Research Article

Soon-Mo Jung*

Hyers-Ulam stability of isometries on bounded domains

https://doi.org/10.1515/math-2021-0063
received May 9, 2021; accepted July 3, 2021

Abstract: More than 20 years after Fickett attempted to prove the Hyers-Ulam stability of isometries defined on bounded subsets of $\mathbb{R}^n$ in 1981, Alestalo et al. [*Isometric approximation*, Israel J. Math. **125** (2001), 61–82] and Väisälä [*Isometric approximation property in Euclidean spaces*, Israel J. Math. **128** (2002), 127] improved Fickett's theorem significantly. In this paper, we will improve Fickett's theorem by proving the Hyers-Ulam stability of isometries defined on bounded subsets of $\mathbb{R}^n$ using a more intuitive and more efficient approach that differs greatly from the methods used by Alestalo et al. and Väisälä.

Keywords: isometry, ε-isometry, Hyers-Ulam stability, bounded domain

MSC 2020: 39B82, 46B04

1 Introduction

In 1940, Ulam gave a lecture at a mathematics club at the University of Wisconsin introducing some important unsolved problems. Then, based on that lecture, he published a book 20 years later (see [1]). A number of unresolved problems are introduced in this book, among which the following question about the Hyers-Ulam stability of group homomorphism is closely related to the subject matter of this paper:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, the following year, Hyers [2] was able to successfully solve Ulam's question about the approximately additive functions, assuming that both G_1 and G_2 were Banach spaces. Indeed, he has proved that if a function $f : G_1 \to G_2$ satisfies inequality $\|f(x + y) - f(x) - f(y)\| \le \varepsilon$ for some $\varepsilon \ge 0$ and all $x, y \in G_1$, then there exists an additive function $A : G_1 \to G_2$ such that $\|f(x) - A(x)\| \le \varepsilon$ for each $x \in G_1$. In this case, the Cauchy additive equation, $f(x + y) = f(x) + f(y)$, is said to have (or satisfy) the Hyers-Ulam stability. In the theorem of Hyers, the relevant additive function $A : G_1 \to G_2$ is constructed from the given function f by using the formula $A(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$. This method is now called the *direct method*.

We use the notations $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ to denote Hilbert spaces over $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. A mapping $f : E \to F$ is said to be an *isometry* if f satisfies

$$\|f(x) - f(y)\| = \|x - y\| \tag{1}$$

for any $x, y \in E$.

*** Corresponding author: Soon-Mo Jung,** Mathematics Section, College of Science and Technology, Hongik University, 30016 Sejong, Republic of Korea, e-mail: smjung@hongik.ac.kr

By considering the definition of Hyers and Ulam [3], for each fixed $\varepsilon \geq 0$, a function $f : E \to F$ is said to be an *ε-isometry* if f satisfies inequality

$$\big| \|f(x) - f(y)\| - \|x - y\| \big| \leq \varepsilon \tag{2}$$

for any $x, y \in E$. If there exists a positive constant K depending on only E and F (independent of f and ε) such that for each ε-isometry $f : E \to F$, there is an isometry $U : E \to F$ satisfying inequality $\|f(x) - U(x)\| \leq K\varepsilon$ for every $x \in E$, then the functional equation (1) is said to have (or satisfy) the *Hyers-Ulam stability*.

To the best of our knowledge, Hyers and Ulam were the first mathematicians to study the Hyers-Ulam stability of isometries (see [3]). Indeed, they were able to prove the stability of isometries based on the properties of the inner product of Hilbert spaces: For each surjective ε-isometry $f : E \to E$ satisfying $f(0) = 0$, there is a surjective isometry $U : E \to E$ satisfying $\|f(x) - U(x)\| \leq 10\varepsilon$ for every $x \in E$. We encourage readers who want to read historically important papers dealing with similar topics to look for the papers [4–7].

In 1978, Gruber [8] proved the following theorem: Assume that E and F are real normed spaces, $f : E \to F$ is a surjective ε-isometry, and that $U : E \to F$ is an isometry satisfying $f(p) = U(p)$ for a $p \in E$. If $\|f(x) - U(x)\| = o(\|x\|)$ as $\|x\| \to \infty$ uniformly, then U is a surjective linear isometry and $\|f(x) - U(x)\| \leq 5\varepsilon$ for all $x \in E$. In particular, if f is continuous, then $\|f(x) - U(x)\| \leq 3\varepsilon$ for each $x \in E$. This Gruber's result was further improved by Gevirtz [9] and by Omladič and Šemrl [10]. There are many other papers related to the stability of isometries, but it is regrettable that due to the restriction of space, they cannot be quoted one by one. Nevertheless, see [11–25] for more general information on the stability of isometries and related topics.

The following Fickett's theorem is an important motive for this paper (see [14]):

Theorem 1.1. (Fickett) *For a fixed integer $n \geq 2$, let D be a bounded subset of $\mathbb{R}^n$ and let $\varepsilon > 0$ be given. If a function $f : D \to \mathbb{R}^n$ satisfies inequality* (2) *for all $x, y \in D$, then there exists an isometry $U : D \to \mathbb{R}^n$ such that*

$$\|f(x) - U(x)\| \leq 27\varepsilon^{1/2^n} \tag{3}$$

for each $x \in D$.

The upper bound associated with inequality (3) in Fickett's theorem becomes very large for any sufficiently small ε in comparison to ε. This is a big drawback of Fickett's theorem. Thus, the work of further improving Fickett's theorem has to be attractive.

After Fickett attempted to prove the Hyers-Ulam stability of isometries defined on a bounded subset of $\mathbb{R}^n$ in 1981, several papers have been published steadily to improve his result over the past 40 years. However, most of the results were not very satisfactory (see [15,16,26–28]). Fortunately, however, Alestalo et al. [29] and Väisälä [25] significantly improved Fickett's result by proving the Hyers-Ulam stability of isometries defined on bounded subsets of $\mathbb{R}^n$.

In this paper, we significantly improve Fickett's theorem by using a more intuitive and more efficient method that is completely different from the methods used by Alestalo et al. and Väisälä. The main idea of this paper is simple. We "solve" inequality (5) without using any special mathematical methods. To "solve" inequality (5) here means to find only necessary conditions, not sufficient conditions for (5). Indeed, we prove the Hyers-Ulam stability of isometries defined on bounded subsets of $\mathbb{R}^n$ for $n \geq 3$ (see Theorem 4.1).

Finally, in Section 6, we will discuss that the main result of this paper has several major advantages over the results of papers [25] and [29].

2 Real version of QR decomposition

An *orthogonal matrix* $\mathbf{Q}$ is a real square matrix whose columns and rows are orthonormal vectors. In other words, a real square matrix $\mathbf{Q}$ is orthogonal if its transpose is equal to its inverse: $\mathbf{Q}^{tr} = \mathbf{Q}^{-1}$, where $\mathbf{Q}^{tr}$ and $\mathbf{Q}^{-1}$ stand for the transpose and the inverse of $\mathbf{Q}$, respectively. As a linear transformation, an orthogonal matrix preserves the inner product of vectors, and therefore acts as an isometry of Euclidean spaces.

Most papers and textbooks that mention *QR decomposition* only prove the complex version of QR decomposition (see [30, Theorem 6.3.7] or [31, Theorem 2.2 in §1]). However, the real version of QR decomposition is required in this paper. Nevertheless, since the proof of the real version is similar to the proof of the complex version, we will omit the proof of the former here.

Theorem 2.1. (QR decomposition) *Every real square matrix* $\mathbf{A}$ *can be decomposed as* $\mathbf{A} = \mathbf{QR}$, *where* $\mathbf{Q}$ *is an orthogonal matrix and* $\mathbf{R}$ *is an upper triangular matrix whose elements are real numbers. In particular, every diagonal element of* $\mathbf{R}$ *is nonnegative.*

We can prove the following lemma using the real version of QR decomposition (Theorem 2.1), and this lemma plays an important role in achieving the final goal of this paper. In practice, using this lemma, we can almost halve the number of unknowns to consider in the main theorem.

Lemma 2.2. *Let* $\mathbb{R}^n$ *be the* n*-dimensional Euclidean space for a fixed integer* $n > 0$. *Assume that* D *is a subset of* $\mathbb{R}^n$ *with* $\{e_1, e_2, \ldots, e_n\} \subset D$, *where* $\{e_1, e_2, \ldots, e_n\}$ *is the standard basis for* $\mathbb{R}^n$. *If* $f : D \to \mathbb{R}^n$ *is a function and every* e_i *is written in column vector, then there exist an orthogonal matrix* $\mathbf{P}$ *and real numbers* e'_{ij} *for* $i, j \in \{1, 2, \ldots, n\}$ *with* $i \geq j$ *such that*

$$\mathbf{P}f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)^{tr}$$

for every $i \in \{1, 2, \ldots, n\}$. *In particular,* $e'_{ii} \geq 0$ *for all* $i \in \{1, 2, \ldots, n\}$.

Proof. Let $f(e_i) = (e_{i1}, e_{i2}, \ldots, e_{in})^{tr}$, written in column vector, for any $i \in \{1, 2, \ldots, n\}$. We now define a matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{pmatrix} = \begin{pmatrix} e_{11} & e_{21} & \cdots & e_{n1} \\ e_{12} & e_{22} & \cdots & e_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1n} & e_{2n} & \ldots & e_{nn} \end{pmatrix}.$$

By Theorem 2.1, there exist an orthogonal matrix $\mathbf{Q}$ and an upper triangular (real) matrix $\mathbf{R}$ such that $\mathbf{A} = \mathbf{QR}$. Thus, we have

$$\mathbf{R} = \mathbf{Q}^{tr}\mathbf{A} = \left(\mathbf{Q}^{tr}f(e_1) \;\; \mathbf{Q}^{tr}f(e_2) \;\; \cdots \;\; \mathbf{Q}^{tr}f(e_n)\right) = \begin{pmatrix} e'_{11} & e'_{21} & \cdots & e'_{n1} \\ 0 & e'_{22} & \cdots & e'_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e'_{nn} \end{pmatrix} \tag{4}$$

for some real numbers $e'_{11}, e'_{21}, e'_{22}, \ldots, e'_{n1}, \ldots, e'_{nn}$, where the diagonal element e'_{ii} is nonnegative for all $i \in \{1, 2, \ldots, n\}$.

Finally, the last two terms of (4) are compared to conclude as follows:

$$\begin{cases} \mathbf{Q}^{tr}f(e_1) = (e'_{11}, 0, 0, \ldots, 0)^{tr}, \\ \mathbf{Q}^{tr}f(e_2) = (e'_{21}, e'_{22}, 0, \ldots, 0)^{tr}, \\ \qquad\qquad \vdots \\ \mathbf{Q}^{tr}f(e_n) = (e'_{n1}, e'_{n2}, e'_{n3}, \ldots, e'_{nn})^{tr}. \end{cases}$$

If we put $\mathbf{P} = \mathbf{Q}^{tr}$, then $\mathbf{P}$ is also an orthogonal matrix. □

3 A preliminary theorem

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$, where n is a fixed integer larger than 2. In this section, D denotes a subset of $\mathbb{R}^n$ that only satisfies $\{0, e_1, e_2, \ldots, e_n\} \subset D$, whether bounded or not.

Lemma 2.2 implies that there exists an orthogonal matrix $\mathbf{Q}$, with which we can express $f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)^{tr}$ with respect to the new basis $\{\mathbf{Q}e_1, \mathbf{Q}e_2, \ldots, \mathbf{Q}e_n\}$ instead of the standard basis $\{e_1, e_2, \ldots, e_n\}$ and $e'_{ii} \geq 0$ for all $i \in \{1, 2, \ldots, n\}$. For example, we can choose the orthogonal matrix $\mathbf{Q}$ given in the proof of Lemma 2.2 for this purpose. Therefore, from now on, we will assume that $f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)$ with $e'_{ii} \geq 0$, written in row vector, without loss of generality.

In the statement of the following theorem, we define σ as if we knew the values of $c_{(i+1)i}$ without knowing their values in advance. However, we note that in the proof of this theorem, we can justify the definition of σ by showing that the value of each $c_{(i+1)i}$ is not greater than 9 (ref. Remark 3.1 (*ii*)). It is noted that the situation is similar for the c_{ii}'s.

Theorem 3.1. *Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$, where $\mathbb{R}^n$ denotes the n-dimensional Euclidean space for a fixed integer $n \geq 3$, let D be a subset of $\mathbb{R}^n$ satisfying $\{0, e_1, e_2, \ldots, e_n\} \subset D$, and let $f : D \to \mathbb{R}^n$ be a function that satisfies $f(0) = 0$ and*

$$\big| \|f(x) - f(y)\| - \|x - y\| \big| \leq \varepsilon \tag{5}$$

for all $x, y \in \{0, e_1, e_2, \ldots, e_n\}$ and for some constant ε with $0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \min_{1 \leq i \leq n} \frac{1}{2c_{ii}}, \frac{1}{12}\right\}$, where σ is defined as $\sigma = \sum_{i=1}^{n-1} c_{(i+1)i}^2$, c_{ii} and $c_{(i+1)i}$ will be determined by the formulas (21) and (23), respectively. Then there exist positive integers c_{ij}, $i, j \in \{1, 2, \ldots, n\}$ with $j \leq i$, such that

$$\begin{cases} -c_{ij}\varepsilon \leq e'_{ij} \leq c_{ij}\varepsilon & (\text{for } i > j), \\ 1 - c_{ii}\varepsilon \leq e'_{ii} \leq 1 + \varepsilon & (\text{for } i = j) \end{cases} \tag{6}$$

and such that the c_{ij} satisfy the equations in (19) for all $i, j \in \{1, 2, \ldots, n\}$ with $j \leq i$.

Proof. (a) Using inequality (5) and by assumption $f(0) = 0$, we have

$$\big| \|f(e_j)\| - 1 \big| \leq \varepsilon \quad \text{and} \quad \big| \|f(e_k) - f(e_\ell)\| - \sqrt{2} \big| \leq \varepsilon$$

for any $j, k, \ell \in \{1, 2, \ldots, n\}$ with $k < \ell$. Since $f(e_j) = (e'_{j1}, \ldots, e'_{jj}, 0, \ldots, 0)$ for all $j \in \{1, 2, \ldots, n\}$ and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$, from the last inequalities we get the following two inequalities, which are equivalent to inequality (5) for $x, y \in \{0, e_1, e_2, \ldots, e_n\}$:

$$(1 - \varepsilon)^2 \leq \sum_{i=1}^{j} e'^2_{ji} \leq (1 + \varepsilon)^2 \tag{7}$$

for each $j \in \{1, 2, \ldots, n\}$ and

$$(\sqrt{2} - \varepsilon)^2 \leq \sum_{i=1}^{k} e'^2_{ki} - \sum_{i=1}^{k} 2e'_{ki}e'_{\ell i} + \sum_{i=1}^{\ell} e'^2_{\ell i} \leq (\sqrt{2} + \varepsilon)^2 \tag{8}$$

for every $k, \ell \in \{1, 2, \ldots, n\}$ with $k < \ell$. From now on, we will prove this theorem by using inequalities (7) and (8) instead of inequality (5).

(b) Now we apply the "main" induction on m to prove the array of equations presented in (19). Proving the array of (19) is the most important and longest part of this proof.

(b.1) According to Lemma 2.2, e'_{11} is a nonnegative real number, so setting $j = 1$ in (7) gives us inequality, $1 - \varepsilon \leq e'_{11} \leq 1 + \varepsilon$, and we select $c_{11} = 1$ as the smallest positive integer that satisfies the following inequality:

$$1 - c_{11}\varepsilon \leq 1 - \varepsilon \leq e'_{11} \leq 1 + \varepsilon. \tag{9}$$

This fact guarantees the existence of c_{11} satisfying the second condition of (6) for $i = j = 1$. (We note that c_{11} must not be necessarily the smallest positive integer satisfying inequality (9), for example, $c_{11} = 2$ is not wrong, but if possible, the smaller the c_{11} is, the better it is.) If we set $j = 2$ in (7) and put $k = 1$ and $\ell = 2$ in (8) and then combine the resulting inequalities, then we get

$$\frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e'_{21} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)}$$

and we can choose $c_{21} = 4$ as the smallest positive integer satisfying the condition

$$-c_{21}\varepsilon \le \frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e'_{21} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)} \le c_{21}\varepsilon. \tag{10}$$

Obviously, this fact confirms the existence of c_{21} satisfying the first condition of (6) for $i = 2$ and $j = 1$. (We note that c_{ij} must not be necessarily the smallest positive integer under given conditions, for example, $c_{21} = 5$ is not bad, but if possible, the smaller the c_{ij} is, the better it is. Indeed, assuming that the c_{ij} are the smallest positive integers that satisfy some given conditions, we can find the unique c_{ij} (for $i > j$), which makes it easier to prove the array of equations presented in (19). This is one of the reasons we want c_{ij} to be the smallest positive integer.)

Furthermore, if we set $j = 3$ in (7) and put $k = 1$ and $\ell = 3$ in (8) and then combine the resulting inequalities, then we get the inner part of the following inequalities:

$$-c_{31}\varepsilon \le \frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e'_{31} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)} \le c_{31}\varepsilon \tag{11}$$

and we select c_{31} as the smallest positive integer that satisfies the outermost inequalities of (11). From this fact, we confirm the existence of c_{31} that satisfies the first condition in (6). Since c_{21} and c_{31} are assumed to be the smallest positive integers satisfying the outermost inequalities of (10) and (11), respectively, we come to the conclusion that $c_{31} = c_{21} = 4$.

Moreover, using (7) with $j = 2$ and by a direct calculation, since $0 < \varepsilon < \frac{1}{12}$, we get

$$1 - 2\varepsilon - \frac{19}{12}\varepsilon + 4\varepsilon^2 < 1 - 2\varepsilon - 19\varepsilon^2 + 4\varepsilon^2 = (1 - \varepsilon)^2 - c_{21}^2\varepsilon^2 \le e'^2_{22} \le (1 + \varepsilon)^2$$

and, if possible, we determine c_{22} as the smallest positive integer which satisfies the condition

$$(1 - c_{22}\varepsilon)^2 \le 1 - 2\varepsilon - \frac{19}{12}\varepsilon + 4\varepsilon^2 \le e'^2_{22} \le (1 + \varepsilon)^2.$$

The last inequalities assure the existence of $c_{22} = 2$ satisfying the second condition of (6) for $i = j = 2$.

In a similar way, putting $k = 2$ and $\ell = 3$ in (8) and a routine calculation show the existence of c_{32} satisfying the first condition of (6), for example, $c_{32} = 6$. Analogously, since $0 < \varepsilon < \frac{1}{12}$, inequality (7) with $j = 3$ yields inequality, $1 - 2\varepsilon - 51\varepsilon^2 \le e'^2_{33} \le (1 + \varepsilon)^2$, and we can choose the smallest positive integer c_{33} that satisfies

$$(1 - c_{33}\varepsilon)^2 \le 1 - 2\varepsilon - 51\varepsilon^2 \le e'^2_{33} \le (1 + \varepsilon)^2,$$

which shows that c_{33} exists that satisfies the second condition of (6). More directly, we can choose $c_{33} = 4$. Therefore, all the integers c_{ij} considered in (b.1) satisfy the conditions in (6) and (19) for $n = 3$. By doing this, we start the induction (with $m = 3$).

(b.2) Induction hypothesis. Let m be some integer satisfying $3 \le m < n$. It is assumed that the smallest positive integers c_{ij}, $i, j \in \{1, 2, \ldots, m\}$ with $j \le i$ were found by the methods we did in the subsection (b.1), and that these positive integers satisfy the following inequalities:

$$\begin{cases} -c_{ij}\varepsilon \le e'_{ij} \le c_{ij}\varepsilon & (\text{for } i > j), \\ 1 - c_{ii}\varepsilon \le e'_{ii} \le 1 + \varepsilon & (\text{for } i = j) \end{cases}$$

as well as the array of equations

$$\begin{cases} c_{m1} = c_{(m-1)1} = c_{(m-2)1} = \ldots = c_{41} = c_{31} = c_{21}, \\ c_{m2} = c_{(m-1)2} = c_{(m-2)2} = \ldots = c_{42} = c_{32}, \\ c_{m3} = c_{(m-1)3} = c_{(m-2)3} = \ldots = c_{43}, \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ c_{m(m-3)} = c_{(m-1)(m-3)} = c_{(m-2)(m-3)}, \\ c_{m(m-2)} = c_{(m-1)(m-2)}, \\ c_{m(m-1)}. \end{cases}$$

The last line in the above array consisting of only $c_{m(m-1)}$ means that there exists the smallest possible positive integer $c_{m(m-1)}$ that satisfies $-c_{m(m-1)}\varepsilon \le e'_{m(m-1)} \le c_{m(m-1)}\varepsilon$.

(b.3) We let $j = m + 1$ in (7) and $\ell = m + 1$ in (8) to get

$$(1-\varepsilon)^2 \le \sum_{i=1}^{m+1} e'^2_{(m+1)i} \le (1+\varepsilon)^2 \tag{12}$$

and

$$(\sqrt{2}-\varepsilon)^2 \le \sum_{i=1}^{k} e'^2_{ki} - \sum_{i=1}^{k} 2e'_{ki}e'_{(m+1)i} + \sum_{i=1}^{m+1} e'^2_{(m+1)i} \le (\sqrt{2}+\varepsilon)^2 \tag{13}$$

for every $k \in \{1, 2, \ldots, m\}$.

Similar to what we did to get (10), inequalities (9), (12), and (13) with $k = 1$ yield the inner ones of the following inequalities:

$$-c_{(m+1)1}\varepsilon \le \frac{-(2c_{11} + 2 + 2\sqrt{2})\varepsilon + c_{11}^2\varepsilon^2}{2(1 - c_{11}\varepsilon)} \le e'_{(m+1)1} \le \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - c_{11}\varepsilon)} \le c_{(m+1)1}\varepsilon, \tag{14}$$

and we find the smallest positive integer $c_{(m+1)1}$ satisfying the outermost inequalities of (14). By comparing both inequalities (10) and (14), we may conclude that $c_{(m+1)1} = c_{21}$, with which we initiate an "inner" induction that is subordinate to the main induction.

(b.3.1) We choose some $k \in \{2, 3, \ldots, m\}$ and assume that $-c_{(m+1)i}\varepsilon \le e'_{(m+1)i} \le c_{(m+1)i}\varepsilon$ and $c_{(m+1)i} = c_{(i+1)i}$ for each $i \in \{1, 2, \ldots, k-1\}$. This is the hypothesis for our inner induction on i that operates inside the main induction on m. Based on this hypothesis, we will prove that there exists a positive integer $c_{(m+1)k}$ that satisfies $-c_{(m+1)k}\varepsilon \le e'_{(m+1)k} \le c_{(m+1)k}\varepsilon$ as well as $c_{(m+1)k} = c_{(k+1)k}$. Roughly speaking, this inner induction works to expand each row of (19) horizontally.

(b.3.2) It follows from (13) that

$$\begin{aligned} &(\sqrt{2}-\varepsilon)^2 - \sum_{i=1}^{k} e'^2_{ki} + \sum_{i=1}^{k-1} 2e'_{ki}e'_{(m+1)i} - \sum_{i=1}^{m+1} e'^2_{(m+1)i} \\ &\le -2e'_{kk}e'_{(m+1)k} \le (\sqrt{2}+\varepsilon)^2 - \sum_{i=1}^{k} e'^2_{ki} + \sum_{i=1}^{k-1} 2e'_{ki}e'_{(m+1)i} - \sum_{i=1}^{m+1} e'^2_{(m+1)i} \end{aligned} \tag{15}$$

for any $k \in \{2, 3, \ldots, m\}$. On the other hand, by (7) and (12), we have

$$(1-\varepsilon)^2 \le \sum_{i=1}^{k} e'^2_{ki} \le (1+\varepsilon)^2 \quad \text{and} \quad (1-\varepsilon)^2 \le \sum_{i=1}^{m+1} e'^2_{(m+1)i} \le (1+\varepsilon)^2$$

for each $k \in \{2, 3, \ldots, m\}$. Moreover, it follows from the hypotheses (b.2) and (b.3.1) that

$$-\sum_{i=1}^{k-1} 2c_{ki}c_{(i+1)i}\varepsilon^2 = -\sum_{i=1}^{k-1} 2c_{ki}c_{(m+1)i}\varepsilon^2 \le \sum_{i=1}^{k-1} 2e'_{ki}e'_{(m+1)i} \le \sum_{i=1}^{k-1} 2c_{ki}c_{(m+1)i}\varepsilon^2 = \sum_{i=1}^{k-1} 2c_{ki}c_{(i+1)i}\varepsilon^2$$

for all $k \in \{2, 3, \ldots, m\}$.

Since $c_{kk}\varepsilon < \frac{1}{2}$ and $e'_{kk} > 0$ by (b.2), we use (15) and the last inequalities to get the inner ones of the following inequalities:

$$\begin{aligned} -c_{(m+1)k}\varepsilon &\le \frac{1}{2e'_{kk}}\left(-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2\sum_{i=1}^{k-1} c_{ki}c_{(i+1)i}\varepsilon^2\right) \\ &\le e'_{(m+1)k} \le \frac{1}{2e'_{kk}}\left((4+2\sqrt{2})\varepsilon + \varepsilon^2 + 2\sum_{i=1}^{k-1} c_{ki}c_{(i+1)i}\varepsilon^2\right) \le c_{(m+1)k}\varepsilon \end{aligned} \tag{16}$$

for all $k \in \{2, 3, \ldots, m\}$ and we can select the smallest positive integer $c_{(m+1)k}$ that satisfies the outermost inequalities of (16). Similarly, by (7) and (8) with $\ell = k+1$, a routine calculation yields

$$\begin{aligned} -c_{(k+1)k}\varepsilon &\le \frac{1}{2e'_{kk}}\left(-(4+2\sqrt{2})\varepsilon + \varepsilon^2 - 2\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon^2\right) \\ &\le e'_{(k+1)k} \le \frac{1}{2e'_{kk}}\left((4+2\sqrt{2})\varepsilon + \varepsilon^2 + 2\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon^2\right) \le c_{(k+1)k}\varepsilon, \end{aligned} \tag{17}$$

where $c_{(k+1)k}$ is the smallest positive integer that satisfies the outmost conditions of (17). We note by (b.2) and (b.3.1) that $c_{(k+1)i} = c_{(i+1)i}$ for every integer i satisfying $0 < i < k$. Comparing (16) and (17), we conclude that $c_{(m+1)k} = c_{(k+1)k}$ for each $k \in \{2, 3, \ldots, m\}$. Furthermore, referring to the subsection (b.3), we see that $c_{(m+1)k} = c_{(k+1)k}$ holds for all $k \in \{1, 2, \ldots, m\}$, which proves the truth of the first column of the array of equations in the subsection (b.3.3).

Moreover, inequality (7) with $j = m+1$ yields

$$(1-\varepsilon)^2 - \sum_{i=1}^{m} e'^2_{(m+1)i} \le e'^2_{(m+1)(m+1)} \le (1+\varepsilon)^2 - \sum_{i=1}^{m} e'^2_{(m+1)i}. \tag{18}$$

Since $0 < \varepsilon < \frac{1}{\sigma}$, $m < n$, and $0 < \varepsilon < \frac{1}{12}$, it follows from (18) and some manipulation that

$$\begin{aligned} (1+\varepsilon)^2 \ge e'^2_{(m+1)(m+1)} &\ge (1-\varepsilon)^2 - \sum_{i=1}^{m} c^2_{(m+1)i}\varepsilon^2 = (1-\varepsilon)^2 - \sum_{i=1}^{m} c^2_{(i+1)i}\varepsilon^2 \\ &\ge (1-\varepsilon)^2 - \sigma\varepsilon^2 > 1 - 3\varepsilon + \varepsilon^2 > 9\varepsilon + \varepsilon^2 > 100\varepsilon^2. \end{aligned}$$

We see that $0 < (1 - c\varepsilon)^2 \le 100\varepsilon^2$ whenever c is a positive integer satisfying $\frac{1}{\varepsilon} - 10 \le c < \frac{1}{\varepsilon}$. This fact shows the existence of $c_{(m+1)(m+1)}$ that satisfies the second condition of (6).

(b.3.3) We just proved in the subsections from (b.2) to (b.3.2) that there exist positive integers c_{ij}, $i, j \in \{1, 2, \ldots, m+1\}$ with $j \le i$, such that

$$\begin{cases} -\, c_{ij}\varepsilon \le e'_{ij} \le c_{ij}\varepsilon & (\text{for } i > j), \\ 1 - c_{ii}\varepsilon \le e'_{ii} \le 1 + \varepsilon & (\text{for } i = j), \end{cases}$$

and the c_{ij}'s satisfy

$$\begin{cases} c_{(m+1)1} = c_{m1} = c_{(m-1)1} = \ldots = c_{41} = c_{31} = c_{21}, \\ c_{(m+1)2} = c_{m2} = c_{(m-1)2} = \ldots = c_{42} = c_{32}, \\ c_{(m+1)3} = c_{m3} = c_{(m-1)3} = \ldots = c_{43}, \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \\ c_{(m+1)(m-2)} = c_{m(m-2)} = c_{(m-1)(m-2)}, \\ c_{(m+1)(m-1)} = c_{m(m-1)}, \\ c_{(m+1)m}. \end{cases}$$

The last row in the above array consisting of only $c_{(m+1)m}$ means that there exists the smallest possible positive integer $c_{(m+1)m}$ that satisfies $-c_{(m+1)m}\varepsilon \le e'_{(m+1)m} \le c_{(m+1)m}\varepsilon$. (We can check in (b.3.2) the truth of equations in the first column of the above array. Moreover, we remember that we have assumed in (b.2) that the rest of equations in the array are true.)

(b.4) Altogether, by the main induction conclusion on m ($3 \le m < n$), we may conclude that there exist positive integers c_{ij}, $i, j \in \{1, 2, \dots, n\}$ with $j \le i$, such that each inequality in (6) holds true and the c_{ij}'s satisfy

$$\begin{cases} c_{n1} = c_{(n-1)1} = c_{(n-2)1} = \dots = c_{41} = c_{31} = c_{21}, \\ c_{n2} = c_{(n-1)2} = c_{(n-2)2} = \dots = c_{42} = c_{32}, \\ c_{n3} = c_{(n-1)3} = c_{(n-2)3} = \dots = c_{43}, \\ \vdots \qquad \vdots \qquad \vdots \\ c_{n(n-3)} = c_{(n-1)(n-3)} = c_{(n-2)(n-3)}, \\ c_{n(n-2)} = c_{(n-1)(n-2)}, \\ c_{n(n-1)}. \end{cases} \tag{19}$$

which completes the first part of our proof. We remark that the last row "$c_{n(n-1)}$" in the above array implies that there is an integer $c_{n(n-1)} > 0$ satisfying $-c_{n(n-1)}\varepsilon \le e'_{n(n-1)} \le c_{n(n-1)}\varepsilon$.

(c) Now we will introduce efficient methods to estimate the positive integers c_{jj} and $c_{k(k-1)}$ for every $j \in \{1, 2, \dots, n\}$ and $k \in \{2, 3, \dots, n\}$.

(c.1) We note that inequality (7) holds true for all $j \in \{1, 2, \dots, n\}$. Since $-c_{ji}\varepsilon \le e'_{ji} \le c_{ji}\varepsilon$ for any $i, j \in \{1, 2, \dots, n\}$ with $i < j$, we determine the c_{jj} as the smallest possible positive integer that satisfies

$$(1 - c_{jj}\varepsilon)^2 \le (1 - \varepsilon)^2 - \sum_{i=1}^{j-1} c_{ji}^2\varepsilon^2 \le (1 - \varepsilon)^2 - \sum_{i=1}^{j-1} e'^2_{ji} \le e'^2_{jj} \le (1 + \varepsilon)^2 \tag{20}$$

for all $j \in \{2, 3, \dots, n\}$. However, since the previous inequality is inefficient in practical calculations, we introduce a more practical inequality even at the expense of the smallest property of c_{jj}. Instead of (20), we will determine the c_{jj} as the smallest positive integer that satisfies the new condition

$$(3c_{jj} - 1)(c_{jj} - 1) \ge \sum_{i=1}^{j-1} c_{(i+1)i}^2 \tag{21}$$

for all $j \in \{2, 3, \dots, n\}$, which proves the existence of c_{jj}. Indeed, since $0 < \varepsilon < \frac{1}{2c_{jj}}$, we have

$$(1 - c_{jj}\varepsilon)^2 = 1 - 2\varepsilon c_{jj} + \varepsilon^2 c_{jj}^2 < 1 - 2\varepsilon c_{jj} + \frac{1}{2}\varepsilon c_{jj} = 1 - \frac{3}{2}\varepsilon c_{jj}.$$

It further follows from (19), (21), and the last inequality that

$$\begin{aligned} (1 - c_{jj}\varepsilon)^2 &< 1 - \frac{3}{2}\varepsilon c_{jj} = 1 - \frac{\varepsilon}{2c_{jj}} \cdot 3c_{jj}^2 \le 1 - \frac{\varepsilon}{2c_{jj}}\left(\sum_{i=1}^{j-1} c_{(i+1)i}^2 - 1 + 4c_{jj}\right) \\ &= 1 - 2\varepsilon - \frac{\varepsilon}{2c_{jj}}\left(\sum_{i=1}^{j-1} c_{(i+1)i}^2 - 1\right) \le 1 - 2\varepsilon - \varepsilon^2\left(\sum_{i=1}^{j-1} c_{(i+1)i}^2 - 1\right) \\ &= (1 - \varepsilon)^2 - \sum_{i=1}^{j-1} c_{(i+1)i}^2\varepsilon^2 = (1 - \varepsilon)^2 - \sum_{i=1}^{j-1} c_{ji}^2\varepsilon^2, \end{aligned}$$

which implies the validity of (20).

(c.2) We note that inequality (8) holds for all $k, \ell \in \{1, 2, \dots, n\}$ with $k < \ell$. If we set $\ell = k + 1$ in (8) and make some manipulations, then we obtain the inner ones of the following inequalities:

$$\begin{aligned} -c_{(k+1)k}\varepsilon &\le \frac{1}{2e'_{kk}}\left(-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 - 2\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon^2\right) \\ &\le e'_{(k+1)k} \le \frac{1}{2e'_{kk}}\left((4 + 2\sqrt{2})\varepsilon + \varepsilon^2 + 2\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon^2\right) \le c_{(k+1)k}\varepsilon \end{aligned} \tag{22}$$

for every $k \in \{1, 2, \ldots, n-1\}$. And then, we choose the smallest positive integer $c_{(k+1)k}$ that satisfies the outermost inequalities of (22).

Inequalities (6) and (22) show the existence of the smallest positive integer $c_{(k+1)k}$ that satisfies

$$\frac{1}{2(1-c_{kk}\varepsilon)}\left(4+2\sqrt{2}+\varepsilon+2\sum_{i=1}^{k-1}c_{ki}c_{(k+1)i}\varepsilon\right) \le c_{(k+1)k} \tag{23}$$

for $k \in \{1, 2, \ldots, n-1\}$. Since $0 < \varepsilon < \frac{1}{2c_{kk}}$, we see that $\frac{1}{2(1-c_{kk}\varepsilon)} < 1$. Furthermore, since $c_{ki} = c_{(k+1)i} = c_{(i+1)i}$ for any $i \in \{1, 2, \ldots, k-1\}$ and $0 < \varepsilon < \frac{1}{\sigma}$, we know that $\sum_{i=1}^{k-1} c_{ki}c_{(k+1)i}\varepsilon \le \sigma\varepsilon < 1$. Thus, we get

$$\frac{1}{2(1-c_{kk}\varepsilon)}\left(4+2\sqrt{2}+\varepsilon+2\sum_{i=1}^{k-1}c_{ki}c_{(k+1)i}\varepsilon\right) < 6+2\sqrt{2}+\varepsilon < 9,$$

which together with (23) assures the existence of $c_{(k+1)k}$ with $0 < c_{(k+1)k} \le 9$ for all $k \in \{1, 2, \ldots, n-1\}$. □

Remark 3.1.

(i) Inequality (5) is a sufficient condition for inequalities in (6), and the inequalities in (6) are necessary conditions for inequality (5).

(ii) In view of the last part of the proof of Theorem 3.1, we see that $0 < c_{(i+1)i} \le 9$ for any $i \in \{1, 2, \ldots, n-1\}$.

(iii) We solve the quadratic inequality (21) with respect to c_{jj} as follows:

$$c_{jj} \ge \frac{1}{3}\left(2+\sqrt{1+3\sum_{i=1}^{j-1}c_{(i+1)i}^2}\right)$$

for all $j \in \{1, 2, \ldots, n\}$.

4 Hyers-Ulam stability of isometries on bounded domains

The following theorem significantly improves Fickett's theorem by demonstrating the Hyers-Ulam stability of isometries on the bounded domains.

As before, let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Based on Lemma 2.2, we can assume that $f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)$ is written in row vector, where $e'_{ii} \ge 0$ for each $i \in \{1, 2, \ldots, n\}$. We denote by $B_d(0)$ the closed ball of radius d and centered at the origin of $\mathbb{R}^n$, *i.e.*, $B_d(0) = \{x \in \mathbb{R}^n : \|x\| \le d\}$.

Theorem 4.1. *Given an integer $n \ge 3$, let D be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$ such that $\{0, e_1, e_2, \ldots, e_n\} \subset D \subset B_d(0)$ for some $d \ge 1$. If a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ and inequality* (5) *for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \min_{1\le i\le n}\frac{1}{2c_{ii}}, \frac{1}{12}\right\}$, where $\sigma = \sum_{i=1}^{n-1}c_{(i+1)i}^2$ and the c_{ij} ($i, j \in \{1, 2, \ldots, n\}$ with $j \le i$) are the positive integers estimated in* Theorem 3.1, *then there exists an isometry $U : D \to \mathbb{R}^n$ such that*

$$\|f(x) - U(x)\| \le \left[\sum_{i=1}^{n}\left(\left(2+\sum_{j=1}^{i}c_{ij}\right)d + 4 + \sum_{j=1}^{i}c_{ij}\right)^2\right]^{1/2}\varepsilon$$

for all $x \in D$.

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Based on Lemma 2.2, it can be assumed that $f(e_i) = (e'_{i1}, e'_{i2}, \ldots, e'_{ii}, 0, \ldots, 0)$, where $e'_{ii} \ge 0$ for each $i \in \{1, 2, \ldots, n\}$. For an arbitrary point $x = (x_1, x_2, \ldots, x_n)$ of D, let $f(x) = (x'_1, x'_2, \ldots, x'_n)$. It follows from (5) that

$$\big|\|f(x)\| - \|x\|\big| \le \varepsilon \quad \text{and} \quad \big|\|f(x) - f(e_j)\| - \|x - e_j\|\big| \le \varepsilon$$

and hence, we have

$$\left| \left(\sum_{i=1}^{n} x_i'^2 \right)^{1/2} - \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \right| \le \varepsilon, \tag{24}$$

$$\left| \left(\sum_{i=1}^{j} (x_i' - e_{ji}')^2 + \sum_{i=j+1}^{n} x_i'^2 \right)^{1/2} - \left(\sum_{i=1}^{n} x_i^2 - 2x_j + 1 \right)^{1/2} \right| \le \varepsilon \tag{25}$$

for all $j \in \{1, 2, \ldots, n\}$.

It follows from (24) that

$$\left| \sum_{i=1}^{n} x_i'^2 - \sum_{i=1}^{n} x_i^2 \right| = \left| \left(\sum_{i=1}^{n} x_i'^2 \right)^{1/2} + \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \right| \left| \left(\sum_{i=1}^{n} x_i'^2 \right)^{1/2} - \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \right| \le (2d+1)\varepsilon \tag{26}$$

since $\sqrt{x_1'^2 + \cdots + x_n'^2} \le d + \varepsilon$, $\sqrt{x_1^2 + \cdots + x_n^2} \le d$, and $0 < \varepsilon < 1$. Similarly, it follows from (25) that

$$\left| \left(\sum_{i=1}^{j} (x_i' - e_{ji}')^2 + \sum_{i=j+1}^{n} x_i'^2 \right) - \left(\sum_{i=1}^{n} x_i^2 - 2x_j + 1 \right) \right| \le (2d+3)\varepsilon \tag{27}$$

for all $j \in \{1, 2, \ldots, n\}$, since $0 < \varepsilon < 1$ and

$$\left(\sum_{i=1}^{j} (x_i' - e_{ji}')^2 + \sum_{i=j+1}^{n} x_i'^2 \right)^{1/2} \le d + 1 + \varepsilon \quad \text{and} \quad \left(\sum_{i=1}^{n} x_i^2 - 2x_j + 1 \right)^{1/2} \le d + 1.$$

We use (27) to get

$$\begin{aligned} &-(2d+3)\varepsilon - \left(\sum_{i=1}^{n} x_i'^2 - \sum_{i=1}^{n} x_i^2 \right) + \sum_{i=1}^{j-1} 2e_{ji}'x_i' + 1 - \sum_{i=1}^{j} e_{ji}'^2 \\ &\le 2x_j - 2e_{jj}'x_j' \le (2d+3)\varepsilon - \left(\sum_{i=1}^{n} x_i'^2 - \sum_{i=1}^{n} x_i^2 \right) + \sum_{i=1}^{j-1} 2e_{ji}'x_i' + 1 - \sum_{i=1}^{j} e_{ji}'^2 \end{aligned} \tag{28}$$

for any $j \in \{1, 2, \ldots, n\}$.

Since $|x_i'| \le \|f(x)\| \le \|x\| + \varepsilon < d + 1$ and by Theorem 3.1, we get

$$-2(d+1)\sum_{i=1}^{j-1} c_{ji}\varepsilon \le \sum_{i=1}^{j-1} 2e_{ji}'x_i' \le 2(d+1)\sum_{i=1}^{j-1} c_{ji}\varepsilon.$$

Moreover, by (7), we have

$$-3\varepsilon \le -2\varepsilon - \varepsilon^2 \le 1 - \sum_{i=1}^{j} e_{ji}'^2 \le 2\varepsilon - \varepsilon^2 \le 3\varepsilon.$$

Therefore, it follows from (26) and (28) that

$$-\left(\left(4 + 2\sum_{i=1}^{j-1} c_{ji} \right) d + 7 + 2\sum_{i=1}^{j-1} c_{ji} \right)\varepsilon \le 2x_j - 2e_{jj}'x_j' \le \left(\left(4 + 2\sum_{i=1}^{j-1} c_{ji} \right) d + 7 + 2\sum_{i=1}^{j-1} c_{ji} \right)\varepsilon$$

for all $j \in \{1, 2, \ldots, n\}$.

We note that $|x_j'| < d + 1$ and $-c_{jj}\varepsilon \le 1 - e_{jj}' \le c_{jj}\varepsilon$ by Theorem 3.1, and since $x_j - e_{jj}'x_j' = (x_j - x_j') + (1 - e_{jj}')x_j'$, we can see that

$$|x_j - x_j'| \le \left(\left(2 + \sum_{i=1}^{j} c_{ji} \right) d + 4 + \sum_{i=1}^{j} c_{ji} \right)\varepsilon \tag{29}$$

for $j \in \{1, 2, \ldots, n\}$.

Since we can select an isometry $U : D \to \mathbb{R}^n$ defined by $U(x) = x = (x_1, x_2, \ldots, x_n)$, we see that

$$\begin{aligned}\|f(x) - U(x)\| &= \|(x_1' - x_1, x_2' - x_2, \ldots, x_n' - x_n)\| = \left(\sum_{j=1}^{n}(x_j' - x_j)^2\right)^{1/2} \\ &\le \left(\sum_{j=1}^{n}\left(\left(2 + \sum_{i=1}^{j}c_{ji}\right)d + 4 + \sum_{i=1}^{j}c_{ji}\right)^2\right)^{1/2}\varepsilon\end{aligned}$$

for all $x \in D$. □

We remark that for any $i, j \in \{1, 2, \ldots, n\}$ with $i \ge j$, each c_{ij} is independent of ε for any "sufficiently" small $\varepsilon > 0$.

For every $j \in \{2, 3, \ldots, n\}$, it follows from Remark 3.1 (ii) that

$$\frac{1}{3}\left(2 + \sqrt{1 + 3\sum_{i=1}^{j-1}c_{(i+1)i}^2}\right) \le \frac{1}{3}(2 + \sqrt{1 + 3 \cdot 9^2(j-1)}) \le \frac{1}{3}(2 + \sqrt{256(j-1)}) \le 6\sqrt{j-1}.$$

Then, in view of Remark 3.1 (iii), we may assume that c_{jj} is the possibly smallest integer that satisfies

$$c_{jj} \ge \begin{cases} 1 & (\text{for } j = 1), \\ 6\sqrt{j-1} & (\text{for } j \in \{2, 3, \ldots, n\}), \end{cases}$$

i.e.,

$$c_{jj} \le 6\sqrt{j-1} + 1 \tag{30}$$

for any $j \in \{1, 2, \ldots, n\}$.

By Remark 3.1 (iii) and (30), we get

$$\sum_{j=1}^{i}c_{ij} \le 9(i-1) + 6\sqrt{i-1} + 1$$

and

$$\left(2 + \sum_{j=1}^{i}c_{ij}\right)d + 4 + \sum_{j=1}^{i}c_{ij} \le (9(i-1) + 6\sqrt{i-1} + 5)(d+1)$$

for some $d \ge 1$. Furthermore, since $n \ge 3$, we have

$$\begin{aligned}&\sum_{i=1}^{n}\left[\left(2 + \sum_{j=1}^{i}c_{ij}\right)d + 4 + \sum_{j=1}^{i}c_{ij}\right]^2 \\ &\le (d+1)^2\sum_{i=1}^{n}(81i^2 - 36i - 20) + 108(d+1)^2\sum_{i=1}^{n}(i-1)\sqrt{i-1} + 60(d+1)^2\sum_{i=1}^{n}\sqrt{i-1} \\ &\le (d+1)^2\sum_{i=1}^{n}(81i^2 - 36i - 20) + 108(d+1)^2\int_1^n x\sqrt{x}\,\mathrm{d}x + 60(d+1)^2\int_1^n \sqrt{x}\,\mathrm{d}x \\ &= (d+1)^2\left(27n^3 + \frac{216}{5}n^{5/2} + 40n^{3/2} - 2n - \frac{416}{5}\right) \\ &\le (d+1)^2\left(27 + \frac{216}{5\sqrt{3}} + \frac{40}{3\sqrt{3}}\right)n^3 \le 64(d+1)^2n^3\end{aligned}$$

for any $i \in \{1, 2, \ldots, n\}$.

From Theorem 4.1 and the explanations described above, we obtain the following corollary.

Corollary 4.2. *Given an integer $n \geq 3$, assume that D is a subset of the n-dimensional Euclidean space $\mathbb{R}^n$ satisfying $\{0, e_1, e_2, \ldots, e_n\} \subset D \subset B_d(0)$ for some $d \geq 1$. Let ε be an arbitrary constant that satisfies $0 < \varepsilon < \min\left\{\frac{1}{\sigma}, \min_{1\leq i\leq n} \frac{1}{2c_{ii}}, \frac{1}{12}\right\}$, where $\sigma = \sum_{i=1}^{n-1} c_{(i+1)i}^2$ and the c_{ij} $(i, j \in \{1, 2, \ldots, n\}$ with $j \leq i)$ are the positive integers estimated in* Theorem 3.1. *If a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ and inequality (5) for all $x, y \in D$, then there exists an isometry $U : D \to \mathbb{R}^n$ that satisfies*

$$\|f(x) - U(x)\| \leq 8(d + 1)n\sqrt{n}\,\varepsilon$$

for all $x \in D$.

5 Examples

Example 5.1. We assume that $n = 4$. We will compute some constants c_{ij} by using the recurrence formulas (21) and (23). First, since $c_{11} = 1$ by (b.1) in the proof of Theorem 3.1, it follows from (23) with $k = 1$ and $0 < \varepsilon < \frac{1}{12}$ that

$$c_{21} \geq \frac{6}{11}\left(4 + 2\sqrt{2} + \frac{1}{12}\right) \geq \frac{1}{2(1 - c_{11}\varepsilon)}(4 + 2\sqrt{2} + \varepsilon),$$

and we can choose $c_{21} = 4$ as the smallest positive integer satisfying the last inequality.

Now, we use (21) with $j = 2$ to obtain

$$(3c_{22} - 1)(c_{22} - 1) \geq c_{21}^2$$

and thus, we select $c_{22} = 3$. We note that $c_{22} = 3$ is larger than the estimate in the proof of Theorem 3.1. This difference is due to the use of formula (21) instead of (20).

By (23) with $k = 2$, we get

$$c_{32} \geq \frac{2}{3}\left(4 + 2\sqrt{2} + \frac{33}{12}\right) \geq \frac{1}{2(1 - c_{22}\varepsilon)}(4 + 2\sqrt{2} + \varepsilon + 2c_{21}^2\varepsilon)$$

and hence, we choose $c_{32} = 7$. Furthermore, it follows from (21) with $j = 3$ that

$$(3c_{33} - 1)(c_{33} - 1) \geq c_{21}^2 + c_{32}^2 = 65$$

and hence, $c_{33} = 6$. In the proof of Theorem 3.1, we estimated $c_{32} = 6$ and $c_{33} = 4$, but in this example, we estimate the larger values for them because we use formula (21) instead of (20).

As $n = 4$, we see that $\sigma \geq \sum_{i=1}^{3} c_{(i+1)i}^2 > c_{21}^2 + c_{32}^2 = 65$ and hence, $\varepsilon < \frac{1}{\sigma} < \frac{1}{65}$. Therefore, we have

$$\frac{1}{2(1 - c_{33}\varepsilon)}\left(4 + 2\sqrt{2} + \varepsilon + 2\sum_{i=1}^{2} c_{3i}c_{4i}\varepsilon\right) < \frac{65}{118}\left(4 + 2\sqrt{2} + \frac{1}{65} + 2\right) < 4.88$$

and thus, using (23) with $k = 3$, we can select $c_{43} = 5$. In view of (21) with $j = 4$, we get

$$(3c_{44} - 1)(c_{44} - 1) \geq c_{21}^2 + c_{32}^2 + c_{43}^2 = 90$$

and hence, $c_{44} = 7$, which comply with the claims of Remark 3.1.

Example 5.2. We assume that $n = 5$. As in Example 5.1 we get the following constants: $c_{11} = 1$, $c_{21} = 4$, $c_{22} = 3$, $c_{31} = 4$, $c_{32} = 7$, $c_{33} = 6$, $c_{41} = 4$, $c_{42} = 7$, $c_{43} = 5$, and $c_{44} = 7$.

As $n = 5$, we see that $\sigma \geq \sum_{i=1}^{4} c_{(i+1)i}^2 > c_{21}^2 + c_{32}^2 + c_{43}^2 = 90$ and hence, $\varepsilon < \frac{1}{\sigma} < \frac{1}{90}$. Thus, we have

$$\frac{1}{2(1 - c_{44}\varepsilon)}\left(4 + 2\sqrt{2} + \varepsilon + 2\sum_{i=1}^{3} c_{4i}c_{5i}\varepsilon\right) < \frac{45}{83}\left(4 + 2\sqrt{2} + \frac{1}{90} + 2\right) < 4.80$$

and hence, using (23) with $k = 4$, we can select $c_{54} = 5$. In view of (21) with $j = 5$, we get

$$(3c_{55} - 1)(c_{55} - 1) \geq c_{21}^2 + c_{32}^2 + c_{43}^2 + c_{54}^2 = 115 \quad \text{and hence} \quad c_{55} = 7.$$

Moreover, due to (19), we have $c_{51} = 4$, $c_{52} = 7$, and $c_{53} = 5$.

Example 5.3. Let $D = \{x \in \mathbb{R}^4 : \|x\| \leq d\}$ for some $d \geq 1$ and let $f : D \to \mathbb{R}^4$ be a function satisfying $f(0) = 0$ and inequality (5) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \frac{1}{90}$. Using Theorem 4.1 and Example 5.1, we can prove that there exists an isometry $U : D \to \mathbb{R}^4$ satisfying

$$\|f(x) - U(x)\| \leq \left[\sum_{i=1}^{4}\left(\left(2 + \sum_{j=1}^{i} c_{ij}\right)d + 4 + \sum_{j=1}^{i} c_{ij}\right)^2\right]^{1/2} \varepsilon = \sqrt{1076d^2 + 2376d + 1316}\,\varepsilon < (33d + 37)\varepsilon$$

for all $x \in D$. This result is within the range predicted by Corollary 4.2.

Example 5.4. Let $D = \{x \in \mathbb{R}^5 : \|x\| \leq d\}$ for some $d \geq 1$ and let $f : D \to \mathbb{R}^5$ be a function satisfying $f(0) = 0$ and inequality (5) for all $x, y \in D$ and some constant ε with $0 < \varepsilon < \frac{1}{115}$. Using Theorem 4.1 and Example 5.2, we can prove that there exists an isometry $U : D \to \mathbb{R}^5$ satisfying

$$\|f(x) - U(x)\| \leq \left[\sum_{i=1}^{5}\left(\left(2 + \sum_{j=1}^{i} c_{ij}\right)d + 4 + \sum_{j=1}^{i} c_{ij}\right)^2\right]^{1/2} \varepsilon = \sqrt{1976d^2 + 4296d + 2340}\,\varepsilon < (45d + 49)\varepsilon$$

for all $x \in D$. This result is within the range expected by Corollary 4.2.

6 Discussions

We expect the Hyers-Ulam stability of isometries defined on bounded domains to have widespread application, but no remarkable results have been published, except for the papers [25,29], for 40 years after Fickett's theorem was published.

Now, among the theorems in [29], the theorem that is most closely related to the subject of this paper is introduced. In fact, the following theorem has been proved relatively simply using the so-called John's method, but the proofs of other theorems in paper [29] are technically very complex.

Theorem 6.1. (Alestalo et al.) *Assume that $n \geq 2$ is an integer, D is a bounded subset of $\mathbb{R}^n$ with $D \subset B_R(0)$ for some $R > 0$, and that D contains $0, ru_1, \ldots, ru_n$, where $0 < r \leq R$ and the vectors $u_1, \ldots, u_n$ are orthonormal. If $f : D \to \mathbb{R}^n$ is an ε-isometry, then there exists an isometry $U : D \to \mathbb{R}^n$ such that*

$$\|f(x) - U(x)\| \leq 10\frac{R}{r}n\sqrt{n}\,\varepsilon$$

for all $x \in D$.

If r in Theorem 6.1 is very small, the upper bound of the inequality presented in Theorem 6.1 becomes very large. This is the weakness of paper [29] compared to the present paper. As we can see, the upper bound of inequality presented in Corollary 4.2 of this paper depends only on R and not on r. (We used d instead of R in Corollary 4.2.) This is one of the advantages of this paper compared to the previous study [29].

A finite sequence $(u_0, u_1, \ldots, u_m)$ of a compact subset D of $\mathbb{R}^n$ is called a maximal sequence in D provided that $h_k := d(u_k, A_{k-1})$ is maximal in D for each $k \in \{1, 2, \ldots, m\}$, where A_{k-1} is the affine subspace (or flat) including $\{u_0, u_1, \ldots, u_{k-1}\}$. Now, among the results proved by Väisälä, the part related to the subject of this paper is introduced in the following theorem.

Theorem 6.2. (Väisälä) *Let D be a compact subset of $\mathbb{R}^n$. If there exist a maximal sequence $(u_0, u_1, \ldots, u_n)$ in D and a constant $c \geq 1$ such that*
(*i*) $|u_k - u_0| \leq ch_k$ *for every* $k \in \{2, 3, \ldots, n\}$;
(*ii*) $D \backslash \{u_1, \ldots, u_n\} \subset B_{ch_n}(u_0)$,

then for every ε-isometry $f : D \to \mathbb{R}^n$, there exist an isometry $U : \mathbb{R}^n \to \mathbb{R}^n$ and a positive constant $c^* = c^*(c, n)$ such that

$$\|f(x) - U(x)\| \leq c^* \varepsilon$$

for all $x \in D$.

According to [25, §4.2], we get the recursion formulas

$$\rho_1 = 7.5, \quad \tau_1 = 25.5, \quad \rho_i = 3c\left(2 + \sum_{j=1}^{i-1} \rho_j\right) + 2c^2 \tau_{i-1}, \quad \tau_i = \tau_{i-1} + 3\rho_i$$

for all integers $i \geq 2$, where $c \geq 1$ is a constant. First, assuming $c = 1$, we calculated ρ_i and τ_i for small natural numbers i using the above formulas, and the results are written in the following table:

i	1	2	3	4	5	$\cdots$
ρ_i	7.5	79.5	795	7,950	79,500	$\cdots$
τ_i	25.5	264	2,649	26,499	264,999	$\cdots$
$c^*(1, i)$	4	>79	>799	>7,990	>79,900	$\cdots$
$8(d+1)i\sqrt{i}$	—	—	<84	128	<179	$\cdots$

The values in the fourth row of the table above are due to the following formula:

$$c^*(1, i) = \sqrt{\sum_{j=1}^{i} \rho_j^2}$$

introduced in the proof of [25, Theorem 4.1]. The values in the last row are due to the formula given in Corollary 4.2 with $d = 1$. Comparing the values in the last two rows of the table above, we can see that our result of this paper is more efficient than that of Väisälä.

7 Conclusion

We emphasize once more that we have made great progress in improving Fickett's theorem using an intuitive method. According to Theorem 4.1 or Corollary 4.2 of this paper, if a function $f : D \to \mathbb{R}^n$ satisfies $f(0) = 0$ as well as inequality (5) for all $x, y \in D$ and for some sufficiently small constant $\varepsilon > 0$, then there exist an isometry $U : D \to \mathbb{R}^n$ and a constant $K > 0$ such that inequality $\|f(x) - U(x)\| \leq K\varepsilon$ holds for all $x \in D$. However, it is impossible to deduce this useful conclusion by using Fickett's theorem. From this point of view, we can say that Fickett's theorem has been remarkably improved in this paper.

Funding information: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1F1A1A01049560). This work was supported by 2021 Hongik University Research Fund.

Conflict of interest: Author states no conflict of interest.

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.
[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[3] D. H. Hyers and S. M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. **51** (1945), 288–292.
[4] D. G. Bourgin, *Approximate isometries*, Bull. Amer. Math. Soc. **52** (1946), 704–714.
[5] D. G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J. **16** (1949), 385–397.
[6] R. D. Bourgin, *Approximate isometries on finite dimensional Banach spaces*, Trans. Amer. Math. Soc. **207** (1975), 309–328.
[7] D. H. Hyers and S. M. Ulam, *Approximate isometries of the space of continuous functions*, Ann. Math. **48** (1947), 285–289.
[8] P. M. Gruber, *Stability of isometries*, Trans. Amer. Math. Soc. **245** (1978), 263–277.
[9] J. Gevirtz, *Stability of isometries on Banach spaces*, Proc. Amer. Math. Soc. **89** (1983), 633–636.
[10] M. Omladič and P. Šemrl, *On non linear perturbations of isometries*, Math. Ann. **303** (1995), 617–628.
[11] J. A. Baker, *Isometries in normed spaces*, Amer. Math. Monthly **78** (1971), no. 6, 655–658.
[12] R. Bhatia and P. Šemrl, *Approximate isometries on Euclidean spaces*, Amer. Math. Monthly **104** (1997), 497–504.
[13] G. Dolinar, *Generalized stability of isometries*, J. Math. Anal. Appl. **242** (2000), 39–56.
[14] J. W. Fickett, *Approximate isometries on bounded sets with an application to measure theory*, Studia Math. **72** (1981), 37–46.
[15] S.-M. Jung, *Hyers-Ulam-Rassias stability of isometries on restricted domains*, Nonlinear Stud. **8** (2001), 125–134.
[16] S.-M. Jung, *Asymptotic properties of isometries*, J. Math. Anal. Appl. **276** (2002), 642–653.
[17] J. Lindenstrauss and A. Szankowski, *Non linear perturbations of isometries*, Astérisque **131** (1985), 357–371.
[18] Th. M. Rassias, *Properties of isometric mappings*, J. Math. Anal. Appl. **235** (1999), 108–121.
[19] Th. M. Rassias, *Isometries and approximate isometries*, Internat. J. Math. Math. Sci. **25** (2001), 73–91.
[20] Th. M. Rassias and C. S. Sharma, *Properties of isometries*, J. Natural Geom. **3** (1993), 1–38.
[21] P. Šemrl, *Hyers-Ulam stability of isometries on Banach spaces*, Aequationes Math. **58** (1999), 157–162.
[22] F. Skof, *Sulle δ-isometrie negli spazi normati*, Rendiconti Di Mat. Ser. VII, Roma **10** (1990), 853–866.
[23] F. Skof, *On asymptotically isometric operators in normed spaces*, Ist. Lomb. Sci. Lett. A **131** (1997), 117–129.
[24] R. L. Swain, *Approximate isometries in bounded spaces*, Proc. Amer. Math. Soc. **2** (1951), 727–729.
[25] J. Väisälä, *Isometric approximation property in Euclidean spaces*, Israel J. Math. **128** (2002), 1–27.
[26] J. Chmielinski and S.-M. Jung, *The stability of the Wigner equation on a restricted domain*, J. Math. Anal. Appl. **254** (2001), 309–320.
[27] G. Choi and S.-M. Jung, *The stability of isometries on restricted domains*, Symmetry **13** (2021), 282, DOI: https://doi.org/10.3390/sym13020282.
[28] S.-M. Jung and B. Kim, *Stability of isometries on restricted domains*, J. Korean Math. Soc. **37** (2000), no. 1, 125–137.
[29] P. Alestalo, D. A. Trotsenko, and J. Väisälä, *Isometric approximation*, Israel J. Math. **125** (2001), 61–82.
[30] H. Anton, *Elementary Linear Algebra* (9th ed.), John Wiley & Sons, New York, 2005.
[31] G. W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, New York, 1990.