculations with a/R as an expansion parameter. The construction of the various differential vector operators is straightforward.[9] The gradient operator, for example, is of the form

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{1}{1 - \frac{r}{R}\cos\theta}\frac{\partial}{\partial z} , \qquad (13)$$

where $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_z$ are unit vectors along the corresponding coordinate directions.

To make analytical progress, we assume that the aspect ratio is large and retain terms only to first order in a/R. The only rippling fields which can be applied by external conductors to counteract the effects of toroidal curvature without destroying the closure of the field lines must be periodic functions of z, say $\vec{B}^{(\delta)}(r,\theta)\sin kz$, where the superscript refers to ripple field components producing field line excursions of order δ, and $k = 2\pi/\lambda$ is the azimuthal wavenumber corresponding to wavelength λ. Since the torus can accommodate only an integral number of wavelengths, k is a discrete parameter given by

$$k_s = \frac{s}{R}, \text{ s integer} . \qquad (14)$$

To first order in a/R, the curvature term in B_z which must be overcome is proportional to a $\cos\theta/R$ and is independent of z:

$$B_z(a,\theta,z) = B^{(o)} \left(1 + \frac{a\cos\theta}{R} + \ldots\right) , \qquad (15)$$

where $B^{(o)}$ is a constant field intensity of zeroth order in a/R. If we superpose this field with a ripple component $\vec{B}^{(\delta)}$ and impose the equilibrium condition by demanding the constancy of B^2 (or $\oint d\ell/B$) over the plasma surface, r = a, then the z-independent term in B^2 immediately forces us to an ordering

$$\left[\frac{B^{(\delta)}}{B^{(o)}}\right]^2 \sim \frac{a}{R} . \qquad (16)$$

Furthermore, the magnetic field outside of the plasma is irrotational so that it can be represented as a gradient of a scalar potential satisfying the Laplace equation. These circumstances lead to the consideration of a model in which the field external to the plasma is given by

$$\begin{aligned}\vec{B}(r, \theta, z) = \; & B^{(o)}\vec{\nabla}\Big\{z + \sum_{\ell,s} \frac{1}{k_s}\Big[C^{(1)}_{\ell s} I_\ell(k_s r) \\ & \cos(\ell\theta + \alpha_{\ell s}) \cos(k_s z + \gamma_{\ell s}) \qquad (17)\\ & + D^{(1)}_{\ell s} K_\ell(k_s r) \\ & \cos\left(\ell\theta + \bar{\alpha}_{\ell s}\right) \cos\left(k_s z + \bar{\gamma}_{\ell s}\right)\Big] + \ldots\Big\} .\end{aligned}$$

Here I_ℓ and K_ℓ are the modified Bessel functions of integral order ℓ, and from Eq. (16) it follows that

$$\left[C^{(1)}_{\ell s}\right]^2 \sim \left[D^{(1)}_{\ell s}\right]^2 \sim C^{(1)}_{\ell s} D^{(1)}_{\ell s} \sim \frac{a}{R} . \qquad (18)$$

The parameters α, $\bar{\alpha}$, γ, $\bar{\gamma}$ are constant phases whose inclusion is demanded by generality but for which we ultimately shall make appropriate choices. In Sections II, III, and IV we assume that there is no magnetic field inside the plasma surface, which represents the case of a perfectly diamagnetic, unity-β plasma. In Section V we treat the more general case of arbitrary β in which field penetrates into the plasma. In the latter case, we still retain the "thin-skin" approximation so that the plasma pressure is constant within a boundary surface, and plasma currents are surface currents flowing on the boundary. The internal field is then given by an expression similar to Eqs. (17) and (18) except that all $D^{(1)}_{\ell s}$ coefficients are set equal to zero. Since $I_\ell(k_s r)$ is regular at r = 0 and becomes large as $r \to \infty$, whereas $K_\ell(k_s r)$ is singular at r = 0 and falls off for large r, the $C^{(1)}_{\ell s}$ terms represent fields created by currents in external conductors while the $D^{(1)}_{\ell s}$ terms are generated by the plasma itself.

It is convenient to introduce a dimensionless parameter

$$\delta_{\ell s} \equiv \frac{C^{(1)}_{\ell s}}{(k_s a)^2 K_\ell(k_s a)} . \tag{19}$$

For the configurations treated, we shall see that $a\delta_{\ell s}$ is the distortion amplitude of the surface of discontinuity generated by the ripple field component associated with ℓ and s. The ordering of Eq. (18) asserts that

$$\delta^2_{\ell s} \sim \frac{a}{R} . \tag{20}$$

A perturbation theory with a/R as expansion parameter is therefore by necessity also a perturbation theory in $\delta_{\ell s}$. From the form of Eqs. (17), (18), and (13) it is evident that we have included terms of first order in a/R in the azimuthal field component B_z but have omitted such curvature corrections from the poloidal components represented by the sum over ℓ and s. The reason is that the present calculation will be limited to second-order terms in $\delta_{\ell s}$ while the neglected terms are, according to Eq. (20), of third order in $\delta_{\ell s}$. The form of these terms is well known,[10,11] however, and the latter could, in principle, be included in a higher-order calculation without difficulty.

Our task, then, is to determine the relations which must exist between the $C^{(1)}_{\ell s}$, $D^{(1)}_{\ell s}$, similar higher-order fields, and the aspect ratio such that Eqs. (7) and (11) are satisfied. In Section II we illustrate the technique by calculating the shape of the boundary surface for an unperturbed straight plasma cylinder with all $C^{(1)}_{\ell s}$ and $D^{(1)}_{\ell s}$ equal to zero except for the term with $\ell = 0$ and $k_s = k$. In this case we may, without loss of generality, set the phases α_o, γ_o, $\bar{\alpha}_o$, and $\bar{\gamma}_o$ equal to zero. We consider the toroidal, unity-β problem in Section III, and in Section IV we show that a solution, convergent with respect to the sum over ℓ, can be found. A discussion of the more general toroidal equilibrium in which a magnetic field exists inside the plasma, so that β is arbitrary, is presented in Section V. In Section VI we estimate the magnitude of the currents necessary to generate the field.

II. THE BUMPY, UNITY-BETA CYLINDER EQUILIBRIUM

We seek solutions of Eqs. (7) and (11) for a cylindrical plasma column of unperturbed radius a, subject only to an $\ell = 0$ rippling field. The magnetic field external to the plasma is then given by

$$\vec{B} = B^{(o)}\vec{\nabla}\left\{z + \frac{1}{k}\left[C^{(1)}I_o(kr) + D^{(1)}K_o(kr)\right]\cos kz + \frac{1}{2k}D^{(2)}K_o(2kr)\sin 2kz + \dots\right\} , \tag{21}$$

where the gradient operator is cylindrical and the coefficient $D^{(2)}$ is of order $\left[C^{(1)}\right]^2$. The particular choice of the second-order field will be justified a posteriori.

To zeroth order our surface of discontinuity is a cylinder $r^{(o)} = a$. We obtain the equation of the surface to first order by integrating the differential equation of motion of a field line,

$$\frac{dr}{B_r} = \frac{dz}{B_z} , \tag{22}$$

obtaining

$$r(z) = a + \frac{1}{k}\left[C^{(1)}I_o'(ka) + D^{(1)}K_o'(ka)\right]\sin kz + \dots \tag{23}$$

for a field line which passes through r = a at z = 0. The primes on the Bessel functions denote derivatives with respect to the argument kr.

We next evaluate the magnetic pressure $B^2(r,z)$ on this surface and adjust the coefficients $C^{(1)}$ and $D^{(1)}$ so that it is constant. To second order in our expansion parameter we obtain

$$\begin{aligned} B^2 = B^{(o)2}\Big\{ & 1 - 2\left[C^{(1)}I_o(ka) + D^{(1)}K_o(ka)\right]\sin kz \\ & - 2k(r-a)\left[C^{(1)}I_o'(ka) + D^{(1)}K_o'(ka)\right]\sin kz \\ & + 2D^{(2)}K_o(2ka)\cos 2kz \\ & + \left[C^{(1)}I_o(ka) + D^{(1)}K_o(ka)\right]^2\sin^2 kz \\ & + \left[C^{(1)}I_o'(ka) + D^{(1)}K_o'(ka)\right]^2\cos^2 kz + \dots\Big\} . \end{aligned} \tag{24}$$

The term containing (r - a) arises from a Taylor-series expansion about the zeroth-order surface; the form for this quantity is given by Eq. (23). Clearly B^2 cannot be constant on the surface unless

$$D^{(1)} = - \frac{I_o(ka)}{K_o(ka)} C^{(1)}. \tag{25}$$

By substituting Eqs. (23) and (25) into Eq. (24), we obtain

$$B^2 = B^{(o)2} \left\{ 1 + C^{(1)2} \left[\frac{I_o'(ka)K_o(ka) - I_o(ka)K_o'(ka)}{K_o(ka)} \right]^2 (\cos^2 kz - 2\sin^2 kz) + 2D^{(2)}K_o(2ka)\cos 2kz + \ldots \right\}. \tag{26}$$

We simplify this expression, as well as the form of Eq. (23), by using the Wronskian relation[12]

$$I_\ell(x)K_\ell'(x) - I_\ell'(x)K_\ell(x) = -\frac{1}{x}. \tag{27}$$

Use of Eqs. (25) and (27) in Eq. (23) yields, for the amplitude of field line excursions in units of radius a, an expression

$$\delta_o = \frac{C^{(1)}}{(ka)^2 K_o(ka)}, \tag{28}$$

which has precisely the form of Eq. (19). By making use of obvious trigonometric identities and by choosing

$$D^{(2)} = - \frac{3(ka)^2\delta_o^2}{4K_o(2ka)}, \tag{29}$$

we may express the magnetic pressure on the surface in the form

$$B^2 = B^{(o)2} \left[1 - \frac{1}{2}(ka)^2\delta_o^2 + \ldots\right] \tag{30}$$

which is constant. An arbitrary combination of second-order $C^{(2)}$ and $D^{(2)}$ terms could equally well have been chosen to eliminate the cos 2kz dependence. The particular choice above requires no contribution from second-order externally imposed fields. Instead, the second-order correction is generated by plasma currents set up by the pressure imbalance on the uncorrected plasma surface, and thus is taken care of "by nature." Such a conclusion, of course, relies tacitly on the stability of the equilibrium configuration.

To show that this surface and field configuration satisfies Eq. (7), we evaluate $\oint d\ell/B$, using Eqs. (21), (23), (25), (27), and (29):

$$\oint \frac{d\ell}{B} = \frac{L}{B^{(o)}} \left[1 + \frac{1}{2}(ka)^2\delta_o^2 + \ldots\right], \tag{31}$$

where L is a periodicity length which, in the present case of an infinitely long cylinder, replaces the length of a closed field line. Eq. (31) tells us that $\oint d\ell/B$ is identical for all field lines lying on the surface of the bympy cylinder. Thus the surface is simultaneously a constant-pressure and a constant-$\oint d\ell/B$ surface, satisfying the equilibrium condition of Eq. (7). While the constancy of $\oint d\ell/B$ in the present example is a trivial consequence of cylindrical symmetry, the analogous result for the toroidal cases will be a crucial feature of our solution.

III. THE TOROIDAL, UNITY-BETA PLASMA COLUMN EQUILIBRIUM

We next analyze a toroidal plasma column with completely excluded field, corresponding to a unity-β value. The superposition of rippling fields will be of the general form described by Eq. (17). The procedure outlined in the Introduction depends upon the generation of terms in the magnetic pressure, B^2, which compensate the a cos θ/R terms associated with toroidal curvature. It is clear that such a cancellation can be achieved by mixing terms with $\ell = 0$ and $\ell = 1$; the z-independent part of the interference term between these two contributions has exactly the desired properties. But in superposing these two terms we automatically introduce cos 2θ contributions, which in turn must be compensated by an $\ell = 2$ term. The latter generates cos 3θ and cos 4θ dependences which then must be cancelled by higher ℓ-value fields, etc. In this fashion we obtain an infinite system of coupled algebraic equations for our field line excursion parameters, $\delta_{\ell s}$, of Eq. (19). Our secondary expansion technique, described in Section IV, will be consistent only if we succeed in finding a solution having a rapidly converging set of parameters $\delta_{\ell s}$ as ℓ increases, so that the retention of only the first few terms will provide a practical solution of sufficient accuracy.

In this section we perform an analysis for the toroidal case analogous to that of Section II for the straight cylinder. The resulting infinite system of equations for the excursion parameters is solved in Section IV, where it is shown that the $\ell = 0, 1, \ldots$ superposition does, indeed, lead to convergence. From an engineering standpoint, it would be desirable to omit the $\ell = 0$ field (for which one pays a relatively high price in energy because of the work done in compressing the plasma) and to use a superposition starting from $\ell = 1$. The equilibrium analysis for this case can still be done, but it also can be shown that the resulting set of $\delta_{\ell s}$ has constant-magnitude contributions from terms of arbitrarily high ℓ. This does not necessarily mean that such a configuration is physically impossible; it merely means that the mathematical requirements for handling it are beyond the scope of our present expansion technique.

We consider the field given by Eq. (17) with additional $D^{(2)}_{\ell s}$ contributions properly chosen to cancel the terms sinusoidal in z which arise in the expression for the magnetic pressure. The procedure is identical to that illustrated in the example of the preceding section. We again restrict our attention to configurations in which the unperturbed, zeroth-order surface of discontinuity corresponds to a toroidal cylinder $r^{(o)} = a$. It is clear that to first order in $\delta_{\ell s}$ the magnetic pressure B^2 will vary as we move along a line of force unless we restrict consideration to fields with

$$\alpha_{\ell s} = \bar{\alpha}_{\ell s}; \; \gamma_{\ell s} = \bar{\gamma}_{\ell s} , \tag{32}$$

and

$$D^{(1)}_{\ell s} = - \frac{I_\ell(k_s a)}{K_\ell(k_s a)} C^{(1)}_{\ell s} , \tag{33}$$

the last being a generalization of Eq. (25). The differential equations of motion for the field lines are of the form

$$\frac{dr}{B_r} = \frac{r d\theta}{B_\theta} = \left(1 - \frac{r \cos\theta}{R}\right) \frac{dz}{B_z} . \tag{34}$$

By substituting the form of Eq. (17) for the field components and using Eqs. (32), (33), and (27), we obtain the equation for the field lines up to first order in our expansion parameter:

$$r = a + a \sum_{\ell=0}^{\infty} \sum_s \delta_{\ell s} \cos(\ell\theta_o + \alpha_{\ell s}) \left[\sin(k_s z + \gamma_{\ell s}) - \sin(k_s z_o + \gamma_{\ell s})\right]; \quad \theta = \theta_o . \tag{35}$$

Here $\delta_{\ell s}$ is given by Eq. (19), and the constants of integration were chosen such that the field line passes through $r = a$, $\theta = \theta_o$ when $z = z_o$. Evaluation of B^2 on the first-order perturbed surface, Eq. (35), leads to the expression

$$\begin{aligned} B^2 = B^{(o)2} \Bigg(& 1 + \frac{2a \cos\theta_o}{R} \\ & - \tfrac{1}{2} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s,t} k_s a^2 \delta_{\ell s} \delta_{mt} \\ & \cos\left(\ell\theta_o + \alpha_{\ell s}\right) \cos\left(m\theta_o + \alpha_{mt}\right) \\ & \Big\{ (2k_s - k_t) \cos\left[(k_s - k_t)z + \gamma_{\ell s} - \gamma_{mt}\right] \\ & - (2k_s + k_t) \cos\left[(k_s + k_t)z + \gamma_{\ell s} + \gamma_{mt}\right] \\ & - 2k_s \cos(k_s z - k_t z_o + \gamma_{\ell s} - \gamma_{mt}) \\ & + 2k_s \cos(k_s z + k_t z_o + \gamma_{\ell s} + \gamma_{mt}) \Big\} + D^{(2)}_{\ell s} \text{terms} \Bigg) . \end{aligned} \tag{36}$$

The various constant parameters which appear in Eq. (36) can be chosen to make the magnetic pressure independent of z. The last two cosine terms involving z_o (the "initial condition" terms) can be eliminated by choosing $k_t z_o + \gamma_{mt}$ to be integral multiples of π for all pairs of indices m and t. Without loss of generality we can take $z_o = 0$ and for simplicity set all $\gamma_{mt} = 0$. Clearly the second-order $D^{(2)}_{\ell s}$ terms, which are quadratic in the $\delta_{\ell s}$ amplitudes, can be chosen to compensate for the remaining z-dependent terms in Eq. (36), i.e., for all terms except $\cos[(k_s - k_t)z]$ with $s = t$. This procedure is completely analogous to the one which led to Eq. (29) for the $D^{(2)}$ coefficient in the cylindrical analysis. Hence nothing is gained by superposing external fields with two or more periodicities in z. We simplify Eq. (36) by restricting the summation to a single k_s value and by dropping the subscript, s, which has now become superfluous (the value of k is still governed by Eq. (14), of course):

$$B^2 = B^{(o)^2}\left(1 + \frac{2r\cos\theta_o}{R} - \frac{(ka)^2}{4}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\delta_\ell\delta_m\left\{\cos\left[(\ell - m)\theta_o + \alpha_\ell - \alpha_m\right] + \cos\left[(\ell + m)\theta_o + \alpha_\ell + \alpha_m\right]\right\} + \dots\right) . \quad (37)$$

If two terms with $\ell = 0$ and $\ell = 1$ were present, then one could set the α_ℓ's equal to zero and take $\delta_o\delta_1 = 2/k^2Ra$ to eliminate the cos θ_o dependence. Then the δ_1^2 term would introduce a cos $2\theta_o$ variation. An $\ell = 2$ contribution can be introduced to cancel this term, thereby generating cos $3\theta_o$ and cos $4\theta_o$ variations which must be compensated in turn. Fortunately, as shown in the next section, the series converges rapidly enough so that only a few terms should be needed for a reasonable given limit on the magnetic pressure fluctuations.

To complete the proof of unity-β, hydromagnetic equilibrium, we need to show that the construction which makes B^2 constant on the surface of discontinuity also causes $\oint d\ell/B$ to be constant, i.e., independent of θ_o. That this is indeed the case follows from direct evaluation:

$$\oint \frac{d\ell}{B} = \int_0^{2\pi R}\left(1 - \frac{a\cos\theta_o}{R}\right)\frac{dz}{B_z}$$

$$= \frac{2\pi R}{B^{(o)}}\left(1 - \frac{2a\cos\theta_o}{R} + \frac{(ka)^2}{4}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\delta_\ell\delta_m\left\{\cos\left[(\ell - m)\theta_o + \alpha_\ell - \alpha_m\right] + \cos\left[(\ell + m)\theta_o + \alpha_\ell + \alpha_m\right]\right\} + \dots\right) . \quad (38)$$

IV. DETERMINATION OF FIELDS FOR THE TOROIDAL, UNITY-BETA PLASMA COLUMN

The determination of the equilibrium field reduces, in view of Eq. (19), to the calculation of a set of δ_ℓ's such that the expression

$$\Xi(\theta_o) \equiv \frac{4\cos\theta_o}{k^2 aR} - \tfrac{1}{2}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\delta_\ell\delta_m\left[\cos(\ell - m)\theta_o + \cos(\ell + m)\theta_o\right], \quad (39)$$

which occurs in both Eqs. (37) and (38), becomes independent of θ_o. For simplicity we have set all the α_ℓ phases equal to zero. In this section we show how to find such a set for engineering design.

We assume that multipole fields of all orders ℓ are present, and that the dominant cancellation of the 4 cos θ_o/k^2aR term will arise from the interference of $\ell = 0$ and $\ell = 1$ fields. To make $\Xi(\theta_o)$ independent of θ_o we demand that the coefficient of each cos $n\theta_o$ term, n = 1, 2, 3, ... , vanish. This leads to the following infinite set of coupled equations for the δ_ℓ:

$$2\delta_0\delta_1 + \delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_4 + \dots = \frac{4}{k^2 aR} ,$$

$$\tfrac{1}{2}\delta_1^2 + 2\delta_0\delta_2 + \delta_1\delta_3 + \delta_2\delta_4 + \delta_3\delta_5 + \dots = 0 ,$$

$$2\delta_0\delta_3 + \delta_1\delta_2 + \delta_1\delta_4 + \delta_2\delta_5 + \delta_3\delta_6 + \dots = 0 , \quad (40)$$

$$\tfrac{1}{2}\delta_2^2 + 2\delta_0\delta_4 + \delta_1\delta_3 + \delta_1\delta_5 + \delta_2\delta_6 + \delta_3\delta_7 + \dots = 0 ,$$

...

$$\tfrac{1}{2}\sum_{j=0}^{n}\delta_{n-j}\,\delta_j + \sum_{j=0}^{\infty}\delta_j\,\delta_{j+n} = 0,$$

where the last line represents the general condition that the coefficient of cos $n\theta_o$ vanish for arbitrary $n > 1$. Let us assume that δ_1 is small compared to δ_o so that we may introduce a small ordering parameter, ϵ:

$$\frac{\delta_1}{\delta_o} = \epsilon << 1. \tag{41}$$

A systematic way of solving the set of coupled equations (40) consists of making an a priori assumption that $\delta_\ell/\delta_o \sim \epsilon^\ell$. Then the set of equations may be solved up to a given order in ϵ by neglecting higher-order terms in a given equation and by truncating the system of equations at the line in which each of the terms exceeds the given order in ϵ. Our system of equations then becomes finite, with the number of unknown quantities, δ_ℓ/δ_o, equal to the number of equations. Finally, one may verify that the solution obtained in this fashion indeed has the property $\delta_\ell/\delta_o \sim \epsilon^\ell$ and converges as one goes to higher order in ϵ.

To illustrate, let us solve Eqs. (40) to the lowest few orders in ϵ. To first order, we keep only δ_1/δ_o, neglect δ_2, δ_3, ... , and cut off the system at the first equation:

$$2\delta_o\delta_1 = \frac{4}{k^2 aR} .$$

This has the solution

$$\frac{\delta_1}{\delta_o} = \frac{2}{k^2 aR\delta_o^2} , \tag{42}$$

$$\delta_2 = \delta_3 = \ldots \cong 0.$$

The ordering parameter, ϵ, is therefore equal to $2/k^2 aR\delta_o^2$, and the field line excursion parameter, δ_o, may be arbitrarily chosen subject only to the constraint $\epsilon << 1$. If the plasma pressure, p, is prescribed, however, then, according to Eq. (11), δ_o satisfies the relationship

$$p = \frac{B^{(o)2}}{8\pi} \left[1 - \tfrac{1}{2}(ka)^2 \delta_o^2\right] ,$$

and, consequently,

$$\epsilon = \frac{a}{R} \frac{1}{1 - \frac{8\pi p}{B^{(o)2}}} . \tag{43}$$

It should be noted, however, that we could have inserted a constant B_z-field of second order in the δ's into Eq. (17)

$$\vec{B} = B^{(o)} \vec{\nabla} \left\{z\left[1 + E^{(2)}\right] + \ldots\right\} ,$$

and chosen its relative magnitude $E^{(2)}$ to be arbitrary. By adjusting $E^{(2)}$ we could then have removed this constraint. Physically, this corresponds to a J_θ-current balancing the second-order term. We shall follow such a procedure in the more general development of Section V, as is evidenced by the form of Eq. (47).

To second order in ϵ, we keep only δ_1/δ_o and δ_2/δ_o, neglect δ_3, δ_4, ... , and truncate the system at the second equation:

$$2\delta_o\delta_1 = \frac{4}{k^2 aR} ,$$

$$\tfrac{1}{2}\delta_1^2 + 2\delta_o\delta_2 = 0 .$$

This has the solution

$$\frac{\delta_1}{\delta_o} = \frac{2}{k^2 aR\delta_o^2} ,$$

$$\frac{\delta_2}{\delta_o} = -\left(\frac{1}{k^2 aR\delta_o^2}\right)^2 \tag{44}$$

$$\delta_3 = \delta_4 = \ldots \cong 0.$$

The quantity δ_1/δ_o experiences no second-order corrections. There is a third-order correction to δ_1/δ_o, however, as may be seen by constructing the solution for δ_1/δ_o, δ_2/δ_o, and δ_3/δ_o up to third order in ϵ:

$$\frac{\delta_1}{\delta_o} = \frac{2}{k^2 aR\delta_o^2} \left[1 + \frac{1}{2}\left(\frac{1}{k^2 aR\delta_o^2}\right)^2\right] ,$$

$$\frac{\delta_2}{\delta_o} = -\left(\frac{1}{k^2 aR\delta_o^2}\right)^2 , \tag{45}$$

$$\frac{\delta_3}{\delta_o} = \left(\frac{1}{k^2 aR\delta_o^2}\right)^3 .$$

Continuing in this fashion, it is easy to see that the dominant contribution to each δ_ℓ/δ_o is of the form $c_\ell\epsilon^\ell$, justifying our a priori assumption. The constants c_ℓ satisfy the recursion relations

$$c_\ell = -\frac{1}{4}\sum_{j=1}^{\ell-1} c_{\ell-j}c_j;\ \ell \geq 2. \tag{46}$$

With $c_1 = 1$, it is easy to obtain the c_ℓ's in sequence. Thus $c_2 = -1/4$, $c_3 = 1/8$, $c_4 = -5/64$, $c_5 = 7/128$, etc. The coefficients decrease so rapidly that our series for the field converges rapidly even when ϵ is not infinitesimal compared to unity. For example, ϵ as large as 3/4 still produces rapid convergence.

Since it is difficult, because of energy considerations, to use coils which provide a large $\ell = 0$ field, the question arises whether one might develop a similar formulation with the toroidal curvature term compensated mainly by an interference between $\ell = 1$ and $\ell = 2$ fields. The straightforward series expansion does not converge in this case, even when δ_2/δ_1 is assumed small. In fact, one can show that $\delta_2 \approx -\delta_4 \approx \delta_6$ ... so that convergence is impossible. As we pointed out in the preceding section, this does not necessarily mean that such a configuration does not exist; it means that our expansion technique cannot be applied usefully.

V. TOROIDAL EQUILIBRIUM FOR ARBITRARY BETA

In this section we generalize the analysis of Sections II and IV to the case in which the magnetic field does not vanish inside the plasma, so that the beta is arbitrary. We retain, however, the "thin-skin" assumption for the structure of the interface between the plasma and the external field.

We seek an equilibrium in which the plasma is bounded by a surface $r(\theta,z) = a + \delta a(\theta,z) + \ldots$, where we take δa to be zero when z is zero. Inside this surface the plasma pressure is constant, and the internal magnetic field is given by

$$\vec{B}_{in} = B_{in}^{(o)}\,\vec{\nabla}\left[z(1+E^{(2)}) + \frac{1}{k}\sum_{\ell=0}^{\infty} A_\ell^{(1)} I_\ell(kr)\cos\ell\theta\cos kz + \frac{1}{2k}\sum_{\ell=0}^{\infty} A_\ell^{(2)} I_\ell(2kr)\cos\ell\theta\sin 2kz + \ldots\right], \tag{47}$$

where we have explicitly retained a second-order correction, $E^{(2)}$, to the unmodulated part of the azimuthal field component. For the magnetic field outside of the plasma we write

$$\vec{B}_{out} = B_{out}^{(o)}\,\vec{\nabla}\left\{z + \frac{1}{k}\sum_{\ell=0}^{\infty}\left[C_\ell^{(1)} I_\ell(kr) + D_\ell^{(1)} K_\ell(kr)\right]\cos\ell\theta\cos kz + \frac{1}{2k}\sum_{\ell=0}^{\infty} D_\ell^{(2)} K_\ell(2kr)\cos\ell\theta\sin 2kz + \ldots\right\}. \tag{48}$$

We might have kept a superposition of fields with different wave numbers and phases, but the discussion of Sections II and III shows that Eqs. (47) and (48) are sufficiently general. It is convenient to define the plasma beta as

$$\beta \equiv \frac{8\pi p}{B_{out}^{(o)2}}, \tag{49}$$

and to introduce the notation

$$I^{(\ell)} \equiv \frac{I_\ell(ka)}{ka I_\ell'(ka)}. \tag{50}$$

The pressure balance condition of Eq. (11) across the perturbed surface becomes, to first order in the ripple amplitude δa,

$$B_{in}^{(o)2} = (1-\beta)\,B_{out}^{(o)2}, \tag{51}$$

$$C_\ell^{(1)} I_\ell(ka) + D_\ell^{(1)} K_\ell(ka) = (1 - \beta)A_\ell^{(1)} I_\ell(ka). \tag{52}$$

By integrating the differential equations (34) for the field lines just inside and just outside of the plasma boundary, we can relate our fields to the shape of the boundary. We find that

$$\delta a(\theta,z) = a \sum_{\ell=0}^{\infty} \delta_\ell \cos\ell\theta \sin kz, \tag{53}$$

where

$$\begin{aligned} \delta_\ell &= \frac{1}{ka} A_\ell^{(1)} I_\ell'(ka), \\ &= \frac{1}{ka}\left[C_\ell^{(1)} I_\ell'(ka) + D_\ell^{(1)} K_\ell'(ka)\right]. \end{aligned} \tag{54}$$

The pressure balance, Eq. (52), may therefore be rewritten in the form

$$C_\ell^{(1)} I_\ell(ka) + D_\ell^{(1)} K_\ell(ka) = (1 - \beta)(ka)^2 \delta_\ell I^{(\ell)} . \tag{55}$$

Equations (54) and (55) furnish a complete specification of the first-order fields in terms of δ_ℓ:

$$\begin{aligned} A_\ell^{(1)} &= \frac{ka\delta_\ell}{I_\ell'(ka)}, \\ C_\ell^{(1)} &= (ka)^2\delta_\ell K_\ell(ka)\left[1 - (1-\beta)\frac{I_\ell(ka)K_\ell'(ka)}{I_\ell'(ka)K_\ell(ka)}\right], \\ D_\ell^{(1)} &= -\beta(ka)^2\delta_\ell I_\ell(ka). \end{aligned} \tag{56}$$

These formulas constitute the generalizations of Eqs. (19) and (33) for the arbitrary-β case. We note that the fields satisfy the requirement that the normal component be continuous across the boundary. The normal component, being given to first order in δ_ℓ by the expression $B_r - B_z\, ka \sum_\ell \delta_\ell \cos\ell\theta \cos kz$, vanishes on each side of the boundary because of Eq. (54).

To construct the equations which couple the various δ_ℓ's we must examine the pressure balance condition to second order. As before, the equality involves two classes of terms, those independent of z and those which vary as sin 2kz. Clearly, the $A_\ell^{(2)}$ and $D_\ell^{(2)}$ coefficients can be chosen to compensate the sinusoidal terms and, at the same time, keep the normal field component continuous across the boundary. Then the pressure balance condition reduces to

$$\begin{aligned} \frac{2(1-\beta)E^{(2)}}{\beta} = \frac{2a\cos\theta}{R} - \frac{1}{4}(ka)^2 \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \delta_\ell\delta_m \Big\{ &\left[1 + (1-\beta)(\ell m + k^2a^2)I^{(\ell)}I^{(m)}\right] \cos(\ell - m)\theta \\ &+ \left[1 - (1-\beta)(\ell m - k^2a^2)I^{(\ell)}I^{(m)}\right] \cos(\ell + m)\theta \Big\}, \end{aligned} \tag{57}$$

which is the generalization of Eq. (37) for the arbitrary-β case.

The second-order correction field, $E^{(2)}$, can be chosen to satisfy the θ-independent part of Eq. (57). The δ_ℓ's can then be adjusted to cancel the cosθ, cos2θ, ... terms.

For the equilibria of interest, we have ka < 1, and nothing is lost by retaining only the lowest-order terms in the small-argument expansion of the Bessel functions

$$I_\ell(x) = \frac{x^\ell}{2^\ell \ell!}\left[1 + \frac{x^2}{4(\ell+1)} + \dots\right]. \tag{58}$$

From Eq. (50) we find the approximate relations

$$I^{(o)} \cong \frac{2}{(ka)^2}$$

and (59)

$$I^{(\ell)} \cong \frac{1}{\ell} \cdot (\ell \neq 0).$$

Equation (57) then reduces to the following infinite set of coupled equations for the δ_ℓ's:

$$2(3 - 2\beta)\delta_0\delta_1 + (2 - \beta)\sum_{\ell=1}^{\infty}\delta_\ell\delta_{\ell+1} = \frac{4}{k^2 aR} ,$$

$$\frac{1}{2}\beta\,\delta_1^2 + 2(2 - \beta)\,\delta_0\delta_2 + (2 - \beta)\sum_{\ell=1}^{\infty}\delta_\ell\delta_{\ell+2} = 0 ,$$

$$\frac{2}{3}(5 - 2\beta)\,\delta_0\delta_3 + \beta\delta_1\delta_2 + (2 - \beta)\sum_{\ell=1}^{\infty}\delta_\ell\delta_{\ell+3} = 0 , \qquad (60)$$

$$\frac{1}{2}\beta\delta_2^2 + (3 - \beta)\delta_0\delta_4 + \beta\delta_1\delta_3 + (2 - \beta)\sum_{\ell=1}^{\infty}\delta_\ell\delta_{\ell+4} = 0 ,$$

. . . .

$$2\left[1 + \frac{2}{n}(1 - \beta)\right]\delta_0\delta_n + \frac{1}{2}\beta\sum_{\ell=1}^{n-1}\delta_\ell\delta_{n-\ell} + (2 - \beta)\sum_{\ell=1}^{\infty}\delta_\ell\delta_{\ell+n} = 0.$$

This set of equations reduces to Eq. (40) for $\beta = 1$. We may systematically solve this system in precisely the same way that we solved Eq. (40), by introducing an ordering parameter, ϵ, such that $\delta_\ell/\delta_0 \approx \epsilon^\ell$ and calculating to any desired order in ϵ. Thus, to third order we obtain the solution

$$\frac{\delta_1}{\delta_o} = \frac{2}{(3-2\beta)k^2 aR\delta_o^2}\left[1 + \frac{\beta}{2(3-2\beta)^3(k^2 aR\delta_o^2)^2}\right] ,$$

$$\frac{\delta_2}{\delta_o} = -\frac{\beta}{(2-\beta)(3-2\beta)^2(k^2 aR\delta^3)^2} , \qquad (61)$$

$$\frac{\delta_3}{\delta_o} = \frac{3\beta^2}{(2-\beta)(5-2\beta)(3-2\beta)^3(k^2 aR\delta_o^2)^3} .$$

These solutions reduce to those of Eq. (45) when $\beta = 1$. Again it is possible to show that no convergent solutions exist if δ_o vanishes.

To quantitatively understand the magnitudes involved, in the graphs of Fig. 3 we plot values of δ_1/δ_0 and δ_2/δ_0 as functions of δ_o, with two values of β as a parameter. The curves are obtained on the basis of reasonable toroidal Scyllac design values, R = 250 cm and a = 0.75 cm, and the results are shown for azimuthal wavelengths λ = 21.6 cm and λ = 13.5 cm.

In Table I we list corresponding values of δ_o (in units of a dimensionless $B_z(\ell=0)$ amplitude on axis) and δ_1 (in units of a dimensionless $B_r(\ell = 1)$ amplitude on axis).

TABLE I. Typical Values of δ_o and δ_1

	λ = 21.6 cm		λ = 13.5 cm	
	$\delta_o/B_{z(\ell=0)}$	$\delta_1/B_{r(\ell=1)}$	$\delta_o/B_{z(\ell=0)}$	$\delta_1/B_{r(\ell=1)}$
β = 1.0	12.49	9.42	6.59	6.24
β = 0.75	1.76	7.43	1.62	4.87

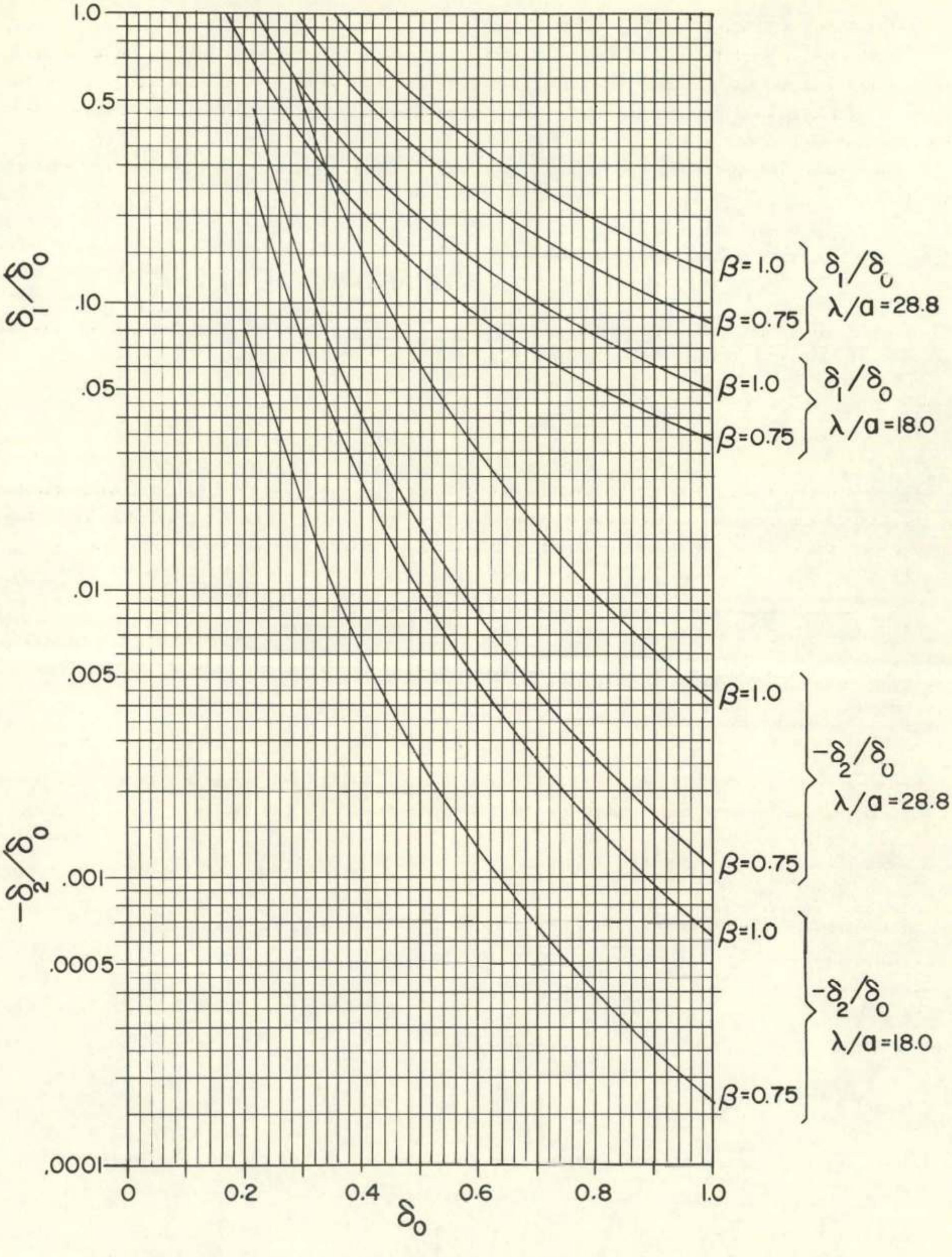

Fig. 3. δ_1/δ_0 and δ_2/δ_0 versus δ_0, with β and λ as parameters.

VI. CURRENT SOURCE STRENGTHS

Although experimental techniques should be used to determine the precise values of the externally applied currents necessary to generate the rippling fields, it seems worthwhile to make analytical estimates not only to obtain general design information but also to check how rapid the series convergence discussed in the last two sections really is.

In order to obtain a ripple component, $\delta_{\ell s}$, on the plasma boundary, we must impose an external field of the form

$$\vec{B}_{out} = B^{(o)}\vec{\nabla}\ \frac{C_{\ell s}}{k_s}\ I_\ell(k_s r)\cos\ell\theta\cos k_s z\ . \tag{62}$$

Let us assume that we accomplish this by means of surface currents on a shell of radius $\rho > a$. The field external to the shell due to this current sheet is of the form

$$\vec{B}_{ext} = B^{(o)}\vec{\nabla}\ \frac{E_{\ell s}}{k_s}\ K_\ell(k_s r)\cos\ell\theta\cos k_s z\ . \tag{63}$$

The boundary conditions at $r = \rho$ are that the normal component of the magnetic field must be continuous and that the jump in the tangential component be proportional to the linear surface current density. Thus, using Eq. (19) for $C_{\ell s}$ and the Wronskian relation Eq. (27), we obtain the surface current density

$$\vec{J}_{\ell s} = \frac{B^{(o)}\delta_{\ell s}(k_s a)^2 K_\ell(k_s a)}{4\pi\ k_s\rho\ K_\ell'(k_s\rho)}\left(\hat{e}_\theta\cos\ell\theta\sin k_s z - \hat{e}_z\ \frac{\ell}{k_s\rho}\ \sin\ell\theta\cos k_s z\right). \tag{64}$$

The coefficient in Eq. (63) is given by the relation

$$E_{\ell s} = \frac{(k_s a)^2\ \delta_{\ell s}\ K_\ell\ (k_s a) I_\ell'(k_s\rho)}{K_\ell'(k_s\rho)}\quad . \tag{65}$$

The currents of Eq. (64) can be quite large, especially if a/ρ is small. If $k_s\rho$ is sufficiently small that the small-argument limit can be used for the Bessel functions, then we obtain

$$\vec{J}_{\ell s} \cong -\ \frac{B^{(o)}\delta_{\ell s}(k_s a)^2}{4\pi\ell}\left(\frac{\rho}{a}\right)^\ell\left(\hat{e}_\theta\cos\ell\theta\sin k_s z - \hat{e}_z\frac{\ell}{k_s\rho}\sin\ell\theta\cos k_s z\right)\ .\quad \ell \neq 0. \tag{66}$$

Thus, the multipole currents do not form a diverging sequence in ℓ, provided that

$$\frac{\delta_{\ell s}}{\delta_{os}} \lesssim \left(\frac{a}{\rho}\right)^\ell\quad , \tag{67}$$

a requirement which can be satisfied by the restriction

$$\rho \lesssim (ka)^2 R\ \delta_o^2\quad . \tag{68}$$

We note that the current necessary to generate the field of Eq. (56) for the general toroidal case is given by

$$\vec{J}_\ell = \frac{B^{(o)}\delta_\ell(ka)^2 K_\ell(ka)}{4\pi k\rho K_\ell'(k\rho)}\left[1 - (1-\beta)\ \frac{I_\ell(ka)K_\ell'(ka)}{I_\ell'(ka)K_\ell(ka)}\right]\left(\hat{e}_\theta\cos\ell\theta\sin kz - \hat{e}_z\ \frac{\ell}{k\rho}\ \sin\ell\theta\cos kz\right). \tag{69}$$

VII. DISCUSSION

We have constructed an analytic series solution of the hydromagnetic equilibrium equations for a plasma with a sharp surface confined in a bumpy torus. For a system of large aspect ratio, and bulges of short wavelength, the series converges very rapidly and only the first few ℓ values (say up to $\ell = 2$) should be needed. An uncomfortable feature of our results is that an $\ell = 0$ field is indispensable; such a field, implying compression of field and plasma, costs much more energy to set up than the higher ℓ fields, which merely distort the surface. From a physical standpoint, however, a system without the $\ell = 0$ component is not excluded, even though it cannot be treated by perturbations about a toroidal cylinder. If the surfaces of constant $\oint d\ell/B$ were not circular at $z = 0$ but had finite ellipticity, then the latter could partially compensate for the δ_1^2 and δ_2 terms which are proportional to $\cos 2\theta$, and, thus, a configuration without $\ell = 0$ fields might exist. The ellipticity is necessarily so large that our expansion technique becomes inapplicable. A more suitable method might be the formulation of the entire problem in terms of separable toroidal-elliptical coordinates, a most formidable mathematical undertaking.

More serious reservations are in order relative to the highly idealized nature of our model, in particular the thin-skin assumption which was dictated by the demands of mathematical tractability. Techniques for an analytic solution of the toroidal, high-beta, diffuse equilibrium problem are not known at present, especially if a realistic self-consistent treatment of the particle orbits and charge separation fields is to be provided.

A third, and crucial, reservation must be made concerning the stability of our equilibrium configuration. The self-adjustment of the column to provide the correct $D_\ell^{(2)}$ z-dependent fields, for example, is predicated upon the equilibrium's being stable. While our configuration can be made stable against general magnetohydrodynamic interchanges, localized ballooning modes are known to develop in the unfavorable region of the bumps, and a normal mode analysis has been made for the modes of higher m-number.[13] The question of stability with respect to a rigid outward (m = 1) displacement is open, but Haas and Wesson[14] have shown that a related configuration is unstable. Haas and Wesson[15] have found that dynamic stabilization is helpful but its efficacy is not yet understood. The subject of microinstabilities for a more realistic plasma model is untouched.

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