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# HYERS-ULAM STABILITY OF TRIGONOMETRIC FUNCTIONAL EQUATIONS

Jeongwook Chang and Jaeyoung Chung

Abstract. In this article we prove the Hyers–Ulam stability of trigonometric functional equations.

## 1. Introduction

In 1940, S. M. Ulam proposed the following problem [18]:

*Let $f$ be a mapping from a group $G_1$ to a metric group $G_2$ with metric $d(\cdot,\cdot)$ such that*

$$d(f(xy),\, f(x)f(y)) \le \varepsilon.$$

*Then does there exist a group homomorphism $L$ and $\delta_\epsilon > 0$ such that*

$$d(f(x),\, L(x)) \le \delta_\epsilon$$

*for all $x \in G_1$?*

This problem was solved affirmatively by D. H. Hyers [11] under the assumption that $G_2$ is a Banach space. In 1978, Th. M. Rassias [16] firstly generalized the above result and since then, stability problems of many other functional equations have been investigated [4, 5, 6, 7, 8, 9, 12, 13, 14, 15]. In 1990, L. Székelyhidi [17] has developed his idea of using invariant subspaces of functions defined on a group or semigroup in connection with stability questions for sine and cosine functional equations. In this paper, employing the idea of L. Székelyhidi [17] we consider the Hyers-Ulam stability problem of the following two trigonometric functional equations

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0, \tag{1.1}$$

$$g(x-y) - g(x)g(y) - f(x)f(y) = 0, \tag{1.2}$$

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where $f, g : G \to \mathbb{C}$ and $G$ is an abelian group divisible by 2. We call $A : G \to \mathbb{C}$ *additive* provided that $A(x+y) = A(x) + A(y)$ for all $x, y \in G$ and call $m : G \to \mathbb{C}$ *exponential* provided that $m(x+y) = m(x)m(y)$ for all $x, y \in G$. We prove as results that if $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \le M$$

for all $x, y \in G$, then $f$, $g$ satisfy one of the followings:

(i) $f = 0, \quad g :$ arbitrary,

(ii) $f$ and $g$ are bounded functions,

(iii) $f(x) = A(x) + B(x)$ and $g(x) = \lambda A(x) + \mu B(x) + 1$,

(iv) $f(x) = \frac{\lambda}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x) + m(-x)) + \frac{\mu}{2}(m(x) - m(-x))$,

where $\lambda, \mu \in \mathbb{C}$, $A$ is an additive function, $m$ is an exponential function and $B$ is a bounded function.

Also we prove that if $f, g : G \to \mathbb{C}$ satisfy the inequality

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \le M$$

for all $x, y \in G$, then $f$, $g$ satisfy one of the followings:

(i) $f$ and $g$ are bounded functions,

(ii) $f(x) = \frac{1}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x) + m(-x))$, where $m$ is an exponential function.

## 2. Stability of the equations

We first discuss the general solutions of the equations (1.1) and (1.2). We refer the reader to Aczél ([1], p. 180) and Aczél–Dombres ([2], pp. 209–217) for the proofs.

**Lemma 2.1.** *Let $G$ be an abelian group divisible by 2. Then the general solutions $f$, $g$ of the equation (1.1) are given by*

$$f(x) = \frac{\lambda}{2}(m(x) - m(-x)),$$
$$g(x) = \frac{1}{2}(m(x) + m(-x)) + \frac{\mu}{2}(m(x) - m(-x)),$$

*or*

$$f(x) = A(x), \quad g(x) = \lambda A(x) + 1,$$

*and the nonconstant general solutions $f$, $g$ of (1.2) are given by*

$$f(x) = \frac{1}{2}(m(x) - m(-x)),$$
$$g(x) = \frac{1}{2}(m(x) + m(-x)),$$

*where $\mu, \lambda \in \mathbb{C}$, $A$ is an additive function and $m$ is an exponential function.*

**Lemma 2.2.** *Let $f, g : G \to \mathbb{C}$ satisfy the inequality; there exists a positive constant $M$ such that*

$$|f(x-y) - f(x)g(y) + g(x)f(y)| \le M \tag{2.1}$$

*for all $x, y \in G$. Then either there exist $\lambda$, $\nu \in \mathbb{C}$, not both zero, and $L > 0$ such that*

$$|\lambda f(x) - \nu g(x)| \le L, \tag{2.2}$$

*or else*

$$f(x-y) - f(x)g(y) + g(x)f(y) = 0 \tag{2.3}$$

*for all $x$, $y \in G$.*

*Proof.* We prove that the equation (2.3) satisfied if the condition (2.2) fails. Assume that $|\lambda f(x) - \nu g(y)| \le L$ for some $L > 0$ implies $\lambda = \nu = 0$. Let

$$F(x, y) = f(x+y) - f(x)g(-y) + g(x)f(-y).$$

Then we can choose $y_1$ satisfying $f(-y_1) \ne 0$. It is easy to show that

$$g(x) = \lambda_0 f(x) + \lambda_1 f(x + y_1) - \lambda_1 F(x, y_1), \tag{2.4}$$

where $\lambda_0 = \frac{g(-y_1)}{f(-y_1)}$ and $\lambda_1 = -\frac{1}{f(-y_1)}$.

By the definition of $F$ and the use of (2.4) we have

$$\begin{aligned}
&f\big((x+y)+z\big) \\
&= f(x+y)g(-z) - g(x+y)f(-z) + F(x+y, z) \\
&= \Big(f(x)g(-y) - g(x)f(-y) + F(x,y)\Big)g(-z) \\
&\quad - \Big(\lambda_0 f(x+y) + \lambda_1 f(x+y+y_1) - \lambda_1 F(x+y, y_1)\Big)f(-z) \\
&\quad + F(x+y, z) \\
&= \Big(f(x)g(-y) - g(x)f(-y) + F(x,y)\Big)g(-z) \\
&\quad - \lambda_0\Big(f(x)g(-y) - g(x)f(-y) + F(x,y)\Big)f(-z) \\
&\quad - \lambda_1\Big(f(x)g(-y-y_1) - g(x)f(-y-y_1) + F(x, y+y_1)\Big)f(-z) \\
&\quad + \lambda_1 F(x+y, y_1)f(-z) + F(x+y, z),
\end{aligned} \tag{2.5}$$

and

$$f\big(x + (y+z)\big) = f(x)g(-y-z) - g(x)f(-y-z) + F(x, y+z). \tag{2.6}$$

It follows from the equations (2.5) and (2.6),

$$
\begin{aligned}
& f(x)\Big(g(-y)g(-z) - \lambda_0 g(-y)f(-z) \\
& - \lambda_1 g(-y-y_1)f(-z) - g(-y-z)\Big) \\
& + g(x)\Big( - f(-y)g(-z) + \lambda_0 f(-y)f(-z) \\
& + \lambda_1 f(-y-y_1)f(-z) + f(-y-z)\Big) \\
= & - F(x,y)g(-z) + \lambda_0 F(x,y)f(-z) + \lambda_1 F(x,y+y_1)f(-z) \\
& - \lambda_1 F(x+y,y_1)f(-z) - F(x+y,z) + F(x,y+z).
\end{aligned}
$$

Since $F$ is a bounded function, if we fix $y$, $z$ the right hand side of the above equation is bounded function of $x$. Thus by the assumption that $|\lambda f(x) - \nu g(y)| \le L$ for some $L > 0$ implies $\lambda = \nu = 0$, the both sides of the above equation become zero. Consequently we have

(2.7)

$$
\begin{aligned}
& | - F(x,y)g(-z) + \Big(\lambda_0 F(x,y) + \lambda_1 F(x,y+y_1) - \lambda_1 F(x+y,y_1)\Big)f(-z)| \\
= & |F(x+y,z) - F(x,y+z)| \le M.
\end{aligned}
$$

Again by the assumption, we have $F(x,y) \equiv 0$. This completes the proof. □

**Theorem 2.3.** *Let $f, g : G \to \mathbb{C}$ satisfy the inequality* (2.1). *Then $f, g$ satisfy one of the followings*:

(i) $f = 0$, $g$ : arbitrary,
(ii) *$f$ and $g$ are bounded functions,*
(iii) $f(x) = A(x) + B(x)$ *and* $g(x) = \lambda A(x) + \mu B(x) + 1$
(iv) $f(x) = \frac{\lambda}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x) + m(-x)) + \frac{\mu}{2}(m(x) - m(-x))$,

*where $\lambda, \mu \in \mathbb{C}$, $A$ is an additive function, $m$ is an exponential function and $B$ is a bounded function.*

*Proof.* First we assume that the inequality (2.2) holds. If $f = 0$, $g$ is arbitrary which is the case (i). If $f$ is a nontrivial bounded function, in view of (2.1) $g$ is also bounded which is the case (ii). If $f$ is unbounded, it follows from (2.2) that $\nu \ne 0$ and

$$g(x) = \mu f(x) + B(x) \tag{2.8}$$

for some $\mu \in \mathbb{C}$ and a bounded function $B$. Putting (2.8) in (2.1) we have

$$|f(x-y) - f(x)B(y) + B(x)f(y)| \le M. \tag{2.9}$$

Replacing $x$ by $y$ and $y$ by $x$ and using the triangle inequality we have

$$|f(x) + f(-x)| \le 2M \tag{2.10}$$

for all $x \in G$. Replacing $x$ by $-x$, $y$ by $-y$ in (2.9) and using the inequality (2.10) we have for some $M_1 > 0$,

$$|f(-x+y) + f(x)B(-y) - B(-x)f(y)| \le M_1. \tag{2.11}$$

Using (2.9), (2.10), (2.11) and the triangle inequality we have

$$|f(x)(B(y)-B(-y))-f(y)(B(x)-B(-x))|\le M_1+3M. \tag{2.12}$$

Since $f$ is unbounded it follows from (2.12) that $B(y)=B(-y)$ for all $y\in G$. Also, in view of (2.9), for fixed $y\in G$, $x\to f(x+y)-f(x)B(-y)$ is a bounded function of $x$. Thus it follows from [10, p. 104, Theorem 5.2] that $B(y)$ is an exponential function. Since $G$ is divisible by 2 we can write $B(x)=B(\frac{x}{2})B(\frac{x}{2})=B(\frac{x}{2})B(-\frac{x}{2})=B(0)$ and that $B(y)\equiv 1$ or 0. Since $f$ is unbounded, we have $B\equiv 1$. Replacing $y$ by $-y$ in (2.9) and using (2.10), we have

$$|f(x+y)-f(x)-f(y)|\le 3M. \tag{2.13}$$

By the well known Hyers-Ulam stability theorem [11], there exists an additive function $A(x)$ such that

$$|f(x)-A(x)|\le 3M, \tag{2.14}$$

which gives the case (iii). Now if the equality (2.3) holds, then by Lemma 2.1, $f,g$ satisfies (iii) or (iv). This completes the proof. □

As a direct consequence of Theorem 2.3 we have the following.

**Corollary 2.4.** *Let $f,g:\mathbb{R}^n\to\mathbb{C}$ be continuous functions satisfying* (2.1). *Then $f$ and $g$ satisfy one of the followings*:
(i) $f\equiv 0$ *and $g$ is arbitrary,*
(ii) *$f$ and $g$ are bounded functions,*
(iii) $f(x)=c\cdot x+r(x),\ g(x)=\lambda(c\cdot x+r(x))+1,$
(iv) $f(x)=\lambda\sin(c\cdot x),\ g(x)=\cos(c\cdot x)+\lambda\sin(c\cdot x)$ *for some $c\in\mathbb{C}^n$, $\lambda\in\mathbb{C}$ and a bounded function $r(x)$.*

*Proof.* The continuous solutions of the equation (1.1) are given by (iv) or $f(x)=c\cdot x,\ g(x)=1+\lambda c\cdot x$. This completes the proof. □

Now we prove the stability of the equation (1.2).

**Lemma 2.5.** *Let $f,g:G\to\mathbb{C}$ satisfy the inequality; there exists a positive constant $M$ such that*

$$|g(x-y)-g(x)g(y)-f(x)f(y)|\le M \tag{2.15}$$

*for all $x,y\in G$. Then either there exist $\lambda$, $\nu\in\mathbb{C}$, not both zero, and $L>0$ such that*

$$|\lambda f(x)-\nu g(x)|\le L, \tag{2.16}$$

*or else*

$$g(x-y)-g(x)g(y)-f(x)f(y)=0 \tag{2.17}$$

*for all $x$, $y\in G$.*

*Proof.* Suppose that, for $L > 0$, $|\lambda f(x) - \nu g(y)| \le L$ does not hold unless $\lambda = \nu = 0$. Note that both $f$ and $g$ are unbounded. Let

$$F(x, -y) = g(x - y) - g(x)g(y) - f(x)f(y). \tag{2.18}$$

Just for convenience, we consider the following equation which is equivalent to (2.18).

$$F(x, y) = g(x + y) - g(x)g(-y) - f(x)f(-y). \tag{2.19}$$

Since $f$ is nonconstant, we can choose $y_1$ satisfying $f(-y_1) \ne 0$. It is easy to show that

$$f(x) = \lambda_0 g(x) + \lambda_1 g(x + y_1) - \lambda_1 F(x, y_1), \tag{2.20}$$

where $\lambda_0 = -\frac{g(-y_1)}{f(-y_1)}$ and $\lambda_1 = \frac{1}{f(-y_1)}$.

By the definition of $F$ and the use of (2.20), we have

$$\begin{aligned}
&g\big((x+y)+z\big) \\
&= g(x+y)g(-z) + f(x+y)f(-z) + F(x+y, z) \\
&= \Big(g(x)g(-y) + f(x)f(-y) + F(x,y)\Big)g(-z) \\
&\quad + \Big(\lambda_0 g(x+y) + \lambda_1 g(x+y+y_1) - \lambda_1 F(x+y, y_1)\Big)f(-z) \\
&\quad + F(x+y, z) \\
&= \Big(g(x)g(-y) + f(x)f(-y) + F(x,y)\Big)g(-z) \\
&\quad + \lambda_0\Big(g(x)g(-y) + f(x)f(-y) + F(x,y)\Big)f(-z) \\
&\quad + \lambda_1\Big(g(x)g(-y-y_1) + f(x)f(-y-y_1) + F(x, y+y_1)\Big)f(-z) \\
&\quad - \lambda_1 F(x+y, y_1)f(-z) + F(x+y, z),
\end{aligned} \tag{2.21}$$

and

$$g\big(x + (y+z)\big) = g(x)g(-y-z) + f(x)f(-y-z) + F(x, y+z). \tag{2.22}$$

By equating the above two equations we have

$$\begin{aligned}
&g(x)\Big(g(-y)g(-z) + \lambda_0 g(-y)f(-z) \\
&\quad + \lambda_1 g(-y-y_1)f(-z) - g(-y-z)\Big) \\
&\quad + f(x)\Big(f(-y)g(-z) + \lambda_0 f(-y)f(-z) \\
&\quad + \lambda_1 f(-y-y_1)f(-z) - f(-y-z)\Big) \\
= &\; -F(x,y)g(-z) - \lambda_0 F(x,y)f(-z) - \lambda_1 F(x, y+y_1)f(-z) \\
&\quad + \lambda_1 F(x+y, y_1)f(-z) - F(x+y, z) + F(x, y+z).
\end{aligned}$$

When $y, z$ are fixed, the right hand side of the above equality is bounded, so we have

(2.23)
$$\begin{aligned} &F(x, y+z) - F(x+y, z) \\ &= F(x,y)g(-z) + \Big(\lambda_0 F(x,y) + \lambda_1 F(x, y+y_1) - \lambda_1 F(x+y, y_1)\Big) f(-z). \end{aligned}$$

Again considering (2.23) as a function of $z$ for all fixed $x$, $y$, we have $F(x, y) \equiv 0$ which is equivalent to (2.17). □

**Theorem 2.6.** *Let $f, g : G \to \mathbb{C}$ satisfy the inequality* (2.15). *Then $f, g$ satisfy one of the followings*:

(i) *$f$ and $g$ are bounded functions,*

(ii) *$f(x) = \frac{1}{2}(m(x) - m(-x))$, $g(x) = \frac{1}{2}(m(x) + m(-x))$, where $m$ is an exponential function.*

*Proof.* First we prove that if the inequalities (2.15) and (2.16) hold, then both $f$ and $g$ are bounded functions. From (2.15), it is impossible that only one of $f$ and $g$ is unbounded. Assume that both $f$ and $g$ are unbounded. In view of (2.16), we can write

$$g = \mu f + B \tag{2.24}$$

for some $\mu \neq 0$ and a bounded function $B$. Putting (2.24) in (2.15) we have

$$|\mu f(x+y) + B(x+y) - \Big(\mu f(x) + B(x)\Big)\Big(\mu f(-y) + B(-y)\Big) + f(x)f(-y)| \leq M.$$

Since $B$ is bounded, we have

$$f(x+y) - \mu^{-1}\Big((\mu^2+1)f(-y) + \mu B(-y)\Big)f(x)$$

is bounded for fixed $y \in G$. Thus it follows from [10, p. 104, Theorem 5.2] that

$$\mu^{-1}\Big((\mu^2+1)f(y) + \mu B(y)\Big) = m(y)$$

for some exponential $m$. Thus if $\mu^2 = -1$, we can write

$$f = \pm i(g - m), \tag{2.25}$$

where $m$ is a bounded exponential function. Putting (2.25) in (2.15) we have

$$|g(x-y) - g(x)m(y) - g(y)m(x)| \leq M \tag{2.26}$$

for all $x$, $y \in G$.

Replacing $x$ by $y$ and $y$ by $x$ and using the triangle inequality we have

$$|g(x) - g(-x)| \leq 2M \tag{2.27}$$

for all $x \in G$. Replacing $x$ by $-x$, $y$ by $-y$ and using the inequality (2.27) we have for some $M_1 > 0$,

$$|g(-x+y) - g(x)m(-y) - m(-x)g(y)| \leq M_1. \tag{2.28}$$

Using (2.26), (2.27), (2.28) and the triangle inequality we have for some $M^* > 0$,

$$|g(x)(m(y) - m(-y)) - g(y)(m(x) - m(-x))| \le M^* + 3M. \tag{2.29}$$

Since $g$ is unbounded and $m$ is bounded, it follows from (2.29) that $m(y) = m(-y)$ for all $y \in G$. Since $G$ is divisible by 2 we have $m \equiv 1$. Putting $y = x$ in (2.26), using the triangle inequality we have $|g(x)| \le \frac{1}{2}(M + |g(0)|)$ for all $x \in G$, which contradicts to the assumption that $f$ and $g$ are unbounded. If $\mu \neq -1$, we have

$$f = \frac{\mu(m-b)}{\mu^2+1}, \quad g = \frac{\mu^2 m + b}{\mu^2+1},$$

which contradicts to the assumption that both $f$ and $g$ are unbounded. If the equation (2.17) holds, then by Lemma 2.1, we have the case (ii). This completes the proof. □

Since every continuous exponential function $m : \mathbb{R}^n \to \mathbb{C}$ is given by $m(x) = e^{c\cdot x}$ for some $c \in \mathbb{C}^n$, we have the following as a direct consequence of Theorem 2.6:

**Corollary 2.7.** *Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be continuous functions satisfying* (2.15). *Then $f$ and $g$ satisfy one of the followings*:
(i) *$f$ and $g$ are bounded measurable functions,*
(ii) $f(x) = \cosh(c \cdot x)$, $g(x) = \sinh(c \cdot x)$, *where $c \in \mathbb{C}^n$.*

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Jeongwook Chang
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
*E-mail address*: jchang@kunsan.ac.kr

Jaeyoung Chung
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
*E-mail address*: jychung@kunsan.ac.kr