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# Hyers–Ulam stability of a coupled system of fractional differential equations of Hilfer–Hadamard type

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**Abstract:** In this paper, existence and uniqueness of solution for a coupled impulsive Hilfer–Hadamard type fractional differential system are obtained by using Kransnoselskii's fixed point theorem. Different types of Hyers–Ulam stability are also discussed. We provide an example demonstrating consistency to the theoretical findings.

**Keywords:** Hilfer-Hadamard type fractional differential equations, Kransnoselskii's fixed point theorem, implicit switched coupled systems, Hyers-Ulam stability

**MSC:** 26A33, 34A08, 34B27

## 1 Introduction

The theory of fractional differential equations (FDEs) is a growing area of research. Recently, it has been realized that FDEs can describe a large number of nonlinear phenomena in different fields of science like physics, chemistry, biology, viscoelasticity, control hypothesis, speculation, fluid dynamics, hydrodynamics, aerodynamics, information processing system networking, notable and picture processing etc. In addition, FDEs can provide marvelous tools for the depiction of memory and inherited properties of many materials and processes. Consequently, FDEs have emerged significant developments and thus important results have reported in recent years [1–17].

One of the most attractive research areas in the field of FDEs which has engrossed great consideration amongst researchers is dedicated to the existence theory of the solutions of fractional models. The aforesaid area has been extensively explored for integer order differential equations (DEs). However, for arbitrary order DEs, there are still many aspects that need further study and research. Different mathematicians explore FDEs in different directions; the reader may see [18–25] and references cited therein. Another imperative and more remarkable area of research which has recently attracted more attention is committed to the stability analysis of DEs of integer and non integer order. The first effort was initiated by Ulam in 1940 and later on confirmed by Hyers in 1941 (see [26]). That's why this type of stability is called Hyers–Ulam (HU) stability. Rassias introduced the Hyers–Ulam–Rassias (HUR) stability. Obloza was the first mathematician who introduced the HU stability for DEs; the reader can consult [27–43] for comprehensive literature. It is to be noted that, the above said areas of interest (existence and stability) have been fabulously deliberated by adapting Riemann–Liouville and Caputo derivatives.

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Recently, significant consideration has been given to the existence of solutions of boundary and initial value problems for FDEs with Hilfer–Hadamard (HH) type fractional derivative. In [44], Abbas *et al.* studied the existence and stability of the solution of FDEs involving HH type derivative given by

$$\begin{cases} {}^{H}D^{\alpha,\beta}u(t) = f(t, u(t)), \ t \in \mathcal{J}, \ \ 0 < \alpha < 1, \ 0 < \beta \le 1, \\ I_{1^+}^{1-\gamma}u(1) = \phi, \ \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

where $\mathcal{J} = (1, T]$ with $T > 1$ and ${}^{H}D^{\alpha,\beta}$ denotes HH fractional derivative of order $\alpha$ and type $\beta$ introduced by Hilfer in [45], $\phi \in \mathcal{R}$, $f : \mathcal{J} \times \mathcal{R} \to \mathcal{R}$ is a continuous function and $I_{1^+}^{1-\gamma}$ is the left–sided mixed Hadamard type integral of order $1 - \gamma$.

In [46], Wang *et al.* investigated existence theory corresponding to solution for the class of FDEs:

$$\begin{cases} {}^{c}D^{\alpha}p(t) - \mathbf{F}_1(t)p(t) = \varphi(t, p(t), q(t)), \quad t \in \mathbf{J} = [0, 1] - \{t_1, t_2, \dots, t_m\}, \\ {}^{c}D^{\beta}q(t) - \mathbf{F}_2(t)q(t) = \psi(t, p(t), q(t)), \\ \lambda p(0) + \xi p'(0) = h(p), \quad \lambda p(1) + \xi p'(1) = g(p), \\ \lambda q(0) + \xi q'(0) = h(q), \quad \lambda q(1) + \xi q'(1) = g(q), \\ \Delta p(t_k) = I_k(p(t_k)), \quad \Delta p'(t_k) = \widetilde{I}_k(p(t_k)), \\ \Delta q(t_k) = I_k(q(t_k)), \quad \Delta q'(t_k) = \widetilde{I}_k(q(t_k)), \ \ 0 < t_k < 1, \end{cases} \tag{1.1}$$

where ${}^{c}D^{\alpha}$ represents the Caputo fractional derivatives of order $\alpha, \beta \in (0, 1]$ for the functions $p$ and $q$ with lower limits $t_k$, $k = 1, 2, \dots, m$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$ and $\mathbf{F}_1(\cdot), \mathbf{F}_2(\cdot)$ are linear and bounded operators on $\mathcal{R}$. Furthermore, $I_k$ and $\widetilde{I}_k$ are the impulsive operators. Also $\phi : \mathcal{C}(\mathbf{J}, \mathcal{R}) \to \mathcal{D}(\chi_1(t))$, $\varphi : \mathcal{C}(\mathbf{J}, \mathcal{R}) \to \mathcal{D}(\chi_2(t))$ are continuous and nonlinear functions. Moreover, $\Delta p(t_k)|_{t_0 \neq t_k} = p(t_k^+) - p(t_k^-)$, $\Delta q(t_k)|_{t_0 \neq t_k} = q(t_k^+) - q(t_k^-)$, $\Delta p'(t_k)|_{t_0 \neq t_k} = p'(t_k^+) - p'(t_k^-)$ and $\Delta q'(t_k)|_{t_0 \neq t_k} = q'(t_k^+) - q'(t_k^-)$, where $p(t_k^+)$, $q(t_k^+)$, $p'(t_k^+)$, $q'(t_k^+)$ and $p(t_k^-)$, $q(t_k^-)$, $p'(t_k^-)$, $q'(t_k^-)$ are right and left limits, respectively. For more details, the reader may see [47–56] and references cited therein.

The objective of this paper is to use the basic concepts mentioned in [44] combined with the methodology applied in [46], to examine the existence and uniqueness as well as different kinds of HU stability for the solutions of coupled impulsive FDEs involving HH type derivative. The proposed system is given by:

$$\begin{cases} {}_{H}D^{p,q}u(t) = f(t, u(t), {}_{H}D^{p,q}v(t)), \quad t \in (1, T], \ T > 1, \ 0 < p < 1, \ 0 < q \le 1, \\ {}_{H}D^{p,q}v(t) = g(t, v(t), {}_{H}D^{p,q}u(t)), \quad \gamma = p + q - pq, \\ I_{1^+}^{1-\gamma}u(1^+) = a, \ I_{1^+}^{1-\gamma}v(1^+) = c, \\ I_{1^+}^{1-\gamma}u(T) = b, \ I_{1^+}^{1-\gamma}v(T) = d, \end{cases} \tag{1.2}$$

where ${}_{H}D^{p,q}$ represents the HH type derivatives for the functions $u$ and $v$ of order $p \in (0, 1)$ and $q \in (0, 1]$ and $I_{1^+}^{1-\gamma}$ is the left–sided mixed Hadamard type integral of order $1 - \gamma$. Let $\mathcal{J} = (1, T]$ with $T > 1$, then $f, g : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are continuous and nonlinear functions on a Banach space $\mathcal{X} := \mathcal{R}$.

This work is outlined as follows: In Section 2, we present some basic notions needed to prove our main results. In Section 3, we setup some adequate conditions that are used to prove the existence-uniqueness and HU stability results of solutions for system (1.2). The established results are illustrated with an example in Section 4.

## 2 Fundamental results

In this section, we introduce basic definitions and lemmas which will be used throughout this manuscript. The notations and terminologies are adopted from [1, 5, 8, 57].

**Definition 2.1.** The fractional order Hadamard type derivative with order $\sigma$ for a function $\theta : [1, \infty) \to \mathcal{X}$ is defined as

$${}_{H}D_{1^+}^{\sigma}\theta(t) = \frac{1}{\Gamma(n-\sigma)}\left(t\frac{d}{dt}\right)^{n}\int_{1}^{t}\left(\log\left(\frac{t}{s}\right)\right)^{n-\sigma-1}\theta(s)\frac{ds}{s}, \ n - 1 < \sigma < n = 1 + \lceil\sigma\rceil,$$

where $\lceil\sigma\rceil$ is the integer part of $\sigma$.

**Definition 2.2.** The fractional order Hadamard type integral with order $\sigma$ for a function $\theta : [1,\infty) \to \mathcal{X}$ is given as

$$I^{\sigma}_{1^+}\theta(t) = \frac{1}{\Gamma(\sigma)} \int\limits_1^t \left(\log\left(\frac{t}{s}\right)\right)^{\sigma-1} \theta(s)\frac{ds}{s}, \sigma > 0,$$

provided that the integral on the right side exists.

**Definition 2.3.** For $\alpha \in (0,1),\ \beta \in (0,1],\ \theta \in L^1\{\mathcal{R}^+\}$ and $I^{(1-\alpha)(1-\beta)}_{1^+} \in \mathcal{C}_{1-\gamma,\log(t)}(\mathcal{J},\mathcal{X})$, the HH type derivative of order $\alpha,\beta$ for a function $\theta$ is defined as

$$D^{\alpha,\beta}_{1^+}\theta(t) = I^{\beta(1-\alpha)}_{1^+}\frac{d}{dt}I^{(1-\alpha)(1-\beta)}_{1^+}\theta(t), t \in \mathcal{J}.$$

**Lemma 2.4.** *Let* $0 < \alpha < 1,\ 0 < \beta \leq 1$. *Then the homogenous DE along with HH fractional order*

$$H^{\alpha,\beta}_{1^+}\theta(t) = 0$$

*has solution of the form*

$$\theta(t) = b_0(\log(t))^{\gamma} + b_1(\log(t))^{\gamma+2\beta-2} + b_2(\log(t))^{\gamma+2(2\beta)-3} + \cdots + b_n(\log(t))^{\gamma+n(2\beta)-(n+1)}.$$

**Theorem 2.5.** *Let* $\mathcal{S} \neq \emptyset$ *be a convex and closed subset of a Banach space* $\mathcal{E}$. *Consider two operators* $\mathbb{G}$ *and* $\mathbb{F}$ *such that*

*(I)* $\mathbb{G}(u,v) + \mathbb{F}(u,v) \in \mathcal{S}$, *where* $(u,v) \in \mathcal{S}$;
*(II)* $\mathbb{G}$ *is a contraction mapping;*
*(III)* $\mathbb{F}$ *is a completely continuous operator.*

*Then the operator system* $\mathbb{G}(u,v) + \mathbb{F}(u,v) = (u,v) \in \mathcal{E}$ *has a solution in* $\mathcal{S}$.

**Definition 2.6.** Consider a Banach space $\mathcal{E}$ such that $\Phi_1,\Phi_2 : \mathcal{E} \to \mathcal{E}$ are two operators. Then the operator system

$$\begin{cases} u(t) = \Phi_1(u,v)(t), \\ v(t) = \Phi_2(u,v)(t) \end{cases} \tag{2.1}$$

is called HU stable if there exist constants $C_i(i = 1,2,3,4) > 0$ for each $\varrho_j(j = 1,2) > 0$ and for each solution $(\widehat{u},\widehat{v}) \in \mathcal{E}$ of the inequalities

$$\begin{cases} \|\widehat{u} - \phi(\widehat{u},\widehat{v})\| \leq \varrho_1, \\ \|\widehat{v} - \varphi(\widehat{u},\widehat{v})\| \leq \varrho_2, \end{cases} \tag{2.2}$$

there exists a solution $(\widetilde{u},\widetilde{v}) \in \mathcal{E}$ of system (2.1), which satisfies the inequalities

$$\begin{cases} \|\widehat{u} - \widetilde{u}\| \leq C_1\varrho_1 + C_2\varrho_2, \\ \|\widehat{v} - \widetilde{v}\| \leq C_3\varrho_1 + C_4\varrho_2. \end{cases} \tag{2.3}$$

**Definition 2.7.** Let $\mu_j$ (for $j = 1,2,\ldots,m$) be the eigenvalues of a matrix $\mathcal{H} \in \mathbb{C}^{m\times m}$. Then the spectral radius $r(\mathcal{H})$ of $\mathcal{H}$ is defined by

$$r(\mathcal{H}) = \max\{|\mu_j|, \quad \text{for} \quad j = 1,2,\ldots,m\}.$$

Furthermore, the system corresponding to $\mathcal{H}$ converges to zero provided that $r(\mathcal{H}) < 1$.

**Theorem 2.8.** *Consider a Banach space $\mathcal{E}$ with operators $\Phi_1, \Phi_2 : \mathcal{E} \to \mathcal{E}$ such that*

$$\begin{cases} \|\Phi_1(u,v) - \Phi_1(\widehat{u},\widehat{v})\| \leq \Lambda_1\|u-\widehat{u}\| + \Lambda_2\|v-\widehat{v}\|, \\ \|\Phi_2(u,v) - \Phi_2(\widehat{u},\widehat{v})\| \leq \Lambda_3\|u-\widehat{u}\| + \Lambda_4\|v-\widehat{v}\|, \end{cases}$$

*if the spectral radius of matrix*

$$\mathcal{H} = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix}$$

*is less than one, then the fixed points corresponding to operational system* (1.2) *are HU stable.*

## 3 Existence, uniqueness and stability results

Here, we discuss the existence, uniqueness and stability of our proposed system. Our first result is stated as follows.

**Theorem 3.1.** *Let $y_1, y_2 \in \mathcal{C}_{1-\gamma,\log}(\mathcal{J},\mathcal{X})$. Then for any $u, v \in \mathcal{C}_{1-\gamma,\log(t)}(\mathcal{J},\mathcal{X})$ have the forms*

$$\begin{aligned} u(t) &= \frac{a(\log(t))^{\gamma-1}}{\Gamma(\gamma)} + (b - a - I^{1-q(1-p)}f(t,u(T),{}_H D^{p,q}v(T))\frac{\Gamma(2q)(\log)^{\gamma+2q-2}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}} \\ &\quad + \frac{1}{\Gamma(p)}\int_1^t \left(\log\left(\frac{t}{s}\right)\right)^{p-1} y_1(s)\frac{ds}{s}, \\ v(t) &= \frac{c(\log(t))^{\gamma-1}}{\Gamma(\gamma)} + (d - c - I^{1-q(1-p)}g(t,v(T),{}_H D^{p,q}u(T)))\frac{\Gamma(2q)(\log)^{\gamma+2q-2}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}} \\ &\quad + \frac{1}{\Gamma(p)}\int_1^t \left(\log\left(\frac{t}{s}\right)\right)^{p-1} y_2(s)\frac{ds}{s}, \end{aligned}$$

*if and only if $u, v$ are the solutions of*

$$\begin{cases} {}_H D^{p,q}u(t) = y_1(t), \quad 0 < p, q \leq 1, \; t \in \mathcal{J}, \\ {}_H D^{p,q}v(t) = y_2(t), \\ I_{1^+}^{1-\gamma}u(1^+) = a, \; I_{1^+}^{1-\gamma}u(T) = b, \\ I_{1^+}^{1-\gamma}v(1^+) = c, \; I_{1^+}^{1-\gamma}v(T) = d. \end{cases} \tag{3.1}$$

*Proof.* Let $u, v \in \mathcal{C}_{1-\gamma,\log(t)}(\mathcal{J},\mathcal{X})$ be a solution of (3.1). Then

$$\begin{cases} {}_H D^{p,q}u(t) = y_1(t), \quad 0 < p \leq 1, 0 < q \leq 1, \; t \in \mathcal{J}, \\ {}_H D^{p,q}v(t) = y_2(t), \\ I_{1^+}^{1-\gamma}u(1^+) = a, \; I_{1^+}^{1-\gamma}u(T) = b, \\ I_{1^+}^{1-\gamma}v(1^+) = c, \; I_{1^+}^{1-\gamma}v(T) = d. \end{cases}$$

Since

$${}_H D^{p,q}u(t) = y_1(t), \quad 0 < p < 1, 0 < q \leq 1, \; t \in \mathcal{J}, \tag{3.2}$$

then by using Lemma 2.4, we have

$$u(t) = b_0(\log t)^{\gamma-1} + b_1(\log t)^{\gamma+2q-2} + \frac{1}{\Gamma(p)}\int_1^t \left(\log\left(\frac{t}{s}\right)\right)^{p-1} y_1(s)\frac{ds}{s}. \tag{3.3}$$

Applying the boundary conditions, we get

$$b_0 = \frac{a}{\Gamma(\gamma)}$$

and

$$b_1 = (b - a - I_{1^+}^{1-q(1-p)} f(t, u(T), {}_H D^{p,q} v(T))) \frac{\Gamma(2q)(\log)^{\gamma+2q-2}}{\Gamma(\gamma + 2q - 1)(\log(T))^{2q-1}}.$$

Therefore (3.3) becomes

$$\begin{aligned} u(t) &= \frac{a(\log(t))^{\gamma-1}}{\Gamma(\gamma)} + (b - a - I_{1^+}^{1-q(1-p)} f(t, u(T), {}_H D^{p,q} v(T))) \frac{\Gamma(2q)(\log)^{\gamma+2q-2}}{\Gamma(\gamma + 2q - 1)(\log(T))^{2q-1}} \\ &\quad + \frac{1}{\Gamma(p)} \int_1^t \left( \log\left(\frac{t}{s}\right)\right)^{p-1} y_1(s) \frac{ds}{s}. \end{aligned}$$

Similarly, we may have

$$\begin{aligned} v(t) &= \frac{c(\log(t))^{\gamma-1}}{\Gamma(\gamma)} + (d - c - I_{1^+}^{1-q(1-p)} g(t, v(T), {}_H D^{p,q} u(T))) \frac{\Gamma(2q)(\log)^{\gamma+2q-2}}{\Gamma(\gamma + 2q - 1)(\log(T))^{2q-1}} \\ &\quad + \frac{1}{\Gamma(p)} \int_1^t \left( \log\left(\frac{t}{s}\right)\right)^{p-1} y_2(s) \frac{ds}{s}. \end{aligned}$$

The proof is completed. □

We make use of the following assumptions:

**(H$_1$)** The functions $f, g : \mathcal{J} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are continuous, $\forall (u, v), (\bar{u}, \bar{v}) \in \mathcal{X} \times \mathcal{X}$ and $t \in \mathcal{J}$, there exist $\mathcal{M}_f, \mathcal{M}_g, \mathcal{M}'_f, \mathcal{M}'_g > 0$ such that

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le \mathcal{M}_f |u - \bar{u}| + \mathcal{M}'_f |v - \bar{v}|,$$

$$|g(t, u, v) - g(t, \bar{u}, \bar{v})| \le \mathcal{M}_g |u - \bar{u}| + \mathcal{M}'_g |v - \bar{v}|;$$

**(H$_2$)** $f, g : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ are completely continuous functions $\forall\, u,\ v \in \mathcal{X}$ and $t \in \mathcal{J}$, there exist nondecreasing continuous linear functions $\mu_f, \mu_g : \mathcal{X} \to \mathcal{X}^+$ such that

$$|f(t, u, v)| \le \mu_f |u| + \mu'_f |v|,$$

$$|g(t, u, v)| \le \mu_g |u| + \mu'_g |v|,$$

where

$$\sup\{\mu_f(t), t \in \mathcal{J}\} = \mu_f, \quad \sup\{\mu_g(t), t \in \mathcal{J}\} = \mu_g,$$
$$\sup\{\mu'_f(t), t \in \mathcal{J}\} = \mu'_f, \quad \sup\{\mu'_g(t), t \in \mathcal{J}\} = \mu'_g;$$

**(H$_3$)** Let $\xi^* = \max\{\xi_1, \xi_2\} < 1$ with

$$\xi_1 = \frac{\big[\Gamma(2q)\Gamma(p+1) + \Gamma(2q - q(1-p))\Gamma(\gamma + 2q - 1)\big] M_f (1 + M'_g)}{(1 - M'_f M'_g \Gamma(2 - q(1-p)))\Gamma(\gamma + 2q - 1)\Gamma(p+1)} (\log(T))^p$$

and

$$\xi_2 = \frac{\big[\Gamma(2q)\Gamma(p+1) + \Gamma(2q - q(1-p))\Gamma(\gamma + 2q - 1)\big] M_g (1 + M'_f)}{(1 - M'_f M'_g \Gamma(2 - q(1-p)))\Gamma(\gamma + 2q - 1)\Gamma(p+1)} (\log(T))^p.$$

Choose a closed ball $\mathcal{E}_r = \left\{(u, v) \in \mathcal{X}, \|(u, v)\|_{1-\gamma,\log(t)} \le r, \|u\|_{1-\gamma,\log(t)} \le \frac{r}{2}, \|v\|_{1-\gamma,\log(t)} \le \frac{r}{2}\right\} \subset \mathcal{X}$, where

$$r \ge \frac{\frac{a+c}{\Gamma(\gamma)} + \frac{((b-a)+(d-c))\Gamma(2q)}{\Gamma(\gamma+2q-1)}}{1 - \frac{(\mu_f(1+\mu'_g)+\mu_g(1+\mu'_f))(\log(T))^p}{2(1-\mu'_f\mu'_g)} \left[ \frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))} + \frac{1}{\Gamma(p+1)} \right]}.$$

It is obvious that $(\mathcal{J}, \mathcal{X})$ is a Banach space with the norm $\|u\| = \max\{|u(t)|, t \in \mathcal{J}\}$ and $(\mathcal{J}, \mathcal{X} \times \mathcal{X})$ is a Banach space with norm $\|(u, v)\| = \|u\| + \|v\|$.

$\mathcal{C}_{1-\gamma,\log(t)}(\mathcal{J},\mathcal{X})$ denote the space of all continuous functions defined as

$$\mathcal{C}_{1-\gamma,\log(t)}(\mathcal{J},\mathcal{X}) = \{x : (1,T] \to \mathcal{X} \mid (\log(\cdot))^{1-\gamma}x(\cdot) \in \mathcal{C}(\mathcal{J},\mathcal{X})\}$$

with norm

$$\|x\|_{\mathcal{C}_{1-\gamma,\log(t)}} = \sup\{|x(t)(\log(t))^{1-\gamma}|,\ t \in \mathcal{J}\}.$$

Define the operators $\mathfrak{F} = (\mathfrak{F}_1, \mathfrak{F}_2)$, $\mathfrak{G} = (\mathfrak{G}_1, \mathfrak{G}_2)$ on $\mathcal{E}_r$ as

$$\begin{cases} \mathfrak{F}_1(u(t)) = \frac{a(\log(t))^{\gamma-1}}{\Gamma(\gamma)} + (b - a - I_{1^+}^{1-q(1-p)} f(t, u(T),_H D^{p,q} v(T))) \frac{\Gamma(2q)(\log(t))^{\gamma+2q-2}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}}, \\ \mathfrak{F}_2(v(t)) = \frac{c(\log(t))^{\gamma-1}}{\Gamma(\gamma)} + (d - c - I_{1^+}^{1-q(1-p)} g(t, v(T),_H D^{p,q} u(T))) \frac{\Gamma(2q)(\log(t))^{\gamma+2q-2}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}} \end{cases} \tag{3.4}$$

and

$$\begin{cases} \mathfrak{G}_1(u(t), v(t)) = \frac{1}{\Gamma(p)} \int_1^t (\log \frac{t}{s})^{p-1} f(s, u(s),_H D^{p,q} v(s)) \frac{ds}{s}, \\ \mathfrak{G}_2(u(t), v(t)) = \frac{1}{\Gamma(p)} \int_1^t (\log \frac{t}{s})^{p-1} g(s, v(s),_H D^{p,q} u(s))) \frac{ds}{s}. \end{cases} \tag{3.5}$$

**Theorem 3.2.** *Let the assumptions* $(\mathbf{H}_1)$*–*$(\mathbf{H}_3)$ *are satisfied. Then problem* (3.1) *has at least one solution.*

*Proof.* For any $(u, v) \in \mathcal{E}_r$, we have

$$\begin{aligned} \|\mathfrak{F}(u,v) + \mathfrak{G}(u,v)\|_{1-\gamma,\log(t)} &\le \|\mathfrak{F}(u,v)\|_{1-\gamma,\log(t)} + \|\mathfrak{G}(u,v)\|_{1-\gamma,\log(t)} \\ &\le \|\mathfrak{F}_1(u)\|_{1-\gamma,\log(t)} + \|\mathfrak{F}_2(v)\|_{1-\gamma,\log(t)} + \|\mathfrak{G}_1(u,v)\|_{1-\gamma,\log(t)} \\ &\quad + \|\mathfrak{G}_2(u,v)\|_{1-\gamma,\log(t)}. \end{aligned} \tag{3.6}$$

Set

$$\begin{aligned} k_f(t) &= f(t, u(t), D^{p,q} v(t)), \\ k_g(t) &= g(t, v(t), D^{p,q} u(t)),\ \forall\, t \in \mathcal{J}. \end{aligned}$$

Thus

$$\begin{aligned} |k_f(t)| &= |f(t, u(t), D^{p,q} v(t))| \\ &\le \mu_f|u| + \mu_f'|D^{p,q} v(t)| \\ &= \mu_f|u| + \mu_f'|k_g(t)| = \mu_f|u| + \mu_f'(|g(t, v(t), D^{p,q} u(t))| \\ &\le \mu_f|u| + \mu_f'(\mu_g|v| + \mu_g'|k_f(t)|) \end{aligned}$$

or

$$|f(t, u(t), D^{p,q} v(t))| \le \frac{\mu_f|u| + \mu_f'\mu_g|v|}{1 - \mu_f'\mu_g'},\ \forall\, t \in \mathcal{J}.$$

Similarly,

$$|g(t, v(t), D^{p,q} u(t))| \le \frac{\mu_g|v| + \mu_g'\mu_f|u|}{1 - \mu_f'\mu_g'},\ \forall\, t \in \mathcal{J}.$$

Next, from (3.4), we get

$$\begin{aligned} |\mathfrak{F}_1 u(t)| &= \left| \frac{a}{\Gamma(\gamma)} + (b - a - I_{1^+}^{1-q(1-p)} f(t, u(T),_H D^{p,q} v(T))) \frac{\Gamma(2q)(\log(t))^{2q-1}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}} \right| \\ &\le \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2q)(\log(t))^{2q-1}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}} (b - a + I_{1^+}^{1-q(1-p)} |f(t, u(T),_H D^{p,q} v(T))|) \end{aligned}$$

$$
\begin{aligned}
&= \frac{a}{\Gamma(\gamma)} + \frac{\Gamma(2q)(\log(t))^{2q-1}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}}\Bigg(b-a+\frac{1}{\Gamma(1-q(1-p))} \\
&\quad \times \int_1^T \left(\log\left(\frac{T}{s}\right)\right)^{(1-q(1-p))-1} |f(t,u(T),{}_H D^{p,q}v(T))|\frac{ds}{s}\Bigg) \\
&\le \frac{a}{\Gamma(\gamma)} + \frac{(b-a)\Gamma(2q)}{\Gamma(\gamma+2q-1)} + \frac{\mu_f\|u\|_{1-\gamma,\log(t)} + \mu_f'\mu_g\|v\|_{1-\gamma,\log(t)}}{1-\mu_f'\mu_g'}\frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)}\frac{(\log(T))^p}{\Gamma(2-q(1-p))} \\
&\le \frac{a}{\Gamma(\gamma)} + \frac{(b-a)\Gamma(2q)}{\Gamma(\gamma+2q-1)} + \frac{\mu_f+\mu_f'\mu_g}{2(1-\mu_f'\mu_g')}\frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)}\frac{(\log(T))^p}{\Gamma(2-q(1-p))}r.
\end{aligned}
$$

Hence

$$
\|\mathfrak{F}_1 u\|_{1-\gamma,\log(t)} \le \frac{a}{\Gamma(\gamma)} + \frac{(b-a)\Gamma(2q)}{\Gamma(\gamma+2q-1)} + \frac{\mu_f+\mu_f'\mu_g}{2(1-\mu_f'\mu_g')}\frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)}\frac{(\log(T))^p}{\Gamma(2-q(1-p))}r.
$$

By similar procedure, we get

$$
\|\mathfrak{F}_2 v\|_{1-\gamma,\log(t)} \le \frac{c}{\Gamma(\gamma)} + \frac{(d-c)\Gamma(2q)}{\Gamma(\gamma+2q-1)} + \frac{\mu_g+\mu_f\mu_g'}{2(1-\mu_f'\mu_g')}\frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)}\frac{(\log(T))^p}{\Gamma(2-q(1-p))}r.
$$

Also, we have

$$
\begin{aligned}
\|\mathfrak{G}_1(u,v)\|_{1-\gamma,\log(t)} &\le \sup_{t\in\mathfrak{J}} \frac{(\log(t))^{1-\gamma}}{\Gamma(p)} \int_1^t \left(\log\left(\frac{t}{s}\right)\right)^{p-1} \left|f(s,u(s),{}_H D^{p,q}v(s))\right|\frac{ds}{s} \\
&\le \frac{(\mu_f+\mu_f'\mu_g)(\log(T))^p}{2(1-\mu_f'\mu_g')\Gamma(p+1)}r
\end{aligned}
$$

and

$$
\|\mathfrak{G}_2(u,v)\|_{1-\gamma,\log(t)} \le \frac{(\mu_g+\mu_g'\mu_f)(\log(T))^p}{2(1-\mu_f'\mu_g')\Gamma(p+1)}r.
$$

Combining all these inequalities and using (3.6), we have

$$
\|\mathfrak{F}(u,v)+\mathfrak{G}(u,v)\|_{1-\gamma,\log(t)} \le r.
$$

Hence, $\mathfrak{F}(u,v)+\mathfrak{G}(u,v)\in\mathcal{E}_r$. Next, for any $t\in\mathfrak{J}$ and $(u,v),(\overline{u},\overline{v})\in\mathcal{X}$, we have

$$
\|(\mathfrak{F}(u,v)-\mathfrak{F}(\overline{u},\overline{v}))\|_{1-\gamma,\log(t)} \le \|(\mathfrak{F}_1(u)-\mathfrak{F}_1(\overline{u}))\|_{1-\gamma,\log(t)} + \|(\mathfrak{F}_2(v)-\mathfrak{F}_2(\overline{v}))\|_{1-\gamma,\log(t)}. \tag{3.7}
$$

Now

$$
\begin{aligned}
\|(\mathfrak{F}_1(u)-\mathfrak{F}_1(\overline{u}))\|_{1-\gamma,\log(t)} &\le \sup_{t\in\mathfrak{J}} \frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)\Gamma(1-q(1-p))} \\
&\quad \times \int_1^t \left(\log\left(\frac{t}{s}\right)\right)^{-q(1-p)} \left|f(s,u(s),{}_H D^{p,q}v(s)) - f(s,\overline{u}(s),{}_H D^{p,q}\overline{v}(s))\right| \\
&\le \frac{\Gamma(2q)(M_f|u-\overline{u}|+M_f'M_g|v-\overline{v}|)}{(1-M_f'M_g')\Gamma(\gamma+2q-1)\Gamma(1-q(1-p))}\int_1^t (\log(\frac{t}{s}))^{-q(1-p)}\frac{ds}{s} \\
&\le \left[\frac{M_f\|u-\overline{u}\|_{1-\gamma,\log(t)} + M_f'M_g\|v-\overline{v}\|_{1-\gamma,\log(t)}}{(1-M_f'M_g')\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}\right]\Gamma(2q)(\log(T))^p,\ t\le T. \qquad (3.8)
\end{aligned}
$$

Similarly,

$$
\|(\mathfrak{F}_2(u)-\mathfrak{F}_2(\overline{u}))\|_{1-\gamma,\log(t)} \le \left[\frac{M_g\|v-\overline{v}\|_{1-\gamma,\log(t)} + M_g'M_f\|u-\overline{u}\|_{1-\gamma,\log(t)}}{(1-M_f'M_g')\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}\right]\Gamma(2q)(\log(T))^p,\ t\le T. \tag{3.9}
$$

Using (3.7), we have

$$\begin{aligned}\|(\mathfrak{F}(u,v)-\mathfrak{F}(\overline{u},\overline{v}))\|_{1-\gamma,\log(t)} &\leq \frac{M_f(1+M'_g)\|u-\overline{u}\|_{1-\gamma,\log(t)}+M_g(1+M'_f)\|v-\overline{v}\|_{1-\gamma,\log(t)}}{(1-M'_fM'_g)\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}\Gamma(2q)(\log(T))^p\\ &\leq \frac{M_f(1+M'_g)\|u-\overline{u}\|_{1-\gamma,\log(t)}\Gamma(2q)(\log(T))^p}{(1-M'_fM'_g)\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}\\ &\quad+\frac{M_g(1+M'_f)\|v-\overline{v}\|_{1-\gamma,\log(t)}\Gamma(2q)(\log(T))^p}{(1-M'_fM'_g)\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}\\ &= \beta_1\|u-\overline{u}\|_{1-\gamma,\log(t)}+\beta_2\|v-\overline{v}\|_{1-\gamma,\log(t)}\\ &\leq \beta\|(u,v)-(\overline{u},\overline{v})\|_{1-\gamma,\log(t)},\end{aligned}$$

or

$$\|\mathfrak{F}(u,v)-\mathfrak{F}(\overline{u},\overline{v})\|_{1-\gamma,\log(t)} \leq \beta\|(u,v)-(\overline{u},\overline{v})\|_{1-\gamma,\log(t)},\ 0<\beta<1.$$

Here $\beta=\max\{\beta_1,\beta_2\}$, where

$$\begin{aligned}\beta_1 &= \frac{M_f(1+M'_g)\Gamma(2q)(\log(T))^p}{(1-M'_fM'_g)\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))},\\ \beta_2 &= \frac{M_g(1+M'_f)\Gamma(2q)(\log(T))^p}{(1-M'_fM'_g)\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}.\end{aligned}$$

Hence $\mathfrak{F}$ is a contraction mapping.

Now, we show that the operator $\mathfrak{G}$ is continuous and compact. Consider a sequence $\xi_n=(u_n,v_n)\in\mathcal{E}_r$ such that $(u_n,v_n)\to(u,v)$ for $n\to\infty\in\mathcal{E}_r$. Therefore, we have

$$\begin{aligned}\|(\mathfrak{G}(u_n,v_n)-\mathfrak{G}(u,v))\|_{1-\gamma,\log(t)} &\leq\|(\mathfrak{G}_1(u_n,v_n)-\mathfrak{G}_1(u,v))\|_{1-\gamma,\log(t)}+\|(\mathfrak{G}_2(u_n,v_n)-\mathfrak{G}_2(u,v))\|_{1-\gamma,\log(t)}\\ &\leq\sup_{t\in\mathfrak{J}}\frac{(\log(t))^{1-\gamma}}{\Gamma(p)}\int_1^t\left(\log\left(\frac{t}{s}\right)\right)^{p-1}\left|f(s,u_n(s),{}_HD^{p,q}v_n(s))-f(s,u(s),{}_HD^{p,q}v(s))\right|\frac{ds}{s}\\ &\quad+\frac{(\log(t))^{1-\gamma}}{\Gamma(p)}\int_1^t\left(\log\left(\frac{t}{s}\right)\right)^{p-1}\left|g(s,v_n(s),{}_HD^{p,q}u_n(s))-g(s,v(s),{}_HD^{p,q}u(s))\right|\frac{ds}{s}.\\ &\leq\left[\frac{M_f(1+M'_g)\|u_n-u\|_{1-\gamma,\log(t)}}{(1-M'_fM'_g)\Gamma(p+1)}+\frac{M_g(1+M'_f)\|v_n-v\|_{1-\gamma,\log(t)}}{(1-M'_fM'_g)\Gamma(p+1)}\right](\log(T))^p,\ t\leq T.\end{aligned}$$

This implies that $\|\mathfrak{G}(u_n,v_n)-\mathfrak{G}(u,v)\|_{\mathcal{C}_{1-\gamma,\log(t)}}\to 0$ as $n\to\infty$. Thus $\mathfrak{G}$ is continuous. To show that the operator $\mathfrak{G}$ is bounded on $\mathcal{E}_r$ we have

$$\begin{aligned}\|\mathfrak{G}(u,v)\|_{1-\gamma,\log(t)} &\leq \|(\mathfrak{G}_1(u,v)(t))\|_{1-\gamma,\log(t)}+\|(\mathfrak{G}_2(u,v)(t))\|_{1-\gamma,\log(t)}\\ &\leq \frac{(\mu_f(1+\mu'_g))+\mu_g(1+\mu'_f)}{2(1-\mu'_f\mu'_g)\Gamma(2-q(1-p))\Gamma(p+1)}(\log(T))^p r,\ t\leq T,\end{aligned}$$

which implies that $\mathfrak{G}$ is uniformly bounded on $\mathcal{E}_r$.

For equicontinuity, take $t_1,t_2\in\mathfrak{J}$ with $t_1<t_2$ and for any $(u,v)\in\mathcal{E}_r\subset\mathcal{X}$, where $\mathcal{E}_r$ is clearly bounded, we obtained

$$\begin{aligned}&\|(\mathfrak{G}(u,v)(t_1)-\mathfrak{G}(u,v)(t_2))\|_{1-\gamma,\log(t)}\\ &\leq\|((\mathfrak{G}_1(u,v)(t_1)-\mathfrak{G}_1(u,v)(t_2)))\|_{1-\gamma,\log(t)}+\|(\mathfrak{G}_2(u,v)(t_1)-\mathfrak{G}_2(u,v)(t_2))\|_{1-\gamma,\log(t)}\\ &\leq\left|\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{p-1}f(s,u(s),{}_HD^{p,q}v(s))\frac{ds}{s}-\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{p-1}f(s,u(s),{}_HD^{p,q}v(s))\frac{ds}{s}\right|\end{aligned}$$

$$+\left|\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{p-1}g(s,v(s),{}_H D^{p,q}u(s))\frac{ds}{s}-\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_1}\left(\log\frac{t_2}{s}\right)^{p-1}g(s,v(s),{}_H D^{p,q}u(s))\frac{ds}{s}\right|$$
$$\leq\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{p-1}|f(s,u(s),{}_H D^{p,q}v(s))|\frac{ds}{s}+\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{p-1}|f(s,u(s),{}_H D^{p,q}v(s))|\frac{ds}{s}$$
$$+\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{p-1}|g(s,v(s),{}_H D^{p,q}u(s))|\frac{ds}{s}+\frac{(\log t)^{1-\gamma}}{\Gamma(p)}\int_1^{t_1}\left(\log\frac{t_2}{s}\right)^{p-1}|g(s,v(s),{}_H D^{p,q}u(s))|\frac{ds}{s}$$
$$\leq\frac{\mu_f(1+\mu_g')+\mu_g(1+\mu_f')}{2(1-\mu_f'\mu_g')}\left[\frac{r}{\Gamma(p+1)}\left(\log(\frac{t_2}{t_1})\right)^p+\frac{r}{\Gamma(p)}\int_1^{t_1}\left|\left(\log(\frac{t_2}{s})\right)^{p-1}-\left(\log(\frac{t_1}{s})\right)^{p-1}\right|\frac{ds}{s}\right].$$

From this, we conclude that $\|\mathfrak{G}(u,v)(t_1)-\mathfrak{G}(u,v)(t_2)\|_{1-\gamma,\log(t)}\to 0$ as $t_1\to t_2$. Therefore $\mathfrak{G}$ is relatively compact on $\mathcal{E}_r$. By Arzelä-Ascolli theorem $\mathfrak{G}$ is compact and, hence, is completely continuous operator. So (3.1) has at least one solution. □

**Theorem 3.3.** *If the assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *are true with* $\xi^\star<1$, *then* (3.1) *has unique solution.*

*Proof.* Define operator $\phi=(\phi_1,\phi_2):\mathcal{X}\to\mathcal{X}$ by

$$\phi(u,v)(t)=(\phi_1(u,v),\phi_2(u,v))(t),\ \forall\, t\in\mathcal{J},$$

where

$$\phi_1(u,v)(t)(\log(t))^{1-\gamma}=\frac{a}{\Gamma(\gamma)}+(b-a-I_{1^+}^{1-q(1-p)}f(t,u(T),{}_H D^{p,q}v(T)))\frac{\Gamma(2q)(\log(t))^{2q-1}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}}+\frac{(\log(t))^{1-\gamma}}{\Gamma(p)}\int_1^t(\log\frac{t}{s})^{p-1}f(s,u(s),{}_H D^{p,q}v(s))\frac{ds}{s}$$

and

$$\phi_2(u,v)(t)(\log(t))^{1-\gamma}=\frac{a}{\Gamma(\gamma)}+(d-c-I_{1^+}^{1-q(1-p)}g(t,v(T),{}_H D^{p,q}u(T)))\frac{\Gamma(2q)(\log(t))^{2q-1}}{\Gamma(\gamma+2q-1)(\log(T))^{2q-1}}+\frac{(\log(t))^{1-\gamma}}{\Gamma(p)}\int_1^t(\log\frac{t}{s})^{p-1}g(s,v(s),{}_H D^{p,q}u(s))\frac{ds}{s}.$$

Now for any $(u,v),(\overline{u},\overline{v})\in\mathcal{X}$, we obtain

$$\|\phi(u,v)-\phi(\overline{u},\overline{v})\|_{1-\gamma,\log(t)}$$
$$\leq\sup_{t\in\mathcal{J}}\frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)\Gamma(1-q(1-p))}\int_1^t(\log\frac{t}{s})^{-q(1-p)}|f(t,u(t),{}_H D^{p,q}v(t))-f(t,\overline{u}(t),{}_H D^{p,q}\overline{v}(t))|\frac{ds}{s}$$
$$+\frac{(\log(t))^{1-\gamma}}{\Gamma(p)}\int_1^t\left(\log\frac{t}{s}\right)^{p-1}|f(t,u(t),{}_H D^{p,q}v(t))-f(t,\overline{u}(t),{}_H D^{p,q}\overline{v}(t))|\frac{ds}{s}$$
$$+\frac{\Gamma(2q)}{\Gamma(\gamma+2q-1)\Gamma(1-q(1-p))}\int_1^t\left(\log\frac{t}{s}\right)^{-q(1-p)}|g(t,v(t),{}_H D^{p,q}u(t))-g(t,\overline{v}(t),{}_H D^{p,q}\overline{u}(t))|\frac{ds}{s}$$
$$+\frac{(\log(t))^{1-\gamma}}{\Gamma(p)}\int_1^t(\log\frac{t}{s})^{p-1}|g(t,v(t),{}_H D^{p,q}u(t))-g(t,\overline{v}(t),{}_H D^{p,q}\overline{u}(t))|\frac{ds}{s}$$

$$
\begin{aligned}
&\leq \left[\frac{(M_f\|u-\overline{u}\|_{1-\gamma,\log(t)} + M_f' M_g\|v-\overline{v}\|_{1-\gamma,\log(t)})\Gamma(2q)}{(1-M_f'M_g')\Gamma(2-q(1-p))\Gamma(\gamma+2q-1)}(\log(T))^p \right.\\
&\quad \left. + \frac{M_f\|u-\overline{u}\|_{1-\gamma,\log(t)} + M_f' M_g\|v-\overline{v}\|_{1-\gamma,\log(t)}}{(1-M_f'M_g')\Gamma(p+1)}(\log(T))^p\right]\\
&\quad + \left[\frac{(M_f M_g'\|u-\overline{u}\|_{1-\gamma,\log(t)} + M_g\|v-\overline{v}\|_{1-\gamma,\log(t)})\Gamma(2q)}{(1-M_f'M_g')\Gamma(2-q(1-p))\Gamma(\gamma+2q-1)}(\log(T))^p \right.\\
&\quad \left. + \frac{M_f M_g'\|u-\overline{u}\|_{1-\gamma,\log(t)} + M_g\|v-\overline{v}\|_{1-\gamma,\log(t)}}{(1-M_f'M_g')\Gamma(p+1)}(\log(T))^p\right]\\
&\leq \frac{\left[\Gamma(2q)\Gamma(p+1) + \Gamma(2q-q(1-p))\Gamma(\gamma+2q-1)\right] M_f(1+M_g')}{(1-M_f'M_g')\Gamma(2-q(1-p)))\Gamma(\gamma+2q-1)\Gamma(p+1)}(\log(T))^p\|u-\overline{u}\|_{1-\gamma,\log(t)}\\
&\quad + \frac{\left[\Gamma(2q)\Gamma(p+1) + \Gamma(2q-q(1-p))\Gamma(\gamma+2q-1)\right] M_g(1+M_f')}{(1-M_f'M_g')\Gamma(2-q(1-p)))\Gamma(\gamma+2q-1)\Gamma(p+1)}(\log(T))^p\|v-\overline{v}\|_{1-\gamma,\log(t)}.
\end{aligned}
$$

Thus $\|(\phi(u,v)-\phi(\overline{u},\overline{v}))\| \leq \xi^\star\|(u,v)-(\overline{u},\overline{v})\|$. Here $1 > \xi^\star = \max\{\xi_1,\xi_2\}$ with

$$
\begin{aligned}
\xi_1 &= \frac{\left[\Gamma(2q)\Gamma(p+1) + \Gamma(2q-q(1-p))\Gamma(\gamma+2q-1)\right] M_f(1+M_g')}{(1-M_f'M_g'\Gamma(2-q(1-p)))\Gamma(\gamma+2q-1)\Gamma(p+1)}(\log(T))^p,\\
\xi_2 &= \frac{\left[\Gamma(2q)\Gamma(p+1) + \Gamma(2q-q(1-p))\Gamma(\gamma+2q-1)\right] M_g(1+M_f')}{(1-M_f'M_g'\Gamma(2-q(1-p)))\Gamma(\gamma+2q-1)\Gamma(p+1)}(\log(T))^p.
\end{aligned}
$$

This implies that the operator $\phi$ is contraction. Therefore (3.1) has a unique solution. □

We complete this section by studying HU stability of the proposed system.

Set

$$
\mathcal{H} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},
$$

where $C_1 = \frac{M_f(1+M_g')\Gamma(2q)(\log(T))^p}{(1-M_f'M_g')\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}$, $C_2 = \frac{M_g(1+M_f')\Gamma(2q)(\log(T))^p}{(1-M_f'M_g')\Gamma(\gamma+2q-1)\Gamma(2-q(1-p))}$, $C_3 = \frac{M_f(1+M_g')}{(1-M_f'M_g')\Gamma(p+1)}(\log(T))^p$, $C_4 = \frac{M_g(1+M_f')}{(1-M_f'M_g')\Gamma(p+1)}(\log(T))^p$.

**Theorem 3.4.** *Suppose that the assumptions* $(\mathbf{H}_1)$–$(\mathbf{H}_3)$ *with* $\xi^\star < 1$ *hold, along with the condition that spectral radius of* $\mathcal{H}$ *is less than one. Then the solution of* (3.1) *is HU stable.*

*Proof.* In view of Theorem 3.3, we have

$$
\begin{cases} \|\phi_1(u,v)-\phi_1(\overline{u},\overline{v})\| \leq C_1\|u-\overline{u}\| + C_2\|v-\overline{v}\|, \\ \|\phi_2(u,v)-\phi_2(\overline{u},\overline{v})\| \leq C_3\|u-\overline{v}\| + C_4\|v-\overline{v}\|. \end{cases} \tag{3.10}
$$

From (3.10), we obtain the following inequality

$$
\|\phi(u,v)-\phi(\overline{u},\overline{v})\| \leq \mathcal{H}\begin{pmatrix} \|u-\overline{u}\| \\ \|v-\overline{v}\| \end{pmatrix}. \tag{3.11}
$$

By the given assumptions, (3.1) converges to zero. Thus by Theorem 2.8, (3.1) is HU stable. □

**Remark 3.5.** This work can be extended to obtain generalized HU, HU–Rassias and generalized HU–Rassias stability by using the same approach.

# 4 An example

To demonstrate our theoretical results, an example is presented as follows.

**Example 4.1.** Consider the following system of fractional order differential equations consisting of HH type fractional derivatives as

$$\begin{cases} {}_H D^{p,q} u(t) = \frac{t+\sin(|u(t)|)+{}_H D^{p,q} v(t)}{10e^{t^2}+1}, \quad t \in (1, e], \\ {}_H D^{p,q} v(t) = \frac{\cos(|v(t)|)+{}_H D^{p,q} u(t)}{20+t^3}, \\ I_{1^+}^{1-\gamma} u(1) = 1 = I_{1^+}^{1-\gamma} v(1), \\ I_{1^+}^{1-\gamma} u(e) = 2 = I_{1^+}^{1-\gamma} v(e). \end{cases} \tag{4.1}$$

Setting

$$f(t, u(t), {}_H D^{p,q} v(t)) = \frac{t + \sin(|u(t)|) + {}_H D^{p,q} v(t)}{10e^{t^2} + 1}$$

and

$$g(t, u(t), {}_H D^{p,q} v(t)) = \frac{\cos(|v(t)|) + {}_H D^{p,q} u(t)}{20 + t^3}.$$

For any $(u, v), (\overline{u}, \overline{v}) \in \mathcal{X}$, we have

$$|f(t, u(t), v(t)) - f(t, \overline{u}(t), \overline{v}(t))| \leq \frac{1}{10e^2}|u - \overline{u}| + \frac{1}{10e^2}|v - \overline{v}|$$

and

$$|g(t, u(t), v(t)) - g(t, \overline{u}(t), \overline{v}(t))| \leq \frac{1}{20}|u - \overline{u}| + \frac{1}{20}|v - \overline{v}|.$$

Here $\mathcal{M}_f = \mathcal{M}'_f = \frac{1}{10e^2}$, $\mathcal{M}_g = \mathcal{M}'_g = \frac{1}{20}$, $T = e$. If we take $p = \frac{2}{3}$, $q = \frac{1}{2}$ then we get $\gamma = \frac{5}{6}$. Upon calculations, we have $\xi^* = 0.0251 < 1$. Therefore, system (4.1) has a unique solution. Furthermore, we observe that

$$\mathcal{H} = \begin{pmatrix} 0.0039 & 0.0142 \\ 0.0031 & 0.0109 \end{pmatrix}$$

and if $\omega_1$ and $\omega_2$ are the eigenvalues, then $\omega_1 = 0.0149$ and $\omega_2 = -0.0001$. Since the spectral radius of $\mathcal{H}$ is less than one. Thus, system (4.1) converges to 0. That is, system (4.1) is HU stable.

# Conclusion

We used Banach contraction principle and Krasnoselskii fixed point theorem to establish sufficient conditions for the existence and uniqueness of the solution of coupled impulsive fractional differential system of HH type given in (1.2). In addition and under particular assumptions and conditions, we have studied the UH stability results of different kinds for the solution of the proposed problem. In view of the results of this paper, we conclude that such a method is very powerful, effectual and suitable for the solution of nonlinear fractional differential equations.

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