Hyers–Ulam Stability of an n–Apollonius type Quadratic Mapping

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Abstract

Let X and Y be linear spaces. It is shown that for a fixed positive integer $n \geq 2$, if a mapping $Q : X \to Y$ satisfies the following functional equation

$$\sum_{i=1}^{n} Q(z - x_i) = \frac{1}{n} \sum_{\substack{1 \leq i,j \leq n \\ j<i}} Q(x_i - x_j) + nQ\Big(z - \frac{1}{n}\sum_{i=1}^{n} x_i\Big) \tag{1}$$

for all $z, x_1, \ldots, x_n \in X$, then the mapping $Q : X \to Y$ is a *quadratic mapping of Apollonius type* and a quadratic mapping. We moreover prove the Hyers–Ulam stability of the functional equation (1) in Banach spaces.

1 Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [22] concerning the stability of group homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the question: When the solutions of an equation differing

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slightly from a given one must be close to the true solution of the given equation. For Banach spaces the Ulam problem was first solved by D.H. Hyers [9] in 1941, which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$\Big\| f(x+y) - f(x) - f(y) \Big\|_Y \le \delta \tag{2}$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\Big\| f(x) - T(x) \Big\|_Y \le \delta$$

for all $x \in X$. Th.M. Rassias [18] succeeded in extending the result of Hyers by weakening the condition for the Cauchy difference to be unbounded. A number of mathematicians were attracted to this result of Th.M. Rassias and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Th.M. Rassias in his 1978 paper is called the *Hyers–Ulam stability.* G.L. Forti [5] and P. Găvruta [8] have generalized the result of Th.M. Rassias, which permitted the Cauchy difference to become arbitrary unbounded. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers $[2, 6, 7, 10 - 16, 18, 19, 20]$. Now, a square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all vectors x, y. The following functional equation, which was motivated by this equation,

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \tag{3}$$

is called a *quadratic functional equation,* and every solution of equation (3) is said to be a *quadratic mapping.*

F. Skof [21] proved the Hyers–Ulam stability of the quadratic functional equation (3) for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. In [4], S. Czerwik proved the Hyers–Ulam stability of the quadratic functional equation. C. Borelli and G.L. Forti [3] generalized the stability result as follows: let G be an abelian group, E a Banach space. Assume that a mapping $f : G \to E$ satisfies the functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to E$ with the properties

$$\|f(x) - Q(x)\| \le \phi(x, x)$$

for all $x \in G$. Jun and Lee [14] proved the Hyers–Ulam stability of the Pexiderized quadratic equation

$$f(x+y) + g(x-y) = 2h(x) + 2k(y)$$

for mappings f, g, h and k.

In an inner product space, the equality

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\Big\|z - \frac{x+y}{2}\Big\|^2 \tag{4}$$

holds, and is called the *Apollonius' identity.* The following functional equation, which was motivated by this equation,

$$Q(z-x) + Q(z-y) = \frac{1}{2}Q(x-y) + 2Q\Big(z - \frac{x+y}{2}\Big), \tag{5}$$

is quadratic (see [17]). For this reason, the functional equation (5) is called a *quadratic functional equation of Apollonius type,* and each solution of the functional equation (5) is said to be a *quadratic mapping of Apollonius type* [12, 17]. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [1].

In [17], C. Park and Th.M. Rassias introduced and investigated a functional equation, which is called the *generalized Apollonius type quadratic functional equation.*

In this paper, employing the above equality (5), for a fixed positive integer $n \geq 2$, we introduce the new functional equation, which is called the *quadratic functional equation of n–Apollonius type* and whose solution of the functional equation is said to be a *quadratic mapping of n–Apollonius type,*

$$\sum_{i=1}^{n} Q(z-x_i) = \frac{1}{n} \sum_{\substack{1\leq i,j\leq n \\ j<i}} Q(x_i - x_j) + nQ\Big(z - \frac{1}{n}\sum_{i=1}^{n} x_i\Big). \tag{6}$$

We introduce the n–Apollonius' identity in an inner product space for a fixed positive integer $n \geq 2$. We show that the quadratic functional equation of n–Apollonius type (6) is quadratic functional equation of Apollonius type, and quadratic. We also prove the Hyers–Ulam stability of quadratic mappings of n–Apollonius type in Banach spaces.

2 n–Apollonius' identity with some properties and quadratic functional equations of n–Apollonius type

The following theorem introduces the n–Apollonius' identity in an inner product space for a fixed positive integer $n \geq 2$.

Theorem 2.1. *(n–Apollonius' identity) Let X be an inner product space with norm $\|.\|$ introduced by its inner product $\langle ., . \rangle$. For a fixed positive integer $n \geq 2$, we have*

$$\sum_{i=1}^{n} \|z-x_i\|^2 = \frac{1}{n} \sum_{\substack{1\leq i,j\leq n \\ j<i}} \|x_i - x_j\|^2 + n\Big\|z - \frac{1}{n}\sum_{i=1}^{n} x_i\Big\|^2 \tag{7}$$

for all $z, x_1, \ldots, x_n \in X$.

Proof. At first we prove that

$$\sum_{\substack{1\leq i,j\leq n\\ j<i}} \|x_i - x_j\|^2 + \Big\|\sum_{i=1}^{n} x_i\Big\|^2 = n\sum_{i=1}^{n}\|x_i\|^2 \tag{8}$$

for all $x_1,\ldots,x_n \in X$. It is clear that

$$\begin{aligned}\sum_{\substack{1\leq i,j\leq n\\ j<i}} \|x_i - x_j\|^2 &= \sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\|x_i-x_j\|^2\\ &= \sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\Big(\|x_i\|^2+\|x_j\|^2-2\Re\langle x_i,x_j\rangle\Big) \qquad (9)\\ &= \sum_{j=1}^{n-1}\Big[\sum_{i=j+1}^{n}\|x_i\|^2+(n-j)\|x_j\|^2\Big]-2\Re\sum_{\substack{1\leq i,j\leq n\\ j<i}}\langle x_i,x_j\rangle,\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{n-1}\Big[\sum_{i=j+1}^{n}\|x_i\|^2+(n-j)\|x_j\|^2\Big] &= \Big[\sum_{i=2}^{n}\|x_i\|^2+(n-1)\|x_1\|^2\Big]\\ &\quad+\cdots+\Big[\sum_{i=j+1}^{n}\|x_i\|^2+(n-j)\|x_j\|^2\Big] \qquad (10)\\ &\quad+\cdots+\Big[\|x_n\|^2+\|x_{n-1}\|^2\Big]\\ &= (n-1)\sum_{i=1}^{n}\|x_i\|^2\end{aligned}$$

for all $x_1,\ldots,x_n \in X$. Also, we have

$$\Big\|\sum_{i=1}^{n} x_i\Big\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x_i,x_j\rangle = \sum_{i=1}^{n}\|x_i\|^2 + 2\Re\sum_{\substack{1\leq i,j\leq n\\ j<i}}\langle x_i,x_j\rangle \tag{11}$$

for all $x_1,\ldots,x_n \in X$. So, we obtain (8) from (9), (10) and (11).

Now, we prove (7). By using (8), a simple computation shows that

$$\begin{aligned}&\frac{1}{n}\sum_{\substack{1\leq i,j\leq n\\ j<i}} \|x_i - x_j\|^2 + n\Big\|z-\frac{\sum_{i=1}^{n}x_i}{n}\Big\|^2\\ &= \frac{1}{n}\sum_{\substack{1\leq i,j\leq n\\ j<i}} \|x_i - x_j\|^2 + \frac{1}{n}\Big\|\sum_{i=1}^{n}x_i\Big\|^2 + n\|z\|^2 - 2\Re\sum_{i=1}^{n}\langle z,x_i\rangle\\ &= \sum_{i=1}^{n}\|x_i\|^2 + n\|z\|^2 - 2\Re\sum_{i=1}^{n}\langle z,x_i\rangle\\ &= \sum_{i=1}^{n}\|z-x_i\|^2\end{aligned}$$

for all $z, x_1,\ldots,x_n \in X$. ∎

Theorem 2.2. *A mapping $Q : X \to Y$ is a quadratic mapping of n–Apollonius type $(n \geq 2)$, i.e., Q satisfies*

$$\sum_{i=1}^{n} Q(z - x_i) = \frac{1}{n} \sum_{\substack{1 \leq i,j \leq n \\ j<i}} Q(x_i - x_j) + nQ\left(z - \frac{1}{n}\sum_{i=1}^{n} x_i\right) \tag{12}$$

for all $z, x_1, \ldots, x_n \in X$, if and only if Q is a quadratic mapping of Apollonius type and quadratic mapping.

Proof. Let $n > 2$. We claim that $Q(0) = 0$. Putting $x_1 = \cdots = x_n = z$ in (12), we get

$$nQ(0) = \frac{n^2 - n}{2n} Q(0) + nQ(0).$$

So $Q(0) = 0$. Putting $x_3 = \cdots = x_n = z$ in (12), we get

$$\begin{aligned} Q(z - x_1) + Q(z - x_2) = \frac{n-2}{n}\Big[Q(z - x_1) + Q(z - x_2)\Big] \\ + \frac{1}{n} Q(x_2 - x_1) + nQ\Big(\frac{2z - x_1 - x_2}{n}\Big). \end{aligned}$$

Therefore, we have

$$\frac{2}{n}\Big[Q(z - x_1) + Q(z - x_2)\Big] = \frac{1}{n} Q(x_2 - x_1) + nQ\Big(\frac{2z - x_1 - x_2}{n}\Big) \tag{13}$$

for all $z, x_1, x_2 \in X$. Replacing x_1 by $-x_1$ and x_2 by x_1 in (13), respectively, we obtain that

$$\frac{2}{n}\Big[Q(z + x_1) + Q(z - x_1)\Big] = \frac{1}{n} Q(2x_1) + nQ\Big(\frac{2z}{n}\Big) \tag{14}$$

for all $z, x_1 \in X$. Letting $x_1 = z$ in (14), we obtain that

$$\frac{1}{n^2} Q(2z) = Q\Big(\frac{2z}{n}\Big) \tag{15}$$

for all $z \in X$. Hence (15) implies that

$$Q\Big(\frac{x}{n}\Big) = \frac{1}{n^2} Q(x), \quad Q\Big(nx\Big) = n^2 Q(x), \quad (x \in X). \tag{16}$$

It follows from (14) and (15) that

$$2\Big[Q(z + x_1) + Q(z - x_1)\Big] = Q(2x_1) + Q(2z) \tag{17}$$

for all $z, x_1 \in X$. Putting $x_1 = 0$ in (17), we get

$$Q(2z) = 4Q(z), \quad (z \in X). \tag{18}$$

It follows from (17) and (18) that

$$Q(z + x_1) + Q(z - x_1) = 2Q(x_1) + 2Q(z) \tag{19}$$

for all $z, x_1 \in X$. Thus $Q : X \to Y$ is a quadratic mapping. So Q is an even mapping and

$$Q(kx) = k^2 Q(x), \quad Q\Big(\frac{x}{k}\Big) = \frac{1}{k^2} Q(x) \tag{20}$$

for all $k \in \mathbb{Z} \setminus \{0\}$ and all $x \in X$. So, we can get from (13) and (20) that

$$Q(z - x_1) + Q(z - x_2) = \frac{1}{2} Q(x_2 - x_1) + 2Q\Big(z - \frac{x_1 + x_2}{2}\Big) \tag{21}$$

for all $z, x_1, x_2 \in X$. Therefore, Q is a quadratic mapping of Apollonius type.

Conversely, if Q is a quadratic mapping, then by [1], there exists a symmetric bi-additive mapping $B : X \times X \to Y$ such that $Q(x) = B(x, x)$ for all $x \in X$. Therefore Q is a quadratic mapping of n–Apollonius type satisfying the Eq. (6). ■

The Apollonius' identity (4), is 2–Apollonius' identity. The n–Apollonius' identity in an inner product space motivates us to introduce *quadratic functional equation of n–Apollonius type* (6). By our notation, a quadratic functional equation of Apollonius type (5) is a quadratic functional equation of 2–Apollonius type.

3 Hyers–Ulam stability of a quadratic mapping of n–Apollonius type

Throughout this section, let X be a normed space with norm $\|.\|_X$ and Y a Banach space with norm $\|.\|_Y$.

For a fixed integer $n \geq 2$ and given a mapping $Q : X \to Y$, we define $D_n Q : X^{n+1} \to Y$ by

$$\begin{aligned} &D_n Q(x_1, x_2, \ldots, x_n, z) \\ &\quad := \sum_{i=1}^{n} Q(z - x_i) - \frac{1}{n} \sum_{\substack{1 \leq i,j \leq n \\ j < i}} Q(x_i - x_j) - nQ\Big(z - \frac{1}{n}\sum_{i=1}^{n} x_i\Big) \end{aligned}$$

Theorem 3.1. *Let l and m be integers with $1 \leq l < m$ and let δ be a nonnegative real number. Suppose that $Q : X \to Y$ is a mapping for which there exists a function $\varphi : X^{m+1} \to [-\delta, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} (\frac{l}{m})^{2i} \varphi\Big(\underbrace{0, \ldots, 0}_{l-times}, (\frac{m}{l})^i z, \ldots, (\frac{m}{l})^i z\Big) < \infty, \tag{22}$$

$$\lim_{i \to \infty} (\frac{l}{m})^{2i} \varphi\Big((\frac{m}{l})^i x_1, \ldots, (\frac{m}{l})^i x_m, (\frac{m}{l})^i z\Big) = 0 \tag{23}$$

and

$$\Big\| D_m Q(x_1, x_2, \ldots, x_m, z) \Big\|_Y \leq \delta + \varphi(x_1, \ldots, x_m, z) \tag{24}$$

for all $z, x_1, \ldots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\Big\| Q(x) - T(x) - \frac{2l^2 - m^2 - m}{2(m^2 - l^2)} Q(0) \Big\|_Y \leq \frac{m\delta}{m^2 - l^2} + \frac{1}{m} \widetilde{\varphi}\Big(\frac{m}{l} x\Big) \tag{25}$$

for all $x \in X$.

Proof. Putting $x_1 = \cdots = x_l = 0$ and $x_{l+1} = \cdots = x_m = z$ in (24), we get that

$$\Big\| lQ(z) + (m-l)Q(0) - \frac{1}{m}\Big[l(m-l)Q(z) + Q(0) + 2Q(0) + \cdots + (l-1)Q(0) + Q(0) + 2Q(0) + \cdots + (m-l-1)Q(0)\Big] - mQ\Big(z - \frac{(m-l)z}{m}\Big)\Big\|_Y \le \delta + \varphi(\underbrace{0,\ldots,0}_{l-times}, \underbrace{z,\ldots,z}_{(m-l)-times}, z).$$

Therefore we have

$$\Big\| \frac{l^2}{m}Q(z) - mQ\Big(\frac{l}{m}z\Big) + \frac{m^2+m-2l^2}{2m}Q(0)\Big\|_Y \le \delta + \varphi(\underbrace{0,\ldots,0}_{l-times}, \underbrace{z,\ldots,z}_{(m-l)-times}, z) \tag{26}$$

for all $z \in X$. Let $\alpha = \frac{l}{m}$ and $\beta = \frac{m^2+m-2l^2}{2m}$. It is clear that $0 < \alpha < 1$. Replacing z by $\frac{1}{\alpha}z$ in (26), we get that

$$\Big\| \alpha^2 Q\Big(\frac{1}{\alpha}z\Big) - Q(z) + \frac{\beta}{m}Q(0)\Big\|_Y \le \frac{1}{m}\Big[\delta + \varphi\Big(\underbrace{0,\ldots,0}_{l-times}, \underbrace{\frac{1}{\alpha}z,\ldots,\frac{1}{\alpha}z}_{(m-l)-times}, \frac{1}{\alpha}z\Big)\Big] \tag{27}$$

for all $z \in X$. Replacing z by $\frac{z}{\alpha^n}$ and multiplying α^{2n} in (27), we get that

$$\Big\| \alpha^{2(n+1)} Q\Big(\frac{1}{\alpha^{n+1}}z\Big) - \alpha^{2n}Q\Big(\frac{1}{\alpha^n}z\Big) + \frac{\beta\alpha^{2n}}{m}Q(0)\Big\|_Y \le \frac{\alpha^{2n}}{m}\Big[\delta + \varphi\Big(\underbrace{0,\ldots,0}_{l-times}, \underbrace{\frac{1}{\alpha^{n+1}}z,\ldots,\frac{1}{\alpha^{n+1}}z}_{(m-l)-times}, \frac{1}{\alpha^{n+1}}z\Big)\Big] \tag{28}$$

for all $z \in X$ and all integers $n \ge 0$. Therefore we have

$$\begin{aligned} &\Big\| \sum_{i=j}^{n} \Big[\alpha^{2(i+1)} Q\Big(\frac{1}{\alpha^{i+1}}z\Big) - \alpha^{2i}Q\Big(\frac{1}{\alpha^i}z\Big) + \frac{\beta\alpha^{2i}}{m}Q(0)\Big]\Big\|_Y \\ &\le \sum_{i=j}^{n} \Big\| \alpha^{2(i+1)} Q\Big(\frac{1}{\alpha^{i+1}}z\Big) - \alpha^{2i}Q\Big(\frac{1}{\alpha^i}z\Big) + \frac{\beta\alpha^{2i}}{m}Q(0)\Big\|_Y \\ &\le \sum_{i=j}^{n} \frac{\alpha^{2i}}{m}\Big[\delta + \varphi\Big(\underbrace{0,\ldots,0}_{l-times}, \underbrace{\frac{1}{\alpha^{i+1}}z,\ldots,\frac{1}{\alpha^{i+1}}z}_{(m-l)-times}, \frac{1}{\alpha^{i+1}}z\Big)\Big] \end{aligned} \tag{29}$$

for all $z \in X$ and integers $n \geq j \geq 0$. Hence we obtain from (29) that

$$\begin{aligned}\Big\|\alpha^{2(n+1)}Q\Big(\frac{1}{\alpha^{n+1}}z\Big)-\alpha^{2j}Q\Big(\frac{1}{\alpha^{j}}z\Big)+\sum_{i=j}^{n}\frac{\beta\alpha^{2i}}{m}Q(0)\Big\|_Y\\ \leq \sum_{i=j}^{n}\frac{\alpha^{2i}}{m}\Big[\delta+\varphi\Big(\underbrace{0,\dots,0}_{l-times},\underbrace{\frac{1}{\alpha^{i+1}}z,\dots,\frac{1}{\alpha^{i+1}}z}_{(m-l)-times},\frac{1}{\alpha^{i+1}}z\Big)\Big]\end{aligned} \tag{30}$$

for all $z \in X$ and all integers $n \geq j \geq 0$. It follows from (22) and (30), that $\{\alpha^{2n}Q(\frac{1}{\alpha^n}z)\}_n$ is a Cauchy sequence in Y for all $z \in X$. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(z) := \lim_{n\to\infty}\alpha^{2n}Q(\frac{1}{\alpha^n}z) \tag{31}$$

for all $z \in X$. Putting $j = 0$ in (30), we obtain

$$\begin{aligned}\Big\|\alpha^{2(n+1)}Q\Big(\frac{1}{\alpha^{n+1}}z\Big)-Q(z)+\sum_{i=0}^{n}\frac{\beta\alpha^{2i}}{m}Q(0)\Big\|_Y\\ \leq \sum_{i=0}^{n}\frac{\alpha^{2i}}{m}\Big[\delta+\varphi\Big(\underbrace{0,\dots,0}_{l-times},\underbrace{\frac{1}{\alpha^{i+1}}z,\dots,\frac{1}{\alpha^{i+1}}z}_{(m-l)-times},\frac{1}{\alpha^{i+1}}z\Big)\Big]\end{aligned} \tag{32}$$

Letting $n \to \infty$ in (32), we get the inequality (25). Now, we show that T is a quadratic mapping of m–Apollonius type. It follows from (23), (24) and (31), that

$$\begin{aligned}&\Big\|\sum_{i=1}^{m}T(z-x_i)-\frac{1}{m}\sum_{\substack{1\leq i,j\leq m\\ j<i}}T(x_i-x_j)-mT\Big(z-\frac{1}{m}\sum_{i=1}^{m}x_i\Big)\Big\|_Y\\ &=\lim_{n\to\infty}\alpha^{2n}\Big\|\sum_{i=1}^{m}Q\Big(\frac{z-x_i}{\alpha^n}\Big)-\frac{1}{m}\sum_{\substack{1\leq i,j\leq m\\ j<i}}Q\Big(\frac{x_i-x_j}{\alpha^n}\Big)-mQ\Big(\frac{z-\frac{1}{m}\sum_{i=1}^{m}x_i}{\alpha^n}\Big)\Big\|_Y\\ &\leq \lim_{n\to\infty}\alpha^{2n}\Big[\delta+\varphi\Big(\frac{x_1}{\alpha^n},\dots,\frac{x_m}{\alpha^n},\frac{z}{\alpha^n}\Big)\Big]=0\end{aligned}$$

for all $z, x_1, \dots, x_m \in X$. So, T is a quadratic mapping of m–Apollonius type. Let U be another quadratic mapping of m–Apollonius type satisfying (25). By Theorem 2.2, U is a quadratic mapping. Then (22), (25) and (31), implies that

$$\begin{aligned}\Big\|U(x)-T(x)\Big\|_Y&=\lim_{n\to\infty}\alpha^{2n}\Big\|U\Big(\frac{1}{\alpha^n}x\Big)-Q\Big(\frac{1}{\alpha^n}x\Big)\Big\|_Y\\ &\leq \lim_{n\to\infty}\frac{\alpha^{2n}}{m}\widetilde{\varphi}\Big(\frac{1}{\alpha^{n+1}}x\Big)\\ &=\frac{1}{m\alpha^2}\lim_{n\to\infty}\sum_{i=n+1}^{\infty}\alpha^{2i}\varphi\Big(\underbrace{0,\dots,0}_{l-times},\frac{1}{\alpha^i}x,\dots,\frac{1}{\alpha^i}x\Big)=0\end{aligned}$$

for all $x \in X$. It proves the uniqueness of T. ∎

Corollary 3.2. *Let l and m be integers with $1 \le l < m$ and let $p_1, \dots, p_{m+1}$ be non-zero real numbers and $\epsilon_1, \dots, \epsilon_{m+1}, \delta$ be nonnegative real numbers such that $p_1, \dots, p_{m+1} < 2,\ p_1, \dots, p_l > 0$. Suppose that $Q : X \to Y$ is a mapping satisfying*

$$\Big\|D_m Q(x_1, x_2, \dots, x_m, z)\Big\|_Y \le \delta + \epsilon_1 \|x_1\|_X^{p_1} + \cdots + \epsilon_m \|x_m\|_X^{p_m} + \epsilon_{m+1} \|z\|_X^{p_{m+1}} \tag{33}$$

for all $z, x_1, \dots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\begin{aligned} &\Big\|Q(x) - T(x) - \frac{2l^2 - m^2 - m}{2(m^2 - l^2)} Q(0)\Big\|_Y \\ &\le \frac{m\delta}{m^2 - l^2} + \frac{1}{m} \sum_{i=l+1}^{m+1} \frac{\epsilon_i}{\alpha^{p_i} - \alpha^2} \|x\|_X^{p_i}. \end{aligned} \tag{34}$$

for all $x \in X$ (for all $x \in X \setminus \{0\}$ when $p_i < 0$ for some $l + 1 \le i \le m + 1$). Moreover, if $0 < p_1, \dots, p_{m+1} < 2$, then

$$\Big\|Q(x) - T(x)\Big\|_Y \le \frac{m^2 - m + |m^2 + m - 2l^2|}{(m^2 - l^2)(m - 1)} \delta + \frac{1}{m} \sum_{i=l+1}^{m+1} \frac{\epsilon_i}{\alpha^{p_i} - \alpha^2} \|x\|_X^{p_i} \tag{35}$$

for all $x \in X$, where $\alpha = \frac{l}{m}$.

Proof. In Theorem 3.1, let

$$\varphi(x_1, \dots, x_m, z) = \epsilon_1 \|x_1\|_X^{p_1} + \cdots + \epsilon_m \|x_m\|_X^{p_m} + \epsilon_{m+1} \|z\|_X^{p_{m+1}}.$$

Then (34) follows from (25). To obtain (35), putting $x_1 = \cdots = x_m = z = 0$, in (33), we get that

$$\|Q(0)\|_Y \le \frac{2\delta}{m - 1}.$$

Then (35) follows from (34). ■

Theorem 3.3. *Let l and m be integers with $1 \le l < m$ and suppose that $Q : X \to Y$ is a mapping for which there exists a function $\varphi : X^{m+1} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} (\frac{m}{l})^{2i} \varphi\Big(\underbrace{0, \dots, 0}_{l-times}, (\frac{l}{m})^i z, \dots, (\frac{l}{m})^i z\Big) < \infty, \tag{36}$$

$$\lim_{i \to \infty} (\frac{m}{l})^{2i} \varphi\Big((\frac{l}{m})^i x_1, \dots, (\frac{l}{m})^i x_m, (\frac{l}{m})^i z\Big) = 0 \tag{37}$$

and

$$\Big\|D_m Q(x_1, x_2, \dots, x_m, z)\Big\|_Y \le \varphi(x_1, \dots, x_m, z) \tag{38}$$

for all $z, x_1, \dots, x_m \in X$. Let $Q(0) = 0$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\Big\|Q(x) - T(x)\Big\|_Y \le \frac{m}{l^2} \widetilde{\varphi}(x) \tag{39}$$

for all $x \in X$.

Proof. Let $\alpha = \frac{l}{m}$. Similar to the proof of Theorem 3.1, putting $x_1 = \cdots = x_l = 0$ and $x_{l+1} = \cdots = x_m = z$ in (38), we get that

$$\left\|Q(z) - \frac{1}{\alpha^2}Q(\alpha z)\right\|_Y \le \frac{1}{m\alpha^2}\varphi(\underbrace{0,\ldots,0}_{l-times}, \underbrace{z,\ldots,z}_{(m-l)-times}, z) \tag{40}$$

for all $z \in X$. Replacing z by $\alpha^n z$ and dividing α^{2n} in (40), we get that

$$\begin{aligned}&\left\|\frac{1}{\alpha^{2(n+1)}}Q(\alpha^{n+1}z) - \frac{1}{\alpha^{2n}}Q(\alpha^n z)\right\|_Y \\ &\le \frac{1}{m\alpha^{2(n+1)}}\varphi(\underbrace{0,\ldots,0}_{l-times}, \underbrace{\alpha^n z,\ldots,\alpha^n z}_{(m-l)-times}, \alpha^n z)\end{aligned} \tag{41}$$

for all $z \in X$ and all integers $n \ge 0$. Therefore we have

$$\begin{aligned}&\left\|\sum_{i=j}^{n}\left[\frac{1}{\alpha^{2(i+1)}}Q(\alpha^{i+1}z) - \frac{1}{\alpha^{2i}}Q(\alpha^i z)\right]\right\|_Y \\ &\le \sum_{i=j}^{n}\left\|\frac{1}{\alpha^{2(i+1)}}Q(\alpha^{i+1}z) - \frac{1}{\alpha^{2i}}Q(\alpha^i z)\right\|_Y \\ &\le \sum_{i=j}^{n}\frac{1}{m\alpha^{2(i+1)}}\varphi(\underbrace{0,\ldots,0}_{l-times}, \underbrace{\alpha^i z,\ldots,\alpha^i z}_{(m-l)-times}, \alpha^i z)\end{aligned} \tag{42}$$

for all $z \in X$ and all integers $n \ge j \ge 0$. Hence we obtain from (42) that

$$\begin{aligned}&\left\|\frac{1}{\alpha^{2(n+1)}}Q(\alpha^{n+1}z) - \frac{1}{\alpha^{2j}}Q(\alpha^j z)\right\|_Y \\ &\le \sum_{i=j}^{n}\frac{1}{m\alpha^{2(i+1)}}\varphi(\underbrace{0,\ldots,0}_{l-times}, \underbrace{\alpha^i z,\ldots,\alpha^i z}_{(m-l)-times}, \alpha^i z)\end{aligned} \tag{43}$$

for all $z \in X$ and all integers $n \ge j \ge 0$. It follows from (36) and (43), that $\{\alpha^{-2n}Q(\alpha^n z)\}_n$ is a Cauchy sequence in Y for all $z \in X$. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(z) := \lim_{n\to\infty}\frac{1}{\alpha^{2n}}Q(\alpha^n z). \tag{44}$$

Putting $j = 0$ in (43) and letting $n \to \infty$, we obtain the inequality (39). The rest of the proof is similar to the proof of Theorem 3.1. ■

Corollary 3.4. *Let l and m be integers with $1 \le l < m$ and let $p_1, \ldots, p_{m+1}$ and $\epsilon_1, \ldots, \epsilon_{m+1}$ be nonnegative real numbers such that $p_1, \ldots, p_{m+1} > 2$. Suppose that $Q : X \to Y$ is a mapping satisfying*

$$\left\|D_m Q(x_1, x_2, \ldots, x_m, z)\right\|_Y \le \epsilon_1\|x_1\|_X^{p_1} + \cdots + \epsilon_m\|x_m\|_X^{p_m} + \epsilon_{m+1}\|z\|_X^{p_{m+1}} \tag{45}$$

for all $z, x_1, \dots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\Big\|Q(x) - T(x)\Big\|_Y \le \frac{1}{m}\sum_{i=l+1}^{m+1}\frac{\epsilon_i}{\alpha^2 - \alpha^{p_i}}\|x\|_X^{p_i} \tag{46}$$

for all $x \in X$, where $\alpha = \frac{l}{m}$.

Proof. In Theorem 3.3, let

$$\varphi(x_1, \dots, x_m, z) = \epsilon_1\|x_1\|_X^{p_1} + \dots + \epsilon_m\|x_m\|_X^{p_m} + \epsilon_{m+1}\|z\|_X^{p_{m+1}}.$$

Putting $x_1 = \dots = x_m = z = 0$ in (45), we get that $Q(0) = 0$. Then (46) follows from (39). ■

Theorem 3.5. *Let m be an even integer and suppose that $Q : X \to Y$ is an even mapping with $Q(0) = 0$ for which there exists a function $\varphi : X^{m+1} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} 4^i\varphi\Big(0, \frac{z}{2^i}, 0, \frac{z}{2^i}, 0, \dots, 0, \frac{z}{2^i}, \frac{z}{2^i}\Big) < \infty, \tag{47}$$

$$\lim_{i\to\infty} 4^i\varphi\Big(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \dots, \frac{x_m}{2^i}, \frac{z}{2^i}\Big) = 0 \tag{48}$$

and

$$\Big\|D_mQ(x_1, x_2, \dots, x_m, z)\Big\|_Y \le \varphi(x_1, \dots, x_m, z) \tag{49}$$

for all $z, x_1, \dots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\Big\|Q(x) - T(x)\Big\|_Y \le \frac{4}{m}\widetilde{\varphi}(x) \tag{50}$$

for all $x \in X$.

Proof. Let $x_1 = x_3 = \dots = x_{m-3} = x_{m-1} = 0$ and $x_2 = x_4 = \dots = x_{m-2} = x_m = z$. Then we have $\sum_{i=1}^m Q(z - x_i) = \frac{m}{2}Q(z)$ and

$$\begin{aligned}\sum_{\substack{1\le i,j\le m\\ j<i}} Q(x_i - x_j) &= \Big[2 + 4 + \dots + (m-4) + (m-2) + \frac{m}{2}\Big]Q(z)\\ &= \frac{m^2}{4}Q(z).\end{aligned}$$

So, (49) implies that

$$\Big\|Q(z) - 4Q\Big(\frac{z}{2}\Big)\Big\|_Y \le \frac{4}{m}\varphi(0, z, 0, \dots, z, 0, z, z) \tag{51}$$

for all $z \in X$. Replacing z by $\frac{z}{2^n}$ and multiplying 4^n in (51), we get that

$$\Big\|4^{n+1}Q\Big(\frac{z}{2^{n+1}}\Big) - 4^nQ\Big(\frac{z}{2^n}\Big)\Big\|_Y \le \frac{4^{n+1}}{m}\varphi\Big(0, \frac{z}{2^n}, 0, \dots, \frac{z}{2^n}, 0, \frac{z}{2^n}, \frac{z}{2^n}\Big) \tag{52}$$

for all $z \in X$ and each integer $n \geq 0$. Therefore we obtain from (52) that

$$\begin{aligned}\Big\|4^{n+1}Q\Big(\frac{z}{2^{n+1}}\Big)-4^kQ\Big(\frac{z}{2^k}\Big)\Big\|_Y &= \Big\|\sum_{i=k}^{n}\Big[4^{i+1}Q\Big(\frac{z}{2^{i+1}}\Big)-4^iQ\Big(\frac{z}{2^i}\Big)\Big]\Big\|_Y\\ &\leq \sum_{i=k}^{n}\Big\|4^{i+1}Q\Big(\frac{z}{2^{i+1}}\Big)-4^iQ\Big(\frac{z}{2^i}\Big)\Big\|_Y \qquad (53)\\ &\leq \sum_{i=k}^{n}\frac{4^{i+1}}{m}\varphi\Big(0,\frac{z}{2^i},0,\ldots,\frac{z}{2^i},0,\frac{z}{2^i},\frac{z}{2^i}\Big)\end{aligned}$$

for all $z \in X$ and all integers $0 \leq k \leq n$. It follows from (47) and (53) that the sequence $\{4^nQ(2^{-n}z)\}_n$ is a Cauchy sequence in Y for all $z \in X$. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(z) := \lim_{n\to\infty} 4^nQ\Big(\frac{z}{2^n}\Big).$$

Now, we show that T is a quadratic functional equation of m–Apollonius type. It follows from the definition of T and (48) and (49) that

$$\begin{aligned}&\Big\|\sum_{i=1}^{m}T(z-x_i)-\frac{1}{m}\sum_{\substack{1\leq i,j\leq m\\ j<i}}T(x_i-x_j)-mT\Big(z-\frac{1}{m}\sum_{i=1}^{m}x_i\Big)\Big\|_Y\\ &=\lim_{n\to\infty}4^n\Big\|\sum_{i=1}^{m}Q\Big(\frac{z-x_i}{2^n}\Big)-\frac{1}{m}\sum_{\substack{1\leq i,j\leq m\\ j<i}}Q\Big(\frac{x_i-x_j}{2^n}\Big)-mQ\Big(\frac{z-\frac{1}{m}\sum_{i=1}^{m}x_i}{2^n}\Big)\Big\|_Y\\ &\leq \lim_{n\to\infty}4^n\varphi\Big(\frac{x_1}{2^n},\ldots,\frac{x_m}{2^n},\frac{z}{2^n}\Big)=0\end{aligned}$$

for all $z, x_1, \ldots, x_m \in X$. Putting $k = 0$ and letting $n \to \infty$ in (53), we obtain (50). To prove the uniqueness of T, let $U : X \to Y$ be another quadratic mapping of m–Apollonius type which satisfies (50). By Theorem 2.2 and (47), we have

$$\begin{aligned}\Big\|U(x)-T(x)\Big\|_Y &= \lim_{n\to\infty}4^n\Big\|U\Big(\frac{x}{2^n}\Big)-Q\Big(\frac{x}{2^n}\Big)\Big\|_Y\\ &\leq \lim_{n\to\infty}\frac{4^{n+1}}{m}\widetilde{\varphi}\Big(\frac{x}{2^n}\Big)\\ &= \frac{4}{m}\lim_{n\to\infty}\sum_{i=n}^{\infty}4^i\varphi\Big(0,\frac{x}{2^i},0,\frac{x}{2^i},0,\ldots,0,\frac{x}{2^i},\frac{x}{2^i}\Big)=0\end{aligned}$$

for all $x \in X$. ■

Remark 3.6. *If we replace the condition* (47) *with*

$$\Phi(z) := \sum_{i=0}^{\infty}4^i\varphi\Big(\frac{z}{2^i},0,\frac{z}{2^i},0,\ldots,\frac{z}{2^i},0,\frac{z}{2^i}\Big) < \infty,$$

then we obtain a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\Big\|Q(x)-T(x)\Big\|_Y \leq \frac{4}{m}\Phi(x)$$

for all $x \in X$.

Corollary 3.7. *Let m be an even integer and and let $p_1, \ldots, p_{m+1}, \epsilon_1, \ldots, \epsilon_{m+1}$ be nonnegative real numbers such that $p_1, \ldots, p_{m+1} > 2$. Suppose that $Q : X \to Y$ is an even mapping satisfying*

$$\left\| D_m Q(x_1, x_2, \ldots, x_m, z) \right\|_Y \le \epsilon_1 \|x_1\|_X^{p_1} + \cdots + \epsilon_m \|x_m\|_X^{p_m} + \epsilon_{m+1} \|z\|_X^{p_{m+1}} \tag{54}$$

for all $z, x_1, \ldots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\begin{aligned}\left\| Q(x) - T(x) \right\|_Y \le \frac{4}{m} \Big[& \frac{2^{p_2}}{2^{p_2} - 4} \epsilon_2 \|x\|^{p_2} + \frac{2^{p_4}}{2^{p_4} - 4} \epsilon_4 \|x\|^{p_4} \\ & + \cdots + \frac{2^{p_m}}{2^{p_m} - 4} \epsilon_m \|x\|^{p_m} + \frac{2^{p_{m+1}}}{2^{p_{m+1}} - 4} \epsilon_{m+1} \|x\|^{p_{m+1}} \Big]\end{aligned} \tag{55}$$

for all $x \in X$.

Proof. Let

$$\varphi(x_1, \ldots, x_m, z) = \epsilon_1 \|x_1\|_X^{p_1} + \cdots + \epsilon_m \|x_m\|_X^{p_m} + \epsilon_{m+1} \|z\|_X^{p_{m+1}}.$$

Putting $x_1 = \cdots = x_m = z = 0$ in (54), we get that $Q(0) = 0$. Therefore the result follows from Theorem 3.5. ■

By Remark 3.6 we have an alternative result of Corollary 3.7.

Theorem 3.8. *Let m be an even integer and let δ be a nonnegative real number. Suppose that $Q : X \to Y$ is an even mapping for which there exists a function $\varphi : X^{m+1} \to [-\delta, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} 4^{-i} \varphi\Big(0, 2^i z, 0, 2^i z, 0, \ldots, 0, 2^i z, 2^i z\Big) < \infty, \tag{56}$$

$$\lim_{i \to \infty} 4^{-i} \varphi\Big(2^i x_1, 2^i x_2, \ldots, 2^i x_m, 2^i z\Big) = 0 \tag{57}$$

and

$$\left\| D_m Q(x_1, x_2, \ldots, x_m, z) \right\|_Y \le \delta + \varphi(x_1, \ldots, x_m, z) \tag{58}$$

for all $z, x_1, \ldots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\left\| T(x) - \frac{1}{4} Q(x) + \frac{m+2}{12m} Q(0) \right\|_Y \le \frac{\delta}{3m} + \frac{1}{4m} \widetilde{\varphi}(2x) \tag{59}$$

for all $x \in X$.

Proof. Similar to the proof of Theorem 3.5, let $x_1 = x_3 = \cdots = x_{m-3} = x_{m-1} = 0$ and $x_2 = x_4 = \cdots = x_{m-2} = x_m = z$. Then we have

$$\sum_{i=1}^{m} Q(z - x_i) = \frac{m}{2} Q(z) + \frac{m}{2} Q(0)$$

and

$$\sum_{\substack{1\le i,j\le m\\ j<i}} Q(x_i - x_j) = \Big[2+4+\cdots+(m-4)+(m-2)+\frac{m}{2}\Big]Q(z) + \Big[2+4+\cdots+(m-4)+(m-2)\Big]Q(0) = \frac{m^2}{4}Q(z) + \frac{m^2-2m}{4}Q(0).$$

So, (58) implies that

$$\Big\|\frac{1}{4}Q(z) - Q\Big(\frac{z}{2}\Big) + \frac{m+2}{4m}Q(0)\Big\|_Y \le \frac{\delta}{m} + \frac{1}{m}\varphi(0,z,0,\dots,z,0,z,z) \tag{60}$$

for all $z\in X$. Replacing z by $2^n z$ and dividing 4^n in (60), we get that

$$\Big\|\frac{1}{4^{n+1}}Q\Big(2^n z\Big) - \frac{1}{4^n}Q\Big(2^{n-1}z\Big) + \frac{m+2}{m4^{n+1}}Q(0)\Big\|_Y \le \frac{\delta}{m4^n} + \frac{1}{m4^n}\varphi(0,2^n z,0,\dots,2^n z,0,2^n z,2^n z) \tag{61}$$

for all $z\in X$ and each integer $n\ge 0$. Therefore we obtain from (61) that

$$\begin{aligned}\Big\|\frac{1}{4^{n+1}}Q\Big(2^n z\Big) - \frac{1}{4^k}Q\Big(2^{k-1}z\Big) + \sum_{i=k}^{n}\frac{m+2}{m4^{i+1}}Q(0)\Big\|_Y &= \Big\|\sum_{i=k}^{n}\Big[\frac{1}{4^{i+1}}Q\Big(2^i z\Big) - \frac{1}{4^i}Q\Big(2^{i-1}z\Big) + \frac{m+2}{m4^{i+1}}Q(0)\Big]\Big\|_Y \\ &\le \frac{\delta}{m}\sum_{i=k}^{n}\frac{1}{4^i} + \frac{1}{m}\sum_{i=k}^{n}4^{-i}\varphi(0,2^i z,0,\dots,2^i z,0,2^i z,2^i z)\end{aligned} \tag{62}$$

for all $z\in X$ and all integers $0\le k\le n$. It follows from (56) and (62) that the sequence $\{4^{-n-1}Q(2^n z)\}_n$ is a Cauchy sequence in Y for all $z\in X$. Since Y is complete, there exists a mapping $T: X\to Y$ defined by

$$T(z) := \lim_{n\to\infty}\frac{1}{4^{n+1}}Q(2^n z).$$

The rest of the proof is similar to the proof of Theorem 3.5. ∎

Remark 3.9. *If we replace the condition* (56) *with*

$$\Phi(z) := \sum_{i=0}^{\infty}4^{-i}\varphi\Big(2^i z,0,2^i z,0,2^i z,\dots,2^i z,0,2^i z\Big) < \infty,$$

then we obtain a unique quadratic mapping of m–Apollonius type $T: X\to Y$ such that

$$\Big\|T(x) - \frac{1}{4}Q(x) + \frac{m+2}{12m}Q(0)\Big\|_Y \le \frac{\delta}{3m} + \frac{1}{4m}\Phi(2x)$$

for all $x\in X$.

Corollary 3.10. *Let m be an even integer and let $p_1, \ldots, p_{m+1}$ be non-zero real numbers and $\delta, \epsilon_1, \ldots, \epsilon_{m+1}$ be nonnegative real numbers such that $p_1, \ldots, p_{m+1} < 2$ and $p_1, p_3, \ldots, p_{m-1} > 0$. Suppose that $Q : X \to Y$ is an even mapping satisfying*

$$\Big\|D_m Q(x_1, x_2, \ldots, x_m, z)\Big\|_Y \le \delta + \epsilon_1 \|x_1\|_X^{p_1} + \cdots + \epsilon_m \|x_m\|_X^{p_m} + \epsilon_{m+1}\|z\|_X^{p_{m+1}} \tag{63}$$

for all $z, x_1, \ldots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\begin{aligned}
&\Big\|T(x) - \frac{1}{4}Q(x) + \frac{m+2}{12m}Q(0)\Big\|_Y \\
&\quad \le \frac{\delta}{3m} + \frac{1}{m}\Big[\frac{2^{p_2}}{4-2^{p_2}}\epsilon_2\|x\|^{p_2} + \frac{2^{p_4}}{4-2^{p_4}}\epsilon_4\|x\|^{p_4} \\
&\quad + \cdots + \frac{2^{p_m}}{4-2^{p_m}}\epsilon_m\|x\|^{p_m} + \frac{2^{p_{m+1}}}{4-2^{p_{m+1}}}\epsilon_{m+1}\|x\|^{p_{m+1}}\Big]
\end{aligned} \tag{64}$$

for all $x \in X$ (for all $x \in X \setminus \{0\}$ when $p_i < 0$ for some $i \in \{2, 4, \cdots, m, m+1\}$). Moreover if $0 < p_1, \ldots, p_{m+1} < 2$, then

$$\begin{aligned}
\Big\|T(x) - \frac{1}{4}Q(x)\Big\|_Y &\le \frac{\delta}{2(m-1)} + \frac{1}{m}\Big[\frac{2^{p_2}}{4-2^{p_2}}\epsilon_2\|x\|^{p_2} + \frac{2^{p_4}}{4-2^{p_4}}\epsilon_4\|x\|^{p_4} \\
&\quad + \cdots + \frac{2^{p_m}}{4-2^{p_m}}\epsilon_m\|x\|^{p_m} + \frac{2^{p_{m+1}}}{4-2^{p_{m+1}}}\epsilon_{m+1}\|x\|^{p_{m+1}}\Big]
\end{aligned} \tag{65}$$

Proof. In Theorem 3.8, let

$$\varphi(x_1, \ldots, x_m, z) = \epsilon_1\|x_1\|_X^{p_1} + \cdots + \epsilon_m\|x_m\|_X^{p_m} + \epsilon_{m+1}\|z\|_X^{p_{m+1}}.$$

Then (64) follows from (59). To obtain (65), putting $x_1 = \cdots = x_m = z = 0$, in (63), we get that

$$\|Q(0)\|_Y \le \frac{2\delta}{m-1}.$$

Then (65) follows from (64). ∎

By Remark 3.9 we have an alternative result of Corollary 3.10.

Theorem 3.11. *Let $m \ge 3$ be an odd integer and let δ be a nonnegative real number. Suppose that $Q : X \to Y$ is an even mapping for which there exists a function $\varphi : X^{m+1} \to [-\delta, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} \alpha^i \varphi\Big(0, \gamma^i z, 0, \gamma^i z, 0, \ldots, 0, \gamma^i z, 0, \gamma^i z\Big) < \infty, \tag{66}$$

$$\lim_{i\to\infty} \alpha^i \varphi\Big(\gamma^i x_1, \gamma^i x_2, \ldots, \gamma^i x_m, \gamma^i z\Big) = 0 \tag{67}$$

and

$$\Big\|D_m Q(x_1, x_2, \ldots, x_m, z)\Big\|_Y \le \delta + \varphi(x_1, \ldots, x_m, z) \tag{68}$$

for all $x_1, \ldots, x_m, z \in X$, *where* $\alpha = \frac{(m+1)^2}{4m^2}$ *and* $\gamma = \frac{2m}{m+1}$. *Then there exists a unique quadratic mapping of* m*–Apollonius type* $T : X \to Y$ *such that*

$$\left\| T(x) - \alpha Q(x) + \frac{\alpha\beta}{1-\alpha} Q(0) \right\|_Y \le \frac{\alpha}{m(1-\alpha)} \delta + \frac{\alpha}{m} \widetilde{\varphi}(\gamma x) \tag{69}$$

for all $x \in X$, *where* $\beta = \frac{m^2-1}{4m^2}$.

Proof. Similar to the proof of Theorem 3.5, let $x_1 = x_3 = \cdots = x_{m-2} = x_m = 0$ and $x_2 = x_4 = \cdots = x_{m-3} = x_{m-1} = z$. Then we have

$$\sum_{i=1}^{m} Q(z - x_i) = \frac{m+1}{2} Q(z) + \frac{m-1}{2} Q(0),$$

$$\begin{aligned} \sum_{\substack{1 \le i,j \le m \\ j < i}} Q(x_i - x_j) &= \Big[2 + 4 + \cdots + (m-3) + (m-1) \Big] Q(z) \\ &\quad + \Big[2 + 4 + \cdots + (m-3) + \frac{m-1}{2} \Big] Q(0) \\ &= \frac{m^2-1}{4} Q(z) + \frac{(m-1)^2}{4} Q(0), \end{aligned}$$

and

$$Q\Big(z - \frac{1}{m} \sum_{i=1}^{m} x_i \Big) = Q\Big(\frac{m+1}{2m} z \Big) = Q\Big(\frac{1}{\gamma} z \Big).$$

So, (68) implies that

$$\left\| \alpha Q(z) - Q\Big(\frac{1}{\gamma} z \Big) + \beta Q(0) \right\|_Y \le \frac{\delta}{m} + \frac{1}{m} \varphi(0, z, 0, \ldots, z, 0, z) \tag{70}$$

for all $z \in X$. Replacing z by $\gamma^n z$ and multiplying α^n in (70), we get that

$$\begin{aligned} &\left\| \alpha^{n+1} Q(\gamma^n z) - \alpha^n Q(\gamma^{n-1} z) + \alpha^n \beta Q(0) \right\|_Y \\ &\le \frac{\alpha^n}{m} \delta + \frac{\alpha^n}{m} \varphi(0, \gamma^n z, 0, \ldots, \gamma^n z, 0, \gamma^n z) \end{aligned} \tag{71}$$

for all $z \in X$ and each integer $n \ge 0$. Therefore we obtain from (71) that

$$\begin{aligned} &\left\| \alpha^{n+1} Q(\gamma^n z) - \alpha^k Q(\gamma^{k-1} z) + \sum_{i=k}^{n} \alpha^i \beta Q(0) \right\|_Y \\ &= \left\| \sum_{i=k}^{n} \Big[\alpha^{i+1} Q(\gamma^i z) - \alpha^i Q(\gamma^{i-1} z) + \alpha^i \beta Q(0) \Big] \right\|_Y \\ &\le \sum_{i=k}^{n} \left\| \alpha^{i+1} Q(\gamma^i z) - \alpha^i Q(\gamma^{i-1} z) + \alpha^i \beta Q(0) \right\|_Y \\ &\le \frac{\delta}{m} \sum_{i=k}^{n} \alpha^i + \frac{1}{m} \sum_{i=k}^{n} \alpha^i \varphi(0, \gamma^i z, 0, \ldots, \gamma^i z, 0, \gamma^i z) \end{aligned} \tag{72}$$

for all $z \in X$ and all integers $0 \leq k \leq n$. Since $0 < \alpha < 1$, it follows from (66) and (72) that the sequence $\{\alpha^{n+1}Q(\gamma^n z)\}_n$ is a Cauchy sequence in Y for all $z \in X$. Since Y is complete, there exists a mapping $T : X \to Y$ defined by

$$T(z) := \lim_{n\to\infty} \alpha^{n+1}Q(\gamma^n z).$$

The rest of the proof is similar to the proof of Theorem 3.5. ■

Corollary 3.12. *Let $m \geq 3$ be an odd integer and let $p_1, p_2, \ldots, p_{m+1}$ be non-zero real numbers and $\delta, \epsilon_1, \ldots, \epsilon_{m+1}$ be nonnegative real numbers such that $p_1, p_2, \ldots, p_{m+1} < 2$ and $p_1, p_3, \ldots, p_m > 0$. Suppose that $Q : X \to Y$ is an even mapping satisfying*

$$\Big\|D_mQ(x_1, x_2, \ldots, x_m, z)\Big\|_Y \leq \delta + \epsilon_1\|x_1\|_X^{p_1} + \cdots + \epsilon_m\|x_m\|_X^{p_m} + \epsilon_{m+1}\|z\|_X^{p_{m+1}} \quad (73)$$

for all $z, x_1, \ldots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\begin{aligned}\Big\|T(x) - \alpha Q(x) + \frac{\alpha\beta}{1-\alpha}Q(0)\Big\|_Y &\leq \frac{\alpha}{m(1-\alpha)}\delta + \frac{1}{m}\bigg[\frac{\gamma^{p_2}}{\gamma^2 - \gamma^{p_2}}\epsilon_2\|x\|^{p_2} + \frac{\gamma^{p_4}}{\gamma^2-\gamma^{p_4}}\epsilon_4\|x\|^{p_4} \\ &+ \cdots + \frac{\gamma^{p_{m-1}}}{\gamma^2 - \gamma^{p_{m-1}}}\epsilon_{m-1}\|x\|^{p_{m-1}} + \frac{\gamma^{p_{m+1}}}{\gamma^2-\gamma^{p_{m+1}}}\epsilon_{m+1}\|x\|^{p_{m+1}}\bigg]\end{aligned} \quad (74)$$

for all $x \in X$ (for all $x \in X \setminus \{0\}$ when $p_i < 0$ for some $i \in \{2, 4, \ldots, m-1, m+1\}$). Moreover, if $0 < p_1, p_2, \ldots, p_{m+1} < 2$, then

$$\begin{aligned}\Big\|T(x) - \alpha Q(x)\Big\|_Y &\leq \bigg[\frac{1}{m(1-\alpha)} + \frac{2\beta}{(m-1)(1-\alpha)}\bigg]\alpha\delta \\ &+ \frac{1}{m}\bigg[\frac{\gamma^{p_2}}{\gamma^2 - \gamma^{p_2}}\epsilon_2\|x\|^{p_2} + \frac{\gamma^{p_4}}{\gamma^2-\gamma^{p_4}}\epsilon_4\|x\|^{p_4} \\ &+ \cdots + \frac{\gamma^{p_{m-1}}}{\gamma^2 - \gamma^{p_{m-1}}}\epsilon_{m-1}\|x\|^{p_{m-1}} + \frac{\gamma^{p_{m+1}}}{\gamma^2-\gamma^{p_{m+1}}}\epsilon_{m+1}\|x\|^{p_{m+1}}\bigg]\end{aligned}$$

Proof. In Theorem 3.11, let

$$\varphi(x_1, \ldots, x_m, z) = \epsilon_1\|x_1\|_X^{p_1} + \cdots + \epsilon_m\|x_m\|_X^{p_m} + \epsilon_{m+1}\|z\|_X^{p_{m+1}},$$

as desired. ■

Theorem 3.13. *Let $m \geq 3$ be an odd integer and let δ be a nonnegative real number. Suppose that $Q : X \to Y$ is an even mapping for which there exists a function $\varphi : X^{m+1} \to [-\delta, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} \alpha^i \varphi\Big(\gamma^i z, 0, \gamma^i z, 0, \ldots, 0, \gamma^i z, \gamma^i z\Big) < \infty, \quad (75)$$

$$\lim_{i\to\infty} \alpha^i \varphi\Big(\gamma^i x_1, \gamma^i x_2, \ldots, \gamma^i x_m, \gamma^i z\Big) = 0 \tag{76}$$

and

$$\Big\| D_m Q(x_1, x_2, \ldots, x_m, z)\Big\|_Y \leq \delta + \varphi(x_1, \ldots, x_m, z) \tag{77}$$

for all $x_1, \ldots, x_m, z \in X$, *where* $\alpha = \frac{(m-1)^2}{4m^2}$ *and* $\gamma = \frac{2m}{m-1}$. *Then there exists a unique quadratic mapping of* m*–Apollonius type* $T : X \to Y$ *such that*

$$\Big\| T(x) - \alpha Q(x) + \frac{\alpha\beta}{1-\alpha} Q(0)\Big\|_Y \leq \frac{\alpha}{m(1-\alpha)}\delta + \frac{\alpha}{m}\widetilde{\varphi}(\gamma x) \tag{78}$$

for all $x \in X$, *where* $\beta = \frac{m^2+4m-1}{4m^2}$.

Proof. Similar to the proof of Theorem 3.5, let $x_1 = x_3 = \cdots = x_{m-2} = x_m = z$ and $x_2 = x_4 = \cdots = x_{m-3} = x_{m-1} = 0$. The rest of the proof is similar to the proof of Theorem 3.11 and we omit it. ■

Corollary 3.14. *Let* $m \geq 3$ *be an odd integer and let* $p_1, p_2, \ldots, p_{m+1}$ *be non-zero real numbers and* $\delta, \epsilon_1, \ldots, \epsilon_{m+1}$ *be nonnegative real numbers such that* $p_1, p_2, \ldots, p_{m+1} < 2$ *and* $p_2, p_4, \ldots, p_{m-1} > 0$. *Suppose that* $Q : X \to Y$ *is an even mapping satisfying*

$$\Big\| D_m Q(x_1, x_2, \ldots, x_m, z)\Big\|_Y \leq \delta + \epsilon_1 \|x_1\|_X^{p_1} + \cdots + \epsilon_m \|x_m\|_X^{p_m} + \epsilon_{m+1}\|z\|_X^{p_{m+1}} \tag{79}$$

for all $z, x_1, \ldots, x_m \in X$. *Then there exists a unique quadratic mapping of* m*–Apollonius type* $T : X \to Y$ *such that*

$$\begin{aligned} &\Big\| T(x) - \alpha Q(x) + \frac{\alpha\beta}{1-\alpha} Q(0)\Big\|_Y \\ &\leq \frac{\alpha}{m(1-\alpha)}\delta + \frac{1}{m}\Bigg[\frac{\gamma^{p_1}}{\gamma^2 - \gamma^{p_1}}\epsilon_2\|x\|^{p_1} + \frac{\gamma^{p_3}}{\gamma^2-\gamma^{p_3}}\epsilon_3\|x\|^{p_3} \\ &+ \cdots + \frac{\gamma^{p_m}}{\gamma^2-\gamma^{p_m}}\epsilon_m\|x\|^{p_m} + \frac{\gamma^{p_{m+1}}}{\gamma^2-\gamma^{p_{m+1}}}\epsilon_{m+1}\|x\|^{p_{m+1}}\Bigg] \end{aligned} \tag{80}$$

for all $x \in X$ *(for all* $x \in X \setminus \{0\}$ *when* $p_i < 0$ *for some* $i \in \{1, 3, \ldots, m, m+1\}$*). Moreover, if* $0 < p_1, p_2, \ldots, p_{m+1} < 2$, *then*

$$\begin{aligned} \Big\| T(x) - \alpha Q(x)\Big\|_Y &\leq \Bigg[\frac{1}{m(1-\alpha)} + \frac{2\beta}{(m-1)(1-\alpha)}\Bigg]\alpha\delta \\ &+ \frac{1}{m}\Bigg[\frac{\gamma^{p_1}}{\gamma^2-\gamma^{p_1}}\epsilon_2\|x\|^{p_1} + \frac{\gamma^{p_3}}{\gamma^2-\gamma^{p_3}}\epsilon_3\|x\|^{p_3} \\ &+ \cdots + \frac{\gamma^{p_m}}{\gamma^2-\gamma^{p_m}}\epsilon_m\|x\|^{p_m} + \frac{\gamma^{p_{m+1}}}{\gamma^2-\gamma^{p_{m+1}}}\epsilon_{m+1}\|x\|^{p_{m+1}}\Bigg]. \end{aligned}$$

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