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Research Article

Hyers-Ulam Stability of Nonhomogeneous Linear Differential Equations of Second Order

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The aim of this paper is to prove the stability in the sense of Hyers-Ulam of differential equation of second order $y'' + p(x)y' + q(x)y + r(x) = 0$. That is, if f is an approximate solution of the equation $y'' + p(x)y' + q(x)y + r(x) = 0$, then there exists an exact solution of the equation near to f.

1. Introduction and Preliminaries

In 1940, Ulam [1] posed the following problem concerning the stability of functional equations: give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [2] when G_1 and G_2 are Banach spaces, and the result of Hyers was generalized by Rassias (see [3]). Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [3–5]).

In connection with the stability of exponential functions, Alsina and Ger [6] remarked that the differential equation $y' = y$ has the Hyers-Ulam stability. More explicitly, they proved that if a differentiable function $y : I \to R$ satisfies $|y'(t) - y(t)| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \le 3\varepsilon$ for every $t \in I$.

The above result of Alsina and Ger has been generalized by Miura et al. [7], by Miura [8], and also by Takahasi et al. [9]. Indeed, they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(t)$, while Alsina and Ger investigated the differential equation $y'(t) = y(t)$.

Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized by Miura et al. [10], by Takahasi et al. [11], and also by Jung

[12]. They dealt with the nonhomogeneous linear differential equation of first order

$$y' + p(t)y + q(t) = 0. \tag{1.1}$$

Jung [13] proved the generalized Hyers-Ulam stability of differential equations of the form

$$ty'(t) + \alpha y(t) + \beta t^r x_0 = 0 \tag{1.2}$$

and also applied this result to the investigation of the Hyers-Ulam stability of the differential equation

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0. \tag{1.3}$$

Recently, Wang et al. [14] discussed the Hyers-Ulam stability of the first-order nonhomogeneous linear differential equation

$$p(x)y' + q(x)y + r(x) = 0. \tag{1.4}$$

They proved the following theorem.

Theorem 1.1 (see [14]). *Let $p(x), q(x)$, and $r(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $|q(x)| \geq \delta$ for all $x \in I$ and some $\delta > 0$ independent of $x \in I$. Then* (1.4) *has the Hyers-Ulam stability.*

The aim of this paper is to investigate the Hyers-Ulam stability of the following linear differential equations of second order under some special conditions:

$$y'' + p(x)y' + q(x)y + r(x) = 0, \tag{1.5}$$

where $y \in C^2(I) = C^2(a, b), -\infty < a < b < +\infty$.

For the sake of convenience, all the integrals in the rest of the work will be viewed as existing. We say that (1.5) has the Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: for every $\varepsilon > 0$, $y \in C^2(I)$, if

$$\left| y'' + p(x)y' + q(x)y + r(x) \right| \leq \varepsilon, \tag{1.6}$$

then there exists some $z \in C^2(I)$ satisfying

$$z'' + p(x)z' + q(x)z + r(x) = 0 \tag{1.7}$$

such that $|y(x) - z(x)| < K\varepsilon$. We call such K a Hyers-Ulam stability constant for (1.5).

2. Main Results

Now, the main results of this work are given in the following theorems.

Theorem 2.1. *If a twice continuously differentiable function* $y : I \to R$ *satisfies the differential inequality*

$$\left|y'' + p(x)y' + q(x)y + r(x)\right| \leq \varepsilon \tag{2.1}$$

for all $t \in I$ *and for some* $\varepsilon > 0$*, and the Riccati equation* $u'(x) + p(x)u(x) - u^2(x) = q(x)$ *has a particular solution, then there exists a solution* $v : I \to R$ *of* (1.5) *such that*

$$\left|y(x) - v(x)\right| \leq K\varepsilon, \tag{2.2}$$

where $K > 0$ *is a constant.*

Proof. Let $\varepsilon > 0$ and $y : I \to R$ be a continuously differentiable function such that

$$\left|y'' + p(x)y' + q(x)y + r(x)\right| \leq \varepsilon. \tag{2.3}$$

We will show that there exists a constant K independent of ε and v such that $|y - v| < K\varepsilon$ for some $v \in C^2(I)$ satisfying $v'' + p(x)v' + q(x)v + r(x) = 0$.

Assume that $c(x)$ is a particular solution of Riccati equation $u'(x) + p(x)u(x) - u^2(x) = q(x)$; if we set

$$g(x) = y'(x) + c(x)y(x), \qquad d(x) = p(x) - c(x), \tag{2.4}$$

then

$$g'(x) = y''(x) + c(x)y'(x) + c'(x)y(x), \tag{2.5}$$

thus

$$\begin{aligned}\left|g'(x) + d(x)g(x) + r(x)\right| &= \left|y''(x) + (c(x) + d(x))y'(x) + \left(c'(x) + d(x)c(x)\right)y(x) + r(x)\right| \\ &= \left|y'' + p(x)y' + q(x)y + r(x)\right| \leq \varepsilon.\end{aligned} \tag{2.6}$$

Using the similar technique in [14], we can prove

$$\begin{aligned} w(x) &= \exp\left\{\int_a^x (-d(s))ds\right\}\left[(g(b)-\varepsilon)\exp\left\{-\int_a^b (-d(x))ds\right\}\right. \\ &\qquad \left. -\int_x^b -r(s)\exp\left\{-\int_a^s (-d(t))dt\right\}ds\right] \\ &= \exp\left\{\int_a^x (-d(s))ds\right\}\left[(g(b)-\varepsilon)\exp\left\{\int_a^b (d(x))ds\right\} + \int_x^b r(s)\exp\left\{\int_a^s (d(t))dt\right\}ds\right] \end{aligned} \tag{2.7}$$

satisfying

$$w'(x) + d(x)w(x) + r(x) = 0, \tag{2.8}$$

and there exists an $L > 0$ such that

$$|g(x) - w(x)| \le L\varepsilon. \tag{2.9}$$

By $g(x) = y'(x) + c(x)y(x)$, we get

$$|y'(x) + c(x)y(x) - w(x)| \le \varepsilon. \tag{2.10}$$

Using the same technique as above, we know that

$$\begin{aligned} v(x) &= \exp\left\{\int_a^x (-c(s))ds\right\}\left[(y(b)-\varepsilon)\exp\left\{-\int_a^b (-c(x))ds\right\}\right. \\ &\qquad \left. -\int_x^b w(s)\exp\left\{-\int_a^s (-c(t))dt\right\}ds\right] \\ &= \exp\left\{\int_a^x (-c(s))ds\right\}\left[(y(b)-\varepsilon)\exp\left\{\int_a^b (c(s))ds\right\} - \int_x^b w(s)\exp\left\{\int_a^s (c(t))dt\right\}ds\right] \end{aligned} \tag{2.11}$$

satisfying

$$v'(x) + c(x)v(x) - w(x) = 0, \tag{2.12}$$

and there exists a $K > 0$ such that

$$|y(x) - v(x)| \le K\varepsilon. \tag{2.13}$$

The desired conclusion is proved. □

Theorem 2.2. *Let $p(x), q(x)$, and $r(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $y_0(x)$ is a nonzero bounded particular solution $p(x)y'' + q(x)y' + r(x)y = 0$. If $y : I \to R$ is a twice continuously differentiable function, which satisfies the differential inequality*

$$\left|p(x)y'' + q(x)y' + r(x)y\right| \leq \varepsilon \tag{2.14}$$

for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $v : I \to R$ such that

$$\left|y(x) - v(x)\right| \leq K\varepsilon, \tag{2.15}$$

where $K > 0$ is a constant, and v satisfies $p(x)v'' + q(x)v' + r(x)v = 0$.

Proof. Setting $y(x) = y_0(x)\int_a^x z(s)ds$, we obtain

$$y'(x) = y_0'(x)\int_a^x z(s)ds + y_0(x)z(x) \tag{2.16}$$

and also

$$y''(x) = y_0''(x)\int_a^x z(s)ds + 2y_0'(x)z(x) + y_0(x)z'(x) \tag{2.17}$$

By a simple calculation, we see that

$$\begin{aligned} \left|p(x)y_0(x)z'(x) + \left[2p(x)y_0'(x) + q(x)y_0(x)\right]z(x)\right| &= \left|p(x)y''(x) + q(x)y'(x) + r(x)y(x)\right| \\ &\leq \varepsilon. \end{aligned} \tag{2.18}$$

Without loss of generality, we may assume that $p(x)y_0(x) > 0$. Using the similar technique in [14], we know that

$$z_1(x) = \exp\left\{-\int_a^x \frac{2p(s)y_0'(s) + q(s)}{y_0(s)}ds\right\}\left[(z(b) - \varepsilon)\exp\left\{\int_a^b \frac{2p(s)y_0'(s) + q(s)}{y_0(s)}ds\right\}\right] \tag{2.19}$$

satisfies

$$p(x)y_1(x)z_1'(x) + \left[2p(x)y_1'(x) + q(x)y_1(x)\right]z_1(x) = 0 \tag{2.20}$$

and also

$$|z(x) - z_1(x)| \leq L\varepsilon \tag{2.21}$$

for some $L > 0$.

From the inequalities $-L\varepsilon \le z(x) - z_1(x) \le L\varepsilon$ and the similar technique in [14], we further get that

$$z_2(x) = \left(\frac{y(b)}{y_1(b)} - \varepsilon\right) - \int_x^b z_1(s)ds \tag{2.22}$$

satisfies

$$z_2(x) - z_1(x) = 0 \tag{2.23}$$

and also

$$|z(x) - z_2(x)| \le Q\varepsilon \tag{2.24}$$

for some $Q > 0$.

Consequently, we have

$$\left|y(x) - z_2(x)y_0(x)\right| \le M\varepsilon \tag{2.25}$$

for some positive constant M.

Define $v(x) = z_2(x)y_0(x)$. It then follows from the above inequality that $|z(x) - v(x)| \le M\varepsilon$ holds for every $x \in I$. We can easily verify that v satisfies $p(x)v'' + q(x)v' + r(x)v = 0$. This completes the proof of our theorem. □

We can prove the following corollaries by using an analogous argument. Hence, we omit the proofs.

Corollary 2.3. *Let $p(x), q(x)$, and $r(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $r^2 + p(x)r + q(x) = 0$. If $y : I \to R$ is a twice continuously differentiable function, which satisfies the differential inequality*

$$\left|p(x)y'' + q(x)y' + r(x)y\right| \le \varepsilon \tag{2.26}$$

for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $v : I \to R$ such that

$$\left|y(x) - v(x)\right| \le K\varepsilon, \tag{2.27}$$

where $K > 0$ is a constant, and v satisfies $p(x)v'' + q(x)v' + r(x)v = 0$.

Corollary 2.4. *Let $p(x), q(x), r(x)$, and $s(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $y_0(x)$ is a nonzero bounded particular solution $p(x)y''' + q(x)y'' + r(x)y' + s(x)y = 0$. If $y : I \to R$ is a twice continuously differentiable function, which satisfies the differential inequality*

$$\left|p(x)y''' + q(x)y'' + r(x)y' + s(x)y\right| \le \varepsilon \tag{2.28}$$

for all $t \in I$ *and for some* $\varepsilon > 0$*, then there exists a solution* $v : I \to R$ *such that*

$$\left|y(x) - v(x)\right| \leq K\varepsilon, \tag{2.29}$$

where $K > 0$ *is a constant, and* v *satisfies* $p(x)v''' + q(x)v'' + r(x)v' + s(x)v = 0$*.*

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