Hyperbolic Equations and Classes of Infinitely Differentiable Functions (*) (**).

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Summary. – *We consider the solvability in the Mandelbrojt classes* $\mathcal{E}\{M_h\}$ *(for the definition see* (6), (7), (8), (9) *below) of the Cauchy problem for hyperbolic equations of the type* $u_{tt} - a(t)u_{xx} = 0$, *where* $a(t)$ *is a strictly positive continuous function. More precisely, we give an example of a function* $a(t)$ *for which the Cauchy problem is not well-posed in any class* $\mathcal{E}\{M_h\}$ *containing a non-trivial function with compact support.*

1. – Introduction.

The object of this paper is the study of the hyperbolic Cauchy problem

$$(1)\qquad \begin{cases} u_{tt} - a(t)u_{xx} = 0 & \text{on } \boldsymbol{R}_x \times [0, T] \\ u(x, 0) = \varphi(x)\,, & u_t(x, 0) = \psi(x) \end{cases}$$

where

(2) $\quad a(t)$ *is a continuous function such that* $a(t) \geqslant \nu > 0$.

One of the most meaningful phenomena concerning problem (1) is the *finite speed of propagation* of the solution: if $\varphi(x)$ and $\psi(x)$ vanish on some $\Omega \subset\subset \boldsymbol{R}_x$, then $u(x, t)$ vanishes on the conoid

$$\Gamma_\Omega = \left\{(x, t):\ \operatorname{dist}(x, \complement\Omega) > \int_0^t \sqrt{a(s)}\, ds\right\}.$$

Now it is known that problem (1) may be not well-posed [1] in the space $\mathcal{E}$ of

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(1) We say that problem (1) is well-posed in a linear topological space $\mathcal{F}$ of functions or functionals on $\boldsymbol{R}_x$ if, for any initial data φ and ψ in $\mathcal{F}$, there exists a unique solution $u \in C^2([0, T], \mathcal{F})$ of (1).

infinitely differentiable functions on $\boldsymbol{R}_x$, while it is always well-posed in the space $\mathcal{A}$ of the real analytic functions on $\boldsymbol{R}_x$ (see th. 10 (iii) and th. 4 (i) of [1]).

Unfortunately, due to the non existence of analytic functions with compact support, the property of finite speed of propagation just given above looses its meaning in the case of analytic solutions.

To overcome this difficulty, one could replace the notion of « analytic function vanishing on some open Ω » by the one of « sequence of analytic functions converging to zero in $\mathcal{A}(\Omega)$ »; alternatively, one could study problem (1) in the space $\mathcal{A}'$ of real analytic functionals.

However, as the classical notion of finite speed of propagation seems to be more natural, we are induced to investigate about a possible well-posedness of problem (1) in some function space $\mathcal{F}$ with

$$\mathcal{A} \subset \mathcal{F} \subset \mathcal{E} \tag{3}$$

such that

(4) $\mathcal{F}$ contains non-trivial functions with compact support.

If (4) holds, the class $\mathcal{F}$ is called *non quasi-analytic.*

The most famous classes verifying {(3), (4)} are the Gevrey classes $\mathcal{E}^s$, $1 < s < \infty$ (²), which arise in a quite natural way in the study of some evolution equations. But, if $a(t)$ is only continuous, problem (1) may be not well-posed in any $\mathcal{E}^s$ with $s > 1$ (see th. 10 (i) of [1]).

Hence, we are forced to look for some non quasi-analytic class $\mathcal{F}$, in which problem (1) is well-posed, such that

$$\mathcal{A} \subset \mathcal{F} \subset \bigcap_{s>1} \mathcal{E}^s . \tag{5}$$

We shall look for such a class $\mathcal{F}$ among the function spaces $\mathcal{E}\{M_h\}$ which have been introduced and developed as a natural extension of the Gevrey classes by HADAMARD, DENJOI, CARLEMAN, OSTROYSKI and, more extensively, by MANDELBROJT (for a detailed bibliography about this subject, we refer to [5] and [4]).

We recall that an infinitely differentiable function φ is said to belong to $\mathcal{E}\{M_h\}$, where $\{M_h\}$ is a sequence of positive numbers, if, for any compact set $K \subset\subset \boldsymbol{R}_x$, there exist C_K and A_K such that

$$|D^{(h)}\varphi(x)| \leqslant C_K A_K^h M_h , \quad \forall x \in K, \ \forall h \geqslant 0 . \tag{6}$$

(²) An infinitely differentiable function φ is said to belong to $\mathcal{E}^s$ if, for any compact subset $K \subset\subset \boldsymbol{R}_x$, there exist C_K and A_K such that

$$|D^{(h)}\varphi(x)| \leqslant C_K A_K^h (h!)^s , \quad \forall x \in K, \ \forall h .$$

In order to ensure that the class $\mathcal{E}\{M_h\}$ be stable under the more usual operations, we shall assume that:

$$M_h^2 \leqslant M_{h-1} \cdot M_{h+1} \qquad (h \geqslant 1) \tag{7}$$

$$M_{h+1} \leqslant A^h M_h \tag{8}$$

$$\sqrt[h]{\frac{M_h}{h!}} \leqslant B \sqrt[k]{\frac{M_k}{k!}} \qquad \text{for } h \leqslant k \tag{9}$$

for some constants A and B.

Assumption (7) (*logarithmical convexity* of $\{M_h\}$) ensures that $\mathcal{E}\{M_h\}$ is stable under *multiplication*, while (8) and (9) ensure respectively the stability under *differentiation* and under *left-composition by an entire function* (see [5] and [6]).

The Denjoi-Carleman theorem states that a class $\mathcal{E}\{M_h\}$ is non quasi-analytic, in the sense of (4), if and only if

$$\sum_{h=1}^{\infty} \frac{M_{h-1}}{M_h} < +\infty \tag{10}$$

(let us observe that, if (10) holds, then $\mathcal{E}\{M_h\} \supset \mathcal{A}$).

We are now in the position to make precise the problem which has originated the present work.

PROBLEM. – i) *Is it possible to find a non quasi-analytic class* $\mathcal{E}\{M_h\}$ *in which every problem like* (1), *with* $a(t)$ *satisfying* (2), *is well-posed?*

Or, at least,

ii) *Is it possible to find, for any* $a(t)$ *satisfying* (2), *a non quasi-analytic class* $\mathcal{E}\{M_h\}$ (*depending on* $a(t)$) *in which problem* (1) *is well-posed?*

The aim of this paper is to give a *negative* answer to both questions; this will result as a consequence of the following

THEOREM 1. – *For any* $T > 0$ *and any pair of* C^∞ *periodic data* φ *and* ψ, *having the same period, which are not both analytic, there exists a strictly positive continuous coefficient* $a(t)$, *defined on* $[0, T]$, *such that problem* (1) *does not admit, as a solution, any distribution* (*and not even any Gevrey ultradistribution*).

Now, if $\mathcal{E}\{M_h\} \supsetneq \mathcal{A}$, there exists a 2π-periodic function $\varphi \in \mathcal{E}\{M_h\} \setminus \mathcal{A}$ (see th. 6.6.III and th. 6.7.III of [5]); therefore, as a consequence of theorem 1, we have the following

COROLLARY 1. – *For any* $\mathcal{E}\{M_h\}$ *which contains* $\mathcal{A}$ *strictly there exists a coefficient* $a(t)$, *verifying* (2), *such that problem* (1) *is not well-posed in* $\mathcal{E}\{M_h\}$.

In other words, question i) has a negative answer. On the other hand, a theorem due to W. Rudin (see [6]) states that every non quasi-analytic class $\mathcal{E}\{M_h\}$ (verifying (7), (8) and (9)) contains the class $\mathcal{E}\{(h\cdot\ln h)^h\}$. Hence, from corollary 1 we deduce that also question ii) has a negative answer, i.e.

Corollary 2. – *There exists a function* $a(t)$ *satisfying* (2) *such that problem* (1) *is not well-posed in any non quasi-analytic class* $\mathcal{E}\{M_h\}$ (*verifying* (7), (8) *and* (9)).

Let us finally observe that we can weaken questions i) and ii) by requiring only that $\mathcal{E}\{M_h\}$ contains $\mathcal{A}$ strictly, instead of supposing that $\mathcal{E}\{M_h\}$ be non quasi-analytic: in this case corollary 1 gives once again a *negative* answer to the first question, whereas the energy estimate (90) of [1] permits us to say that the second question has an *adfirmative* answer.

2. – The proof of theorem 1.

As in the counter-examples of [1] (see also [3] and [2]), the construction of the coefficient $a(t)$ will be based upon the function

$$\alpha(\varepsilon;\tau) = 1 - 4\varepsilon\sin 2\tau - \varepsilon^2(1-\cos 2\tau)^2 . \tag{11}$$

Let us observe that $\frac{1}{2}\leqslant\alpha(\varepsilon;\tau)\leqslant 2$ for $0<\varepsilon\leqslant 1/10$.

The interest of this function consists for us in the fact that the ordinary problem

$$\begin{cases} \dfrac{d^2w}{d\tau^2} + \alpha(\varepsilon;\tau)w = 0 \\ w(0)=0; \qquad w'(0)=C \end{cases} \tag{12}$$

has the following explicit solution:

$$w(\tau) = C\sin\tau\cdot\exp\{\varepsilon(\tau - \tfrac{1}{2}\sin 2\tau)\} . \tag{13}$$

In order to construct the wished coefficient $a(t)$, we shall consider, in dependence on the initial data φ and ψ, a double sequence of subintervals of $[0,T]$, $J_k=[t''_{k-1}, t'_k]$, $I_k=[t'_k, t''_k]$, with

$$0 = t''_0 < t'_1 < t''_1 < \ldots \tag{14}$$

$$\sup_k t'_k = \sup_k t''_k = T_* \leqslant T \tag{15}$$

$$t''_k - t'_k = \varrho_k , \tag{16}$$

so that

$$[0, T] = J_1 \cup I_1 \cup J_2 \cup I_2 \cup \ldots \cup [T_*, T] .$$

Then the function $a(t)$ will have the following form:

$$a(t) \equiv 1 \quad \text{on } \Big(\bigcup_{k=1}^{\infty} J_k\Big) \cup [T_*, T] \tag{17}$$

$$a(t) = \alpha\big(\varepsilon_k;\, h_k(t - t_k')\big) \quad \text{on } I_k , \tag{18}$$

where $\alpha(\varepsilon; \tau)$ is defined by (11), while $\{\varepsilon_k\}$, $\{\varrho_k\}$ are two sequences of positive real numbers and $\{h_k\}$ is a sequence of natural numbers (depending on φ and ψ). All these sequences will be specified later, and they will be chosen in such a way that

$$\tfrac{1}{2} \leqslant a(t) \leqslant 2 , \quad \forall t \in [0, T] \tag{19}$$

$$a(t) \in C([0, T]) \tag{20}$$

$$a(t) \quad \text{is lipschitz-continuous on } [0, T_* - \delta] \text{ for any } \delta > 0 . \tag{21}$$

Now, going into the details of the proof, we assume (without loss of generality) that the assigned initial data $\varphi(x)$ and $\psi(x)$ are 2π-periodic, and hence we write

$$\varphi(x) = \sum_{h=-\infty}^{+\infty} a_h \exp(ihx) ; \qquad \psi(x) = \sum_{h=-\infty}^{+\infty} b_h \exp(ihx) . \tag{22}$$

In virtue of (19) and (21), problem (1) will have a (unique) solution $u(x, t)$ on $\boldsymbol{R}_x \times [0, T_*[$; as a consequence of (22), this solution may be written as follows:

$$u(x, t) = \sum_{h=-\infty}^{+\infty} v_h(t) \exp(ihx) \tag{23}$$

where

$$\begin{cases} v_h''(t) + h^2 a(t) v_h(t) = 0 \\ v_h(0) = a_h , \quad v_h'(0) = b_h . \end{cases} \tag{24}$$

Our aim is to prove that there exists an increasing sequence of positive real numbers $\{t_k\}$ converging to T_* such that the sequence of functions $\{u(\cdot, t_k)\}$ is unbounded in every space of Gevrey ultradistributions, more precisely such that

$$\sup_k |v_{h_k}(t_k)| \exp\big(-(h_k)^{1/s}\big) = +\infty , \quad \forall s > 1 \tag{25}$$

where $\{h_k\}$ is the sequence which appears in (18).

But (25) holds if and only if

$$\sup_k |\mathrm{Re}\, v_{h_k}(t_k)| \exp\left(-(h_k)^{1/s}\right) = +\infty, \qquad \forall s > 1$$

or

$$\sup_k |\mathrm{Im}\, v_{h_k}(t_k)| \exp\left(-(h_k)^{1/s}\right) = +\infty, \qquad \forall s > 1;$$

therefore, splitting any $v_h(t)$ into its real and imaginary part and taking into account the reality of $a(t)$, we can suppose without loss of generality that

(26) the Fourier coefficients a_h and b_h of (22) are *real*.

We now choose the sequences $\{\varepsilon_k\}$, $\{\varrho_k\}$ and $\{h_k\}$: $\{\varepsilon_k\}$ and $\{\varrho_k\}$ are two sequences of positive real numbers such that

$$\text{(27)} \qquad \begin{cases} \varepsilon_k \to 0 \ (k \to \infty), \qquad \varepsilon_k \leqslant 1/10, \qquad \forall k, \\ \sum_{k=1}^{\infty} \varrho_k \leqslant T/2 \end{cases}$$

while $\{h_k\}$ is a strictly increasing sequence of positive integers (depending on φ, ψ, $\{\varepsilon_k\}$ and $\{\varrho_k\}$) such that, when $k \to \infty$,

$$\text{(28)} \qquad (h_k^2 a_{h_k}^2 + b_{h_k}^2) \exp(\varepsilon_k \varrho_k h_k) \to +\infty$$

$$\text{(29)} \qquad \frac{\varepsilon_k \varrho_k h_k}{2} - (h_k)^{1/s} - T h_{k-1} \to +\infty, \qquad \forall s > 1.$$

We remark that, once $\{\varepsilon_k\}$ and $\{\varrho_k\}$ have been fixed (with $\varepsilon_k \varrho_k \to 0$), the existence of a sequence $\{h_k\}$ such that (28) holds is equivalent to the fact that $\varphi(x)$ and $\psi(x)$ are not both analytic functions, while (29) may be always obtained passing to some subsequence of h_k.

For technical reasons that will be clear later, we would like that

$$\text{(30)} \qquad \frac{h_k \varrho_k}{\pi} \quad \text{is an integer for any } k;$$

in order to obtain (30), it is sufficient to replace ϱ_k by $\tilde{\varrho}_k = \varrho_k - \pi\theta_k/h_k$, for some $\theta_k \in [0, 1]$. Since this slight modification does not affect (27), (28) and (29), we shall denote $\tilde{\varrho}_k$ again by ϱ_k.

Later on, we shall use an immediate consequence of (30), namely the inequality

$$\text{(31)} \qquad \frac{\pi}{h_k} \leqslant \varrho_k.$$

Now we construct, by induction on k, the sequences $\{t_k'\}$, $\{t_k''\}$, and hence the intervals $J_k = [t_{k-1}', t_k']$ and $I_k = [t_k', t_k'']$, as well as the coefficient $a(t)$ on $J_k \cup I_k$.

For $k = 1$ we take t_1' equal to the first positive zero of the function

$$\xi_1(t) = a_{h_1} \cos(h_1 t) + \frac{1}{h_1} \cdot b_{h_1} \sin(h_1 t)\,; \tag{32}$$

(t_1' exists, since a_{h_1} and b_{h_1} are real numbers; see (26)). Moreover, we take

$$t_1'' = t_1' + \varrho_1 \tag{33}$$

and finally we define $a(t)$ on $J_1 \cup I_1$ by means of (17) and (18).

Assume now that $t_1', t_1'', t_2', \ldots, t_{k-1}', t_{k-1}''$ and $a(t)|_{[0,t_{k-1}'']}$ have been constructed, so that also the functions $v_h(t)$ of (24) are well defined on $[0, t_{k-1}'']$. Then, we take t_k' equal to the first zero greater than t_{k-1}'' of the function

$$\xi_k(t) = v_{h_k}(t_{k-1}'') \cos\big(h_k(t - t_{k-1}'')\big) + \frac{1}{h_k} v_{h_k}'(t_{k-1}'') \sin\big(h_k(t - t_{k-1}'')\big)\,; \tag{34}$$

moreover, we take

$$t_k'' = t_k' + \varrho_k \tag{35}$$

and finally we define $a(t)$ on $J_k \cup I_k$ by means of (17) and (18).

We remark that $T_* \leqslant T$; indeed, the length of J_k may be estimated by

$$t_k' - t_{k-1}'' \leqslant \frac{\pi}{h_k} \tag{36}$$

and, therefore, $T_* = \sum_{k=1}^{\infty} [(t_k'' - t_k') + (t_k' - t_{k-1}'')] \leqslant \sum_{k=1}^{\infty} \varrho_k + \pi \sum_{k=1}^{\infty} 1/h_k \leqslant T$, where, in the last inequality, we have used (27) and (31).

In such a way, we have constructed $a(t)$ on $[0, T_*]$; if we set, in conformity with (17), $a(t) \equiv 1$ on $[T_*, T]$, we see that $a(t)$ verifies (19), (20) and (21).

Now we define the sequence

$$t_k = t_k'' - \frac{\pi}{2h_k} \tag{37}$$

and we prove that (25) holds for this sequence, giving the explicit expression of $v_{h_k}(t)$ on $J_k \cup I_k$.

In the interval J_k, we have $a(t) \equiv 1$, and therefore $v_{h_k}(t)$ coincides with the function $\xi_k(t)$ of (34). By the very definition of t_k', we obtain in particular that

$$v_{h_k}(t_k') = 0\,. \tag{38}$$

As regards the expression of $v_{h_k}(t)$ on I_k, taking (18) into account, we perform the change of variable

$$\tau = h_k(t - t'_k) \qquad (t \in I_k) \tag{39}$$

and we set

$$v_{h_k}(t) = w_k(\tau) \qquad (t \in I_k) . \tag{40}$$

Hence, we see by (24) and (38) that $w_k(\tau)$ coincides with the solution of problem (12) with $\varepsilon = \varepsilon_k$ and $C = (1/h_k)v'_{h_k}(t'_k)$; thus, by (13), we have

$$v_{h_k}(t) = \frac{1}{h_k} v'_{h_k}(t'_k) \sin\big(h_k(t - t'_k)\big) \exp\left\{\varepsilon_k\left[h_k(t - t'_k) - \frac{1}{2}\sin\big(2h_k(t - t'_k)\big)\right]\right\} . \tag{41}$$

Now, in virtue of (31), t_k belongs to I_k (see (37)), and (41) gives

$$|v_{h_k}(t_k)| = \frac{1}{h_k}|v'_{h_k}(t'_k)| \exp\left(\varepsilon_k \varrho_k h_k - \frac{\varepsilon_k \pi}{2}\right) . \tag{42}$$

In order to obtain (25) we estimate $|v'_{h_k}(t_k)|$ from above: to this end we consider the *energy* function

$$E_{h_k}(t) = h_k^2 a(t)|v_{h_k}(t)|^2 + |v'_{h_k}(t)|^2 . \tag{43}$$

Then, by (43) and (24) we easily get

$$E_{h_k}(t'_k) \geqslant E_{h_k}(0) \exp\left\{-\int_0^{t'_k} \frac{|a'(s)|}{a(s)}\, ds\right\} , \tag{42}$$

by which, using the fact that $a(t) \geqslant \frac{1}{2}$ and $|a'(t)| \leqslant h_{k-1}$ on $[0, t'_k]$, we obtain

$$E_{h_k}(t'_k) \geqslant E_{h_k}(0) \exp(-2Th_{k-1}) . \tag{44}$$

Now

$$E_{h_k}(0) = (h_k^2 a_{h_k}^2 + b_{h_k}^2) \tag{45}$$

while, by means of (38), we get

$$E_{h_k}(t'_k) = |v'_{h_k}(t'_k)|^2 . \tag{46}$$

Substituting (45) and (46) into (44) we get

$$|v'_{h_k}(t'_k)| \geqslant (h_k^2 a_{h_k}^2 + b_{h_k}^2)^{\frac{1}{2}} \exp(-Th_{k-1}) ; \tag{47}$$

then, taking (47) into account, (42) gives

$$(48) \qquad |v_{h_k}(t_k)|\cdot \exp\{-(h_k)^{1/s}\} \geqslant \\ \geqslant (h_k^2 a_{h_k}^2 + b_{h_k}^2)^{1/2}\cdot \exp\left\{\varepsilon_k \varrho_k h_k - (h_k)^{1/s} - T h_{k-1} - \ln h_k - \frac{\varepsilon_k \pi}{2}\right\}.$$

Using (28) and (29), by (48) we get (25), i.e. the conclusion of theorem 1. ■

REMARK. – The conclusion of theorem 1 can be used to prove that, for any $T > 0$ and any pair of C^∞ initial data φ and ψ with *compact support*, there exists a coefficient $a(t)$ verifying (2) for which the solution of problem (1) blows up on $[0, T]$.

Indeed, assume that the supports of $\varphi(x)$ and $\psi(x)$ are included in the interval $[-L, L]$ and set

$$\tilde{\varphi}(x) = \sum_{h=-\infty}^{+\infty} \varphi\big(x + 2h(L + \sqrt{2}\,T)\big)$$
$$\tilde{\psi}(x) = \sum_{h=-\infty}^{+\infty} \psi\big(x + 2h(L + \sqrt{2}\,T)\big).$$

The functions $\tilde{\varphi}$ and $\tilde{\psi}$ are $2(L + \sqrt{2}\,T)$-periodic, hence we can find the corresponding coefficient $\tilde{a}(t)$ in the sense of theorem 1. Owing to the finite speed of propagation (which may be estimated by means of (19)) we easily get that theorem 1 holds for problem 1 with $a(t) = \tilde{a}(t)$ and initial data $\varphi(x)$ and $\psi(x)$.

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