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Hyperbolic and Parabolic Packings*

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Abstract. The contacts graph, or nerve, of a packing, is a combinatorial graph that describes the combinatorics of the packing. Let G be the 1-skeleton of a triangulation of an open disk. G is said to be CP parabolic (resp. CP hyperbolic) if there is a locally finite disk packing P in the plane (resp. the unit disk) with contacts graph G. Several criteria for deciding whether G is CP parabolic or CP hyperbolic are given, including a necessary and sufficient combinatorial criterion. A criterion in terms of the random walk says that if the random walk on G is recurrent, then G is CP parabolic. Conversely, if G has bounded valence and the random walk on G is transient, then G is CP hyperbolic.

We also give a new proof that G is either CP parabolic or CP hyperbolic, but not both. The new proof has the advantage of being applicable to packings of more general shapes. Another new result is that if G is CP hyperbolic and D is any simply connected proper subdomain of the plane, then there is a disk packing P with contacts graph G such that P is contained and locally finite in D.

1. Introduction

We consider packings of compact connected sets in the plane $\mathbb{C} = \mathbb{R}^2$ or in the Riemann sphere $\hat{\mathbb{C}} = S^2$.

Given an indexed packing $P = (P_v\colon v \in V)$, its *contact graph*, or *nerve* $G = G(P)$, is defined as follows. The set of vertices of G is V, the indexing set for P, and an

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edge $[v,u]$ appears in G precisely when the sets P_v and P_u intersect. Thus G encodes some of the combinatorics of P. If all the sets P_v are smooth disks[1] in $\mathbb{C}$, then it is easy to see that the contacts graph is planar.

The circle-packing theorem [16] says that for any finite planar graph G there is some packing of (geometric) disks in the plane whose contacts graph is G. This fantastic theorem has received much attention since Thurston conjectured that the Riemann map from a simply connected domain to the unit disk can be approximated using circle packings with prescribed nerves. The conjecture was later proved by Rodin and Sullivan [20]. Some proofs of the circle-packing theorem appear in [1], [2], [28, Chapter 13], [18], [10], [4], [13], [6], [7], [21], [24], and [23].

Here, we are concerned with infinite packings. Suppose, for example, that G is a *disk triangulation graph*; that is, the 1-skeleton of a triangulation of an open topological disk. By taking a Hausdorff limit of packings corresponding to finite subgraphs of G, an infinite packing P of disks in $\mathbb{C}$ whose contacts graph is G can be obtained. A few questions then naturally arise about the properties of P. Can P be bounded? Can P be locally finite in the plane? (This means that every compact subset of the plane intersects finitely many of the sets in the packing.) To what extent is P unique?

It is not hard to see that (still assuming G to be a disk triangulation graph) there is a unique open topological disk $D \subset \hat{\mathbb{C}}$ such that P is contained in D and is locally finite in D. The boundary of D is just the set of accumulation points of P.[2] This D is called the *carrier* of P, and is denoted carr(P).

It was proved in [15] that P can be chosen such that carr(P) is the plane or the unit disk $U = \{z \in \mathbb{C}\colon |z| < 1\}$. Beardon and Stephenson [3] have obtained this result under the additional assumption that G has bounded valence.[3] There is a strong uniqueness statement valid when carr(P) $= \mathbb{C}$: any other disk packing $P' \subset \hat{\mathbb{C}}$ with nerve G is the image of P under a Möbius transformation [22], [15]. (The Möbius group is the group generated by inversions in circles. It is six dimensional.) In particular, it follows that there cannot be two disk packings P, P' with carr(P) $=$ $\mathbb{C}$, carr(P') $= U$, and $G = G(P) = G(P')$. If carr(P) $= U$, then there is a weaker form of uniqueness: any disk packing P' with carr(P') $= U$ that has nerve G is the image of P under a Möbius transformation.

All this parallels neatly with the analytic theory. The existence of a locally finite packing in U or $\mathbb{C}$ is a discrete analog of the uniformization theorem, which says that any simply connected noncompact Riemann surface is conformally equivalent to $\mathbb{C}$ or U. The parallels of the uniqueness statements are that any conformal map from the plane into the sphere or from U onto U is a Möbius transformation.

Let us say that a disk triangulation graph G is *CP parabolic* (resp. *CP hyperbolic*)

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[1] The term *disk* means a geometric disk, a *topological disk* means a set homeomorphic to a compact disk, and a *smooth disk* is a topological disk with C^1 boundary.

[2] A point z is an accumulation point of P if every neighborhood of z intersects infinitely many sets in P.

[3] The *valence* or *degree* of a vertex is the number of neighbors it has. "G has bounded valence" means that there is some $C < \infty$ such that every vertex has valence less than C.

if there is a disk packing P with contacts graph G and carrier $\text{carr}(P) = \mathbb{C}$ (resp. $\text{carr}(P) = U$).

We introduce the notion of a VEL parabolic graph. VEL parabolicity is a combinatorial property, which is defined using Cannon's vertex extremal length [8]. The precise definitions appear later. A graph which is not VEL parabolic is called VEL hyperbolic. We prove that a disk triangulation graph is CP parabolic iff it is VEL parabolic. This gives a complete combinatorial characterization of the "CP type" of any disk triangulation graph.

Using this equivalence of CP parabolic and VEL parabolic, we prove:

1.1. Theorem. *Let G be a disk triangulation graph. If the random walk on G is recurrent, then G is CP parabolic. Conversely, if the degrees of the vertices in G are bounded and the random walk on G is transient, then G is CP hyperbolic.*

It will be shown that there are CP parabolic disk triangulation graphs on which the random walk is transient.

We also give new proofs to the above-quoted results that every disk triangulation graph is either CP parabolic, or CP hyperbolic, but not both. The results here actually generalize these theorems, since the proofs apply not only to packings by geometric disks, but to more general sets. In order to state some of our results, we introduce the notion of fat sets. Heuristically, a set is fat if its area is roughly proportional to the square of its diameter, and this property also holds locally. The precise definition is:

Definitions [26]. The open disk with center x and radius r is denoted $D(x,r)$. Let $\tau > 0$. A measurable set $X \subset \hat{\mathbb{C}}$ is *τ-fat* if, for every $x \in X$, $x \neq \infty$, and for every $r > 0$ such that $D(x,r)$ does not contain X, the inequality

$$\text{area}(X \cap D(x,r)) \geq \tau \, \text{area}(D(x,r))$$

holds. A packing $P = (P_v \colon v \in V)$ is *fat* if there is some $\tau > 0$ such that each P_v is τ-fat.

For example, any smooth disk is τ-fat for some $\tau > 0$, and it is not hard to see that K-quasi-disks are $\tau(K)$-fat [26].

We can now state:

1.2. Theorem. *Let $G = (V, E)$ be a disk triangulation graph, and for each $v \in V$ let $Q_v \subset \mathbb{C}$ be a smooth compact topological disk. Suppose that there is some $\tau > 0$ such that each Q_v is τ-fat. Let $D \subset \mathbb{C}$ be a simply connected domain, and suppose that $D \neq \mathbb{C}$ (resp. $D = \mathbb{C}$) if G is VEL hyperbolic (resp. VEL parabolic). Then there is a packing $P = (P_v \colon v \in V)$ in D, which is locally finite in D, whose contacts graph is G, and such that P_v is homothetic to Q_v for each $v \in V$.*[4]

Conversely, suppose that $P = (P_v \colon v \in V)$ is a fat packing in $\hat{\mathbb{C}}$ of smooth disks whose nerve is G. Then G is VEL parabolic if and only if $\hat{\mathbb{C}}$-carr(P) consists of a single point.

[4] "P_v is homothetic to Q_v" means that there are $a_v > 0$ and $b_v \in \mathbb{C}$ such that $P_v = a_v Q_v + b_v$.

From [24] we know that given a finite planar graph $G^* = (V^*, E^*)$ and a smooth disk $Q_v^* \subset \mathbb{C}$ for each $v \in V^*$, there is a packing $P^* = (P_v^*: v \in V^*)$, with $G(P^*) = G^*$ and P_v^* homothetic to Q_v^* for each $v \in V^*$. This constitutes the "finite case" for the existence part in Theorem 1.2. The basic innovation here is the control one gets on carr(P). The situation where the Q_v are disks, G is VEL hyperbolic, and D is an arbitrary simply connected proper subdomain of $\mathbb{C}$ seems interesting in itself.

Although the equivalence of CP parabolicity to VEL parabolicity gives a complete characterization for disk triangulation graphs, it is quite natural to ask for other criteria. It has been shown by Beardon and Stephenson [5] that if every vertex in G has degree greater than 7, then G is CP hyperbolic, while if every vertex has degree at most 6, G is CP parabolic. We show that if finitely many vertices in G have valence greater than 6, then G is CP parabolic, while if the lower average valence (see Section 10 for the definition) in G is greater than 6, G is CP hyperbolic.

From Rodin and Sullivan's proof of the length–area lemma [20], it follows that if $\gamma_1, \gamma_2, \ldots$ is a sequence of nested simple closed paths in G and $\sum_j 1/|\gamma_j| = \infty$, then G is not CP hyperbolic. This can be seen as a criterion for CP parabolicity. In Section 9 we present a criterion of CP hyperbolicity based on a perimetric inequality in G. There will also be a somewhat restricted converse to this criterion, which is in the spirit of Rodin and Sullivan's length–area lemma.

The interested reader may wish to consult Soardi's paper [27], which studies problems related to those discussed here.

2. Discrete Extremal Length

In this section, we define discrete extremal length. Later, a brief discussion of the history of these definitions appears. We have chosen to start with an abstract notion, and then specialize to more geometric situations.

Combinatorial Extremal Length. Let Γ be a nonempty collection of nonempty subsets of some set X. A (discrete) *metric* on X is a function $m: X \to [0, \infty)$. The *area* of m is just the square of the L^2 norm of m:

$$\text{area}(m) = \|m\|^2 = \sum_{x \in X} m(x)^2.$$

The collection of all metrics m on X with $0 < \text{area}(m) < \infty$ is denoted $\mathscr{M}(X)$. Given a set $A \subset X$, we define the *length* of A in the metric m to be

$$L_m(A) = \sum_{x \in A} m(x).$$

This is also called the m-length of A. If Γ is a collection of subsets of X, we define its m-length to be the least m-length of a set in Γ:

$$L_m(\Gamma) = \inf_{A \in \Gamma} L_m(A).$$

Finally, the *extremal length* of Γ is defined as

$$\mathrm{EL}(\Gamma) = \sup\left\{ \frac{L_m(\Gamma)^2}{\mathrm{area}(m)} : m \in \mathscr{M}(X) \right\}.$$

This is a number in $[0, \infty]$. Note that the ratio $L_m(\Gamma)^2/\mathrm{area}(m)$ does not change if we multiply m by a positive constant. Also note that $\mathrm{EL}(\Gamma)$ does not depend on X; that is, the value of $\mathrm{EL}(\Gamma)$ does not change if we replace X with any other set that contains every $A \in \Gamma$.

The verification of the following simple monotonicity property of extremal length is left to the reader.

2.1. Monotonicity Property. *If each* $\gamma \in \Gamma$ *contains some* $\gamma' \in \Gamma'$, *then* $\mathrm{EL}(\Gamma) \geq \mathrm{EL}(\Gamma')$.

At least when X is finite, a geometric interpretation can be given to $\mathrm{EL}(\Gamma)$. Consider the Euclidean space $\mathbb{R}^X$ of all functions $f \colon X \to \mathbb{R}$. For each subset γ of X, let $\chi_\gamma \in \mathbb{R}^X$ be defined by $\chi_\gamma(x) = 1$ for $x \in \gamma$ and $\chi_\gamma(x) = 0$ otherwise. Now let Γ be, as before, a collection of subsets of X.

2.2. Theorem (Geometric Description of Extremal Length). *Let* m_0 *be the point of least norm in the convex hull of* $\{\chi_\gamma \colon \gamma \in \Gamma\}$. *Then*

$$\mathrm{EL}(\Gamma) = \frac{L_{m_0}(\Gamma)^2}{\mathrm{area}(m_0)} = \|m_0\|^2.$$

We do not use this theorem. The simple proof is left to the reader.

Extremal Length in Graphs. In the following, $G = (V, E)$ is a locally finite connected graph. It will always be a simple graph; that is, each edge has two distinct vertices, and there is at most one edge joining any two vertices.

A *path* γ in G is a finite or infinite sequence $(v_0, v_1, \dots)$ of vertices such that $[v_i, v_{i+1}] \in E$ for every $i = 0, 1, \dots$. The edges and vertices of γ are denoted by $E(\gamma) = \{[v_i, v_{i+1}] \colon i = 0, 1, \dots\}$ and $V(\gamma) = \{v_0, v_1, \dots\}$, respectively. Likewise, for Γ a set of paths in G, we set $V(\Gamma) = \{V(\gamma) \colon \gamma \in \Gamma\}$ and $E(\Gamma) = \{E(\gamma) \colon \gamma \in \Gamma\}$. A set $A \subset V$ of vertices is said to be *connected* if, for every $v, w \in A$, there is a path γ in G from v to w with $V(\gamma) \subset A$. (We allow trivial paths, paths that contain only one vertex.)

Given subsets $A, B \subset V$, we let $\Gamma(A, B) = \Gamma_G(A, B)$ denote the set of all paths in G with initial point in A and terminal point in B. We let $\Gamma_{\mathrm{V}}(A, B)$ (resp. $\Gamma_{\mathrm{E}}(A, B)$) denote the sets of vertices (resp. edges) of such paths:

$$\Gamma_{\mathrm{V}}(A, B) = \{V(\gamma) \colon \gamma \in \Gamma(A, B)\},$$

$$\Gamma_{\mathrm{E}}(A, B) = \{E(\gamma) \colon \gamma \in \Gamma(A, B)\}.$$

A function $m: V \to [0, \infty)$ is called a *v-metric* on G, and a function $m: E \to [0, \infty)$ is called an *e-metric*. When m is a v-metric (resp. an e-metric) we use $L_m(\gamma)$ as a shorthand for $L_m(V(\gamma))$ (resp. $L_m(E(\gamma))$).

The *vertex extremal length* VEL and *edge extremal length* EEL between A and B are defined by

$$\mathrm{VEL} = \mathrm{VEL}_G(A, B) = \mathrm{EL}(\Gamma_{\mathrm{V}}(A, B)),$$

$$\mathrm{EEL} = \mathrm{EEL}_G(A, B) = \mathrm{EL}(\Gamma_{\mathrm{E}}(A, B)).$$

To make the definition of $\mathrm{VEL}(A, B)$ more explicit, we have

$$\mathrm{VEL}(A, B) = \sup_m \inf_\gamma \frac{L_m(\gamma)^2}{\mathrm{area}(m)} = \sup_m \inf_\gamma \frac{\left(\sum_{v \in V(\gamma)} m(v)\right)^2}{\sum_{v \in V} m(v)^2}.$$

Here m runs over $\mathscr{M}(V)$ and γ runs over $\Gamma_G(A, B)$.

These definitions give two discrete analogs for the classical notion of extremal length. (Reference [17] is a good introduction to continuous extremal length.) As we will see below, both are useful. The edge extremal length was introduced by Duffin, who showed in [12] that $\mathrm{EEL}(A, B)$ is equal to the electrical resistance between A and B, if each edge in G is considered to be a resistor with unit resistance. The vertex extremal length was introduced by Cannon [8]. Cannon's motivation was to obtain criteria for deciding when a group can be made to act conformally on the Riemann sphere $\hat{\mathbb{C}}$. Later it was discovered [9], [25] that extremal metrics of vertex extremal length (that is, metrics realizing the supremum in the definition of the extremal length) give square tilings of rectangles with prescribed contacts.

An infinite path γ in G is *transient* if it contains infinitely many distinct vertices. The set of transient paths in G that have an initial point in A is denoted by $\Gamma(A, \infty)$. The edge and vertex extremal length from A to ∞ are defined as

$$\mathrm{EEL}(A, \infty) = \mathrm{EL}(\{E(\gamma): \gamma \in \Gamma(A, \infty)\}),$$

$$\mathrm{VEL}(A, \infty) = \mathrm{EL}(\{V(\gamma): \gamma \in \Gamma(A, \infty)\}).$$

Of course, this makes sense only for infinite G.

For a v-metric or e-metric m, we let $d_m(A, B)$ (resp. $d_m(A, \infty)$) denote the distance from A to B (resp. to ∞) in the metric m; that is,

$$d_m(A, B) = L_m(\Gamma(A, B)) = \inf\{L_m(\gamma): \gamma \in \Gamma(A, B)\},$$

$$d_m(A, \infty) = L_m(\Gamma(A, \infty)) = \inf\{L_m(\gamma): \gamma \in \Gamma(A, \infty)\}.$$

An infinite graph G is *VEL parabolic* if $\mathrm{VEL}(\{v\}, \infty) = \infty$ for some $v \in V$. Otherwise, G is *VEL hyperbolic*. Similarly, G is *EEL parabolic* if $\mathrm{EEL}(\{v\}, \infty) = \infty$ for some $v \in V$, and is *EEL hyperbolic*, otherwise.

2.3. Remark. If $\mathrm{VEL}(\{v\},\infty) = \infty$, then a finite area v-metric $m \in \mathcal{M}(V)$ exists such that $d_m(\{v\},\infty) = \infty$. To see this, just take $m(v) = \sum_{j=1}^{\infty} m_j(v)$, where the metrics m_j satisfy $A(m_j) < 2^{-j}$ and $L_{m_j}(\Gamma(\{v\},\infty)) = 1$.

2.4. Exercise. Try to determine $\mathrm{VEL}(A,B)$, $\mathrm{EEL}(A,B)$, and whether G is EEL or VEL parabolic for examples of your choice.

2.5. Exercise. Let G be an infinite connected graph, and let $A \subset V$ be finite and nonempty. Show that $\mathrm{EEL}(A,\infty) = \infty$ iff G is EEL parabolic, and that $\mathrm{VEL}(A,\infty) = \infty$ iff G is VEL parabolic.

While the VEL type (whether parabolic or hyperbolic) is more relevant to packings, the EEL type is closely related to random walks and electricity. We do not introduce the terminology of electrical networks here, but remark that a graph G is electrically parabolic if the electric resistance to infinity in the graph is infinite. (See [11].)

The following theorem is known.

2.6. Theorem. *Let $G = (V, E)$ be a locally finite connected graph. The following are equivalent*:

1. *G is EEL parabolic.*
2. *G is electrically parabolic.*
3. *The simple random walk on G is recurrent.*

The equivalence of (1) and (2) is essentially contained in [12], while the equivalence of (2) and (3) is given in [11]. Also see Section 4 of [29] regarding Theorem 2.6 and further equivalent properties.

In Section 8 we see that VEL and EEL parabolicity are closely related.

3. The Packing Type and Vertex Extremal Length

3.1. Type Characterization Theorem. *Let $P = (P_v\colon v \in V)$ be a fat packing of (compact connected) sets in the Riemann sphere $\hat{\mathbb{C}}$, and let $G = (V, E)$ be the contacts graph of P. Assume that G is locally finite and connected.*

1. *If P is locally finite in $\hat{\mathbb{C}} - \{p\}$, where p is some point in $\hat{\mathbb{C}}$, then G is VEL parabolic.*
2. *Conversely, suppose that each P_v is a smooth disk and that G is a disk triangulation graph, which is VEL parabolic. Then P is locally finite in $\hat{\mathbb{C}} - \{p\}$ for some point $p \in \hat{\mathbb{C}}$.*

The following results about fat sets prove useful.

3.2. Observation. *Let F be a τ-fat set, $\tau > 0$. Then*

$$\operatorname{area}(D(z,3r) \cap F) \geq \pi\tau \operatorname{diameter}(D(z,r) \cap F)^2$$

holds for every $z \in \mathbb{C}$, $r > 0$.

Proof. Let $x, y \in D(z,r) \cap F$. It is clear that $D(x,|y-x|) \subset D(z,3r)$. By the τ-fatness of F, we then have

$$\operatorname{area}(D(z,3r) \cap F) \geq \operatorname{area}(D(x,|y-x|) \cap F) \geq \pi\tau|y-x|^2.$$

The observation follows. □

The following lemma appears in [26].

3.3. Lemma. *There is a positive function $\tau^*: (0,\infty) \to (0,\infty)$ such that for every $\tau > 0$, for every τ-fat set $A \subset \hat{\mathbb{C}}$, and for every Möbius transformation $\varphi: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ the set $\varphi(A)$ is $\tau^*(\tau)$-fat.*

A central ingredient in the proof of 3.1 is the following lemma, which will also be useful later.

3.4. Lemma. *Let $P = (P_v: v \in V)$ be a fat packing in $\hat{\mathbb{C}}$. Let $G = (V, E)$ denote the contacts graph of P, and assume that G is locally finite. Suppose that z is an accumulation point of the packing P that does not belong to $\bigcup_{v \in V} P_v$. Let K be a compact set in $\hat{\mathbb{C}}$ that does not contain z. For every $A \subset \hat{\mathbb{C}}$ let $V(A)$ denote the set of vertices $v \in V$ such that P_v intersects A. Then*

$$\sup\{\operatorname{VEL}_G(V(K), V(W)) : W \text{ is open and } z \in W\} = \infty.$$

In the following, $C(z,r) = \partial D(z,r)$ denotes the circle with center z and radius r.

Proof. Let $\tau > 0$ be such that all the sets P_v are τ-fat. Suppose first that $z \neq \infty$. We now establish that a neighborhood of z is disjoint from $\bigcup_{v \in V(K)} P_v$. Let $R > 0$ be smaller than the distance from z to K, and let V' be the set of $v \in V$ such that P_v intersects both circles $C(z,R)$ and $C(z,R/2)$. Since the distance from $C(z,R)$ to $C(z,R/2)$ is $R/2$, for each $v \in V(C(z,R)) \cap V(C(z,R/2))$, we have $\operatorname{diameter}(D(z,R) \cap P_v) > R/2$. Therefore, Observation 3.2 shows that $\operatorname{area}(D(z,3R) \cap P_v) \geq \pi\tau R^2/4$. In particular, we see that $V(C(z,R)) \cap V(C(z,R/2))$ is finite. This implies that there is an $r_1 \in (0,R/2)$ such that $V(D(z,r_1))$ is disjoint from $V(C(z,R)) \cap V(C(z,R/2))$. Then it follows that $D(z,r_1)$ is disjoint from $\bigcup_{v \in V(K)} P_v$.

We define inductively a sequence $r_1 > r_2 > \cdots$ of positive numbers. The first number in this sequence, r_1, has been defined already. Suppose that $n > 1$, and that $r_1, \ldots, r_{n-1}$ have been defined. Let $r_n \in (0, r_{n-1}/2)$ be sufficiently small so that $V(C(z,r_n)) \cap V(C(z,r_{n-1}/2)) = \emptyset$. The argument above shows that such an r_n exists.

For each n let A_n be the closed annulus bounded by $C(z, r_n)$ and $C(z, r_n/2)$. Define a v-metric m on G by setting

$$m(v) = \sum_{n=1}^{\infty} \frac{\text{diameter}(P_v \cap A_n)}{nr_n},$$

for each $v \in V$. By the construction of the sequence r_n, at most one term in this sum is nonzero. Using this and Observation 3.2, we get an estimate for the area of m, as follows:

$$\begin{aligned}
\text{area}(m) &= \sum_{v \in V} \left(\sum_{n=1}^{\infty} \frac{\text{diameter}(P_v \cap A_n)}{nr_n} \right)^2 \\
&= \sum_{n=1}^{\infty} \sum_{v \in V} \frac{\text{diameter}(P_v \cap A_n)^2}{n^2 r_n^2} \\
&\leq \sum_{n=1}^{\infty} \sum_{v \in V} \frac{\text{diameter}(P_v \cap D(z, r_n))^2}{n^2 r_n^2} \\
&\leq \sum_{n=1}^{\infty} \sum_{v \in V} \frac{\pi^{-1}\tau^{-1}\, \text{area}(P_v \cap D(z, 3r_n))}{n^2 r_n^2} \\
&\leq \pi^{-1}\tau^{-1} \sum_{n=1}^{\infty} \frac{\text{area}(D(z, 3r_n))}{n^2 r_n^2} \\
&= 9\tau^{-1} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\end{aligned}$$

Fix a positive integer N, and consider some path γ from $V(K)$ to $V(D(z, r_N))$. For each integer $n \in [1, N)$ the union $\bigcup_{v \in \gamma} P_v$ is a connected set that intersects the two circles $C(z, r_n)$ and $C(z, r_n/2)$ forming the boundary of A_n. Therefore, for such n, $\sum_{v \in \gamma} \text{diameter}(P_v \cap A_n) \geq r_n/2$. This then implies $L_m(\gamma) \geq \frac{1}{2} \sum_{n=1}^{N-1} 1/n$, which tends to infinity as $N \to \infty$. Since $\text{area}(m) < \infty$, we get $\text{VEL}(V(K), V(D(z, r_N))) \to \infty$, as $N \to \infty$, which proves the lemma in case $z \neq \infty$.

It is easy to modify the above argument to deal with the case $z = \infty$. The numbers $r_1, r_2, \ldots$ must satisfy in this case $D(0, r_1) \supset \bigcup \{P_v \colon v \in V(K)\}$, $r_{n+1} > 2r_n$, and $V(C(0, 2r_n)) \cap V(C(0, r_{n+1})) = \emptyset$. The annulus A_n is defined as the annulus whose boundary is $C(0, r_n) \cup C(0, 2r_n)$. The rest of the proof remains essentially the same. Alternatively, using Lemma 3.3, the case $z = \infty$ can be reduced to the case $z = 0$. □

With this lemma, the proof of the first part of 3.1 is easy.

Proof of 3.1(1). Suppose that P is locally finite in $\hat{\mathbb{C}} - \{p\}$. Pick some $v_0 \in V$. Applying Lemma 3.4 with $K = P_{v_0}$, $z = p$, we see that G is VEL parabolic. □

4. Some Topological Lemmas

In this section we gather a few elementary topological lemmas, which will be needed below. The reader is advised to skip the proofs, and perhaps return to them later.

The following two lemmas will enable us to infer topological information of a packing from the combinatorics of the contacts graph.

4.1. Neighbors Separation Lemma. *Let $P = (P_v\colon v \in V)$ be a packing of smooth disks in $\hat{\mathbb{C}}$, and suppose that the contacts graph $G = (V, E)$ of P is a disk triangulation graph. Let $v_0 \in V$ be some vertex, and let $N \subset V - \{v_0\}$ be the set of neighbors of v_0. Then there is a Jordan curve $\gamma \subset \bigcup_{v \in N} P_v - P_{v_0}$ that separates P_{v_0} from $\bigcup_{v \in V-(N \cup \{v_0\})} P_v$ in $\hat{\mathbb{C}}$.*[5]

The same conclusion holds if G is the (finite) 1-skeleton of a triangulation of a closed disk that has v_0 and all its neighbors as interior vertices.

Proof. Note that the assumptions that each P_v is smooth imply that the intersection of any three sets in P is empty. Let f be an embedding of G in $\hat{\mathbb{C}}$ such that the image of any edge $[v_1, v_2]$ is contained in $P_{v_1} \cup P_{v_2}$ and is disjoint from all the other sets in the packing. (To make an explicit construction of such an embedding, for each $v \in V$ let $h_v\colon \overline{U} \to P_v$ be a homeomorphism from the closed unit disk $\overline{U} \subset \mathbb{C}$ onto P_v, and for every edge $[v_1, v_2] \in E$ let p_{v_1, v_2} be some point in the intersection $P_{v_1} \cap P_{v_2}$. We may then take $f([v_1, v_2]) = \{h_{v_1}(t h_{v_1}^{-1}(p_{v_1, v_2}))\colon 0 \le t \le 1\} \cup \{h_{v_2}(t h_{v_2}^{-1}(p_{v_1, v_2}))\colon 0 \le t \le 1\}$.)

In the following we think of G as the 1-skeleton of a triangulation T of an (open or closed) disk. Let $[v_1, v_2, v_3]$ be any triangle in the interior of T. For $j = 1, 2, 3$, let V_j be the neighbors of v_j in G. Clearly, for each $j = 1, 2, 3$ the set $V_j' = V_j - \{v_1, v_2, v_3\}$ is connected as a set of vertices. Since any two of the sets V_1', V_2', V_3' intersect, the union $V' = V_1' \cup V_2' \cup V_3'$ is connected. Since G is connected and any path from a vertex $v \in V - \{v_1, v_2, v_3\}$ to a vertex in $\{v_1, v_2, v_3\}$ must intersect V', it follows that any two vertices in $G - \{v_1, v_2, v_3\}$ can be connected by a path in $G - \{v_1, v_2, v_3\}$. If $(u_1, u_2, u_3, \ldots, u_n)$ is a path in $G - \{v_1, v_2, v_3\}$, then the path $\bigcup_{j=1}^{n-1} f([u_j, u_{j+1}])$ is disjoint from $P_{v_1} \cup P_{v_2} \cup P_{v_3}$, and intersects both P_{u_1} and P_{u_n}. We therefore conclude that every $P_v, v \in V - \{v_1, v_2, v_3\}$, is contained in the same connected component of $\hat{\mathbb{C}} - (f([v_1, v_2]) \cup f([v_2, v_3]) \cup f([v_3, v_1]))$. The set $f([v_1, v_2]) \cup f([v_2, v_3]) \cup f([v_3, v_1])$ is a simple closed curve; we let D_{v_1, v_2, v_3} denote the component of $\hat{\mathbb{C}} - (f([v_1, v_2]) \cup f([v_2, v_3]) \cup f(v_3, v_1]))$ that is disjoint from $\bigcup_{v \in V - \{v_1, v_2, v_3\}} P_v$. Now consider two distinct triangles $[v_1, v_2, v_3], [w_1, w_2, w_3]$ in T. The intersection of the two Jordan curves $\partial D_{v_1, v_2, v_3}, \partial D_{w_1, w_2, w_3}$ is either empty, or consists of a single point, or a single arc. Therefore, the sets $D_{v_1, v_2, v_3}, D_{w_1, w_2, w_3}$ are either disjoint, or one is contained in the other. Suppose, without loss of generality, that $w_1 \notin \{v_1, v_2, v_3\}$. Then $\partial D_{w_1, w_2, w_3}$ intersects P_{w_1}, which is disjoint from the closure of D_{v_1, v_2, v_3}. We conclude that D_{w_1, w_2, w_3} is not contained in D_{v_1, v_2, v_3}. Similarly, D_{v_1, v_2, v_3} is not contained in D_{w_1, w_2, w_3}. Hence $D_{w_1, w_2, w_3} \cap D_{v_1, v_2, v_3} = \varnothing$.

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[5] A Jordan curve is a simple closed curve.

Let $n_0, n_2, \ldots, n_{k-1}$ be the neighbors of v_0 in circular order around v_0, and let γ be the Jordan curve $\gamma = \bigcup_{j=0}^{k-1} f([n_j, n_{j+1}])$, where we take $n_k = n_0$. The curve γ is contained in $\bigcup_{v \in N} P_v$ and is disjoint from P_{v_0}. We say that two distinct triangles $[v_1, v_2, v_3], [w_1, w_2, w_3]$ in T *neighbor* if they share an edge. If $[v_1, v_2, v_3]$ is a triangle of T that does not contain v_0 but neighbors with a triangle containing v_0, say with $[v_0, n_j, n_{j+1}]$, then D_{v_1, v_2, v_3} and $D_{v_0, n_j, n_{j+1}}$ lie on opposite sides of the arc $f([n_j, n_{j+1}])$. Consequently, D_{v_1, v_2, v_3} is not in the same connected component of $\hat{\mathbb{C}} - \gamma$ as P_{v_0}. If $[v_1, v_2, v_3]$ and $[w_1, w_2, w_3]$ are two neighboring triangles that do not contain v_0, then it is clear that D_{v_1, v_2, v_3} and D_{w_1, w_2, w_3} are in the same connected component of $\hat{\mathbb{C}} - \gamma$. Hence it easily follows that for every triangle $[v_1, v_2, v_3]$ that does not contain v_0 the set D_{v_1, v_2, v_3} is disjoint from the connected component of $\hat{\mathbb{C}} - \gamma$ that contains P_{v_0}. This implies that γ separates P_{v_0} from $\bigcup_{v \in V - (N \cup \{v_0\})} P_v$, and the lemma follows since $\gamma \subset \bigcup_{v \in N} P_v - P_{v_0}$. □

4.2. Corollary. *Let G be a disk triangulation graph, and let P be a packing of smooth disks in $\hat{\mathbb{C}}$ with $G(P) = G$. Let Z be the set of accumulation points of P. Then there is a connected component D of $\hat{\mathbb{C}} - Z$ that contains P, P is locally finite in D, and D is a topological disk.*

This D is called the *carrier* of P, $D = \mathrm{carr}(P)$.

The verification of Corollary 4.2 is left to the reader.

4.3. Lemma. *Let $P = (P_v \colon v \in V)$ and $G = (V, E)$ be as in Lemma 4.1, and let $v \in V$, $C \subset V - \{u\}$. Suppose that C is finite and u is contained in a finite component of $G - C$. Then $\bigcup_{v \in C} P_v$ separates P_u from the set of accumulation points of P.*

Proof. Let V_0 be the set of vertices that are contained in the same connected component of $G - C$ as u is, and let $K \subset \hat{\mathbb{C}} - \bigcup_{v \in C} P_v$ be a connected set that intersects P_u. For $w \in V$, let $N(w) \subset V - \{w\}$ denote the neighbors of w in G. From Lemma 4.1 we know that for each $w \in V_0$ there is a Jordan curve $\gamma_w \subset \bigcup_{v \in N(w)} P_v - P_w$ that separates P_w from $\bigcup_{v \in V - (N(w) \cup \{w\})} P_v$. Let Q_w denote the component of $\hat{\mathbb{C}} - \gamma_w$ that contains P_w, and let $Q = \bigcup_{v \in V_0} Q_v$. Suppose that $p \in K \cap \partial Q_w$, where $w \in V_0$. Then $p \in K \cap \gamma_w$. Since K is disjoint from $\bigcup_{v \in C} P_v$ and $\gamma_w \subset \bigcup_{v \in N(w)} P_v$, we conclude that $p \in Q_{w'}$ with $w' \in V_0$. Thus $\partial Q_w \cap K \subset Q$, for every $w \in V_0$. Since V_0 is finite, we have $\partial Q \subset \bigcup_{v \in V_0} \partial Q_v$. The above implies that $\partial Q \cap K \subset Q$, and because Q is open, $\partial Q \cap K = \emptyset$. Hence $Q \cap K$ is a relatively open and relatively closed subset of K. As $Q \cap K \neq \emptyset$ and K is connected, we conclude that $K \subset Q$. Because each Q_v intersects finitely many of the sets in the packing P, the lemma follows. □

4.4. Connected Cut Lemma. *Let $G = (V, E)$ be the 1-skeleton of a triangulation T of a simply connected surface S. Let $A, B \subset V$ be two disjoint connected sets of vertices. Suppose that $X \subset V$ intersects every path joining A and B. Then there is a connected subset of X that intersects every such path.*

The lemma is surely known, though we have not been able to locate a reference. Since the proof of Alexander's lemma in [19] can be modified to establish 4.4, we do not include a proof here.

5. Duality

The following theorem appears in [25].

5.1. Duality Theorem. *Let $G = (V, E)$ be a finite connected graph, and let A, B be two nonempty subsets of V. Let $\Gamma = \Gamma(A, B)$ denote the set of all paths from A to B in G. Let Γ^* denote the collection of all sets $C \subset V$ with the property that each $\gamma \in \Gamma$ intersects C. Then*

$$\mathrm{EL}(\Gamma^*) = \mathrm{EL}(\Gamma)^{-1}.$$

A related duality theorem can be found in [9].

We need only the inequality $\mathrm{EL}(\Gamma^*) \leq \mathrm{EL}(\Gamma)^{-1}$, but in the following slightly more general setting, where the graph is infinite.

5.2. Proposition. *Let $G = (V, E)$ be a connected graph, possibly infinite, let $A, B \subset V$ be two nonempty subsets. Let Γ be either $V(\Gamma(A, B))$ or $V(\Gamma(A, \infty))$. Denote by Γ^* the collection of all subsets $\gamma^* \subset V$ such that γ^* intersects every $\gamma \in \Gamma$. Then*

$$\mathrm{EL}(\Gamma^*)\,\mathrm{EL}(\Gamma) \leq 1.$$

Proof. If $\mathrm{EL}(\Gamma) = 0$, there is nothing to prove. So assume that $m \in \mathcal{M}(V)$ satisfies $L_m(\gamma) \geq L > 0$, for some L and every $\gamma \in \Gamma$. For $v \in V$ let the *height* of v be defined as

$$h(v) = \inf\{L_m(\gamma) \colon \gamma \text{ is a path from } A \text{ to } v\}.$$

For $t \in \mathbb{R}$, let V_t denote the set of vertices $v \in V$ such that $h(v) - m(v) \leq t \leq h(v)$. Since the m-length of every path in Γ is at least L, it is easy to see that $V_t \in \Gamma^*$ for $t \in [0, L)$.

Now let $m^* \in \mathcal{M}(V)$, and set $L^* = L_{m^*}(\Gamma^*) = \inf\{L_{m^*}(\gamma^*) \colon \gamma^* \in \Gamma^*\}$. Since $V_t \in \Gamma^*$ for $t \in [0, L)$, we have

$$L^*L \leq \int_0^L L_{m^*}(V_t)\,dt = \int_0^L \sum_{v \in V_t} m^*(v)\,dt.$$

For any $v \in V$, the set of t such that $v \in V_t$ is an interval of length $m(v)$. Therefore, the above inequality yields

$$L^*L \leq \sum_{v \in V} m^*(v)m(v) \leq \|m^*\|\,\|m\|.$$

This gives

$$\frac{L^{*2}}{\text{area}(m^*)}\frac{L^2}{\text{area}(m)} \le 1,$$

which proves the proposition. □

Suppose now that Γ is some finite nonempty collection of finite nonempty subsets of some set X. Let Γ^* denote the collection of all subsets of X that intersect each $\gamma \in \Gamma$. It is not difficult to see that $\text{EL}(\Gamma)\text{EL}(\Gamma^*) \ge 1$. (Consider the geometric interpretation, Theorem 2.2, of combinatorial extremal length.) The example $\Gamma = \{\{1,2\}, \{2,3\}, \{3,1\}\}$, $\Gamma^* = \Gamma$, shows that the inequality may be strict. Hence duality fails in the purely combinatorial setting.

6. CP Hyperbolic Implies VEL Hyperbolic

Proof of 3.1 (continued). It remains to prove the second part of the theorem. We now adopt the assumptions of 3.1(2). Let Z be the set of accumulation points of P. Our immediate goal is to verify that Z is connected. Let $V_1 \subset V_2 \subset \cdots$ be a sequence of finite subsets of V such that $V = \bigcup_n V_n$. For each n, let Q_n be the set of vertices in the infinite connected component of $G - V_n$, and let $\hat{Q}_n$ denote the closure of $\bigcup_{v \in Q_n} P_v$. Clearly, we have $\hat{Q}_1 \supset \hat{Q}_2 \supset \cdots$, and each set $\hat{Q}_n$ is compact and connected. Note that $Z = \bigcap_n \hat{Q}_n$. Since a nested intersection of compact connected sets is connected, it follows that Z is connected.

Let $u \in V$ be some vertex. Normalizing with a Möbius transformation, we assume that $\{z \in \mathbb{C}\colon |z| \ge 1\} \cup \{\infty\}$ is contained in P_u. Lemma 3.3 shows that this does not involve any loss of generality.

Let m be the v-metric on G defined by

$$m(v) = \begin{cases} \text{diameter}(P_v) & \text{for} \quad v \ne u, \\ 0 & \text{for} \quad v = u. \end{cases}$$

Let $\tau > 0$ be such that each P_v is τ-fat. Since $P_v \subset D(0,1)$ for $v \ne u$, we have

$$\begin{aligned} \text{area}(m) &= \sum_{v \in V - \{u\}} \text{diameter}(P_v)^2 \\ &\le \pi^{-1}\tau^{-1} \sum_{v \in V - \{u\}} \text{area}(P_v) \\ &\le \pi^{-1}\tau^{-1}\text{area}(D(0,1)) < \infty. \end{aligned}$$

Let C be any finite subset of $V - \{u\}$ such that u is disjoint from the infinite component of $G - C$. From Lemma 4.3 it follows that the union $\bigcup_{v \in C} P_v$ separates P_u from Z. This clearly implies that

$$\sum_{v \in C} m(v) \geq \text{diameter}(Z). \tag{6.1}$$

Since G is VEL parabolic and $A(m) < \infty$, Proposition 5.2 implies that

$$\inf_S \sum_{v \in S} m(v) = 0,$$

where the infimum runs over all sets $S \subset V$ such that u is not in an infinite component of $G - S$. However, every such S contains a finite $C \subset S$ such that u is not in the infinite component of $G - C$ (for example, the neighbors of the component of $G - S$ containing u). Therefore, (6.1) shows that diameter$(Z) = 0$, as required. □

We easily get the following generalization of 3.1(2).

6.1. Theorem. *Let $\tau > 0$, let $P = (P_v\colon v \in V)$ be a packing of τ-fat sets in the Riemann sphere $\hat{\mathbb{C}}$, and let $G = (V, E)$ be the contacts graph of P. Assume that G is connected, locally finite, and VEL parabolic. Also suppose that for each $u \in V$ there is a Jordan curve $\gamma \subset \bigcup_{v \in N(u)} P_v - P_u$ that separates P_u from $\bigcup_{v \in V-(N(u) \cup \{u\})} P_v$ in $\hat{\mathbb{C}}$. Here $N(u)$ denotes the set of neighbors of u. Then the set of accumulation points of P has zero length. If G has one end this set consists of a single point.*

We recall that a graph $G = (V, E)$ has one end iff $G - K$ has one infinite component for every finite $K \subset V$.

For example, if P is a locally finite tiling of a domain $\Omega \subset \mathbb{C}$ by compact squares, and if the contacts graph is connected, then $\partial\Omega$ has zero length. In this case the contacts graph does not have to be planar, since four squares may meet at a point.

Proof. The proof is essentially the same as for 3.1(2). Note that the assumptions that the sets P_v are smooth and that G is a disk triangulation graph were used only in the proof of Lemma 4.1. Since we are assuming the conclusion of this lemma here, these assumptions are not needed. If G has one end, then it is easy to see that the set of accumulation points of P is connected. The proof of 3.1(2) shows that in our present case for every $\varepsilon > 0$ the set of accumulation points of P is covered by a finite collection of sets such that the sum of their diameters is less than ε. This clearly implies Theorem 6.1. □

7. Uniformizations of Packings

7.1. Uniformization Theorem. *Let $G = (V, E)$ be a disk triangulation graph, for each $v \in V$ let $Q_v \subset \mathbb{C}$ be a smooth disk, and let $D \subset \mathbb{C}$ be a simply connected domain.*

Assume that there is a $\tau > 0$ such that Q_v is τ-fat for each $v \in V$. Also suppose that $D \neq \mathbb{C}$ (resp. $D = \mathbb{C}$) if G is VEL hyperbolic (resp. parabolic). Then there is a packing $P = (P_v \colon v \in V)$ with $\operatorname{carr}(P) = D$ *whose contacts graph is G, and such that P_v is homothetic to Q_v for each $v \in V$.*

We note that the continuous analogue of this theorem appears in [26]. The proof is also similar.

Proof. Let T be the triangulation of a disk that has G as its 1-skeleton. Let $T^1 \subset T^2 \subset T^3 \subset \cdots$ be an exhaustion of T. By this we mean that $T = \bigcup_j T^j$, and each T^j is a finite triangulation of a disk (with boundary). It is easy to see that such an exhaustion exists. We also require, without loss of generality, that T^1 has some interior vertex, say v_0. For each $j = 1, 2, \ldots,$ let $G^j = (V^j, E^j)$ denote the 1-skeleton of T^j.

Suppose, without loss of generality, that $0 \in D$ and 0 is in the interior of Q_{v_0}. Let D^j be a sequence of smooth Jordan domains[6] in $\mathbb{C}$ such that $0 \in D^1 \subset D^2 \subset \cdots$ and $D = \bigcup_j D^j$. From the packing theorems of [24] we know that for each $j = 1, 2, \ldots$ there is a packing $P^j = (P_v^j \colon v \in V^j)$ in the closure of D^j, such that each P_v^j is homothetic to Q_v, the sets P_v^j are tangent to ∂D^j when v is a boundary vertex of T^j, and $P_{v_0}^j$ has the form $t_j Q_{v_0}$ for some $t_j > 0$. Let $\{j(k)\}$ be a subsequence of $\{1, 2, \ldots\}$ such that the Hausdorff limit

$$\tilde{P}_v = \lim_{k \to \infty} \frac{1}{\operatorname{diameter}\left(P_{v_0}^{j(k)}\right)} P_v^{j(k)} \tag{7.1}$$

exists for every $v \in V$. The Hausdorff limit is taken in $\hat{\mathbb{C}}$; that is, *a priori* we must allow for the possibility that ∞ is contained in some $\tilde{P}_v$.

We show now that the sets $\tilde{P}_v$ do not degenerate to single points and do not contain ∞. The set $\tilde{P}_{v_0}$ certainly is OK, since it contains 0, has diameter 1, and is homothetic to Q_{v_0}, by construction. Let u be any neighbor of v_0. Since $\tilde{P}_u$ is a Hausdorff limit of sets homothetic to Q_u, which is smooth, $\tilde{P}_u$ is either homothetic to Q_u, or is a single point, or a half-plane, or $\tilde{P}_u = \hat{\mathbb{C}}$. The last case is clearly impossible, since the interior of $\tilde{P}_u$ does not intersect $\tilde{P}_{v_0}$. It is also clear that $\tilde{P}_u$ intersects $\tilde{P}_{v_0}$ but does not intersect its interior.

Let $u_1, u_2, \ldots, u_n$ be the neighbors of v_0, in circular order. For every j such that v_0 and all its neighbors are in the interior of T^j, the set $P_{u_1}^j \cup \cdots \cup P_{u_n}^j$ contains a Jordan curve that separates $P_{v_0}^j$ from ∞. (This follows from Lemma 4.1.) Therefore, for at least two neighbors u of v_0 the sets $\tilde{P}_u$ contain more than a single point. Suppose, for example, that $\tilde{P}_{u_1}$ is a single point p, and that $\tilde{P}_{u_n}$ is not a single point. Let m be the largest number in $\{1, 2, \ldots, n\}$ such that $\tilde{P}_{u_r} = \{p\}$ for each $r < m$ in $\{1, 2, \ldots, n\}$. Since at least two $\tilde{P}_{u_i}$ do not degenerate to points, $m < n$. It is clear that each $\tilde{P}_{u_i}$ intersects $\tilde{P}_{u_{i+1}}$ and that $\tilde{P}_{u_1}$ intersects $\tilde{P}_{u_n}$. Therefore, the three smooth sets $\tilde{P}_{v_0}, \tilde{P}_{u_n}, \tilde{P}_{u_m}$ contain the point p. This implies that the interiors of two

[6] A smooth Jordan domain is the interior of a smooth disk.

of these sets must intersect, which is clearly impossible. Thus we conclude that none of the sets $\tilde{P}_{u_i}$ consists of a single point, and that the ratios diameter$(P^j_{v_0})/$diameter$(P^j_{u_i})$ are bounded from above. (The reader may wish to compare the above argument with the Ring Lemma of [20].)

Is it possible that $\tilde{P}_{u_1}$ is a half-plane? To see that it is not, consider a Hausdorff limit of the packings $(h_j(P^j_v)\colon v \in V^j)$, where h_j is the homothety that takes $P^j_{u_1}$ to Q_{u_1}. The same argument as above, but with the roles of u_1 and v_0 switched, shows then that the ratios diameter$(P^j_{u_1})/$diameter$(P^j_{v_0})$ are bounded from above. Similarly, for every edge $[u, w]$ the ratio diameter$(P^j_u)/$diameter(P^j_w) is bounded independently of j. Since G is connected, this also holds when $u, w \in V$ are not neighbors. Therefore, each set $\tilde{P}_v$ is not a half-plane, nor a point, and thus is homothetic to Q_v.

If G is VEL parabolic, then part (2) of 3.1 implies that $\tilde{P}$ is locally finite in $\hat{\mathbb{C}} - \{p\}$ for some $p \in \hat{\mathbb{C}}$. It is easy to see that $p = \infty$, and thus $\tilde{P}$ is locally finite in $\mathbb{C}$. This completes the proof in the case that G is VEL parabolic.

Now suppose that G is VEL hyperbolic. The set $P^j_{v_0}$ is contained in $D \subsetneqq \mathbb{C}$ and has the form $t_j Q_{v_0}$, $t_j > 0$. Since 0 is an interior point of Q_{v_0}, this implies that the sequence t_j is bounded from above, and hence diameter$(P^j_{v_0})$ is bounded from above. By passing to a subsequence of $j(k)$, if necessary, assume that $t = \lim_{k\to\infty}$ diameter$(P^j_{v_0}) \in [0, \infty)$ exists. We have established above that for any $v, w \in V$ the ratios diameter$(P^j_v)/$diameter(P^j_w) remain bounded as $j \to \infty$. Consider the Hausdorff limits

$$P_v = \lim_{k\to\infty} P^{j(k)}_v. \tag{7.2}$$

If $t = 0$, then, because G is connected, it follows that $P_v = \{0\}$ for each v, and in particular the limits (7.2) exist. If $t > 0$, then comparing with (7.1), we conclude again that these limits exist, and that each P_v is homothetic to Q_v.

We now prove that each P_v is contained in D. Consider some vertex $v \in V$, and let $N(v)$ denote the neighbors of v. By Lemma 4.1, for each j sufficiently large (so that $N(v)$ is contained in the interior of T^j) there is a Jordan curve in $\bigcup_{u\in N(v)} P^j_u - P^j_v$ that separates P^j_v from ∂D^j. Assuming that $t > 0$, since for any fixed u the sets P^j_u vary within a compact collection of homotheties of Q_u, the above implies that the distance from P^j_v to ∂D^j is bounded from below independently of j. Therefore, $P_v \subset D$. The same conclusion is true, of course, if $t = 0$, because then $P_v = \{0\}$. So we have established that the packing P is contained in D.

Clearly, the interiors of the sets P_v are disjoint, and $P_v \cap P_w \neq \emptyset$ whenever $[v, w] \in E$. Therefore, the proof will be complete once we show that the packing $P = (P_v\colon v \in V)$ is locally finite in D. (This will also rule out the possibility $t = 0$, $P_v = \{0\}$.) That is actually the most significant part of the proof. It turns out that the packing $\tilde{P}$ is useful to proving this property of P.

Let F be some compact connected subset of D that contains 0. We prove that F intersects finitely many sets in P, and this shows that P is locally finite in D. Let F' be any compact connected subset of D that contains F in its interior. Let z be some accumulation point of $\tilde{P}$. From Lemma 4.3 we know that $\tilde{P}$ is disjoint from its accumulation points. By Theorem 3.1, z is not the only accumulation point of $\tilde{P}$.

Therefore, there is a compact connected set K that intersects $\tilde{P}_{v_0}$, contains an accumulation point of $\tilde{P}$, and is disjoint from z. In the following, for a set $X \subset \hat{\mathbb{C}}$, let $\tilde{V}(X)$ denote the set of $v \in V$ such that $\tilde{P}_v$ intersects X. Since K is connected and contains an accumulation point of $\tilde{P}$, it is clear that each component of $\tilde{V}(K)$ is infinite. (This follows from Lemma 4.3.)

We let ε be a small positive number whose value is determined below. By Lemma 3.4, there is some open set $W = W(z, K, \varepsilon)$ containing z such that

$$\mathrm{VEL}_G(\tilde{V}(K), \tilde{V}(W)) > \frac{1}{\varepsilon}.$$

Without loss of generality, we assume that W is connected. Then every component of $\tilde{V}(W)$ is infinite.

Assume for the moment that D has finite area. We show that if ε is chosen sufficiently small, then P_v is disjoint from F for every $v \in \tilde{V}(W)$. Let C be some component of $\tilde{V}(W)$. Let j be sufficiently large so that C intersects V^j, and let C^j be any component of $C \cap V^j$. Since every component of $\tilde{V}(W)$ is infinite, C is infinite, and therefore C^j must contain boundary vertices of T^j. Let H^j be the component of $\tilde{V}(K) \cap V^j$ that contains v_0. The above argument tells us that H^j contains boundary vertices of T^j. Let Γ^{*j} denote the family of all subsets of V^j that intersect every path in $\Gamma_{G^j}(H^j, C^j)$. Proposition 5.2 now implies that

$$\mathrm{EL}(\Gamma^{*j}) \le \mathrm{VEL}_{G^j}(H^j, C^j)^{-1} \le \mathrm{VEL}_G(\tilde{V}(K), \tilde{V}(W))^{-1} < \varepsilon. \tag{7.3}$$

Consider the v-metric $m = m_j$ that assigns to each $v \in V^j$ the diameter of P_v^j. By the τ-fatness of the sets Q_v, we have

$$\mathrm{area}(P_v^j) \ge \tau\pi m(v)^2.$$

This implies

$$\mathrm{area}(m) \le \tau^{-1}\pi^{-1}\, \mathrm{area}(D) < \infty.$$

Inequality (7.3) now implies that there is some $\gamma^* \in \Gamma^{*j}$ such that

$$L_m(\gamma^*) < \sqrt{\frac{\varepsilon\, \mathrm{area}(D)}{\tau\pi}}.$$

We now choose ε to be sufficiently small so that the right-hand side of the above inequality is smaller than $d(F', \partial D)/2$, half the distance from F' to ∂D. So we have

$$L_m(\gamma^*) < \frac{d(F', \partial D)}{2}. \tag{7.4}$$

Since the sets C^j and H^j are connected, Lemma 4.4 implies that there is a $\gamma_1^* \in \Gamma^{*j}$ that is connected and is contained in γ^*. Let $Y^j = \bigcup_{v \in \gamma_1^*} P_v^j$. Because Y^j is connected, we may estimate its diameter as follows:

$$\operatorname{diameter}(Y^j) \leq \sum_{v \in \gamma^*} \operatorname{diameter}(P_v^j) = L_m(\gamma^*) < \frac{d(F', \partial D)}{2}.$$

Recall that C^j and H^j both contain boundary vertices of T^j. Since γ_1^* separates H^j from C^j, it too must contain boundary vertices. This implies that Y^j intersects ∂D^j. We now assume that j is sufficiently large so that $d(F', \partial D^j) > d(F', \partial D)/2$. Since $\operatorname{diameter}(Y^j) < d(F', \partial D)/2 < d(F', \partial D^j)$, and Y^j intersects ∂D^j, it is clear that Y^j does not intersect F'. Since γ_1^* separates H^j from C^j in G^j, it is clear that $Y^j \cup \partial D^j$ separates $\bigcup_{v \in C^j} P_v^j$ from $P_{v_0}^j$. Since $Y^j \cup \partial D^j$ does not intersect F', which is connected and intersects $P_{v_0}^j$, it follows that $\bigcup_{v \in C^j} P_v^j$ is disjoint from F'. Recall that C^j is any component of $C \cap V^j$, and C is any component of $\tilde{V}(W)$. Therefore, for any $v \in \tilde{V}(W)$, if j is sufficiently large so that $v \in V^j$ and $d(F', \partial D^j) > d(F', \partial D)/2$, then $P_v^j \cap F' = \varnothing$. Taking limits, it follows that P_v is disjoint from the interior of F', which contains F, and so $P_v \cap F = \varnothing$.

We summarize our conclusions as follows. For every accumulation point z of $\tilde{P}$ there is a neighborhood W_z of z such that $P_v \cap F = \varnothing$ for every $v \in \tilde{V}(W_z)$. Let W^* be the union of all W_z, over all accumulation points z of $\tilde{P}$. Then W^* is an open set that contains the accumulation points of $\tilde{P}$. Consequently, $V - \tilde{V}(W^*)$ is finite. Since $F \cap P_v = \varnothing$ for all $v \in \tilde{V}(W^*)$, only finitely many sets in the packing P intersect F. Hence P is locally finite in D.

This concludes the proof in the case that D has finite area. When D has infinite area, the same proof is valid when the spherical metric of $\hat{\mathbb{C}}$ is used in place of the flat metric of $\mathbb{C}$. The only fact to note is that there is some τ_1, which depends only on τ, such that the spherical area of P_v^j is at least τ_1 times the square of the spherical diameter of P_v^j. This follows easily from Lemma 3.3. Thus the proof is complete. □

We can now prove:

7.2. Theorem. *A disk triangulation graph is CP parabolic iff it is VEL parabolic. A disk triangulation graph is CP hyperbolic iff it is VEL hyperbolic.*

Proof of Theorems 1.2 and 7.2. These follow immediately from 3.1 and 7.1. □

8. VEL Parabolicity, EEL Parabolicity, and Recurrence

We have seen that the VEL type of a disk triangulation graph is equal to its CP type, and now we establish the connection between VEL and EEL type. Through Theorem 2.6, this relates the CP type of graph to well-studied notions.

8.1. Theorem. *Let $G = (V, E)$ be a locally finite graph. If G is EEL parabolic, then it is also VEL parabolic. Conversely, if G has bounded valence and is VEL parabolic, then it is EEL parabolic.*

Proof. Suppose that M is an e-metric on G. Define a v-metric m on G by

$$m(v) = \max\{M([v,u]) : [v,u] \in E\}.$$

If γ is any transient path in G, then a simple diagonalization argument shows that there is a path $\gamma' = (v_1, v_2, \ldots)$ with distinct vertices that are all in γ. Thus

$$L_m(\gamma) \geq L_m(\gamma') = \sum_{j=1}^{\infty} m(v_j) \geq \sum_{j=1}^{\infty} M([v_j, v_{j+1}]) = L_M(\gamma'). \tag{8.1}$$

For each $v \in V$ let $e(v)$ denote an edge e of G containing v that maximizes $M(e)$ among such edges. Clearly, each $e \in E$ is equal to $e(v)$ for at most two vertices v. Using this, we get

$$\begin{aligned} \operatorname{area}(m) &= \sum_{v \in V} m(v)^2 = \sum_{v \in V} M(e(v))^2 \\ &\leq 2 \sum_{e \in E} M(e)^2 = 2 \operatorname{area}(M). \end{aligned} \tag{8.2}$$

Together with (8.1) this establishes that an EEL parabolic graph is VEL parabolic.

To prove the opposite implication, assume that there is a global bound k on the valence of any vertex $v \in V$. Let m be some v-metric on G. Define an e-metric M by $M([u, v]) = \max(m(u), m(v))$. It is easy to establish that for any path γ we have $L_M(\gamma) \geq L_m(\gamma)$. Moreover, since each vertex is incident with at most k edges, a calculation similar to (8.2) gives $\operatorname{area}(M) \leq k \operatorname{area}(m)$. These inequalities show that a bounded valence VEL parabolic graph is EEL parabolic, and the proof of the theorem is complete. □

8.2. Theorem. *There is a disk triangulation graph which is CP and VEL parabolic, but EEL hyperbolic and transient.*

This shows that the bounded valence requirement in the second part of Theorem 8.1 is essential.

Proof. Let T be a triangulation of an open disk. It is not hard to see that by adding vertices and edges inside the triangular faces of T a new triangulation T^* whose 1-skeleton G^* is transient can be obtained. On the other hand, G^* is VEL parabolic iff the 1-skeleton of T is VEL parabolic. The details are left to the reader. □

Proof (of 1.1). Follows immediately from Theorems 8.1 and 7.2. □

9. Perimetric Inequalities and the Type

9.1. Theorem. *Let $G = (V, E)$ be a locally finite, infinite, connected graph, let W_0 be a finite nonempty set of vertices of G, and let $g: [0, \infty) \to (0, \infty)$ be some nondecreasing function.*

(1) *If G is VEL parabolic and satisfies the perimetric inequality*

$$|\partial W| \geq g(|W|) \tag{9.1}$$

for every finite connected vertex set $W \supset W_0$, then

$$\sum_{n=1}^{\infty} \frac{1}{g(n)^2} = \infty. \tag{9.2}$$

Here ∂W denotes the set of vertices that are not in W but neighbor with some vertex in W, and $|A|$ denotes the cardinality of a set A.

(2) *If (9.2) holds, and*

$$|\partial W_k| \leq g(|W_k|) \tag{9.3}$$

is valid for every $k = 0, 1, 2, \ldots,$ where W_k is defined inductively by $W_{k+1} = W_k \cup \partial W_k$, then G is VEL parabolic.

We remark that part (1) and its proof are analogous to a criterion of Grigor'yan for the hyperbolicity of a Riemannian manifold [14]. Part (2) can be viewed as a generalization of the Rodin–Sullivan length–area lemma [20].

Proof. Assume that G is VEL parabolic. Let m be some v-metric on G with $\text{area}(m) < \infty$ and $d_m(W_0, \infty) = \infty$. (See Remark 2.3.) We also assume, without loss of generality, that $m(v) > 0$ for each $v \in V$. For each $v \in V$, let I_v be the interval

$$I_v = [d_m(W_0, v) - m(v), d_m(W_0, v)].$$

For $h \in [0, \infty)$ set

$$V_h = \{v \in V: h \in I_v\},$$

$$w(h) = \sum \{m(v): v \in V_h\},$$

$$Y_h = \{v \in V: I_v \subset [0, h]\},$$

$$n(h) = |Y_h|.$$

It is easy to see that $V_h = \partial Y_h$, and therefore

$$|V_h| \geq g(n(h)). \tag{9.4}$$

It turns out that $n(h)$ is not convenient to work with, since it is not smooth enough. We therefore define

$$s_v(h) = \frac{\text{length}(I_v \cap [0,h])}{m(v)} \quad \text{for} \quad v \in V,$$

$$s(h) = \sum_{v \in V} s_v(h).$$

Note that $s_v(h)$ is equal to 0 for $h \le \min I_v$, $s_v(h) = 1$ for $h \ge \max I_v$, and s_v is linear in I_v. Since $d_m(W_0, \infty) = \infty$, it follows that for every $h \in [0,\infty)$ there are finitely many v such that I_v intersects $[0,h]$. Therefore $s(h)$ is a piecewise linear function. It should be thought of as a smoothed version of $n(h)$.

Now set

$$f(x) = \min\left(g\left(\frac{x}{2}\right), \frac{x}{2}\right). \tag{9.5}$$

Let $h \in [0,\infty)$. If $|V_h| \ge s(h)/2$, then

$$|V_h| \ge f(s(h)). \tag{9.6}$$

Suppose that $|V_h| < s(h)/2$. Then we have $n(h) = |Y_h| \ge s(h) - |V_h| > s(h)/2$. Consequently

$$|V_h| \ge g(n(h)) \ge g\left(\frac{s(h)}{2}\right) \ge f(s(h)),$$

and we conclude that (9.6) holds in any case.

We are now ready to do some real work. At points h where $s(h)$ is differentiable, we have

$$\frac{ds}{dh}(h) = \sum_{v \in V_h} s_v'(h) = \sum_{v \in V_h} \frac{1}{m(v)}.$$

Therefore, using the Cauchy–Schwarz inequality (or the inequality between the arithmetic and harmonic means) and (9.6), we get

$$\frac{ds}{dh} \ge \frac{|V_h|^2}{\sum \{m(v) : v \in V_h\}} \ge \frac{f(s(h))^2}{w(h)}.$$

This gives

$$\frac{ds}{f(s)^2} \ge \frac{dh}{w(h)}.$$

Integrating for h in some interval $[a, b]$, $0 < a < b < \infty$, and using Cauchy–Schwarz again, we get

$$\int_{s(a)}^{s(b)} \frac{ds}{f(s)^2} \geq \int_a^b \frac{dh}{w(h)} \geq \frac{(b-a)^2}{\int_a^b w(h)\,dh}. \tag{9.7}$$

Note that

$$\int_0^\infty w(h)\,dh = \int_0^\infty \sum_{v \in V_h} m(v)\,dh = \sum_{v \in V} \int_{I_v} m(v)\,dh = \sum_{v \in V} m(v)^2 = \text{area}(m) < \infty.$$

Therefore, letting $b \to \infty$ in (9.7), we get

$$\int_{s(a)}^{\infty} \frac{ds}{f(s)^2} = \infty.$$

Since

$$\frac{1}{f(s)^2} = \max\left(\frac{1}{g(s/2)^2}, \frac{4}{s^2}\right) \leq \frac{1}{g(s/2)^2} + \frac{4}{s^2},$$

we find that $\int_0^\infty g(s)^{-2}\,ds = \infty$, which implies (9.2). This proves part (1).

To establish part (2), now set $n_k = |W_k|$, and assume that (9.3) holds. Let N be some positive integer, and define a v-metric m on G by

$$m(v) = \begin{cases} g(n_k)^{-1} & \text{for} \quad v \in \partial W_k, \quad k \leq N, \\ 0 & \text{otherwise}. \end{cases}$$

We have

$$d_m(W_0, \infty) \geq d_m(W_0, \partial W_N) \geq \sum_{k=0}^{N} g(n_k)^{-1}.$$

On the other hand,

$$\text{area}(m) \leq \sum_{k=0}^{N} \frac{|\partial W_k|}{g(n_k)^2} \leq \sum_{k=0}^{N} g(n_k)^{-1}.$$

Since the above are valid for each N, we get

$$\text{VEL}(W_0, \infty) \geq \sum_{k=0}^{\infty} g(n_k)^{-1}. \tag{9.8}$$

Note that

$$n_{k+1} = |W_{k+1}| = |W_k \cup \partial W_k| \leq |W_k| + |\partial W_k| \leq n_k + g(n_k).$$

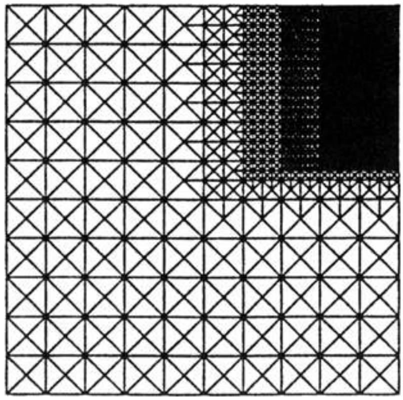

Fig. 9.1. A parabolic graph with exponential growth.

Using this and the monotonicity of g, we obtain

$$\frac{1}{g(n_k)} \geq \frac{1}{n_{k+1} - n_k} \sum_{n=n_k}^{n_{k+1}-1} \frac{1}{g(n)} \geq \sum_{n=n_k}^{n_{k+1}-1} \frac{1}{g(n_k)} \frac{1}{g(n)} \geq \sum_{n=n_k}^{n_{k+1}-1} \frac{1}{g(n)^2}.$$

This implies

$$\sum_{k=0}^{\infty} g(n_k)^{-1} \geq \sum_{n=n_0}^{\infty} \frac{1}{g(n)^2}.$$

Now part (2) follows from (9.8). □

There is a certain asymmetry in the two parts of Theorem 9.1. While the first part examines the relation between the size of the boundary of W and the size of W for every finite connected vertex set $W \supset W_0$, the second part does this only for the sets W_k. This difference is essential; that is, part (1) fails if (9.1) is only assumed for the sets W_k. Figure 9.1 gives a disk triangulation graph, which is essentially equivalent to a graph constructed by Soardi [27], with the following properties:

(1) G is VEL parabolic.
(2) The maximum degree in G is 8.
(3) $|\partial W_k| \geq C|W_k|$ for some $C > 0$ and all $k = 0, 1, \ldots$. The constant C does depend on the choice of W_0, but not on k.

Property (3) clearly implies that G has exponential growth; i.e., $|W_k| \geq (1 + C)^k$.

10. Determining the Type from the Valences

A natural question is, can the type of a graph be determined from the valences of its vertices. Suppose for a moment that G is a disk triangulation graph. It is known [5] that if all the vertices of G have degree greater than 6, then G is not CP parabolic, and if all the vertices of G have degree at most 6, then G is CP parabolic. The following two theorems generalize these results.

10.1. Theorem. *Let G be the 1-skeleton of an infinite triangulation of a surface, and suppose that at most finitely many vertices in G have degree greater than 6. Then G is VEL parabolic, EEL parabolic, and recurrent.*

Due to Theorem 7.2 this implies that a disk triangulation graph with finitely many vertices of degree greater than 6 is CP parabolic.

Proof. The method of proof is to show that the rate of growth of G is too slow for G to be hyperbolic.

Given a set $W \subset V$, we let ∂W denote the set of $v \in V - W$ that neighbor with some vertex in W, and let $\tilde{\partial} W$ be the set of vertices in ∂W that neighbor with some vertex in $V - (W \cup \partial W)$. If K is finite, then ∂K is a finite set of vertices, and K is disjoint from the infinite components of $V - \partial K$.

Let $K_0 \subset V$ be a finite nonempty set of vertices that contains all the vertices in V of degree greater than 6. We define the sequence $K_1, K_2, \dots$ inductively by setting

$$K_{n+1} = K_n \cup \partial K_n .$$

Note that each K_n is finite, and each vertex in $\tilde{\partial} K_n$ has at least one neighbor in K_n and at least one neighbor outside of K_{n+1}. Let C_n denote the set of vertices in ∂K_n that have precisely one neighbor in K_n, and let $D_n = \partial K_n - C_n$. Consider some $v \in \tilde{\partial} K_n$. Let $u_1, \dots, u_m$ be its neighbors, in circular order, and set $u_0 = u_m$. Since v has a neighbor in K_n, we assume without loss of generality that it is $u_0 = u_m$. Let $j \in \{1, \dots, m-1\}$ be such that $u_j \notin K_{n+1}$. Since $v \in \tilde{\partial} K_n$, such a j exists. Let a be the least index in $\{1, \dots, j\}$ such that $u_a \notin K_{n+1}$, and let b be the maximal index in $\{j, \dots, m-1\}$ such that $u_b \notin K_{n+1}$. Since u_{a-1} neighbors with v and with u_a, and $u_{a-1} \in K_{n+1}$, it is clear that $u_a \in D_{n+1}$ and $u_{a-1} \in \partial K_n$. Similarly, $u_{b+1} \in \partial K_n$ and $u_b \in D_{n+1}$. By construction, $\{u_a, \dots, u_b\}$ contains all the neighbors of v in ∂K_{n+1}.

Suppose for a moment that $v \in D_n \cap \tilde{\partial} K_n$. We know that v has at most six neighbors. Of these, at least two are in K_n, and at least two are in ∂K_n, namely, u_{a-1} and u_{b+1}. If $a \neq b$, then v has at least two neighbors in D_{n+1}, namely, u_a, u_b. As $2 + 2 + 2 = 6$, we see that when $a \neq b$, v has precisely two neighbors in D_{n+1} and no neighbors in C_{n+1}. If $a = b$, then $u_a = u_b$ is the only neighbor of v in ∂K_{n+1}, and this neighbor is in D_{n+1}. We conclude that a vertex in D_n neighbors with at most two vertices in D_{n+1} and with no vertices in C_{n+1}.

The above reasoning also shows that a vertex in C_n neighbors with at most three vertices in ∂K_{n+1}, of which at most one is in C_{n+1}. One conclusion that we get is

$$|C_{n+1}| \le |C_n|. \tag{10.1}$$

Let m_{n+1} denote the number of edges between K_{n+1} and D_{n+1}. On the one hand, $m_{n+1} \ge 2|D_{n+1}|$, because every vertex in D_{n+1} has at least two neighbors in K_{n+1}.

On the other hand, the only vertices in K_{n+1} that neighbor with D_{n+1} are in $D_n \cup C_n$, the vertices in D_n have at most two neighbors in D_{n+1}, and the vertices in C_n have at most three neighbors in D_{n+1}. Therefore,

$$2|D_{n+1}| \le m_{n+1} \le 2|D_n| + 3|C_n|,$$

which gives

$$|D_{n+1}| \le |D_n| + \frac{3|C_n|}{2}. \tag{10.2}$$

Using induction and inequalities (10.1) and (10.2), we see that

$$|C_n| \le |C_0|, \qquad |D_n| \le |D_0| + \frac{3n|C_0|}{2}.$$

Therefore,

$$|\partial K_n| = |C_n \cup D_n| \le |D_0| + (2n+1)|C_0|. \tag{10.3}$$

Let m be the v-metric on G defined by $m(v) = 1/(n \log n)$ for $v \in \partial K_n$, $n > 1$, and $m(v) = 0$ for $v \notin \bigcup_{n>1} \partial K_n$. Since ∂K_n intersects every transient path meeting K_0, we see that $d_m(K_0, \infty) \ge \sum_{n>1} 1/(n \log n) = \infty$. On the other hand, (10.3) implies that area$(m) < \infty$. Hence G is VEL parabolic. From Theorems 8.1 and 2.6 it follows that G is EEL parabolic and recurrent. □

Let G be a disk triangulation graph. For $v \in V$, let deg(v) denote the degree of v in G. The average valence of a finite nonempty set of vertices W is just

$$\mathrm{av}(W) = \frac{1}{|W|} \sum_{v \in W} \deg(v).$$

The *lower average valence* of G is defined to be

$$\mathrm{lav}(G) = \sup_{W_0} \inf_{W \supset W_0} \mathrm{av}(W);$$

where W and W_0 are nonempty finite connected sets of vertices. (The authors do not know if this notion appears in the literature.)

10.2. Theorem. *Let G be a locally finite connected planar graph, and suppose that* lav$(G) > 6$. *Then G is VEL hyperbolic, and therefore EEL hyperbolic and transient.*

Note that the lower average valence of the hexagonal grid is 6.

Beardon and Stephenson [5] have shown that if every vertex of G has degree at least 7, then G is not CP parabolic. The above theorem is a generalization of this result.

Proof. In any finite planar graph G^* with vertex set V^*, the average valence satisfies

$$\mathrm{av}(V^*) < 6. \tag{10.4}$$

This is a well-known fact but for the convenience of the nonexpert readers, we give the proof here. Let n, e, f be the number of vertices, edges, and faces of the graph (which is embedded in the plane). The Euler formula gives $n + f = e + 2$, and the inequality $3f \leq 2e$ holds if $f > 1$, since every face must have at least three edges on its boundary, and each edge is on the boundary of at most two faces. From these it follows that $n > e/3$ (actually it is this inequality which we need later). However, $\mathrm{av}(V^*) = 2e/n$, since every edge is counted exactly twice in the sum $\Sigma_{v \in V^*} \deg(v)$. This establishes $\mathrm{av}(V^*) < 6$.

We now return to the infinite graph G. Let W_0 be a finite connected nonempty set of vertices such that $\mathrm{av}(W) > C > 6$ for some constant C and every finite connected set of vertices $W \supset W_0$. Consider such a W, and let G^* be the restriction of G to $W \cup \partial W$; that is, the vertices of G^* are $W \cup \partial W$, and an edge of G appears in G^* iff both its endpoints are in $W \cup \partial W$. Denote by n and e the number of vertices and edges in G^*, respectively. Then, clearly, $2e \geq |W| \mathrm{av}(W)$, and therefore, by the previous paragraph,

$$|W| + |\partial W| = |W \cup \partial W| = n > \frac{e}{3} \geq \frac{|W| \mathrm{av}(W)}{6} > \left(\frac{C}{6}\right)|W|.$$

This gives

$$|\partial W| > g(|W|)$$

with $g(x) = (C - 6)x/6$. Now, since $\Sigma_{n=1}^{\infty} g(n)^{-2} < \infty$, part (1) of Theorem 9.1 shows that G must be VEL hyperbolic, and the proof is complete. □

It would be interesting to narrow the wide gap between Theorems 10.1 and 10.2. Suppose, for example, that G is a bounded valence disk triangulation graph and that v_0 is some vertex in G. Let $k_n = \Sigma_v (6 - \deg(v))$, where the sum extends over all vertices v at distance at most n from v_0. Can criteria for the type of G based on the sequence $\{k_n\}$ be given? For example, if k_n is bounded, does it follow that G is VEL parabolic?

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