

Research Article

Hyers-Ulam Stability of Nonhomogeneous Linear Differential Equations of Second Order

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The aim of this paper is to prove the stability in the sense of Hyers-Ulam of differential equation of second order $y'' + p(x)y' + q(x)y + r(x) = 0$. That is, if f is an approximate solution of the equation $y'' + p(x)y' + q(x)y + r(x) = 0$, then there exists an exact solution of the equation near to f .

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1. Introduction and Preliminaries

In 1940, Ulam [1] posed the following problem concerning the stability of functional equations: give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [2] when G_1 and G_2 are Banach spaces, and the result of Hyers was generalized by Rassias (see [3]). Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [3–5]).

In connection with the stability of exponential functions, Alsina and Ger [6] remarked that the differential equation $y' = y$ has the Hyers-Ulam stability. More explicitly, they proved that if a differentiable function $y : I \rightarrow R$ satisfies $|y'(t) - y(t)| \leq \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \rightarrow R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \leq 3\varepsilon$ for every $t \in I$.

The above result of Alsina and Ger has been generalized by Miura et al. [7], by Miura [8], and also by Takahasi et al. [9]. Indeed, they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(t)$, while Alsina and Ger investigated the differential equation $y'(t) = y(t)$.

Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized by Miura et al. [10], by Takahasi et al. [11], and also by Jung

[12]. They dealt with the nonhomogeneous linear differential equation of first order

$$y' + p(t)y + q(t) = 0. \quad (1.1)$$

Jung [13] proved the generalized Hyers-Ulam stability of differential equations of the form

$$ty'(t) + \alpha y(t) + \beta t^r x_0 = 0 \quad (1.2)$$

and also applied this result to the investigation of the Hyers-Ulam stability of the differential equation

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0. \quad (1.3)$$

Recently, Wang et al. [14] discussed the Hyers-Ulam stability of the first-order nonhomogeneous linear differential equation

$$p(x)y' + q(x)y + r(x) = 0. \quad (1.4)$$

They proved the following theorem.

Theorem 1.1 (see [14]). *Let $p(x)$, $q(x)$, and $r(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $|q(x)| \geq \delta$ for all $x \in I$ and some $\delta > 0$ independent of $x \in I$. Then (1.4) has the Hyers-Ulam stability.*

The aim of this paper is to investigate the Hyers-Ulam stability of the following linear differential equations of second order under some special conditions:

$$y'' + p(x)y' + q(x)y + r(x) = 0, \quad (1.5)$$

where $y \in C^2(I) = C^2(a, b)$, $-\infty < a < b < +\infty$.

For the sake of convenience, all the integrals in the rest of the work will be viewed as existing. We say that (1.5) has the Hyers-Ulam stability if there exists a constant $K > 0$ with the following property: for every $\varepsilon > 0$, $y \in C^2(I)$, if

$$|y'' + p(x)y' + q(x)y + r(x)| \leq \varepsilon, \quad (1.6)$$

then there exists some $z \in C^2(I)$ satisfying

$$z'' + p(x)z' + q(x)z + r(x) = 0 \quad (1.7)$$

such that $|y(x) - z(x)| < K\varepsilon$. We call such K a Hyers-Ulam stability constant for (1.5).

2. Main Results

Now, the main results of this work are given in the following theorems.

Theorem 2.1. *If a twice continuously differentiable function $y : I \rightarrow \mathbb{R}$ satisfies the differential inequality*

$$|y'' + p(x)y' + q(x)y + r(x)| \leq \varepsilon \quad (2.1)$$

for all $t \in I$ and for some $\varepsilon > 0$, and the Riccati equation $u'(x) + p(x)u(x) - u^2(x) = q(x)$ has a particular solution, then there exists a solution $v : I \rightarrow \mathbb{R}$ of (1.5) such that

$$|y(x) - v(x)| \leq K\varepsilon, \quad (2.2)$$

where $K > 0$ is a constant.

Proof. Let $\varepsilon > 0$ and $y : I \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$|y'' + p(x)y' + q(x)y + r(x)| \leq \varepsilon. \quad (2.3)$$

We will show that there exists a constant K independent of ε and v such that $|y - v| < K\varepsilon$ for some $v \in C^2(I)$ satisfying $v'' + p(x)v' + q(x)v + r(x) = 0$.

Assume that $c(x)$ is a particular solution of Riccati equation $u'(x) + p(x)u(x) - u^2(x) = q(x)$; if we set

$$g(x) = y'(x) + c(x)y(x), \quad d(x) = p(x) - c(x), \quad (2.4)$$

then

$$g'(x) = y''(x) + c(x)y'(x) + c'(x)y(x), \quad (2.5)$$

thus

$$\begin{aligned} |g'(x) + d(x)g(x) + r(x)| &= |y''(x) + (c(x) + d(x))y'(x) + (c'(x) + d(x)c(x))y(x) + r(x)| \\ &= |y'' + p(x)y' + q(x)y + r(x)| \leq \varepsilon. \end{aligned} \quad (2.6)$$

Using the similar technique in [14], we can prove

$$\begin{aligned} w(x) &= \exp\left\{\int_a^x (-d(s))ds\right\} \left[(g(b) - \varepsilon) \exp\left\{-\int_a^b (-d(x))ds\right\} \right. \\ &\quad \left. - \int_x^b r(s) \exp\left\{-\int_a^s (-d(t))dt\right\} ds \right] \\ &= \exp\left\{\int_a^x (-d(s))ds\right\} \left[(g(b) - \varepsilon) \exp\left\{\int_a^b (d(x))ds\right\} + \int_x^b r(s) \exp\left\{\int_a^s (d(t))dt\right\} ds \right] \end{aligned} \quad (2.7)$$

satisfying

$$w'(x) + d(x)w(x) + r(x) = 0, \quad (2.8)$$

and there exists an $L > 0$ such that

$$|g(x) - w(x)| \leq L\varepsilon. \quad (2.9)$$

By $g(x) = y'(x) + c(x)y(x)$, we get

$$|y'(x) + c(x)y(x) - w(x)| \leq \varepsilon. \quad (2.10)$$

Using the same technique as above, we know that

$$\begin{aligned} v(x) &= \exp\left\{\int_a^x (-c(s))ds\right\} \left[(y(b) - \varepsilon) \exp\left\{-\int_a^b (-c(x))ds\right\} \right. \\ &\quad \left. - \int_x^b w(s) \exp\left\{-\int_a^s (-c(t))dt\right\} ds \right] \\ &= \exp\left\{\int_a^x (-c(s))ds\right\} \left[(y(b) - \varepsilon) \exp\left\{\int_a^b (c(s))ds\right\} - \int_x^b w(s) \exp\left\{\int_a^s (c(t))dt\right\} ds \right] \end{aligned} \quad (2.11)$$

satisfying

$$v'(x) + c(x)v(x) - w(x) = 0, \quad (2.12)$$

and there exists a $K > 0$ such that

$$|y(x) - v(x)| \leq K\varepsilon. \quad (2.13)$$

The desired conclusion is proved. \square

Theorem 2.2. Let $p(x), q(x)$, and $r(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $y_0(x)$ is a nonzero bounded particular solution $p(x)y'' + q(x)y' + r(x)y = 0$. If $y : I \rightarrow \mathbb{R}$ is a twice continuously differentiable function, which satisfies the differential inequality

$$|p(x)y'' + q(x)y' + r(x)y| \leq \varepsilon \quad (2.14)$$

for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $v : I \rightarrow \mathbb{R}$ such that

$$|y(x) - v(x)| \leq K\varepsilon, \quad (2.15)$$

where $K > 0$ is a constant, and v satisfies $p(x)v'' + q(x)v' + r(x)v = 0$.

Proof. Setting $y(x) = y_0(x) \int_a^x z(s) ds$, we obtain

$$y'(x) = y_0'(x) \int_a^x z(s) ds + y_0(x)z(x) \quad (2.16)$$

and also

$$y''(x) = y_0''(x) \int_a^x z(s) ds + 2y_0'(x)z(x) + y_0(x)z'(x) \quad (2.17)$$

By a simple calculation, we see that

$$\begin{aligned} |p(x)y_0(x)z'(x) + [2p(x)y_0'(x) + q(x)y_0(x)]z(x)| &= |p(x)y''(x) + q(x)y'(x) + r(x)y(x)| \\ &\leq \varepsilon. \end{aligned} \quad (2.18)$$

Without loss of generality, we may assume that $p(x)y_0(x) > 0$. Using the similar technique in [14], we know that

$$z_1(x) = \exp \left\{ - \int_a^x \frac{2p(s)y_0'(s) + q(s)}{y_0(s)} ds \right\} \left[(z(b) - \varepsilon) \exp \left\{ \int_a^b \frac{2p(s)y_0'(s) + q(s)}{y_0(s)} ds \right\} \right] \quad (2.19)$$

satisfies

$$p(x)y_1(x)z_1'(x) + [2p(x)y_1'(x) + q(x)y_1(x)]z_1(x) = 0 \quad (2.20)$$

and also

$$|z(x) - z_1(x)| \leq L\varepsilon \quad (2.21)$$

for some $L > 0$.

From the inequalities $-L\varepsilon \leq z(x) - z_1(x) \leq L\varepsilon$ and the similar technique in [14], we further get that

$$z_2(x) = \left(\frac{y(b)}{y_1(b)} - \varepsilon \right) - \int_x^b z_1(s) ds \quad (2.22)$$

satisfies

$$z_2(x) - z_1(x) = 0 \quad (2.23)$$

and also

$$|z(x) - z_2(x)| \leq Q\varepsilon \quad (2.24)$$

for some $Q > 0$.

Consequently, we have

$$|y(x) - z_2(x)y_0(x)| \leq M\varepsilon \quad (2.25)$$

for some positive constant M .

Define $v(x) = z_2(x)y_0(x)$. It then follows from the above inequality that $|z(x) - v(x)| \leq M\varepsilon$ holds for every $x \in I$. We can easily verify that v satisfies $p(x)v'' + q(x)v' + r(x)v = 0$. This completes the proof of our theorem. \square

We can prove the following corollaries by using an analogous argument. Hence, we omit the proofs.

Corollary 2.3. *Let $p(x), q(x)$, and $r(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $r^2 + p(x)r + q(x) = 0$. If $y : I \rightarrow R$ is a twice continuously differentiable function, which satisfies the differential inequality*

$$|p(x)y'' + q(x)y' + r(x)y| \leq \varepsilon \quad (2.26)$$

for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $v : I \rightarrow R$ such that

$$|y(x) - v(x)| \leq K\varepsilon, \quad (2.27)$$

where $K > 0$ is a constant, and v satisfies $p(x)v'' + q(x)v' + r(x)v = 0$.

Corollary 2.4. *Let $p(x), q(x), r(x)$, and $s(x)$ be continuous real functions on the interval $I = (a, b)$ such that $p(x) \neq 0$ and $y_0(x)$ is a nonzero bounded particular solution $p(x)y''' + q(x)y'' + r(x)y' + s(x)y = 0$. If $y : I \rightarrow R$ is a twice continuously differentiable function, which satisfies the differential inequality*

$$|p(x)y''' + q(x)y'' + r(x)y' + s(x)y| \leq \varepsilon \quad (2.28)$$

for all $t \in I$ and for some $\varepsilon > 0$, then there exists a solution $v : I \rightarrow R$ such that

$$|y(x) - v(x)| \leq K\varepsilon, \quad (2.29)$$

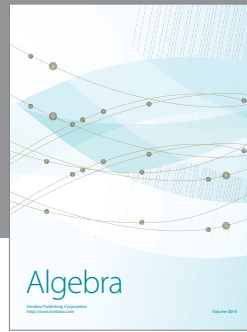
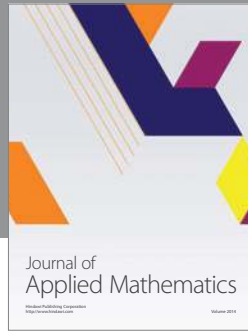
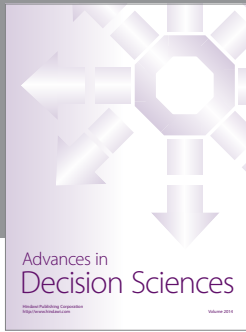
where $K > 0$ is a constant, and v satisfies $p(x)v''' + q(x)v'' + r(x)v' + s(x)v = 0$.

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