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HYPERBOLIC RELAXATION OF REACTION-DIFFUSION EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. Under consideration is the hyperbolic relaxation of a semilinear reaction-diffusion equation

$$\varepsilon u_{tt} + u_t - \Delta u + f(u) = 0$$

on a bounded domain $\Omega \subset \mathbb{R}^3$ with $\varepsilon \in (0,1]$ and the prescribed dynamic condition

$$\partial_{\mathbf{n}} u + u + u_t = 0$$

on the boundary $\Gamma := \partial\Omega$. We also consider the limit parabolic problem ($\varepsilon = 0$) with the same dynamic boundary condition. Each problem is well-posed in a suitable phase space where the global weak solutions generate a Lipschitz continuous semiflow which admits a bounded absorbing set. Because of the nature of the boundary condition, fractional powers of the Laplace operator are not well-defined. The precompactness property required by the hyperbolic semiflows for the existence of the global attractors is gained through the approach of Pata and Zelik (2006). In this case, the optimal regularity for the global attractors is also readily established. In the parabolic setting, the regularity of the global attractor is necessary for the semicontinuity result. After fitting both problems into a common framework, a proof of the upper-semicontinuity of the family of global attractors is given at $\varepsilon = 0$. Finally, we also establish the existence of a family of exponential attractors.

Contents

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1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma := \partial\Omega$ of class C^2. We consider the *hyperbolic* relaxation of a semilinear reaction diffusion equation

$$\varepsilon u_{tt} + u_t - \Delta u + f(u) = 0 \tag{1.1}$$

in $(0,\infty) \times \Omega$ where $\varepsilon \in [0,1]$. The equation is endowed with the dynamic boundary condition

$$\partial_{\mathbf{n}} u + u + u_t = 0 \tag{1.2}$$

on $(0,\infty) \times \Gamma$ and with the initial conditions

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \text{ in } \Omega. \tag{1.3}$$

For the nonlinear term f, we assume that $f \in C^2(\mathbb{R})$ and that there is a constant $\ell \geq 0$ such that for all $s \in \mathbb{R}$ the following growth and sign conditions are satisfied:

$$|f''(s)| \leq \ell(1+|s|) \tag{1.4}$$

and

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda, \tag{1.5}$$

where $\lambda > 0$ is the best Sobolev/Poincaré-type constant

$$\lambda \int_\Omega u^2 \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Gamma u^2 \mathrm{d}S. \tag{1.6}$$

Finally, assume that there is $\vartheta > 0$ such that for all $s \in \mathbb{R}$,

$$f'(s) \geq -\vartheta. \tag{1.7}$$

Notice that the derivative $f = F^{'}$ of the double-well potential $F(u) = \frac{1}{4}u^4 - ku^2$, $k > 0$, satisfies assumptions (1.4), (1.5), and (1.7). The first two assumptions made here on the nonlinear term, (1.4) and (1.5), are the same assumptions made on the nonlinear term in [16], [41], and [51], for example ([41] additionally assumes $f(0) = 0$). The third assumption (1.7) appears in [14], [23], [27], and [44]; the bound is utilized to obtain the precompactness property for the semiflow associated with evolution equations when dynamic boundary conditions present a difficulty (e.g., here, fractional powers of the Laplace operator subject to (1.2) are undefined). It is worth mentioning that (1.5) can also be replaced by a less general (but still widely used in the literature) condition

$$\liminf_{|s|\to\infty} f^{'}(s) \geq -\lambda,$$

in which case (1.7) is automatically satisfied. Furthermore, assumption (1.4) implies that the growth condition for f is the critical case since $\Omega \subset \mathbb{R}^3$. Such assumptions are common when one is investigating the existence of a global attractor or the existence of an exponential attractor for a partial differential equation of evolution.

Of course, when (1.1) is equipped with Dirichlet, Neumann, or periodic boundary conditions, (1.6) simplifies. Moreover, if (1.1) is equipped with a Robin boundary condition, then an estimate like (1.6) holds, but λ possesses an explicit description as the first eigenvalue of the Laplacian with respect to the Robin boundary condition. The relation between the dynamic condition (1.2) with the acoustic boundary condition is discussed below. The hyperbolic equation (1.1) is a well-known nonlinear wave equation motivated from (relativistic) quantum mechanics (cf., e.g., [3, 15, 36, 49]). However, as mentioned, most sources study the asymptotic behavior of (1.1) with a static boundary condition such as Dirichlet, Neumann, periodic, or Robin. One of the goals of this paper is to extend some results concerning the asymptotic behavior of (1.1), now with the dynamic boundary condition (1.2). The corresponding linear case for (1.1)–(1.3) is treated in [46]. The existence of the global attractor for a linear damped wave equation with a nonlinear dynamic boundary condition is considered in [53]. More general systems, with supercritical nonlinear sources on both the interior and the boundary, are considered in [2, 9–12]. These contributions devote their attention mainly to issues like Hadamard local wellposedness, global existence, blow-up, and non-existence theorems, as well as estimates on the uniform energy dissipation rates for the appropriate classes of solutions. We also refer the reader to [13] for a unified overview of these results.

Our main goal is to compare the hyperbolic relaxation problem (1.1)–(1.3) to that of the limit parabolic equation where, for $\varepsilon = 0$, we have the reaction-diffusion equation

$$u_t - \Delta u + f(u) = 0 \tag{1.8}$$

in $(0, \infty) \times \Omega$ with the dynamic boundary condition

$$\partial_{\mathbf{n}} u + u + u_t = 0 \tag{1.9}$$

on $(0, \infty) \times \Gamma$ and the initial conditions

$$u(0, x) = u_0(x) \text{ in } \Omega,\ \ u(0, x) = \gamma_0(x) \text{ on } \Gamma. \tag{1.10}$$

For the sake of simplicity, we shall restrict our attention only to linear boundary conditions of the form (1.9) even though our framework can easily allow for a complete treatment of nonlinear dynamic boundary conditions (see Remark 3.19; cf. also [14], [23], [27]).

Because of its importance in the physical sciences and the development of mathematical physics, the reaction-diffusion equation (1.8) and its asymptotic behavior are well known to the literature. Many of the books referenced above contain a treatment on the parabolic semilinear reaction-diffusion equation (1.8) with *static* boundary conditions. In particular, the Chaffee-Infante reaction-diffusion equation with $f(u) = u^3 - ku$, $k > 0$, and Dirichlet boundary conditions can be found in [47, Section 11.5]. A discussion on the structure of the associated global attractor can also be found there. Additionally, the Chaffee-Infante equation and its hyperbolic relaxation, again with Dirichlet boundary conditions, are discussed in [40, Chapters 3–5].

Recently there has been a great amount of research taking place in the area of partial differential equations of evolution type, subject to dynamic boundary conditions. Boundary conditions of the form (1.9) arise for many known equations of mathematical physics. This can especially be seen by the many applications given to heat control problems, phase-transition phenomena, Stefan problems, some models in climatology, and many others. Without being too exhaustive we refer the reader to [24, 25, 29] for more details about the system (1.8)–(1.10) and a more complete list of references. A version of equation (1.2), but with nonlinear dissipation on the boundary, already appears in the literature; we refer to [17, 18]. There the authors are able to show the existence of a global attractor without the presence of the weak interior damping term u_t, by assuming that f is *subcritical.* One motivation for considering a boundary condition like (1.2) comes from mechanical considerations: there is frictional damping on the boundary Γ that is linearly proportional to the velocity u_t. In [51], the convergence, as time goes to infinity, of unique global strong solutions of (1.1)–(1.3) to a single equilibrium is established provided that f is also real analytic. Note that the set of equilibria for (1.1)–(1.3) may form a continuum so that, in general, guaranteeing this convergence is a highly nontrivial matter. The second motivation comes from thermodynamics. Suppose that we want to consider heat flow in a metal. The standard derivation of the heat equation is always based on the idea that "heat in equals heat out" over a region $\overline{\Omega}$. But the classical approach ignores the contribution of heat sources located on the boundary Γ, by taking into account only heat sources/sinks which are present inside the region (in our case, $-f(u)$ is treated as a source within Ω). A new derivation of the heat equation in the presence of heat sources/sinks located at Γ, assuming the Fourier law of cooling states (i.e., the heat flux $\overrightarrow{q}$ is directly proportional to the temperature gradient, $\overrightarrow{q} = -\nabla u$), was given in [31], and it has lead to the precise formulation of the system in (1.8)–(1.10). However, the derivation in [31] suffers from an important drawback which cannot be ignored: initial perturbations in (1.8) propagate with infinite speed. This means that the presence of a heat source located at Γ is instantaneously felt by all observers in Ω, no matter how far away from Γ they happen to be. This behavior can be traced to the "parabolic" character of Fourier's law. Thus, in many relevant phenomena the system (1.8)–(1.10) can become a bad approximation (see, e.g., [1], [35] for many examples). In order to overcome these problems, a generalization of the standard Fourier law must be considered, leading to a new formulation for which the heat flux $\overrightarrow{q}$ obeys the so-called Maxwell–Cattaneo heat conduction law:

$$\varepsilon\partial_t\overrightarrow{q} + \overrightarrow{q} = -\nabla u, \tag{1.11}$$

in $(0,\infty)\times\Omega$. Note that the Fourier law is obtained from (1.11) when $\varepsilon = 0$. This expression for the heat flux $\overrightarrow{q}$ leads to the hyperbolic equation (1.1), which entails that u propagates at finite speed. It is also worth mentioning that one can write (1.11) in the equivalent form of

$$\overrightarrow{q}(t,x) = -\int_0^\infty \Theta_\varepsilon(t-s)\nabla u(s,x)\,\mathrm{d}s,\ \Theta_\varepsilon(t) := \frac{1}{\varepsilon}e^{-\frac{t}{\varepsilon}}. \tag{1.12}$$

This points to a situation in which the (past) thermal memory of the material plays a role, but its relevance goes down quickly as we move to the past. Finally, it may be worth mentioning that the form of flux $\overrightarrow{q}$ assumed in (1.12), in which Θ_ε is assumed to be a *generic* memory kernel, also yields the following problem:

$$u_t = \int_0^\infty \Theta_\varepsilon \left(t-s\right) \left(\Delta u\left(s\right) - f\left(u\left(s\right)\right)\right) \mathrm{d}s. \tag{1.13}$$

In this case, $\Theta_\varepsilon\left(s\right) = \varepsilon^{-1}\Theta\left(s/\varepsilon\right)$ and $\Theta : (0,\infty) \to (0,\infty)$ is a given (smooth) summable and convex (hence decreasing) relaxation kernel. A complete treatment of equation (1.13), endowed with the dynamic boundary condition (1.9), will be the subject of further investigation in the future.

It may also be interesting to note that the dynamic boundary condition given in (1.2) can be recovered, in some sense, from the linear acoustic boundary condition,

$$\begin{cases} m\delta_{tt} + \delta_t + \delta = -u_t & \text{on } (0,T)\times\Gamma, \\ \partial_{\mathbf{n}} u = \delta_t & \text{on } (0,T)\times\Gamma. \end{cases} \tag{1.14}$$

Here the unknown $\delta = \delta(t,x)$ represents the *inward* "displacement" of the boundary Γ reacting to a pressure described by $-u_t$. The first equation $(1.14)_1$ describes the spring-like effect in which Γ (and δ) interacts with $-u_t$, and the second equation $(1.14)_2$ is the continuity condition: velocity of the boundary displacement δ agrees with the normal derivative of u. Together, (1.14) describes Γ as a locally reactive surface. The term $m = m(x)$ represents mass, so in a mass*less* system, the inertial term disappears. In the case when δ can be modeled by u near the boundary (i.e., if $\delta \sim u$ near Γ), we arrive at the boundary condition described by (1.2). In applications, the unknown u may be taken as a velocity potential of some fluid or gas in Ω that was disturbed from its equilibrium. The acoustic boundary condition was rigorously described by Beale and Rosencrans in [6] and [7]. Various recent sources investigate the wave equation equipped with acoustic boundary conditions, [19, 26, 42, 50]. However, more recently, it has been introduced as a dynamic boundary condition for problems that study the asymptotic behavior of weakly damped wave equations; see [23] and [48].

The aim of this paper is to extend the asymptotic results for dissipative wave equations (1.1) and reaction-diffusion equations (1.8) with the dynamic boundary condition (1.2), in terms of a perturbation problem, and ultimately discuss the continuity of the attracting sets generated by these problems. Due to the nature of the boundary condition imposed for the model problem (1.1), we are unable to prove the existence of global attractors for the hyperbolic relaxation problem through the compactness argument which is typical for damped wave equations with static boundary conditions, such as Dirichlet, Neumann, periodic, or Robin boundary conditions (cf., e.g., [40, 49, 52]). The problem arises from our lack of defining fractional powers of the Laplacian with respect to the boundary condition (1.2). This situation takes place because of the permanence of the u_t term on Γ, which in turn means the "Laplacian" is not selfadjoint. Thus, for example, the model problem does not enjoy an explicit Poincaré inequality found with a Fourier series, nor the existence of a local weak solution found with a typical Galerkin basis. Local solutions will be sought with semigroup methods that rely on monotone operator techniques as in [16]. Then estimates are applied to extend the local solutions to global ones and the existence of an absorbing set is determined. For the hyperbolic relaxation problem

(1.1)–(1.3), we obtain the relatively compact part in the decomposition of the solution by following the approach in [44].

The main novelties of the present paper with respect to previous results on the damped wave equation (1.1) are the following:

- We extend the results on the existence of global attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in(0,1]}$ for the damped wave equation (1.1) with a critical nonlinearity and a "dynamic" boundary condition instead of the usual Dirichlet boundary condition (see, e.g., [33], [34]). This is achieved through the decomposition method exploited in [44] which allows us to establish that $\mathcal{A}_\varepsilon$ has also *optimal* regularity (see Theorem 3.18).
- We show that a certain family $\left\{\widetilde{\mathcal{A}}_\varepsilon\right\}_{\varepsilon\in[0,1]}$ of compact sets, which is topologically conjugated to $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ in a precise way, is also upper-semicontinuous as ε goes to zero. Roughly speaking, we show that these sets $\widetilde{\mathcal{A}}_\varepsilon$ converge to the "lifted" global attractor $\widetilde{\mathcal{A}}_0$ associated with the parabolic problem. The argument utilizes the sequential characterization of the global attractor (cf., e.g., [40, Proposition 2.15]). The main difficulty comes from the fact that the phase spaces for the perturbed and unperturbed equations are not the same; indeed, solutions of the hyperbolic problem are defined for $(u_0,u_1)\in H^{s+1}(\Omega)\times H^s(\Omega)$, $s\in\{0,1\}$, while solutions of the parabolic problem make sense only in spaces like $L^2(\Omega)\times L^2(\Gamma)$ and $H^{s+1}(\Omega)\times H^{s+1/2}(\Gamma)$, respectively (see (1.10)). Thus, previous constructions obtained for parabolic equations with Dirichlet boundary conditions cannot be applied and have to be adapted.
- We prove the existence of a family of exponential attractors $\{\mathcal{M}_\varepsilon\}$, $\varepsilon\in(0,1]$, which entails that $\mathcal{A}_\varepsilon$ is also finite dimensional even in the critical case. We recall that the same result was shown in [16] for the wave equation (i.e., (1.1) without any damping in Ω) subject to the boundary condition (1.9). Unfortunately, we are unable to show that this dimension is uniform with respect to $\varepsilon>0$ as ε goes to zero. Some other open questions are formulated at the end of the article.

The article is organized as follows. The limit ($\varepsilon=0$) reaction-diffusion problem is discussed in Section 2. The section is mostly devoted to citing the already known main results of the parabolic problem: the existence and uniqueness of global solutions in an appropriate phase space (see Theorem 2.3), the definition of the (Lipschitz) semiflow, the existence and regularity of the global attractor (see Theorem 2.6). Section 3 contains our treatment of the hyperbolic relaxation problem, for all $\varepsilon\in(0,1]$. We discuss the existence and uniqueness of solutions defined for all positive times in Section 3.2 (see Theorem 3.6). The solutions generate a semiflow on the phase space, and thanks to the continuous dependence estimate, we know that the semiflow is locally Lipschitz continuous. The existence of a bounded absorbing set is also shown (see Lemma 3.10). The global attractor and its properties are established in Section 3.3, while the upper-semicontinuous result is established in Section 3.4. The existence of exponential attractors for the hyperbolic problem is presented in Section 4. The statement of a Grönwall-type inequality, used frequently in the estimates, is included in the appendix.

2. The limit parabolic problem. In this short section, we recall some results for the limit parabolic problem (1.8)–(1.10), i.e., (1.1)–(1.3) with $\varepsilon = 0$. Unlike the hyperbolic problem, a full general treatment of the limit parabolic problem with dynamic boundary conditions already appears in the literature (cf., e.g., [24, 25, 29, 39] and references there in); in particular, this section will summarize some of the main results from [25]. It should be noted for the interest of the reader that all formal calculations made with the weak solutions of the parabolic problem can be rigorously justified using the Galerkin discretization scheme that appears, for instance, in [29, Theorem 2.6]. Indeed, it is through the use of the Galerkin approximations that the existence of weak solutions for the parabolic problem is shown. The solution operator associated with the parabolic problem generates a locally Lipschitz continuous semiflow on the appropriate phase space. We also know that this semiflow admits a connected global attractor that is bounded in a more *regular* phase space. It follows that solutions, when restricted to the global attractor, are in fact strong solutions, exhibiting further regularity that will become *essential* when we later consider the continuity properties of the family of global attractors produced by the hyperbolic relaxation problem ($\varepsilon > 0$) and the limit parabolic problem ($\varepsilon = 0$).

We need to introduce some notation and definitions. From now on, we denote by $\|\cdot\|$ and $\|\cdot\|_k$ the norms in $L^2(\Omega)$ and $H^k(\Omega)$, respectively. We use the notation $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_k$ to denote the products on $L^2(\Omega)$ and $H^k(\Omega)$, respectively. For the boundary terms, $\|\cdot\|_{L^2(\Gamma)}$ and $\langle\cdot,\cdot\rangle_{L^2(\Gamma)}$ denote the norm and, respectively, product on $L^2(\Gamma)$. We will require the norm in $H^k(\Gamma)$ to be denoted by $\|\cdot\|_{H^k(\Gamma)}$, where $k \geq 1$. The $L^p(\Omega)$ norm, $p \in (0,\infty]$, is denoted by $|\cdot|_p$. The dual pairing between $H^1(\Omega)$ and its dual $(H^1(\Omega))^*$ is denoted by (u,v). We denote the measure of the domain Ω by $|\Omega|$. In many calculations, functional notation indicating dependence on the variable t is dropped; for example, we will write u in place of $u(t)$. Throughout the paper, $C \geq 0$ will denote a *generic* constant, while $Q : \mathbb{R}_+ \to \mathbb{R}_+$ will denote a *generic* increasing function. All these quantities, unless explicitly stated, are *independent* of ε. Further dependencies of these quantities will be specified on occurrence.

The following inequalities are straightforward consequences of the Poincaré-type inequality (1.6) and assumptions (1.5) and (1.7). From (1.5) it follows that, for some constants $\mu \in (0,\lambda]$ and $c_1 = c_1(f,|\Omega|) \geq 0$ and for all $\xi \in H^1(\Omega)$,

$$\begin{aligned} \langle f(\xi),\xi\rangle &\geq -(\lambda-\mu)\|\xi\|^2 - c_1 \\ &\geq -\frac{(\lambda-\mu)}{\lambda}\left(\|\nabla\xi\|^2 + \|\xi\|^2_{L^2(\Gamma)}\right) - c_1. \end{aligned} \tag{2.1}$$

Let $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$. For some constant $c_2 = c_2(f,|\Omega|) \geq 0$ and for all $\xi \in H^1(\Omega)$,

$$\begin{aligned} \int_\Omega F(\xi)\mathrm{d}x &\geq -\frac{\lambda-\mu}{2}\|\xi\|^2 - c_2 \\ &\geq -\frac{\lambda-\mu}{2\lambda}\|\xi\|_1^2 - c_2. \end{aligned} \tag{2.2}$$

See [17, p. 1913] for an explicit proof of (2.2). The proof of (2.1) is similar. Finally, using (1.7) and integration by parts on $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$, we have the upper bound

$$\begin{aligned}\int_\Omega F(\xi)\mathrm{d}x &\leq \langle f(\xi), \xi\rangle + \frac{\vartheta}{2}\|\xi\|^2 \\ &\leq \langle f(\xi), \xi\rangle + \frac{\vartheta}{2\lambda}\|\xi\|_1^2.\end{aligned} \tag{2.3}$$

The natural energy phase space for the limit parabolic problem (1.8)–(1.10) is the space

$$Y = L^2(\Omega) \times L^2(\Gamma),$$

which is Hilbert when equipped with the norm whose square is given by, for all $\zeta = (u, \gamma) \in Y$,

$$\|\zeta\|_Y^2 := \|u\|^2 + \|\gamma\|_{L^2(\Gamma)}^2.$$

It is well known that the Dirichlet trace map $\mathrm{tr_D} : C^\infty\left(\overline{\Omega}\right) \to C^\infty(\Gamma)$, defined by $\mathrm{tr_D}(u) = u_{|\Gamma}$, extends to a linear continuous operator $\mathrm{tr_D} : H^r(\Omega) \to H^{r-1/2}(\Gamma)$, for all $r > 1/2$, which is onto for $1/2 < r < 3/2$. This map also possesses a bounded right inverse $\mathrm{tr_D^{-1}} : H^{r-1/2}(\Gamma) \to H^r(\Omega)$ such that $\mathrm{tr_D}\left(\mathrm{tr_D^{-1}}\psi\right) = \psi$, for any $\psi \in H^{r-1/2}(\Gamma)$. Identifying each function $\psi \in C\left(\overline{\Omega}\right)$ with the vector $V = (\psi, \mathrm{tr_D}(\psi)) \in C\left(\overline{\Omega}\right) \times C(\Gamma)$, it follows that $C\left(\overline{\Omega}\right)$ is a dense subspace of $Y = L^2(\Omega) \times L^2(\Gamma)$ (see, e.g., [43, Lemma 2.1]). Also, we introduce the subspaces of $H^r(\Omega) \times H^{r-1/2}(\Gamma)$, for every $r > 1/2$,

$$\mathcal{V}^r := \left\{(u, \gamma) \in H^r(\Omega) \times H^{r-1/2}(\Gamma) : \gamma = \mathrm{tr_D}(u)\right\},$$

and we note that we have the following dense and compact embeddings $\mathcal{V}^{r_1} \hookrightarrow \mathcal{V}^{r_2}$, for any $r_1 > r_2 > 1/2$. The linear subspace $\mathcal{V}^r$ is densely and compactly embedded into Y, for any $r > 1/2$. We emphasize that $\mathcal{V}^r$ is not a product space and that, due to the boundedness of the trace operator $\mathrm{tr_D}$, the space $\mathcal{V}^r$ is topologically isomorphic to $H^r(\Omega)$ in the obvious way. Thus, we can identify each $u \in H^r(\Omega)$ with a pair $(u, \mathrm{tr_D}(u)) \in \mathcal{V}^r$. Finally, note that both spaces $H^r(\Omega)$ and $\mathcal{V}^r$ are normed spaces with equivalent norms.

The following definition of weak solution to problem (1.8)–(1.10) is taken from [29] (see, e.g., [24, Definition 2.1] for the more general case).

DEFINITION 2.1. Let $T > 0$ and $(u_0, \gamma_0) \in Y = L^2(\Omega) \times L^2(\Gamma)$. The pair $\zeta(t) = (u(t), \gamma(t))$ is said to be a (global) *weak solution* of (1.8)–(1.10) on $[0, T]$ if, for almost all $t \in (0, T]$, $\gamma(t) = u_{|\Gamma}(t)$ and ζ fulfills

$$\begin{aligned}\zeta &\in C([0,T]; Y) \cap L^2\left(0, T; \mathcal{V}^1\right), \\ \partial_t\zeta &\in L^2(0, T; \left(\mathcal{V}^1\right)^*),\ u \in H^1_{\mathrm{loc}}\left((0, T]; L^2(\Omega)\right), \\ \gamma &\in H^1_{\mathrm{loc}}\left((0, T]; L^2(\Gamma)\right)\end{aligned}$$

such that the following identity holds: for almost all $t \in [0, T]$ and for all $\xi = (\chi, \psi) \in \mathcal{V}^1$,

$$(\partial_t\zeta, \xi)_{(\mathcal{V}^1)^*, \mathcal{V}^1} + \langle\nabla u, \nabla\chi\rangle + \langle f(u), \chi\rangle + \langle u, \psi\rangle_{L^2(\Gamma)} = 0. \tag{2.4}$$

Moreover,

$$\zeta(0) = (u_0, \gamma_0) =: \zeta_0 \text{ a.e. in } Y.$$

The map $\zeta = (u, \gamma)$ is a weak solution on $[0, \infty)$ (i.e., *a global weak solution*) if it is a weak solution on $[0, T]$, for all $T > 0$.

REMARK 2.2. It is important to observe, for the weak solutions of Definition 2.1, that $\gamma_0 = u_{|\Gamma}(0)$ need not be the trace of $u_0 = u_{|\Omega}(0)$ at the boundary, and so in this context the boundary equation (1.9) is interpreted as an additional parabolic equation, now acting on the boundary Γ. However, the weak solution does fulfill $\gamma(t) = \mathrm{tr_D} u(t)$, for almost all $t > 0$.

The existence part of the (global) weak solutions is from [29, Theorem 2.6], and the continuous dependence with respect to the initial data ζ_0, local Lipschitz continuity on Y, uniformly in t on compact intervals, and the uniqueness of the weak solutions follow from [24, Proposition 2.8] (cf. also [29, Lemma 2.7]).

THEOREM 2.3. Assume (1.4), (1.5), and (1.7) hold. For each $\zeta_0 = (u_0, \gamma_0) \in Y$, there exists a unique global weak solution in the sense of Definition 2.1. Moreover, the following estimate holds: for all $t \geq 0$,

$$\|\zeta(t)\|_Y^2 + \int_t^{t+1} \|\zeta(s)\|_{\mathcal{V}^1}^2 \mathrm{d}s \leq C\|\zeta_0\|_Y^2 e^{-\rho t} + C, \tag{2.5}$$

for some positive constants $\rho, C > 0$. Furthermore, let $\zeta(t) = (u(t), \gamma(t))$ and $\theta(t) = (\chi(t), \psi(t))$ denote the corresponding weak solutions with initial data $\zeta_0 = (u_0, \gamma_0)$ and $\theta_0 = (\chi_0, \psi_0)$, respectively. Then, for all $t \geq 0$,

$$\|\zeta(t) - \theta(t)\|_Y \leq Ce^{\nu t}\|\zeta_0 - \theta_0\|_Y, \tag{2.6}$$

where $C = C(R) > 0$ is such that $\|\zeta_0\|_Y \leq R, \|\theta_0\|_Y \leq R$.

Proof. Since the proofs in [29], [24], [25] involve quite different assumptions on the nonlinearity other than the ones in the statement of the theorem, we will sketch a short proof of (2.5). This is the main estimate on which the proof for the existence of a weak solution is based (of course, (2.5) can be rigorously justified using a suitable Galerkin discretization scheme). To this end, testing (2.4) with ζ and appealing to (2.1), we deduce the following inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\zeta(t)\|_Y^2 + \left(1 - \frac{\lambda - \mu}{\lambda}\right)\left(\|\nabla u(t)\|^2 + \|u(t)\|_{L^2(\Gamma)}^2\right) \leq C, \tag{2.7}$$

for all $t \geq 0$, where we recall that $\mu \in (0, \lambda]$. Exploiting now the continuous embedding $\mathcal{V}^1 \hookrightarrow Y$, (2.5) follows from the application of Gronwall's inequality (see Proposition 5.1 in the appendix) to (2.7). The claim is proven. $\square$

REMARK 2.4. Theorem 2.3 still holds if we keep (1.7) and we drop the assumptions (1.4) and (1.5) and replace them by the following:

$$\eta_1 |y|^p - C_f \leq f(y)\, y \leq \eta_2 |y|^p + C_f, \tag{2.8}$$

for some $\eta_1, \eta_2 > 0$, $C_f \geq 0$, and any $p > 2$. In this case, the same weak formulation (2.4) must be satisfied a.e. on $[0, T]$, for all $\xi = (\chi, \psi) \in \mathcal{V}^1$, with $\chi \in L^p(\Omega)$ (see, e.g., [24,25]). Finally, we note that without assumption (1.7), the uniqueness of weak solutions (given in Definition 2.1) is not known in general (see [24]).

COROLLARY 2.5. Let the assumptions of Theorem 2.3 be satisfied. We can define a strongly continuous semigroup

$$S_0(t) : Y \to Y$$

by setting, for all $t \geq 0$,

$$S_0(t)\zeta_0 := \zeta(t)$$

where $\zeta(t) = (u(t), u_{|\Gamma}(t))$ is the unique weak solution to problem (1.8)–(1.10).

The existence of a bounded absorbing set in $\mathcal{V}^1$ was shown for the first time in [29, Theorem 2.8] and the existence of the global attractor for (1.8)–(1.10) can be found in [24, 25]. The following theorem concerns the existence and regularity of the global attractor $\mathcal{A}_0$ admitted by the semiflow S_0 and is taken from [25, Theorem 2.3]. The proof relies on a uniform estimate which states that problem (1.8)–(1.10) possesses the $Y - \mathcal{V}^2$ smoothing property and exploits (2.5).

THEOREM 2.6. The semiflow S_0 possesses a connected global attractor $\mathcal{A}_0$ in Y, which is a bounded subset of $\mathcal{V}^2$. The global attractor $\mathcal{A}_0$ contains only strong solutions. Finally, S_0 also admits an exponential attractor $\mathcal{M}_0$ which is bounded in $\mathcal{V}^2$ and compact in Y.

REMARK 2.7. The boundedness of $\mathcal{A}_0$ in $\mathcal{V}^2$, shown in [25, Theorem 2.3], is essential for the proof of the continuity property at $\varepsilon = 0$ of the global attractors associated with problem (1.1)–(1.3). The last assertion follows from results in [28, Theorem 4.2], where (1.8)–(1.10) is a special case of a phase-field system endowed with dynamic boundary conditions.

3. The hyperbolic relaxation problem. In this section, we study the hyperbolic relaxation problem (1.1)-(1.3) with $\varepsilon \in (0, 1]$. Our first goal is to prove the existence of a global attractor for (1.1)-(1.3). As indicated in [51], semigroup methods are applied to obtain local mild solutions whereby a suitable estimate is used to extend the solution to a global one. We will offer a detailed presentation on the well-posedness of the hyperbolic relaxation problem in this section for the reader's convenience. The solution operators define a semiflow on the phase space and because of the continuous dependence estimate on the solutions, the semiflow is locally Lipschitz continuous, uniformly in t on compact intervals. Further estimates are used to establish the existence of an absorbing set for the semiflow. As discussed above, we will follow the decomposition method in [44] to obtain the existence of the global attractor in $H^1(\Omega) \times L^2(\Omega)$ for the corresponding semiflow S_ε, for each $\varepsilon \in (0, 1]$. The (optimal) regularity result for the global attractors $\mathcal{A}_\varepsilon$ and a proof of their continuity properties conclude the section.

3.1. *The functional framework.* Here we consider the functional setup associated with problem (1.1)–(1.3). The finite energy phase space for the hyperbolic relaxation problem is the space

$$\mathcal{H}_\varepsilon = H^1(\Omega) \times L^2(\Omega).$$

The space $\mathcal{H}_\varepsilon$ is Hilbert when endowed with the ε-weighted norm whose square is given by, for $\varphi = (u, v) \in \mathcal{H}_\varepsilon = H^1(\Omega) \times L^2(\Omega)$,

$$\|\varphi\|^2_{\mathcal{H}_\varepsilon} := \|u\|^2_1 + \varepsilon\|v\|^2 = \left(\|\nabla u\|^2 + \|u\|^2_{L^2(\Gamma)}\right) + \varepsilon\|v\|^2.$$

As introduced in [51] (cf. also [16]), $\Delta_{\mathrm{R}} : L^2(\Omega) \to L^2(\Omega)$ is the Robin-Laplacian operator with domain

$$D(\Delta_{\mathrm{R}}) = \{u \in H^2(\Omega) : \partial_{\mathbf{n}} u + u = 0 \text{ on } \Gamma\}.$$

Easy calculations show that the operator Δ_{R} is selfadjoint and positive. The Robin-Laplacian is extended to a continuous operator $\Delta_{\mathrm{R}} : H^1(\Omega) \to \left(H^1(\Omega)\right)^*$, defined by, for all $v \in H^1(\Omega)$,

$$(-\Delta_{\mathrm{R}} u, v) = \langle \nabla u, \nabla v \rangle + \langle u, v \rangle_{L^2(\Gamma)}.$$

Next, [16, 51] also define the Robin map $R : H^s(\Gamma) \to H^{s+(3/2)}(\Omega)$ by

$$Rp = q \text{ if and only if } \Delta q = 0 \text{ in } \Omega \text{ and } \partial_{\mathbf{n}} q + q = p \text{ on } \Gamma.$$

The adjoint of the Robin map satisfies, for all $v \in H^1(\Omega)$,

$$R^* \Delta_{\mathrm{R}} v = -v \text{ on } \Gamma.$$

Define the closed subspace of $H^2(\Omega) \times H^1(\Omega)$,

$$\mathcal{D}_\varepsilon := \{(u, v) \in H^2(\Omega) \times H^1(\Omega) : \partial_{\mathbf{n}} u + u = -v \text{ on } \Gamma\},$$

endowed with norm whose square is given by, for all $\varphi = (u, v) \in \mathcal{D}_\varepsilon$,

$$\|\varphi\|^2_{\mathcal{D}_\varepsilon} := \|u\|_2^2 + \|v\|_1^2.$$

Let $D(A_\varepsilon) = \mathcal{D}_\varepsilon$ (note that ε-dependance does not enter through the norm of $\mathcal{D}_\varepsilon$, but rather in the definition of A_ε below). Define the linear unbounded operator $A_\varepsilon : D(A_\varepsilon) \to \mathcal{H}_\varepsilon$ by

$$A_\varepsilon := \begin{pmatrix} 0 & 1 \\ \frac{1}{\varepsilon}\Delta_{\mathrm{R}} & \frac{1}{\varepsilon}(\Delta_{\mathrm{R}} R \operatorname{tr}_{\mathrm{D}} - 1) \end{pmatrix},$$

where $\operatorname{tr}_{\mathrm{D}}$ denotes the Dirichlet trace operator (i.e., $\operatorname{tr}_{\mathrm{D}}(v) = v|_\Gamma$). Notice that if $(u, v) \in \mathcal{D}_\varepsilon$, then $u + R\operatorname{tr}_{\mathrm{D}}(v) \in D(\Delta_{\mathrm{R}})$. By the Lumer-Phillips theorem (cf., e.g., [45, Theorem I.4.3]) and the Lax-Milgram theorem, it is not hard to see that, for all $\varepsilon \in (0, 1]$, the operator A_ε, with domain $\mathcal{D}_\varepsilon$, is an infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{H}_\varepsilon$, denoted by $e^{A_\varepsilon t}$.

Define the map $\mathcal{F} : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ by

$$\mathcal{F}(\varphi) := \begin{pmatrix} 0 \\ -\frac{1}{\varepsilon} f(u) \end{pmatrix}$$

for all $\varphi = (u, v) \in \mathcal{H}_\varepsilon$. Since $f : H^1(\Omega) \to L^2(\Omega)$ is locally Lipschitz continuous [52, cf., e.g., Theorem 2.7.13], it follows that the map $\mathcal{F} : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ is as well.

The hyperbolic relaxation problem (1.1)–(1.3) may be put into the abstract form in $\mathcal{H}_\varepsilon$, for $\varphi(t) = (u(t), u_t(t))^{\mathrm{tr}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = A_\varepsilon \varphi(t) + \mathcal{F}(\varphi(t)); \ \varphi(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}. \tag{3.1}$$

LEMMA 3.1. For each $\varepsilon \in (0, 1]$, the adjoint of A_ε, denoted by A_ε^*, is given by

$$A_\varepsilon^* := -\begin{pmatrix} 0 & 1 \\ \frac{1}{\varepsilon}\Delta_{\mathrm{R}} & -\frac{1}{\varepsilon}(\Delta_{\mathrm{R}} R \operatorname{tr}_{\mathrm{D}} - 1) \end{pmatrix},$$

with domain

$$D(A_\varepsilon^*) := \{(\chi, \psi) \in H^2(\Omega) \times H^1(\Omega) : \partial_{\mathbf{n}}\chi + \chi = -\psi \text{ on } \Gamma\}.$$

Proof. The proof is a calculation similar to, e.g., [5, Lemma 3.1]. $\square$

3.2. *Well-posedness for the hyperbolic relaxation problem.* The notion of weak solution to problem (1.1)–(1.3) is as follows (see [4]).

DEFINITION 3.2. A function $\varphi = (u, u_t) : [0, T] \to \mathcal{H}_\varepsilon$ is a weak solution of (3.1) on $[0, T]$ if and only if $\mathcal{F}(\varphi(\cdot)) \in L^1(0, T; \mathcal{H}_\varepsilon)$ and φ satisfies the variation of constants formula, for all $t \in [0, T]$,

$$\varphi(t) = e^{A_\varepsilon t}\varphi_0 + \int_0^t e^{A_\varepsilon(t-s)}\mathcal{F}(\varphi(s))\mathrm{d}s.$$

It can be easily shown that the notion of weak solution given in Definition 3.2 is also equivalent to the following notion of a weak solution (see, e.g., [5, Definition 3.1 and Proposition 3.5]).

DEFINITION 3.3. Let $T > 0$ and $(u_0, u_1) \in \mathcal{H}_\varepsilon$. A map $\varphi = (u, u_t) \in C([0, T]; \mathcal{H}_\varepsilon)$ is a *weak solution* of (3.1) on $[0, T]$ if for each $\theta = (\chi, \psi) \in D(A_\varepsilon^*)$ the map $t \mapsto \langle \varphi(t), \theta \rangle_{\mathcal{H}_\varepsilon}$ is absolutely continuous on $[0, T]$ and satisfies, for almost all $t \in [0, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \varphi(t), \theta \rangle_{\mathcal{H}_\varepsilon} = \langle \varphi(t), A_\varepsilon^*\theta \rangle_{\mathcal{H}_\varepsilon} + \langle \mathcal{F}(\varphi(t)), \theta \rangle_{\mathcal{H}_\varepsilon}. \tag{3.2}$$

The map $\varphi = (u, u_t)$ is a weak solution on $[0, \infty)$ (i.e., a *global weak solution*) if it is a weak solution on $[0, T]$, for all $T > 0$.

The above definitions are equivalent to the standard concept of a weak (distributional) solution to (1.1)–(1.3).

DEFINITION 3.4. Let $\varepsilon \in (0, 1]$. A function $\varphi = (u, u_t) : [0, T] \to \mathcal{H}_\varepsilon$ is a weak solution of (3.1) (and thus of (1.1)–(1.3)) on $[0, T]$ if

$$\varphi = (u, u_t) \in C([0, T]; \mathcal{H}_\varepsilon),\ u_t \in L^2([0, T] \times \Gamma)$$

and, for each $\psi \in H^1(\Omega)$, $(u_t, \psi) \in C^1([0, T])$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon u_t(t), \psi) + \langle \nabla u(t), \nabla \psi \rangle + \langle u_t(t), \psi \rangle + \langle u_t(t) + u(t), \psi \rangle_{L^2(\Gamma)} = -\langle f(u(t)), \psi \rangle, \tag{3.3}$$

for almost all $t \in [0, T]$.

Indeed, by [5, Lemma 3.3] we have that $f : H^1(\Omega) \to L^2(\Omega)$ is sequentially weakly continuous and continuous, on account of the assumptions (1.4) and (1.5). Moreover, $(\varphi_t, \theta) \in C^1([0, T])$ for all $\theta \in D(A_\varepsilon^*)$, and (3.2) is satisfied. The assertion in Definition 3.4 follows then from the explicit characterization of $D(A_\varepsilon^*)$ and from [5, Proposition 3.4].

Finally, the notion of strong solution to problem (1.1)–(1.3) is as follows.

DEFINITION 3.5. Let $\varphi_0 = (u_0, u_1) \in \mathcal{D}_\varepsilon$, $\varepsilon > 0$, i.e., $(u_0, u_1) \in H^2(\Omega) \times H^1(\Omega)$ such that it satisfies the compatibility condition

$$\partial_{\mathbf{n}}u_0 + u_0 + u_1 = 0 \text{ on } \Gamma.$$

A *function* $\varphi(t) = (u(t), u_t(t))$ *is called a (global) strong solution if it is a weak solution in the sense of Definition* 3.4 *and if it satisfies the following regularity properties:*

$$\begin{aligned} &\varphi \in L^\infty(0,\infty;\mathcal{D}_\varepsilon),\ \varphi_t \in L^\infty(0,\infty;\mathcal{H}_\varepsilon),\\ &u_{tt} \in L^\infty(0,\infty;L^2(\Omega)),\ \ u_{tt} \in L^2(0,\infty;L^2(\Gamma)). \end{aligned} \tag{3.4}$$

Therefore, $\varphi(t) = (u(t), u_t(t))$ satisfies the equations (1.1)–(1.3) almost everywhere; i.e., it is a strong solution.

We can now state the main theorems of this section.

THEOREM 3.6. Assume (1.4) and (1.5) hold. For each $\varepsilon \in (0,1]$ and $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$, there exists a unique global weak solution $\varphi = (u, u_t) \in C([0,\infty);\mathcal{H}_\varepsilon)$ to (1.1)–(1.3). In addition,

$$\partial_{\mathbf{n}} u \in L^2_{\mathrm{loc}}([0,\infty)\times\Gamma) \text{ and } u_t \in L^2_{\mathrm{loc}}([0,\infty)\times\Gamma). \tag{3.5}$$

For each weak solution, the map

$$t \mapsto \|\varphi(t)\|^2_{\mathcal{H}_\varepsilon} + 2\int_\Omega F(u(t))\mathrm{d}x \tag{3.6}$$

is $C^1([0,\infty))$ and the energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\varphi(t)\|^2_{\mathcal{H}_\varepsilon} + 2\int_\Omega F(u(t))\mathrm{d}x\right\} = -2\|u_t(t)\|^2 - 2\|u_t(t)\|^2_{L^2(\Gamma)} \tag{3.7}$$

holds (in the sense of distributions) a.e. on $[0,\infty)$. Furthermore, let $\varphi(t) = (u(t), u_t(t))$ and $\theta(t) = (v(t), v_t(t))$ denote the corresponding weak solution with initial data $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$ and $\theta_0 = (v_0, v_1) \in \mathcal{H}_\varepsilon$, respectively, such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \le R$, $\|\theta_0\|_{\mathcal{H}_\varepsilon} \le R$. Then there exists a constant $\nu_1 = \nu_1(R) > 0$ such that, for all $t \ge 0$,

$$\begin{aligned} &\|\varphi(t)-\theta(t)\|^2_{\mathcal{H}_\varepsilon} + \int_0^t \left(\|u_t(\tau)-v_t(\tau)\|^2 + \|u_t(\tau)-v_t(\tau)\|^2_{L^2(\Gamma)}\right)\mathrm{d}\tau \\ &\le e^{\nu_1 t}\|\varphi_0-\theta_0\|^2_{\mathcal{H}_\varepsilon}. \end{aligned} \tag{3.8}$$

THEOREM 3.7. For each $\varepsilon \in (0,1]$ and $(u_0, u_1) \in \mathcal{D}_\varepsilon$, problem (1.1)–(1.3) possesses a unique global strong solution in the sense of Definition 3.5.

REMARK 3.8. The proof of Theorem 3.7 is outlined in [51] (cf. also [16]) when $\varepsilon = 1$.

Proof of Theorem 3.6. We only give a sketch of the proof.

STEP 1. As discussed in the previous section, for each $\varepsilon \in (0,1]$, the operator A_ε with domain $D(A_\varepsilon) = \mathcal{D}_\varepsilon$ is an infinitesimal generator of a strongly continuous semigroup of contractions on $\mathcal{H}_\varepsilon$, and the map $\mathcal{F} : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ is locally Lipschitz continuous. Therefore, by [52, Theorem 2.5.4], for any $\varepsilon \in (0,1]$ and for any $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$, there is a $T^* = T^*(\|\varphi_0\|_{\mathcal{H}_\varepsilon}) > 0$ such that the abstract problem (3.1) admits a unique local weak solution on $[0, T^*)$ satisfying

$$\varphi \in C([0,T^*);\mathcal{H}_\varepsilon).$$

The next step is to show that $T^*(\|\varphi_0\|_{\mathcal{H}_\varepsilon}) = \infty$. Since the map (3.6) is absolutely continuous on $[0, T^*)$ (cf., e.g., [5, Theorem 3.1]), then integration of the energy equation

(3.7) over $(0,t)$ yields, for all $t\in[0,T^*)$,

$$\begin{aligned}\|\varphi(t)\|^2_{\mathcal{H}_\varepsilon}+2\int_\Omega F(u(t))\mathrm{d}x+2\int_0^t\|u_t(\tau)\|^2\mathrm{d}\tau+2\int_0^t\|u_t(\tau)\|^2_{L^2(\Gamma)}\mathrm{d}\tau\\ =\|\varphi_0\|^2_{\mathcal{H}_\varepsilon}+2\int_\Omega F(u_0)\mathrm{d}x.\end{aligned} \tag{3.9}$$

Applying inequality (2.2) to (3.9) and applying (2.3) to the integral on the right-hand side, we find that there is a function $Q(\|\varphi_0\|_{\mathcal{H}_\varepsilon})>0$ such that, for all $t\in[0,T^*)$,

$$\|\varphi(t)\|_{\mathcal{H}_\varepsilon}\le Q(\|\varphi_0\|_{\mathcal{H}_\varepsilon}). \tag{3.10}$$

Since the bound on the right-hand side of (3.10) is independent of $t\in[0,T^*)$, $T^*(\|\varphi_0\|_{\mathcal{H}_\varepsilon})$ can be extended indefinitely, and therefore, for each $\varepsilon\in(0,1]$, we have that $T^*(\|\varphi_0\|_{\mathcal{H}_\varepsilon})=\infty$.

We now show the boundary property (3.5). Applying (2.2), (2.3), and (3.10) to identity (3.9), we obtain a bound of the form, for all $\varphi_0\in\mathcal{H}_\varepsilon$ and $t\ge0$, in which

$$\int_0^t\|u_t(\tau)\|^2_{L^2(\Gamma)}\mathrm{d}\tau\le Q(\|\varphi_0\|_{\mathcal{H}_\varepsilon}).$$

It follows that $u_t\in L^2_{\mathrm{loc}}([0,\infty)\times\Gamma)$. By the trace theorem, $u\in L^\infty(0,\infty;H^1(\Omega))\hookrightarrow L^\infty(0,\infty;L^2(\Gamma))$, so $u\in L^2_{\mathrm{loc}}([0,\infty)\times\Gamma)$. Comparison with (1.2) yields that $\partial_{\mathbf{n}}u\in L^2_{\mathrm{loc}}([0,\infty)\times\Gamma)$.

STEP 2. To show that the continuous dependence estimate (3.8) holds, consider the difference $z(t):=u(t)-v(t)$, $t\ge0$. We easily get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(z,z_t)\|^2_{\mathcal{H}_\varepsilon}+2\|z_t\|^2+2\|z_t\|^2_{L^2(\Gamma)}=2\langle f(v)-f(u),z_t\rangle. \tag{3.11}$$

Since $f:H^1(\Omega)\to L^2(\Omega)$ is locally Lipschitz continuous,

$$2|\langle f(v)-f(u),z_t\rangle|\le Q(R)\|z\|_1^2+\|z_t\|^2, \tag{3.12}$$

where $R>0$ is such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon}\le R$, $\|\theta_0\|_{\mathcal{H}_\varepsilon}\le R$. Combining (3.11) and (3.12) produces, for almost all $t\ge0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(z,z_t)\|^2_{\mathcal{H}_\varepsilon}\le Q(R)\|(z,z_t)\|^2_{\mathcal{H}_\varepsilon}. \tag{3.13}$$

Hence (3.8) follows immediately from (3.13), using the standard Gronwall lemma. This completes the proof of the theorem. □

In view of Theorem 3.6, the following is immediate.

COROLLARY 3.9. Let the assumptions of Theorem 3.6 be satisfied. Then, for each $\varepsilon\in(0,1]$ we can define a strongly continuous semigroup

$$S_\varepsilon(t):\mathcal{H}_\varepsilon\to\mathcal{H}_\varepsilon,$$

by setting, for all $t\ge0$,

$$S_\varepsilon(t)\varphi_0=\varphi(t)=(u(t),u_t(t)),$$

where $\varphi(t)$ is the unique weak solution to problem (1.1)–(1.3).

3.3. *The global attractor $\mathcal{A}_\varepsilon$ in $\mathcal{H}_\varepsilon$.* In this section, we aim to show the existence of a global attractor and prove some additional regularity properties. We point out that all the computations we will perform below can be rigorously justified by means of an approximation procedure which relies upon the result in Theorem 3.7. Indeed, we shall use the usual procedure of approximating weak solutions by strong solutions and then pass to the limit by using density theorems in the final estimates (see, also, [16]). Thus, in what follows we can proceed formally.

We begin our analysis with a uniform estimate for the weak solutions of Theorem 3.6. The estimate provides the existence of a bounded absorbing set $\mathcal{B}_\varepsilon \subset \mathcal{H}_\varepsilon$, for the semiflow S_ε, for each $\varepsilon \in (0,1]$.

LEMMA 3.10. For all $\varepsilon \in (0,1]$ and $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$, there exist a positive function Q, constants $\omega_0 > 0$, $P_0 > 0$, all independent of ε, such that $\varphi(t)$ satisfies, for all $t \geq 0$,

$$\|\varphi(t)\|^2_{\mathcal{H}_\varepsilon} \leq Q(\|\varphi_0\|_{\mathcal{H}_\varepsilon}) e^{-\omega_0 t} + P_0. \tag{3.14}$$

Consequently, the ball $\mathcal{B}_\varepsilon$ in $\mathcal{H}_\varepsilon$,

$$\mathcal{B}_\varepsilon := \{\varphi \in \mathcal{H}_\varepsilon : \|\varphi\|_{\mathcal{H}_\varepsilon} \leq P_0 + 1\}, \tag{3.15}$$

is a bounded absorbing set in $\mathcal{H}_\varepsilon$ for the dynamical system $(S_\varepsilon(t), \mathcal{H}_\varepsilon)$.

Proof. Let $\varepsilon \in (0,1]$ and $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$. For $\alpha > 0$ yet to be chosen, multiply (1.1) by αu in $L^2(\Omega)$. Adding the result to the energy equation (3.7) above yields the differential identity, which holds for almost all $t \geq 0$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\varphi\|^2_{\mathcal{H}_\varepsilon} + \alpha\varepsilon\langle u_t, u\rangle + 2\int_\Omega F(u)\,\mathrm{d}x\right\} \\ &\quad + (2-\varepsilon\alpha)\|u_t\|^2 + \alpha\langle u_t, u\rangle + \alpha\|u\|_1^2 \\ &\quad + 2\|u_t\|^2_{L^2(\Gamma)} + \alpha\langle u_t, u\rangle_{L^2(\Gamma)} + \alpha\langle f(u), u\rangle = 0. \end{aligned} \tag{3.16}$$

For each $\varepsilon \in (0,1]$, define the functional $E_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}$ by

$$E_\varepsilon(\varphi(t)) = \|\varphi(t)\|^2_{\mathcal{H}_\varepsilon} + \alpha\varepsilon\langle u_t(t), u(t)\rangle + 2\int_\Omega F(u(t))\mathrm{d}x. \tag{3.17}$$

It is not hard to see that the map $t \mapsto E_\varepsilon(\varphi(t))$ is $C^1([0,\infty))$; this essentially follows from equation (3.6) of Theorem 3.6. First, we estimate, for all $\eta > 0$,

$$\alpha|\langle u_t, u\rangle_{L^2(\Gamma)}| \leq \alpha\eta\|u_t\|^2_{L^2(\Gamma)} + \frac{\alpha}{4\eta}\|u\|^2_{L^2(\Gamma)}, \tag{3.18}$$

and with (2.1), we have,

$$\alpha|\langle f(u), u\rangle| \geq -\frac{\alpha(\lambda-\mu)}{\lambda}\|u\|_1^2 - \alpha C. \tag{3.19}$$

Combining (3.16) with (3.18) and (3.19) gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon + (2-\alpha)\varepsilon\|u_t\|^2 + \alpha\langle u_t, u\rangle \\ &+ \alpha\left(1 - \frac{1}{4\eta} - \frac{\lambda-\mu}{\lambda}\right)\|u\|_1^2 + (2-\alpha\eta)\|u_t\|^2_{L^2(\Gamma)} \\ &\leq \alpha C. \end{aligned} \tag{3.20}$$

Hence, for any $\eta > \frac{\lambda}{4\mu}$ and any $0 < \alpha < \min\{2, \frac{2}{\eta}\}$, we have $2 - \eta > 0$ and $2 - \alpha\eta > 0$,

$$\omega_0 := \min\left\{2 - \alpha, \alpha\left(\frac{\mu}{\lambda} - \frac{1}{4\eta}\right)\right\} > 0,$$

and estimate (3.20) becomes, for almost all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon + \omega_0 E_\varepsilon + (2 - \alpha\eta)\|u_t\|^2_{L^2(\Gamma)} \leq C_\alpha. \tag{3.21}$$

Applying Gronwall's inequality (see, e.g., [44, Lemma 5]; cf. also Proposition 5.1 in the appendix) to (3.21) produces, for all $t \geq 0$,

$$E_\varepsilon\left(\varphi\left(t\right)\right) \leq E_\varepsilon\left(\varphi\left(0\right)\right) e^{-\omega_0 t} + C. \tag{3.22}$$

We now apply (2.2) to (3.17) to attain the bound

$$E_\varepsilon\left(\varphi\right) \geq \varepsilon\left(1 - \frac{\alpha}{2}\right)\|u_t\|^2 + \left(1 - \frac{\alpha}{2\lambda} - \frac{\lambda - \mu}{\lambda}\right)\|u\|^2_1 - C. \tag{3.23}$$

After updating the smallness condition on α to $0 < \alpha < \min\{2, \frac{2}{\eta}, 2\mu\}$, we see that for

$$\omega_1 := \min\left\{1 - \frac{\alpha}{2}, 1 - \frac{\alpha}{2\lambda} - \frac{\lambda - \mu}{\lambda}\right\} > 0,$$

we have, for all $t \geq 0$,

$$E_\varepsilon(\varphi(t)) \geq \omega_1\|\varphi(t)\|^2_{\mathcal{H}_\varepsilon} - C. \tag{3.24}$$

On the other hand, by estimating in a similar fashion, using (2.2), it follows for all $t \geq 0$ that

$$E_\varepsilon(\varphi(t)) \leq Q\left(\|\varphi(t)\|_{\mathcal{H}_\varepsilon}\right). \tag{3.25}$$

Thus, estimate (3.14) follows now from (3.24), (3.25), and (3.22). The assertion (3.15) is an immediate consequence of (3.14). This concludes the proof. ☐

REMARK 3.11. The following bounds are an immediate consequence of estimate (3.14):

$$\limsup_{t\to\infty}\|\varphi(t)\|^2_{\mathcal{H}_\varepsilon} \leq P_0 \tag{3.26}$$

and

$$\int_0^\infty \left(\|u_t\left(\tau\right)\|^2 + \|u_t\left(\tau\right)\|^2_{L^2(\Gamma)}\right)\mathrm{d}\tau \leq Q\left(\|\varphi_0\|_{\mathcal{H}_\varepsilon}\right). \tag{3.27}$$

The last bound is found by integrating the energy equation (3.7) with respect to t over $(0, \infty)$ and estimating the result with (1.4), (1.5), (2.2), (2.3), and (3.26).

REMARK 3.12. Note that the last assumption (1.7) (which is $f^{'}\left(s\right) \geq -\vartheta$, for all $s \in \mathbb{R}$) is nowhere needed in the proofs of Theorem 3.6, Theorem 3.7 (cf. [51, Theorem 1.1 and Lemma 2.2]), and Lemma 3.10. It will only become important later (see (3.28)) when we establish the optimal regularity of the global attractor for the hyperbolic problem (1.1)–(1.3).

The semiflow S_ε admits a bounded absorbing set $\mathcal{B}_\varepsilon$ in $\mathcal{H}_\varepsilon$. To obtain a global attractor, it suffices to prove that the semiflow admits a decomposition into the sum of two operators, $S_\varepsilon = Z_\varepsilon + K_\varepsilon$, where $Z_\varepsilon = (Z_\varepsilon(t))_{t\geq 0}$ and $K_\varepsilon = (K_\varepsilon(t))_{t\geq 0}$ are not necessarily semiflows, but operators that are uniformly decaying to zero and uniformly compact for large t, respectively. To obtain the compactness property for the operator K_ε, recall that, when fractional powers of the Laplacian are well-defined, one usually multiplies the

PDE by the solution and a suitable fractional power of the Laplacian (i.e., $(-\Delta)^s u$ for some $s > 0$) and then estimates using a stronger norm while keeping in mind the uniform bound on u and the null initial conditions. However, in our case, the dynamic boundary condition does not allow us to proceed with the usual argument to obtain the relative compactness of K_ε. This is because the Laplacian equipped with the dynamic boundary condition (1.2) is not selfadjoint or positive. In turn, we cannot apply the standard spectral theory to define fractional powers of the Laplacian. So to obtain the relative compactness of K_ε, we follow the approach in [44]. The main tool is to differentiate the equations with respect to time and obtain uniform estimates for the new equations. Such strategies also proved useful when dealing with a damped wave equation with acoustic boundary conditions [23] or a wave equation with a nonlinear dynamic boundary condition [16], and hyperbolic relaxation of a Cahn-Hilliard equation with dynamic boundary conditions [14], [27].

Following an approach similar to the one taken in the above references, first define

$$\psi(s) := f(s) + \beta s \tag{3.28}$$

for some constant $\beta \geq \vartheta$ to be determined later (in this case, $\psi'(s) \geq 0$ thanks to assumption (1.7)). Set $\Psi(s) := \int_0^s \psi(\sigma)\mathrm{d}\sigma$. Let $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$. Then rewrite the hyperbolic relaxation problem into the system of equations in v and w, where $v + w = u$,

$$\begin{cases} \varepsilon v_{tt} + v_t - \Delta v + \psi(u) - \psi(w) = 0 & \text{in } (0,\infty) \times \Omega, \\ \partial_{\mathbf{n}} v + v + v_t = 0 & \text{on } (0,\infty) \times \Gamma, \\ v(0) = u_0,\ v_t(0) = u_1 + f(0) - \beta u_0 & \text{in } \Omega \end{cases} \tag{3.29}$$

and

$$\begin{cases} \varepsilon w_{tt} + w_t - \Delta w + \psi(w) = \beta u & \text{in } (0,\infty) \times \Omega, \\ \partial_{\mathbf{n}} w + w + w_t = 0 & \text{on } (0,\infty) \times \Gamma, \\ w(0) = 0,\ w_t(0) = -f(0) + \beta u_0 & \text{in } \Omega. \end{cases} \tag{3.30}$$

In view of Lemmas 3.13 and 3.15 below, we define the one-parameter family of maps $K_\varepsilon(t) : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ by

$$K_\varepsilon(t)\varphi_0 := (w(t), w_t(t)),$$

where (w, w_t) is a solution of (3.30). With such w, we may define a second function (v, v_t) as the solution of (3.29). Through the dependence of v on w and $\varphi_0 = (u_0, u_1)$, the solution of (3.29) defines a one-parameter family of maps $Z_\varepsilon(t) : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ defined by

$$Z_\varepsilon(t)\varphi_0 := (v(t), v_t(t)).$$

Notice that if v and w are solutions to (3.29) and (3.30), respectively, then the function $u := v + w$ is a solution to the original hyperbolic relaxation problem (1.1)–(1.3).

The first lemma shows that the operators K_ε are bounded in $\mathcal{H}_\varepsilon$, uniformly with respect to ε. The result essentially follows from the existence of a bounded absorbing set $\mathcal{B}_\varepsilon$ in $\mathcal{H}_\varepsilon$ for S_ε (recall (3.26)).

Lemma 3.13. *Assume (1.4), (1.5), and (1.7) hold. For each $\varepsilon \in (0,1]$ and $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$, there exists a unique global weak solution $(w, w_t) \in C([0,\infty); \mathcal{H}_\varepsilon)$ to problem (3.30) satisfying*

$$\partial_{\mathbf{n}} w \in L^2_{\mathrm{loc}}([0,\infty) \times \Gamma) \text{ and } w_t \in L^2_{\mathrm{loc}}([0,\infty) \times \Gamma). \tag{3.31}$$

Moreover, for all $\varphi_0 \in \mathcal{H}_\varepsilon$ with $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \le R$ for all $\varepsilon \in (0,1]$, it follows for all $t \ge 0$ that

$$\|K_\varepsilon(t)\varphi_0\|_{\mathcal{H}_\varepsilon} \le Q(R). \tag{3.32}$$

The following result will be useful later on.

LEMMA 3.14. For all $\varepsilon \in (0,1]$ and $\eta > 0$, there is a function $Q_\eta(\cdot) \sim \eta^{-1}$ such that for every $0 \le s \le t$ and $\varphi_0 = (u_0, u_1) \in \mathcal{B}_\varepsilon$,

$$\int_s^t \left(\|w_t(\tau)\|^2 + \|u_t(\tau)\|^2\right) \mathrm{d}\tau \le \frac{\eta}{2}(t-s) + Q_\eta(R), \tag{3.33}$$

where $R > 0$ is such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \le R$, for all $\varepsilon \in (0,1]$.

Proof. Let $\varepsilon \in (0,1]$ and $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$, with $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \le R$. Adding the identity

$$-2\beta \frac{\mathrm{d}}{\mathrm{d}t}\langle u, w\rangle = -2\beta\langle u_t, w\rangle - 2\beta\langle u, w_t\rangle$$

to equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|(w,w_t)\|^2_{\mathcal{H}_\varepsilon} + 2\int_\Omega \Psi(w)\mathrm{d}x\right\} + 2\|w_t\|^2 + \|w_t\|^2_{L^2(\Gamma)} = 2\beta\langle u, w_t\rangle \tag{3.34}$$

produces, for almost all $t \ge 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|(w,w_t)\|^2_{\mathcal{H}_\varepsilon} + 2\int_\Omega \Psi(w)\mathrm{d}x - 2\beta\langle u, w\rangle\right\} + 2\|w_t\|^2 + \|w_t\|^2_{L^2(\Gamma)} \\ = -2\beta\langle u_t, w\rangle.\end{aligned} \tag{3.35}$$

Using (3.32), we estimate, for all $\eta > 0$,

$$2\beta|\langle u_t, w\rangle| \le \eta + Q_\eta(R)\|u_t\|^2. \tag{3.36}$$

For each $\varepsilon \in (0,1]$, define the functional $W_\varepsilon : H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$,

$$W_\varepsilon(t) := \|(w(t), w_t(t))\|^2_{\mathcal{H}_\varepsilon} + 2\int_\Omega \Psi(w(t))\mathrm{d}x - 2\beta\langle u(t), w(t)\rangle.$$

Because of (2.3), (1.4), (1.5), (3.28), (3.26), and (3.32), we can easily check that for all $t \ge 0$ and $\varepsilon \in (0,1]$,

$$|W_\varepsilon(t)| \le Q(R). \tag{3.37}$$

We now combine (3.35) and (3.36) together as, for all $\eta > 0$ and for almost all $t \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}W_\varepsilon + 2\|w_t\|^2 + \|w_t\|^2_{L^2(\Gamma)} + 2\|u_t\|^2 \le \eta + (Q_\eta(R) + 2)\|u_t\|^2. \tag{3.38}$$

Integrating (3.38) over $(0,t)$ and recalling (3.37) and (3.27) gives the desired estimate in (3.33). This proves the claim. $\Box$

The next result shows that the operators Z_ε are uniformly decaying to zero in $\mathcal{H}_\varepsilon$.

LEMMA 3.15. Assume (1.4), (1.5), and (1.7) hold. For each $\varepsilon \in (0,1]$ and $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$, there exists a unique global weak solution $(v, v_t) \in C([0,\infty); \mathcal{H}_\varepsilon)$ to problem (3.29) satisfying

$$\partial_{\mathbf{n}} v \in L^2_{\mathrm{loc}}([0,\infty) \times \Gamma) \text{ and } v_t \in L^2_{\mathrm{loc}}([0,\infty) \times \Gamma). \tag{3.39}$$

Moreover, for all $\varphi_0 \in \mathcal{D}_\varepsilon$ with $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \le R$ for all $\varepsilon \in (0,1]$, there exists $\omega > 0$, independent of ε, such that, for all $t \ge 0$,

$$\|Z_\varepsilon(t)\varphi_0\|_{\mathcal{H}_\varepsilon} \le Q(R)e^{-\omega t}. \tag{3.40}$$

Proof. In a similar fashion to the arguments in Section 3.2, the existence of a global weak solution as well as (3.39) can be found. Because of (3.26) and (3.32), we know that the functions $(u(t), u_t(t))$ and $(w(t), w_t(t))$ are uniformly bounded in $\mathcal{H}_\varepsilon$ with respect to t and ε. It remains to show that (3.40) holds.

Let $\varepsilon \in (0,1]$ and $\varphi_0 = (u_0, u_1) \in \mathcal{H}_\varepsilon$, with $R > 0$ such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \le R$. Observe that

$$\begin{aligned} 2\langle \psi(u) - \psi(w), v_t\rangle = \frac{\mathrm{d}}{\mathrm{d}t}&\{2\langle \psi(u) - \psi(w), v\rangle - \langle \psi'(u)v, v\rangle\} \\ &- 2\langle(\psi'(u) - \psi'(w))w_t, v\rangle + \langle \psi''(u)u_t, v^2\rangle. \end{aligned} \tag{3.41}$$

Multiply the first equation of (3.29) by $2v_t + \alpha v$ in $L^2(\Omega)$, for $\alpha > 0$ to be chosen later. We find that, with (3.41),

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon\|v_t\|^2 + \alpha\varepsilon\langle v_t, v\rangle + \|v\|_1^2 + 2\langle \psi(u) - \psi(w), v\rangle - \langle \psi'(u)v, v\rangle\right\} \\ &\quad + (2 - \alpha\varepsilon)\|v_t\|^2 + \alpha\langle v_t, v\rangle + \alpha\|v\|_1^2 + 2\|v_t\|^2_{L^2(\Gamma)} + \alpha\langle v_t, v\rangle_{L^2(\Gamma)} \\ &\quad + \alpha\langle \psi(u) - \psi(w), v\rangle = 2\langle(\psi'(u) - \psi'(w))w_t, v\rangle - \langle \psi''(u)u_t, v^2\rangle. \end{aligned} \tag{3.42}$$

For each $\varepsilon \in (0,1]$, define the functional

$$V_\varepsilon : H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$$

by

$$\begin{aligned} V_\varepsilon(t) :=& \varepsilon\|v_t(t)\|^2 + \alpha\varepsilon\langle v_t(t), v(t)\rangle + \|v(t)\|_1^2 \\ &+ 2\langle \psi(u(t)) - \psi(w(t)), v(t)\rangle - \langle \psi'(u(t))v(t), v(t)\rangle. \end{aligned}$$

As with the functional E_ε above, the map $t \mapsto V_\varepsilon(t)$ is $AC(\mathbb{R}_{\ge 0}; \mathbb{R}_{\ge 0})$. We now will show that, given that $(u, u_t), (w, w_t) \in \mathcal{H}_\varepsilon$ are uniformly bounded with respect to t and ε, there are constants $C_1, C_2 > 0$ independent of t and ε (possibly depending on $R > 0$) in which, for all $(v, v_t) \in \mathcal{H}_\varepsilon$,

$$C_1\|(v, v_t)\|^2_{\mathcal{H}_\varepsilon} \le V_\varepsilon \le C_2\|(v, v_t)\|^2_{\mathcal{H}_\varepsilon}. \tag{3.43}$$

We begin by estimating the products in V_ε that involve ψ; with (1.4), (1.5), the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, and (3.26), it follows that

$$\begin{aligned} |\langle \psi'(u)v, v\rangle| &\le C_\Omega\left(1 + \|u\|_1^2\right)\|v\|_1\|v\| \\ &\le \frac{1}{2}\|v\|_1^2 + Q(R)\|v\|^2. \end{aligned} \tag{3.44}$$

From assumption (1.7) and the definition of ψ, cf. (3.28),

$$2\langle \psi(u) - \psi(w), v\rangle \ge 2(\beta - \vartheta)\|v\|^2. \tag{3.45}$$

Hence, for β sufficiently large, $\beta \geq \left(C\left(R\right)+2\vartheta\right)/2$, the combination of (3.44) and (3.45) produces

$$\begin{aligned} 2\langle \psi(u)-\psi(w), v\rangle - \langle \psi'(u)v, v\rangle &\geq 2(\beta-\vartheta)\|v\|^2 - \frac{1}{2}\|v\|_1^2 - C(R)\|v\|^2 \\ &\geq -\frac{1}{2}\|v\|_1^2. \end{aligned}$$

Then we attain the lower bound on V_ε,

$$V_\varepsilon \geq \left(1-\frac{\alpha}{2}\right)\varepsilon\|v_t\|^2 + \left(\frac{1}{2}-\frac{\alpha}{2\lambda}\right)\|v\|_1^2.$$

So for $0<\alpha<\min\{2,\lambda\}$, set

$$\omega_2 := \min\left\{1-\frac{\alpha}{2}, \frac{1}{2}-\frac{\alpha}{2\lambda}\right\} > 0;$$

then, for all $t \geq 0$, we have that

$$V_\varepsilon(t) \geq \omega_2\|(v(t), v_t(t))\|_{\mathcal{H}_\varepsilon}^2. \tag{3.46}$$

Now by the (local) Lipschitz continuity of f and the uniform bounds on u and w, it is easy to check that

$$2\langle \psi(u)-\psi(w), v\rangle \leq 2\|\psi(u)-\psi(w)\|\|v\| \leq Q(R)\|v\|_1^2.$$

Also, using (1.4), (1.5), and the bound (3.26), it follows that

$$|\langle \psi'(u)v, v\rangle| \leq Q(R)\|v\|_1^2. \tag{3.47}$$

Thus, the assertion in (3.43) holds. Exploiting the fact that

$$\alpha|\langle v_t, v\rangle_{L^2(\Gamma)}| \leq \frac{\alpha}{2}\|v_t\|_{L^2(\Gamma)}^2 + \frac{\alpha}{2}\|v\|_{L^2(\Gamma)}^2,$$

we see that (3.42) becomes

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon + (2-\alpha)\varepsilon\|v_t\|^2 + \alpha\langle v_t, v\rangle + \alpha\|\nabla v\|^2 + \frac{\alpha}{2}\|v\|_{L^2(\Gamma)}^2 \\ &\quad + \left(2-\frac{\alpha}{2}\right)\|v_t\|_{L^2(\Gamma)}^2 + \alpha\langle \psi(u)-\psi(w), v\rangle - \langle \psi'(u)v, v\rangle \\ &\quad \leq -\langle \psi'(u)v, v\rangle + 2\langle (\psi'(u)-\psi'(w))w_t, v\rangle - \langle \psi''(u)u_t, v^2\rangle. \end{aligned} \tag{3.48}$$

Recall that $0<\alpha<\min\{2,\lambda\}$, so when we set

$$\omega_3 := \min\left\{2-\alpha, 1, \frac{\alpha}{2}\right\} > 0,$$

we write (3.48) as

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon + \omega_3 V_\varepsilon \leq -\langle \psi'(u)v, v\rangle + 2\langle (\psi'(u)-\psi'(w))w_t, v\rangle - \langle \psi''(u)u_t, v^2\rangle. \tag{3.49}$$

Using the uniform bound on u and w (recall assumptions (1.4), (1.5), (3.26), and (3.32)), there is a positive function $Q_\eta(R)>0$, depending on η, such that, for all $\eta>0$,

$$\begin{aligned} \left|\left\langle \left(\psi'(u)-\psi^{'}(w)\right) w_t, v\right\rangle\right| &\leq C_\Omega\left(1+\|u\|_1+\|w\|_1\right)\|w_t\|\,\|v\|_1^2 \\ &\leq \frac{\eta}{2}\|v\|_1^2 + Q_\eta\left(R\right)\|w_t\|^2 V_\varepsilon. \end{aligned} \tag{3.50}$$

The last inequality in the above estimate follows from (3.46). In a similar fashion we estimate using assumption (1.4) and the bound (3.26):

$$\begin{aligned}|\langle\psi^{''}(u)u_t, v^2\rangle| &\leq C_\Omega\left(1+\|u\|_1\right)\|u_t\|\,\|v\|_1^2 \\ &\leq \frac{\eta}{2}\|v\|_1^2 + Q_\eta(R)\|u_t\|^2 V_\varepsilon.\end{aligned} \tag{3.51}$$

Applying (3.44) to (3.49) and inserting (3.50) and (3.51) into (3.49), we then have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon + \omega_3 V_\varepsilon - \eta\|v\|_1^2 \leq Q_\eta(R)\left(\|u_t\|^2 + \|w_t\|^2\right)V_\varepsilon. \tag{3.52}$$

There is a sufficiently small η, precisely $0<\eta<\omega_3/2$, so that (3.52) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\varepsilon + \eta V_\varepsilon \leq Q_\eta(R)\left(\|u_t\|^2 + \|w_t\|^2\right)V_\varepsilon. \tag{3.53}$$

At this point, we remind the reader of Lemma 3.14. Applying a suitable Gronwall-type inequality (see, e.g., [44, Lemma 5]; cf. also Proposition 5.1 in the appendix) to (3.53) yields

$$V_\varepsilon(t) \leq V_\varepsilon(0)e^{Q_\eta(R)}e^{-\eta t/2}. \tag{3.54}$$

By virtue of (3.43), for all $\varepsilon\in(0,1]$,

$$\begin{aligned}V_\varepsilon(0) &\leq Q(R)\|(v(0), v_t(0))\|_{\mathcal{H}_\varepsilon}^2 \\ &\leq Q(R)\left(\|u_0\|_1^2 + \varepsilon\|u_1 + f(0) - \beta u_0\|^2\right) \\ &\leq Q(R),\end{aligned}$$

for some positive function Q independent of ε. Therefore (3.54) shows that the operators Z_ε are uniformly decaying to zero. □

The following lemma establishes the uniform compactness of the operators K_ε.

Lemma 3.16. For all $\varphi_0=(u_0,u_1)\in\mathcal{H}_\varepsilon$ such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon}\leq R$ for all $\varepsilon\in(0,1]$, the following estimate holds:

$$\|K_\varepsilon(t)\varphi_0\|_{\mathcal{D}_\varepsilon} \leq Q(R),$$

for all $t\geq 0$. Furthermore, the operators K_ε are uniformly compact in $\mathcal{H}_\varepsilon$.

Proof. Let $\varepsilon\in(0,1]$ and let $\varphi_0=(u_0,u_1)\in\mathcal{H}_\varepsilon$ with $R>0$ such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon}\leq R$. Differentiate (3.30) with respect to t and set $h=w_t$. Then h satisfies the equations

$$\begin{cases}\varepsilon h_{tt} + h_t - \Delta h + \psi'(w)h = \beta u_t & \text{in } (0,\infty)\times\Omega, \\ \partial_{\mathbf{n}}h + h + h_t = 0 & \text{on } (0,\infty)\times\Gamma, \\ h(0) = w_t(0),\ h_t(0) = w_{tt}(0) & \text{in } \Omega.\end{cases} \tag{3.55}$$

Note that, by the choice of data in (3.30), we actually have $h(0) = -f(0) + \beta u_0$ and $h_t(0) = 0$. Multiply the first equation of (3.55) by $2h_t + \alpha h$, where $\alpha > 0$ is yet to be determined, and integrate over Ω. Adding the result to the identity

$$2\langle\psi'(w)h, h_t\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle\psi'(w)h, h\rangle - \langle\psi''(w)w_t, h^2\rangle$$

produces

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon\|h_t\|^2+\alpha\varepsilon\langle h_t,h\rangle+\|h\|_1^2+\langle\psi'(w)h,h\rangle\right\}\\
&\quad+(2-\alpha\varepsilon)\|h_t\|^2+\alpha\langle h_t,h\rangle+\alpha\|h\|_1^2\\
&\quad+2\|h_t\|^2_{L^2(\Gamma)}+\alpha\langle h_t,h\rangle_{L^2(\Gamma)}+\alpha\langle\psi'(w)h,h\rangle\\
&\quad=\langle\psi''(w)w_t,h^2\rangle+2\beta\langle u_t,h_t\rangle+\alpha\beta\langle u_t,w_t\rangle.
\end{aligned}\tag{3.56}$$

For each $\varepsilon\in(0,1]$, define the functional

$$\Psi_\varepsilon : H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)\to\mathbb{R}$$

by

$$\Psi_\varepsilon(t):=\varepsilon\|h_t(t)\|^2+\alpha\varepsilon\langle h_t(t),h(t)\rangle+\|h(t)\|_1^2+\langle\psi'(w(t))h(t),h(t)\rangle.\tag{3.57}$$

The map $t\mapsto\Psi_\varepsilon(t)$ is $AC(\mathbb{R}_{\geq 0};\mathbb{R}_{\geq 0})$. Because of the bound given in Lemma 3.13, we obtain the estimate similar to (3.47)

$$\alpha|\langle\psi'(w)h,h\rangle|\leq\alpha Q(R)\|h\|_1^2.\tag{3.58}$$

Obviously, we have

$$\alpha\varepsilon|\langle h_t,h\rangle|\leq\frac{\alpha\varepsilon}{2}\|h_t\|^2+\frac{\alpha}{2\lambda}\|h\|_1^2.\tag{3.59}$$

After combining (3.57)–(3.59), we find

$$\Psi_\varepsilon\geq\left(1-\frac{\alpha}{2}\right)\varepsilon\|h_t\|^2+\left(1-\frac{\alpha}{2\lambda}-\alpha Q(R)\right)\|h\|_1^2.$$

Hence, when

$$0<\alpha<\min\left\{2,\left(\frac{1}{2\lambda}+Q(R)\right)^{-1}\right\},$$

we have

$$\omega_4(R):=\min\left\{1-\frac{\alpha}{2},1-\frac{\alpha}{2\lambda}-\alpha Q(R)\right\}>0;$$

thus, for all $t\geq 0$

$$\Psi_\varepsilon(t)\geq\omega_4\|(h(t),h_t(t))\|^2_{\mathcal{H}_\varepsilon}.\tag{3.60}$$

On the other hand, again with (3.58),

$$\Psi_\varepsilon\leq\left(1+\frac{\alpha}{2}\right)\varepsilon\|h_t\|^2+\left(1+\frac{\alpha}{2\lambda}+\alpha Q(R)\right)\|h\|_1^2,$$

and with

$$\omega_5(R):=\max\left\{1+\frac{\alpha}{2},1+\frac{\alpha}{2\lambda}+\alpha Q(R)\right\},$$

an upper bound for Ψ_ε is given by, for all $t\geq 0$,

$$\Psi_\varepsilon(t)\leq\omega_5\|(h(t),h_t(t))\|^2_{\mathcal{H}_\varepsilon}.\tag{3.61}$$

Using the bounds found in (3.26) and (3.32), we estimate the following terms from (3.56): for all $\eta>0$,

$$\alpha|\langle h_t,h\rangle_{L^2(\Gamma)}|\leq\alpha\eta\|h_t\|^2_{L^2(\Gamma)}+\frac{\alpha}{4\eta}\|h\|^2_{L^2(\Gamma)}\tag{3.62}$$

and

$$\begin{aligned}
2\beta|\langle u_t,h_t\rangle|+\alpha\beta|\langle u_t,w_t\rangle|&\leq Q(R)\|h_t\|+Q(R)\\
&\leq\eta\|h_t\|^2+Q_\eta(R).
\end{aligned}\tag{3.63}$$

Also, similar to (3.51), but when we now employ (3.60), we have that, for all $\eta > 0$,

$$\langle \psi''(w)w_t, h^2\rangle \leq Q_\eta(R)\|w_t\|\Psi_\varepsilon. \tag{3.64}$$

Combine (3.62)–(3.64) with (3.56) and obtain the following estimate (note that when $2-\alpha-\eta>0$, we have $(2-\alpha-\eta)\varepsilon < 2-\alpha\varepsilon-\eta$):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_\varepsilon + (2-\alpha-\eta)\varepsilon\|h_t\|^2 + \alpha\langle h_t, h\rangle + \alpha\|\nabla h\|^2 + \alpha\left(1-\frac{1}{4\eta}\right)\|h\|^2_{L^2(\Gamma)} \\ + (2-\alpha\eta)\|h_t\|^2_{L^2(\Gamma)} + \alpha\langle\psi'(w)h, h\rangle \leq Q_\eta(R)\|w_t\|\Psi_\varepsilon + Q_\eta(R).\end{aligned} \tag{3.65}$$

With some $\frac{1}{4} < \eta < 2$ now fixed, then, for

$$0 < \alpha < \min\left\{2-\eta, 1, \frac{2}{\eta}\right\} \text{ and } \omega_6 := 1 - \frac{1}{4\eta},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_\varepsilon + \omega_6\Psi_\varepsilon + (2-\alpha\eta)\|h_t\|^2_{L^2(\Gamma)} \leq Q\left(R\right)\|w_t\|\Psi_\varepsilon + Q(R). \tag{3.66}$$

An immediate consequence of (3.27) is the bound on the integral

$$\int_0^\infty \|w_t(\tau)\|^2\mathrm{d}\tau \leq Q\left(R\right).$$

Applying a suitable version of the Gronwall inequality (see, e.g., [32, Lemma 2.2]; cf. also Proposition 5.1 in the appendix) it follows that

$$\Psi_\varepsilon(t) \leq Q\left(R\right)\Psi_\varepsilon(0)e^{-\omega_6 t/2} + Q\left(R\right). \tag{3.67}$$

Using (3.60) and (3.61) and the fact that $\Psi_\varepsilon(0) \leq \omega_5\|(h(0), h_t(0))\|^2_{\mathcal{H}_\varepsilon} \leq Q\left(R\right)$, we arrive at the bound

$$\|w_t(t)\|_1^2 + \varepsilon\|w_{tt}(t)\|^2 \leq Q\left(R\right), \tag{3.68}$$

for all $t \geq 0$, $\varepsilon \in (0,1]$, and $\varphi_0 \in \mathcal{H}_\varepsilon$, with $R>0$ such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \leq R$.

In order to bound $\|(w, w_t)\|_{\mathcal{D}_\varepsilon}$, we need to bound the term $\|w\|_2$. We have due to standard elliptic regularity theory (see, e.g., [38, Theorem II.5.1]) that

$$\|w(t)\|_2 \leq C\left(\|\Delta w(t)\| + \|\partial_{\mathbf{n}} w(t)\|_{H^{1/2}(\Gamma)}\right). \tag{3.69}$$

Thus, using the first equation of (3.30), the bounds (3.26), (3.32), and (3.68), and also (1.4), (1.5), and (3.28), we have

$$\|\Delta w\left(t\right)\| \leq \sqrt{\varepsilon}\|w_{tt}\left(t\right)\| + \|w_t\left(t\right)\| + \|\psi(w\left(t\right))\| + \beta\|u\left(t\right)\| \leq Q(R). \tag{3.70}$$

Also, by (3.68), we have that $w_t \in L^\infty\left(\mathbb{R}_{\geq 0}, H^{1/2}(\Gamma)\right)$. Thus, from the second equation of (3.30),

$$\|\partial_{\mathbf{n}} w\left(t\right)\|_{H^{1/2}(\Gamma)} \leq \|w\left(t\right)\|_{H^{1/2}(\Gamma)} + \|w_t\left(t\right)\|_{H^{1/2}(\Gamma)} \leq Q(R). \tag{3.71}$$

Combining (3.70) and (3.71) with (3.69) together with applying (3.68) proves that for all $t \geq 0$,

$$\|(w(t), w_t(t))\|_{\mathcal{D}_\varepsilon} \leq Q(R).$$

It follows that the operators K_ε are uniformly compact (with $t_c = 0$). □

Next, we will discuss regularity properties of the weak solutions.

THEOREM 3.17. For each $\varepsilon \in (0,1]$, there exists a closed and bounded subset $\mathcal{C}_\varepsilon \subset \mathcal{D}_\varepsilon$ such that for every nonempty bounded subset $B \subset \mathcal{H}_\varepsilon$,

$$\operatorname{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)B, \mathcal{C}_\varepsilon) \leq Q(\|B\|_{\mathcal{H}_\varepsilon})e^{-\omega t}, \tag{3.72}$$

where Q and $\omega > 0$ are independent of ε.

Proof. Let $\varepsilon \in (0,1]$. Define the subset $\mathcal{C}_\varepsilon$ of $\mathcal{D}_\varepsilon$ by

$$\mathcal{C}_\varepsilon := \{\varphi \in \mathcal{D}_\varepsilon : \|\varphi\|_{\mathcal{D}_\varepsilon} \leq Q(R)\},$$

where $Q(R) > 0$ is the function from Lemma 3.16 and $R > 0$ is such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon} \leq R$. Now let $\varphi_0 = (u_0, u_1) \in \mathcal{B}_\varepsilon$ (endowed with the same topology of $\mathcal{H}_\varepsilon$). Then, for all $t \geq 0$ and for all $\varphi_0 \in \mathcal{B}_\varepsilon$, $S_\varepsilon(t)\varphi_0 = Z_\varepsilon(t)\varphi_0 + K_\varepsilon(t)\varphi_0$, where $Z_\varepsilon(t)$ is uniformly and exponentially decaying to zero by Lemma 3.15, and, by Lemma 3.16, $K_\varepsilon(t)$ is uniformly bounded in $\mathcal{D}_\varepsilon$. In particular,

$$\operatorname{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)\mathcal{B}_\varepsilon, \mathcal{C}_\varepsilon) \leq Q(R)e^{-\omega t}.$$

(Recall that $\omega > 0$ is independent of ε due to Lemma 3.15.)

Recall that, by Lemma 3.10, we already know that for each $\varepsilon \in (0,1]$ and for every nonempty bounded subset B of $\mathcal{H}_\varepsilon$,

$$\operatorname{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)B, \mathcal{B}_\varepsilon) \leq Q(R)e^{-\omega_0 t},$$

for all $t \geq 0$. In light of these estimates, (3.72) can now be accomplished by appealing to the transitivity property of the exponential attraction (see, e.g., [22, Theorem 5.1]). Note that (3.72) entails that $\mathcal{C}_\varepsilon$ is a compact attracting set in $\mathcal{H}_\varepsilon$ for $S_\varepsilon(t)$. The proof is finished. □

By standard arguments of the theory of attractors (see, e.g., [34, 49]), the existence of a compact global attractor $\mathcal{A}_\varepsilon \subset \mathcal{C}_\varepsilon$ for the semigroup $S_\varepsilon(t)$ follows.

THEOREM 3.18. For each $\varepsilon \in (0,1]$, the semiflow S_ε generated by the solutions of the hyperbolic relaxation problem (1.1)–(1.3) admits a unique global attractor

$$\mathcal{A}_\varepsilon = \omega(\mathcal{B}_\varepsilon) := \bigcap_{s \geq t} \overline{\bigcup_{t \geq 0} S_\varepsilon(t)\mathcal{B}_\varepsilon}^{\mathcal{H}_\varepsilon}$$

in $\mathcal{H}_\varepsilon$. Moreover, the following hold:

(i) For each $t \geq 0$, $S_\varepsilon(t)\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon$.

(ii) For every nonempty bounded subset B of $\mathcal{H}_\varepsilon$,

$$\lim_{t \to \infty} \operatorname{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)B, \mathcal{A}_\varepsilon) = 0. \tag{3.73}$$

(iii) The global attractor $\mathcal{A}_\varepsilon$ is bounded in $\mathcal{D}_\varepsilon$ and trajectories on $\mathcal{A}_\varepsilon$ are strong solutions.

REMARK 3.19. We can extend all the results in Sections 3.2 and 3.3 (with the appropriate modifications; see [14, 27]) to the case when the linear boundary condition (1.2) is replaced by

$$\partial_{\mathbf{n}} u + g(u) + u_t = 0 \text{ on } (0, \infty) \times \Gamma$$

such that $g \in C^2(\mathbb{R})$ satisfies

$$|g''(s)| \le C_g \left(1 + |s|^2\right), \ g'(s) \ge -\theta_g, \ g(s)\, s \ge s^2 - C_g',$$

for all $s \in \mathbb{R}$ and some constants $C_g > 0$, $C_g' \ge 0$.

3.4. *The upper-semicontinuity of $\mathcal{A}_\varepsilon$ for the singularly perturbed problem.* This section contains one of the main results of the paper, the proof of the upper-semicontinuity of the family of global attractors given by the model problems for $\varepsilon \in [0,1]$. Recall that the case $\varepsilon = 0$, the limit parabolic problem, admits a global attractor $\mathcal{A}_0$ that is bounded in $\mathcal{V}^2$. Naturally, we will study the continuity at $\varepsilon = 0$. For $\varepsilon \in (0,1]$, we know that the hyperbolic relaxation problem admits a global attractor $\mathcal{A}_\varepsilon$ in $\mathcal{D}_\varepsilon$. However, the spaces involved with the parabolic problem invoke the trace of the solution on the boundary Γ, whereas the spaces involved with the hyperbolic relaxation problem do *not* contain prescribed traces. Before we lift the global attractor $\mathcal{A}_0$ for the parabolic problem into the finite energy phase space for the hyperbolic relaxation problem, we need to make an extension of $\mathcal{H}_\varepsilon$ so that it also includes the information of the traces of u and u_t.

To begin, we recall that the natural phase space for the parabolic problem (1.8)–(1.10) is $Y = L^2(\Omega) \times L^2(\Gamma)$, while the finite energy phase space for the hyperbolic relaxation problem (1.1)–(1.3) is $\mathcal{H}_\varepsilon = H^1(\Omega) \times L^2(\Omega)$. Thus, we need to find a suitable extension of the phase space for the hyperbolic relaxation problem so that, when we lift the parabolic problem, both problems will be situated in the *same* framework. A natural way to make this extension is to introduce the space

$$\mathcal{X}_0 = H^1(\Omega) \times L^2(\Gamma)$$

and then the *extended* phase space for the hyperbolic relaxation problem

$$\mathcal{X}_\varepsilon = \mathcal{X}_0 \times Y = H^1(\Omega) \times L^2(\Gamma) \times L^2(\Omega) \times L^2(\Gamma).$$

The space $\mathcal{X}_\varepsilon$ is Hilbert when endowed with the ε-weighted norm whose square is given by, for all $\zeta = (u, \gamma, v, \delta) \in \mathcal{X}_\varepsilon$,

$$\|\zeta\|^2_{\mathcal{X}_\varepsilon} := \|u\|_1^2 + \|\gamma\|^2_{L^2(\Gamma)} + \varepsilon\|v\|^2 + \varepsilon\|\delta\|^2_{L^2(\Gamma)}.$$

It is then in the space $\mathcal{X}_\varepsilon$ that we can lift $\mathcal{A}_0$ and estimate the Hausdorff semidistance between (an extension of) $\mathcal{A}_\varepsilon$ and $\mathcal{L}\mathcal{A}_0$ (for a proper lifting map $\mathcal{L}$) with the new extended topology. However, it must be noted that the lifted attractor $\mathcal{L}\mathcal{A}_0$ is *not* necessarily a global attractor when set in the extended phase space. Finally, the topology that we will use to show the convergence of the attractors at $\varepsilon = 0$ will be defined with the four-component norm of $\mathcal{X}_\varepsilon$.

For both problems, we also recall that trajectories on the attractor are strong solutions due to the regularity results obtained in Sections 2 and 3.3 (see Theorems 2.6 and 3.17). The regularized phase space $\mathcal{D}_\varepsilon$ for the hyperbolic relaxation problem is isomorphically extended to

$$\begin{aligned}\widetilde{\mathcal{D}}_\varepsilon := \{(u,\gamma,v,\delta) \in H^2(\Omega) \times H^{3/2}(\Gamma) \times H^1(\Omega) \times H^{1/2}(\Gamma) : \gamma = \mathrm{tr_D}(u),\\ \delta = \mathrm{tr_D}(v),\ \partial_{\mathbf{n}} u + \gamma = -\delta \text{ on } \Gamma\}. \quad (3.74)\end{aligned}$$

Of course, $\widetilde{\mathcal{D}}_\varepsilon \subset \mathcal{V}^2 \times \mathcal{V}^1$ and the injection $\widetilde{\mathcal{D}}_\varepsilon \hookrightarrow \mathcal{X}_\varepsilon$ is compact. Recall that, for each $(u_0, u_1) \in \mathcal{D}_\varepsilon$, problem (1.1)–(1.3) generates a dynamical system $(S_\varepsilon(t), \mathcal{D}_\varepsilon)$ of strong solutions (cf. Theorem 3.7; see also [51]). By appealing once more to the continuity of the trace map $\mathrm{tr_D} : H^s(\Omega) \to H^{s-1/2}(\Gamma)$, $s > 1/2$, and exploiting the results in Section 3.2, it is not difficult to realize that we can extend the semiflow $S_\varepsilon(t)$ to a strongly continuous semigroup

$$\widetilde{S}_\varepsilon(t) : \widetilde{\mathcal{D}}_\varepsilon \to \widetilde{\mathcal{D}}_\varepsilon \tag{3.75}$$

such that $\widetilde{S}_\varepsilon(t)$ is also Lipschitz continuous in $\widetilde{\mathcal{D}}_\varepsilon$, endowed with the metric topology of $\mathcal{V}^2 \times \mathcal{V}^1$ (see Lemma 3.20 below). Recall that, by definition for $p, q \geq 1$,

$$\begin{aligned}&\mathcal{V}^p \times \mathcal{V}^q \\ &= \left\{ (u, \gamma, v, \delta) \in H^p(\Omega) \times H^{p-1/2}(\Gamma) \times H^q(\Omega) \times H^{q-1/2}(\Gamma) : \gamma = \mathrm{tr_D}(u), \delta = \mathrm{tr_D}(v) \right\};\end{aligned}$$

see Section 2 (as before, $\mathcal{V}^p \times \mathcal{V}^q$ is topologically isomorphic to $H^p(\Omega) \times H^q(\Omega)$).

LEMMA 3.20. Let $\varphi_0, \theta_0 \in \widetilde{\mathcal{D}}_\varepsilon$ be such that $\|\varphi_0\|_{\widetilde{\mathcal{D}}_\varepsilon} \leq R$, and $\|\theta_0\|_{\widetilde{\mathcal{D}}_\varepsilon} \leq R$, for every $\varepsilon \in (0, 1]$. Then the following estimate holds:

$$\left\| \widetilde{S}_\varepsilon(t)\varphi_0 - \widetilde{S}_\varepsilon(t)\theta_0 \right\|_{\widetilde{\mathcal{D}}_\varepsilon} \leq \frac{Q(R)}{\sqrt{\varepsilon}} e^{\nu_1 t} \|\varphi_0 - \theta_0\|_{\widetilde{\mathcal{D}}_\varepsilon}, \tag{3.76}$$

where $Q(R) > 0$ and $\nu_1 > 0$ are independent of $\varepsilon > 0$.

Proof. Let

$$\varphi(t) = (u_1(t), u_{1|\Gamma}(t), \partial_t u_1(t), \partial_t u_{1|\Gamma}(t))$$

and

$$\theta(t) = (u_2(t), u_{2|\Gamma}(t), \partial_t u_2(t), \partial_t u_{2|\Gamma}(t))$$

denote the corresponding strong solutions with initial data φ_0 and θ_0, respectively. Then the difference $u(t) := u_1(t) - u_2(t)$ satisfies

$$\begin{cases} -\Delta u(t) = f'(u_2(t)) - f'(u_1(t)) - u_t(t) - \varepsilon u_{tt}(t), & \text{a.e. in } \mathbb{R}_+ \times \Omega, \\ \partial_{\mathbf{n}} u(t) + u(t) = -u_t(t), & \text{a.e. in } \mathbb{R}_+ \times \Gamma, \end{cases} \tag{3.77}$$

subject to the initial condition

$$u(0) = u_1(0) - u_2(0).$$

Setting $v := \partial_t u_1 - \partial_t u_2$, we have $(v_t, \psi) \in C^1([0, T])$ for every $\psi \in H^1(\Omega)$ (see the definition of strong solution). Then v solves the identity

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon v_t(t), \psi) + \langle \nabla v(t), \nabla\psi \rangle + \langle v_t(t), \psi \rangle + \langle v_t(t) + v(t), \psi \rangle_{L^2(\Gamma)} \\ &= -\langle f'(u_1(t)) - f'(u_2(t))\, u_1(t), \psi \rangle - \langle f'(u_2(t))\, u(t), \psi \rangle,\end{aligned}$$

for almost all $t \in [0, T]$. Testing with $\psi = v_t$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{ \varepsilon \|v_t\|^2 + \|\nabla v\|^2 + \|v\|^2_{L^2(\Gamma)} \right\} + \|v_t\|^2 + \|v_t\|^2_{L^2(\Gamma)} \\ &= -\langle f'(u_1) - f'(u_2)\, u_1, v_t \rangle - \langle f'(u_2)\, u, v_t \rangle.\end{aligned} \tag{3.78}$$

We can bound the terms on the right-hand side in a standard way:

$$\langle f'(u_1) - f'(u_2)\, u_1, v_t \rangle + \langle f'(u_2)\, u, v_t \rangle \le Q(|u_i|_\infty) \|u\|^2 + \frac{1}{2} \|v_t\|^2$$

(which follows easily on account of the fact that $\|(u_i(t), \partial_t u_i(t))\|_{\mathcal{D}_\varepsilon} \le R$, $i = 1, 2$, and the embedding $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$). Then we insert them into (3.78). By virtue of (3.8) we get

$$\begin{aligned} &\varepsilon \|v_t(t)\|^2 + \left(\|\nabla v(t)\|^2 + \|v(t)\|^2_{L^2(\Gamma)} \right) \\ &\le Q(R) e^{\nu_1 t} \|\varphi_0 - \theta_0\|^2_{\mathcal{X}_\varepsilon} + \varepsilon \|v_t(0)\|^2 + \left(\|\varphi_0 - \theta_0\|^2_{\widetilde{\mathcal{D}}_\varepsilon} \right), \end{aligned} \tag{3.79}$$

for almost all $t \in [0, T]$. It remains to note that, from (3.77), it follows for every $\varepsilon \in (0, 1]$ that

$$\begin{aligned} \varepsilon \|v_t(0)\|^2 &\le \frac{1}{\varepsilon} \left(\|\Delta u(0)\|^2 + \|f'(u_2(0)) - f'(u_1(0))\|^2 + \|u_t(0)\|^2 \right) \\ &\le \frac{Q(R)}{\varepsilon} \|\varphi_0 - \theta_0\|^2_{\widetilde{\mathcal{D}}_\varepsilon}. \end{aligned} \tag{3.80}$$

Summing up, we obtain from (3.79) and (3.80) that

$$\varepsilon \|v_t(t)\|^2 + \|v(t)\|_1^2 \le \frac{Q(R)}{\varepsilon} e^{\nu_1 t} \|\varphi_0 - \theta_0\|^2_{\widetilde{\mathcal{D}}_\varepsilon}. \tag{3.81}$$

We can now bound the term $\|u_1(t) - u_2(t)\|_2$. As before, due to standard elliptic regularity theory, we have in (3.77), using (3.81), that

$$\begin{aligned} \|u(t)\|_2^2 &\le C \left(\varepsilon^2 \|v_t(t)\|^2 + \|f'(u_2(t)) - f'(u_1(t))\|^2 + \|v(t)\|_1^2 \right) \\ &\le Q(R) \|\varphi_0 - \theta_0\|^2_{\widetilde{\mathcal{D}}_\varepsilon}. \end{aligned} \tag{3.82}$$

Finally, (3.81) and (3.82) together with the fact that the trace map $H^s(\Omega) \to H^{s-1/2}(\Gamma)$, $s > 1/2$, is bounded yields the desired inequality (3.76). ☐

By Lemma 3.20, the family of global attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in (0,1]} \subset \mathcal{D}_\varepsilon$ can be naturally extended to the family of compact sets $\left\{\widetilde{\mathcal{A}}_\varepsilon\right\}_{\varepsilon \in (0,1]}$,

$$\widetilde{\mathcal{A}}_\varepsilon = \left\{ (u, \gamma, v, \delta) \in \widetilde{\mathcal{D}}_\varepsilon : (u, v) \in \mathcal{A}_\varepsilon \right\}, \tag{3.83}$$

which are bounded in $\widetilde{\mathcal{D}}_\varepsilon$ and compact in $\mathcal{X}_\varepsilon$. Note that we do not claim that $\widetilde{\mathcal{A}}_\varepsilon$ is a global attractor for $(\widetilde{S}_\varepsilon(t), \mathcal{X}_\varepsilon)$; see Remark 3.21 below. Also, it is in the space $\mathcal{V}^2 \times Y \subset \mathcal{X}_\varepsilon$ where we lift the parabolic problem. Since the global attractor $\mathcal{A}_0$ for (1.8)–(1.10) is a bounded subset of the space $\mathcal{V}^2 \subset C(\overline{\Omega}) \times C(\Gamma)$ (since $\Omega \subset \mathbb{R}^3$), the canonical extension map

$$\mathcal{E} : \mathcal{V}^2 \to Y \tag{3.84}$$

is well-defined with

$$(u, u_{|\Gamma}) \mapsto (\Delta u - f(u), -\partial_{\mathbf{n}} u - u_{|\Gamma}), \tag{3.85}$$

and so the corresponding lift map

$$\mathcal{L} : \mathcal{V}^2 \to \mathcal{V}^2 \times Y \tag{3.86}$$

is defined by

$$(u,u_{|\Gamma})\mapsto(u,u_{|\Gamma},\Delta u-f(u),-\partial_{\mathbf{n}}u-u_{|\Gamma}). \tag{3.87}$$

Let $\mathcal{A}_0$ denote the global attractor of the limit parabolic problem (see Theorem 2.6) and let $\widetilde{\mathcal{A}}_\varepsilon$, $\varepsilon\in(0,1]$, be the sets defined in (3.83). Define the family of compact sets in $\mathcal{X}_\varepsilon$ by

$$\mathbb{A}_\varepsilon:=\begin{cases}\widetilde{\mathcal{A}}_0:=\mathcal{L}\mathcal{A}_0 & \text{for } \varepsilon=0,\\ \widetilde{\mathcal{A}}_\varepsilon & \text{for } \varepsilon\in(0,1].\end{cases} \tag{3.88}$$

REMARK 3.21. The compact set $\widetilde{\mathcal{A}}_\varepsilon$ is *not* a global attractor for $\widetilde{S}_\varepsilon(t)$ acting on the phase-space $\mathcal{X}_\varepsilon$ since traces of functions in $L^2(\Omega)$ are not well-defined in $L^2(\Gamma)$. By construction (3.83), $\widetilde{\mathcal{A}}_\varepsilon$ is only topologically conjugated to the global attractor $\mathcal{A}_\varepsilon$ associated with the dynamical system $(S_\varepsilon,\mathcal{H}_\varepsilon)$.

The main result of this section can now be stated as follows.

THEOREM 3.22. The family $\{\mathbb{A}_\varepsilon\}_{\varepsilon\in[0,1]}$, defined by (3.88), is upper-semicontinuous at $\varepsilon=0$ in the topology of $\mathcal{X}_1$. More precisely,

$$\lim_{\varepsilon\to0}\operatorname{dist}_{\mathcal{X}_1}(\mathbb{A}_\varepsilon,\mathbb{A}_0):=\lim_{\varepsilon\to0}\sup_{a\in\widetilde{\mathcal{A}}_\varepsilon}\inf_{b\in\widetilde{\mathcal{A}}_0}\|a-b\|_{\mathcal{X}_1}=0. \tag{3.89}$$

Proof. Our proof essentially follows the classical argument in [33, 34] and also [40, Theorem 3.31]. Of course, modifications are required to account for the terms on the boundary. Let $\zeta=(u,\gamma,v,\delta)\in\widetilde{\mathcal{A}}_\varepsilon$ and $\bar\zeta=(\bar u,\bar\gamma,\bar v,\bar\delta)\in\widetilde{\mathcal{A}}_0$. We need to show that

$$\begin{aligned}\sup_{(u,\gamma,v,\delta)\in\widetilde{\mathcal{A}}_\varepsilon}\ \inf_{(\bar u,\bar\gamma,\bar v,\bar\delta)\in\widetilde{\mathcal{A}}_0}&\Big(\|u-\bar u\|_1^2+\|\gamma-\bar\gamma\|^2_{L^2(\Gamma)}\\ &+\|v-\bar v\|^2+\|\delta-\bar\delta\|^2_{L^2(\Gamma)}\Big)^{1/2}\to0 \text{ as } \varepsilon\to0.\end{aligned} \tag{3.90}$$

Assuming to the contrary that (3.90) did not hold, then there exist $\eta_0>0$ and sequences $(\varepsilon_n)_{n\in\mathbb{N}}\subset(0,1]$, $(\zeta_n)_{n\in\mathbb{N}}=((u_n,\gamma_n,v_n,\delta_n))_{n\in\mathbb{N}}\subset\widetilde{\mathcal{A}}_{\varepsilon_n}$ such that $\varepsilon_n\to0$ and for all $n\in\mathbb{N}$,

$$\inf_{(\bar u,\bar\gamma,\bar v,\bar\delta)\in\widetilde{\mathcal{A}}_0}\Big(\|u_n-\bar u\|_1^2+\|\gamma_n-\bar\gamma\|^2_{L^2(\Gamma)}+\|v_n-\bar v\|^2+\|\delta_n-\bar\delta\|^2_{L^2(\Gamma)}\Big)\geq\eta_0^2. \tag{3.91}$$

By Theorem 3.17, the compact sets $\widetilde{\mathcal{A}}_{\varepsilon_n}$ are bounded in the space $\widetilde{\mathcal{D}}_1$ (see (3.74) with $\varepsilon=1$) and we have the following uniform bound for some positive constant $C>0$ independent of n:

$$\|u_n\|_2^2+\|\gamma_n\|^2_{H^{3/2}(\Gamma)}+\|v_n\|_1^2+\|\delta_n\|^2_{H^{1/2}(\Gamma)}\leq C.$$

This means that there is a weakly converging subsequence of $(\zeta_n)_{n\in\mathbb{N}}$ (not relabeled) that converges to some $(u^*,\gamma^*,v^*,\delta^*)$ weakly in $\widetilde{\mathcal{D}}_1$. By the compactness of the embedding $\widetilde{\mathcal{D}}_1\hookrightarrow\mathcal{X}_1$, the subsequence converges strongly in $\mathcal{X}_1$. It now suffices to show that $(u^*,\gamma^*,v^*,\delta^*)\in\widetilde{\mathcal{A}}_0$ since this is a contradiction to (3.91).

With each $\zeta_n=(u_n,\gamma_n,v_n,\delta_n)\in\widetilde{\mathcal{A}}_{\varepsilon_n}$, then, for each $n\in\mathbb{N}$, there is a complete orbit

$$(u^n(t),u^n_{|\Gamma}(t),u^n_t(t),u^n_{t|\Gamma}(t))_{t\in\mathbb{R}}=(\widetilde{S}_{\varepsilon_n}(t)(u_n,\gamma_n,v_n,\delta_n))_{t\in\mathbb{R}}$$

contained in $\widetilde{\mathcal{A}}_{\varepsilon_n}$ and passing through $(u_n, \gamma_n, v_n, \delta_n)$ where

$$(u^n(0), u^n_{|\Gamma}(0), u^n_t(0), u^n_{t|\Gamma}(0)) = (u_n, \gamma_n, v_n, \delta_n)$$

(cf., e.g., [40, Proposition 2.39]).

In view of the regularity $\widetilde{\mathcal{A}}_{\varepsilon_n} \subset \widetilde{\mathcal{D}}_1$ (see (3.4)), we obtain the uniform bounds:

$$\varepsilon_n \|u^n_{tt}(t)\|^2 + \|u^n_t(t)\|_1^2 + \|u^n_t(t)\|^2_{H^{1/2}(\Gamma)} + \|u^n(t)\|_2^2 + \|u^n(t)\|^2_{H^{3/2}(\Gamma)} \leq C, \tag{3.92}$$

where the constant $C > 0$ is independent of t and ε_n. Now, for all $T > 0$, the functions u^{ε_n}, $u^{\varepsilon_n}_{|\Gamma}$, $u^{\varepsilon_n}_t$, $u^{\varepsilon_n}_{t|\Gamma}$, and $\sqrt{\varepsilon_n} u^{\varepsilon_n}_{tt}$ are, respectively, bounded in $L^\infty(-T, T; H^2(\Omega))$, $L^\infty(-T, T; H^{3/2}(\Gamma))$, $L^\infty(-T, T; H^1(\Omega))$, $L^\infty(-T, T; H^{1/2}(\Gamma))$, and $L^\infty(-T, T; L^2(\Omega))$. Thus, there is a function u and a subsequence (not relabeled) in which

$$u^{\varepsilon_n} \rightharpoonup u \text{ in } L^\infty(-T, T; H^2(\Omega)) \text{ (weakly*)}, \tag{3.93}$$

$$u^{\varepsilon_n}_{|\Gamma} \rightharpoonup u_{|\Gamma} \text{ in } L^\infty(-T, T; H^{3/2}(\Gamma)) \text{ (weakly*)}, \tag{3.94}$$

$$u^{\varepsilon_n}_t \rightharpoonup u_t \text{ in } L^\infty(-T, T; H^1(\Omega)) \text{ (weakly*)}, \tag{3.95}$$

$$u^{\varepsilon_n}_{t|\Gamma} \rightharpoonup u_{t|\Gamma} \text{ in } L^\infty(-T, T; H^{1/2}(\Gamma)) \text{ (weakly*)}, \tag{3.96}$$

$$\varepsilon_n u^{\varepsilon_n}_{tt} \to 0 \text{ in } L^\infty(-T, T; L^2(\Omega)) \text{ (strongly)}. \tag{3.97}$$

The above convergence properties yield

$$u^{\varepsilon_n} \to u \text{ in } C(-T, T; H^1(\Omega)) \text{ (strongly)} \tag{3.98}$$

due to the embedding

$$\{u \in L^\infty(-T, T; H^2(\Omega)) : u_t \in L^\infty(-T, T; H^1(\Omega))\} \hookrightarrow C(-T, T; H^{2-\eta}(\Omega)), \tag{3.99}$$

which is compact for every $\eta \in (0, 1)$ (see, e.g., [37]). The strong property (3.98) allows us to identify the correct limit in the nonlinear term when $\varepsilon_n \to 0$. Moreover, from (3.93) and the fact that $H^2(\Omega) \hookrightarrow C^0\left(\overline{\Omega}\right)$, it follows that

$$\begin{aligned} \sup_{t \in [-T,T]} \|f(u^{\varepsilon_n}) - f(u)\|^2 &\leq \sup_{t \in [-T,T]} Q_*\left(|u^{\varepsilon_n}(t)|_\infty, |u(t)|_\infty\right) \|u^{\varepsilon_n}(t) - u(t)\|^2 \\ &\leq C(\Omega) \sup_{t \in [-T,T]} \|u^{\varepsilon_n}(t) - u(t)\|^2, \end{aligned} \tag{3.100}$$

for some positive (increasing) function $Q_* : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, independent of n and ε_n. By virtue of (3.98) it is then easy to see that

$$f(u^{\varepsilon_n}) \to f(u) \text{ in } C(-T, T; L^2(\Omega)) \text{ (strongly)}.$$

It follows that u is a weak solution of the limit parabolic problem on $\mathbb{R}$. In particular, $(u_n, \gamma_n) = (u^n(0), u^n_{|\Gamma}(0)) \to (u(0), u_{|\Gamma}(0))$ in $\mathcal{V}^1$. Hence, we have that $(u(0), u_{|\Gamma}(0)) = (u^*, \gamma^*)$, and therefore $(u(0), u_{|\Gamma}(0)) \in \mathcal{V}^2$. As $(u, u_{|\Gamma})$ is a complete orbit through (u^*, γ^*), it follows that $(u^*, \gamma^*) \in \mathcal{A}_0$. It remains to show that $v^* = \Delta u^* - f(u^*)$ and $\delta^* = -\partial_{\mathbf{n}} \gamma^* - \gamma^*$, in which case $(u^*, \gamma^*, v^*, \delta^*) \in \widetilde{\mathcal{A}}_0$.

Now by (3.97) and (3.92), it follows that

$$\|\varepsilon_n u^n_{tt}(0)\| = \sqrt{\varepsilon_n} \|\sqrt{\varepsilon_n} u^n_{tt}(0)\| \leq \sqrt{\varepsilon_n} C,$$

and so $\varepsilon_n u^n_{tt}(0) \to 0$ in $L^2(\Omega)$ as $\varepsilon_n \to 0$. With this at hand,

$$\begin{aligned} u^n_t(0) &= -\varepsilon_n u^n_{tt}(0) + \Delta u^n(0) - f(u^n(0)) \\ &= -\varepsilon_n u^n_{tt}(0) + \Delta u^* - f(u^*), \end{aligned}$$

so that

$$u^n_t(0) \rightharpoonup \Delta u^* - f(u^*) \text{ in } L^2(\Omega) \text{ (weakly).} \tag{3.101}$$

Since $u^n_t(0) = v^n$, with (3.101) we have that

$$v^* = \Delta u^* - f(u^*). \tag{3.102}$$

Similarly, since

$$u^n_{t|\Gamma}(0) = -\partial_{\mathbf{n}} u^n(0) - u^n_{|\Gamma}(0)$$

and since $u^n_{|\Gamma}(0) = \gamma^*$ and $u^n_{t|\Gamma}(0) = \delta^*$,

$$\delta^* = -\partial_{\mathbf{n}}\gamma^* - \gamma^*. \tag{3.103}$$

We know $(u^*, \gamma^*) \in \mathcal{A}_0$, so (3.102) and (3.103) imply that $(u^*, \gamma^*, v^*, \delta^*) \in \widetilde{\mathcal{A}}_0$, in contradiction to (3.91). This proves the assertion and completes the proof. □

4. Exponential attractors. Exponential attractors (sometimes called inertial sets) are positively invariant sets possessing finite fractal dimension that attract bounded subsets of the phase space exponentially fast. It can readily be seen that when both a global attractor $\mathcal{A}$ and an exponential attractor $\mathcal{M}$ exist, then $\mathcal{A} \subseteq \mathcal{M}$, and so the global attractor is also finite dimensional. The existence of an exponential attractor depends on certain properties of the semigroup, namely, the smoothing property for the difference of any two trajectories and the existence of a more regular bounded absorbing set in the phase space (see, e.g., [20], [21]).

The main result of this section is the following.

THEOREM 4.1. For each $\varepsilon \in (0, 1]$, the dynamical system $(S_\varepsilon, \mathcal{H}_\varepsilon)$ associated with (1.1)–(1.3) admits an exponential attractor $\mathcal{M}_\varepsilon$ compact in $\mathcal{H}_\varepsilon$ and bounded in $\mathcal{C}_\varepsilon$. Moreover, the following are true:

(i) For each $t \geq 0$, $S_\varepsilon(t)\mathcal{M}_\varepsilon \subseteq \mathcal{M}_\varepsilon$.

(ii) The fractal dimension of $\mathcal{M}_\varepsilon$ with respect to the metric $\mathcal{H}_\varepsilon$ is finite, namely,

$$\dim_F (\mathcal{M}_\varepsilon, \mathcal{H}_\varepsilon) \leq C_\varepsilon < \infty,$$

for some positive constant C_ε which *depends* on ε.

(iii) There exist $\varrho > 0$ and a positive nondecreasing function Q_ε such that, for all $t \geq 0$,

$$\mathrm{dist}_{\mathcal{H}_\varepsilon}(S_\varepsilon(t)B, \mathcal{M}_\varepsilon) \leq Q_\varepsilon(\|B\|_{\mathcal{H}_\varepsilon})e^{-\varrho t},$$

for every nonempty bounded subset B of $\mathcal{H}_\varepsilon$.

REMARK 4.2. Above,

$$\dim_{\mathrm{F}}(\mathcal{M}_\varepsilon, \mathcal{H}_\varepsilon) := \limsup_{r\to 0} \frac{\ln \mu_{\mathcal{H}_\varepsilon}(\mathcal{M}_\varepsilon, r)}{-\ln r} < \infty,$$

where $\mu_{\mathcal{H}_\varepsilon}(\mathcal{Z}, r)$ denotes the minimum number of r-balls from $\mathcal{H}_\varepsilon$ required to cover $\mathcal{Z} \subset \mathcal{H}_\varepsilon$.

COROLLARY 4.3. It is true that

$$\dim_{\mathrm{F}}(\mathcal{A}_\varepsilon, \mathcal{H}_\varepsilon) \le \dim_{\mathrm{F}}(\mathcal{M}_\varepsilon, \mathcal{H}_\varepsilon).$$

As a consequence, $\mathcal{A}_\varepsilon$ has finite fractal dimension which depends on $\varepsilon > 0$.

REMARK 4.4. Unfortunately, we cannot show that the fractal dimension of $\mathcal{M}_\varepsilon$ is uniform with respect to $\varepsilon > 0$ (see the subsequent lemmas).

The proof of Theorem 4.1 follows from the application of an abstract result tailored specifically to our needs (see, e.g., [21, Proposition 1], [22], [30]; cf. also Remark 4.10 below).

PROPOSITION 4.5. Let $(S_\varepsilon, \mathcal{H}_\varepsilon)$ be a dynamical system for each $\varepsilon > 0$. Assume the following hypotheses hold:

(C1) There exists a bounded absorbing set $\mathcal{B}^1_\varepsilon \subset \mathcal{D}_\varepsilon$ which is positively invariant for $S_\varepsilon(t)$. More precisely, there exists a time $t_1 > 0$, which *depends* on $\varepsilon > 0$, such that

$$S_\varepsilon(t)\mathcal{B}^1_\varepsilon \subset \mathcal{B}^1_\varepsilon$$

for all $t \ge t_1$ where $\mathcal{B}^1_\varepsilon$ is endowed with the topology of $\mathcal{H}_\varepsilon$.

(C2) There is $t^* \ge t_1$ such that the map $S_\varepsilon(t^*)$ admits the decomposition, for each $\varepsilon \in (0,1]$ and for all $\varphi_0, \theta_0 \in \mathcal{B}^1_\varepsilon$,

$$S_\varepsilon(t^*)\varphi_0 - S_\varepsilon(t^*)\theta_0 = L_\varepsilon(\varphi_0, \theta_0) + R_\varepsilon(\varphi_0, \theta_0)$$

where, for some constants $\alpha^* \in (0, \frac{1}{2})$ and $\Lambda^* = \Lambda^*(\Omega, t^*) \ge 0$ with Λ^* depending on $\varepsilon > 0$, the following hold:

$$\|L_\varepsilon(\varphi_0, \theta_0)\|_{\mathcal{H}_\varepsilon} \le \alpha^* \|\varphi_0 - \theta_0\|_{\mathcal{H}_\varepsilon} \tag{4.1}$$

and

$$\|R_\varepsilon(\varphi_0, \theta_0)\|_{\mathcal{D}_\varepsilon} \le \Lambda^* \|\varphi_0 - \theta_0\|_{\mathcal{H}_\varepsilon}. \tag{4.2}$$

(C3) The map

$$(t, U) \mapsto S_\varepsilon(t)U : [t^*, 2t^*] \times \mathcal{B}^1_\varepsilon \to \mathcal{B}^1_\varepsilon$$

is Lipschitz continuous on $\mathcal{B}^1_\varepsilon$ in the topology of $\mathcal{H}_\varepsilon$.

Then $(S_\varepsilon, \mathcal{H}_\varepsilon)$ possesses an exponential attractor $\mathcal{M}_\varepsilon$ in $\mathcal{B}^1_\varepsilon$.

We now show that the assumptions (C1)–(C3) hold for $(S_\varepsilon(t), \mathcal{H}_\varepsilon)$. We begin with a higher-order dissipative estimate in the norm of $\mathcal{D}_\varepsilon$.

LEMMA 4.6. Condition (C1) holds for fixed $\varepsilon \in (0,1]$.

Proof. The proof is very similar to the proof of Lemma 3.16. Indeed, let $\varepsilon \in (0,1]$, $\varphi_0 = (u_0, u_1) \in \mathcal{D}_\varepsilon$, and $\varphi(t) = S_\varepsilon(t)\varphi_0$. In this setting, we differentiate (1.1)–(1.3) with respect to t and let $h = u_t$. We set β in (3.28) to be $\beta = \vartheta$ where we recall that $\vartheta > 0$ is due to assumption (1.7). Then we easily obtain the analogue of the differential inequality (3.67) except that the size of the initial data now depends on the norm of $\mathcal{D}_\varepsilon$, i.e., $\varphi_0 = (u_0, u_1) \in \mathcal{D}_\varepsilon$ (here the initial conditions are not necessarily equal to zero).

Thus, after applying (3.60) and (3.61), there exist a positive and nondecreasing function Q and a constant $C>0$ such that

$$\|(h(t),h_t(t))\|^2_{\mathcal{H}_\varepsilon} \leq Q(\|\,(h(0),h_t(0))\,\|_{\mathcal{H}_\varepsilon})e^{-\omega_6 t/2}+C\,(R) \tag{4.3}$$

(Q, ω_6, and C are independent of ε) with $R>0$ such that $\|\varphi_0\|_{\mathcal{H}_\varepsilon}\leq R$. Arguing as in Theorem 3.16 by exploiting H^2-elliptic regularity theory, we also deduce

$$\|\varphi\,(t)\|^2_{\mathcal{D}_\varepsilon} \leq Q_\varepsilon(\|\varphi_0\|_{\mathcal{D}_\varepsilon})e^{-\omega_6 t/2}+C\,(R)\,, \tag{4.4}$$

for some new function Q_ε which depends on $\varepsilon>0$. Indeed, using the equations (1.1)–(1.3), it is not difficult to show that $\|\,(h(0),h_t(0))\,\|_{\mathcal{H}_\varepsilon}\leq \frac{C}{\sqrt{\varepsilon}}\|\varphi_0\|_{\mathcal{D}_\varepsilon}$, from (4.4). Consequently, there exists $R_1>0$ (independent of time, $\varepsilon>0$, and initial data) such that $S_\varepsilon\,(t)$ possesses an absorbing ball $\mathcal{B}^1_\varepsilon = B_{\mathcal{D}_\varepsilon}\,(R_1)$ of radius R_1 centered at 0, which is bounded in $\mathcal{D}_\varepsilon$. This establishes condition (C1). □

REMARK 4.7. Unfortunately, the bound in the space $\mathcal{D}_\varepsilon$ is not uniform as $\varepsilon\to 0^+$. Indeed the function $Q_\varepsilon\,(\cdot)$ in (4.4) blows up as $\varepsilon\to 0^+$. Finally, arguing in a standard way as in Theorem 3.17, $\mathcal{B}^1_\varepsilon$ is in fact exponentially attracting in $\mathcal{H}_\varepsilon$.

LEMMA 4.8. Condition (C2) holds for each fixed $\varepsilon\in(0,1]$.

Proof. Let $\varepsilon\in(0,1]$. Let $\varphi_0,\theta_0\in\mathcal{B}^1_\varepsilon$. Define the pair of trajectories, for $t\geq 0$, $\varphi(t)=S_\varepsilon(t)\varphi_0=(u(t),u_t(t))$ and $\theta(t)=S_\varepsilon(t)\theta_0=(v(t),v_t(t))$. For each $t\geq 0$, decompose the difference $\bar\zeta(t):=\varphi(t)-\theta(t)$ with $\bar\zeta_0:=\varphi_0-\theta_0$ as follows:

$$\bar\zeta(t)=\bar\varphi(t)+\bar\theta(t)$$

where $\bar\varphi(t)=(\bar u(t),\bar u_t(t))$ and $\bar\theta(t)=(\bar v(t),\bar v_t(t))$ are solutions of the problems

$$\begin{cases} \varepsilon\bar u_{tt}+\bar u_t-\Delta\bar u=0 & \text{in } (0,\infty)\times\Omega,\\ \partial_{\mathbf{n}}\bar u+\bar u+\bar u_t=0 & \text{on } (0,\infty)\times\Gamma,\\ \bar\varphi(0)=\varphi_0-\theta_0 & \text{in } \Omega \end{cases} \tag{4.5}$$

and

$$\begin{cases} \varepsilon\bar v_{tt}+\bar v_t-\Delta\bar v=f(v)-f(u) & \text{in } (0,\infty)\times\Omega,\\ \partial_{\mathbf{n}}\bar v+\bar v+\bar v_t=0 & \text{on } (0,\infty)\times\Gamma,\\ \bar\theta(0)=0 & \text{in } \Omega. \end{cases} \tag{4.6}$$

By estimating along the usual lines, multiplying $(4.5)_1$ by $2\bar u_t+\bar u$ in $L^2(\Omega)$, we easily obtain the differential inequality, for almost all $t\geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{N}_\varepsilon+\omega_7\mathcal{N}_\varepsilon\leq 0, \tag{4.7}$$

for some positive constant ω_7 sufficiently small and independent of ε and for

$$\mathcal{N}_\varepsilon=\mathcal{N}_\varepsilon(\bar\varphi(t)):=\varepsilon\|\bar u_t(t)\|^2+\varepsilon\langle\bar u_t(t),\bar u(t)\rangle+\|\nabla\bar u(t)\|^2+\|\bar u(t)\|^2_{L^2(\Gamma)}. \tag{4.8}$$

Obviously, $\mathcal{N}_\varepsilon$ is the square of an equivalent norm on $\mathcal{H}_\varepsilon$; i.e., there is a constant $C>0$, independent of ε, such that

$$C^{-1}\|\bar\varphi\|^2_{\mathcal{H}_\varepsilon}\leq\mathcal{N}_\varepsilon(\bar\varphi)\leq C\|\bar\varphi\|^2_{\mathcal{H}_\varepsilon}. \tag{4.9}$$

Following (4.7) and (4.9), we have that, for all $t\geq 0$,

$$\|\bar\varphi(t)\|^2_{\mathcal{H}_\varepsilon}\leq C\|\bar\varphi_0\|^2_{\mathcal{H}_\varepsilon}e^{-\omega_7 t}. \tag{4.10}$$

Set $t^* := \max\{t_1, \frac{1}{\omega_7}\ln(4C)\}$. Then, for all $t \geq t^*$, (4.1) holds with $L_\varepsilon = \bar{\varphi}(t^*)$ and

$$\alpha^* = Ce^{-\omega_7 t^*} < \frac{1}{2}.$$

We now show that (4.2) holds for some $\Lambda^* \geq 0$. First we observe that

$$\begin{aligned} 2\langle f(v) - f(u), \bar{v}_{tt}\rangle_{L^2(\Gamma)} &= \frac{\mathrm{d}}{\mathrm{d}t} 2\langle f(v) - f(u), \bar{v}_t\rangle_{L^2(\Gamma)} \\ &\quad - 2\langle (f'(v) - f'(u))v_t, \bar{v}_t\rangle_{L^2(\Gamma)} + 2\langle f'(u)z_t, \bar{v}_t\rangle_{L^2(\Gamma)}. \end{aligned} \tag{4.11}$$

Next we differentiate the second equation of (4.6) with respect to t, multiply the first equation of (4.6) by $2(-\Delta)\bar{v}_t$ in $L^2(\Omega)$, and insert (4.11) into the result to produce the differential identity, which holds for almost all $t \geq 0$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon\|\bar{v}_t\|_1^2 + \|\bar{v}_t\|_{L^2(\Gamma)}^2 + \|\Delta\bar{v}\|^2 + 2\langle f(u) - f(v), \bar{v}_t\rangle_{L^2(\Gamma)}\right\} \\ &\quad + 2\varepsilon\|\bar{v}_{tt}\|_{L^2(\Gamma)}^2 + 2\|\bar{v}_t\|_1^2 \\ &\quad = 2\langle (f'(v) - f'(u))\nabla v, \nabla\bar{v}_t\rangle - 2\langle f'(u)\nabla z, \nabla\bar{v}_t\rangle \\ &\quad + 2\langle f(v) - f(u), \bar{v}_t\rangle_{L^2(\Gamma)} - 2\langle (f'(v) - f'(u))v_t, \bar{v}_t\rangle_{L^2(\Gamma)} + 2\langle f'(u)z_t, \bar{v}_t\rangle_{L^2(\Gamma)}. \end{aligned} \tag{4.12}$$

Recall that $z := u - v$ denotes the difference of any two weak solutions of (1.1)–(1.3) and is estimated in (3.8). Arguing, for instance, as in [23, (6.11)–(6.13)], we estimate the products on the right-hand side of (4.12), for all $t \in (0, t^*)$, using (1.4), Lemma 4.6, and the embedding $H^2(\Omega) \hookrightarrow C^0(\overline{\Omega})$, as follows:

$$\begin{aligned} 2|\langle (f'(u) - f'(v))\nabla v, \nabla\bar{v}_t\rangle| &\leq C\left(1 + \|u\|_1 + \|v\|_1\right)\|z\|_1\|v\|_2\|\nabla\bar{v}_t\| \\ &\leq C_\varepsilon(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}^2 + \frac{1}{4}\|\nabla\bar{v}_t\|^2, \end{aligned} \tag{4.13}$$

$$\begin{aligned} 2|\langle f'(u)\nabla z, \nabla\bar{v}_t\rangle| &\leq C\left(1 + \|u\|_2^2\right)\|\nabla z\|\|\nabla\bar{v}_t\| \\ &\leq C_\varepsilon(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}^2 + \frac{1}{4}\|\nabla\bar{v}_t\|^2, \end{aligned} \tag{4.14}$$

$$\begin{aligned} 2|\langle f(u) - f(v), \bar{v}_t\rangle_{L^2(\Gamma)}| &\leq C\|z\|_1\|\bar{v}_t\|_{L^2(\Gamma)} \\ &\leq C(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}^2 + \frac{1}{4}\|\bar{v}_t\|_{L^2(\Gamma)}^2, \end{aligned} \tag{4.15}$$

$$\begin{aligned} 2|\langle (f'(u) - f'(v))v_t, \bar{v}_t\rangle_{L^2(\Gamma)}| &\leq C\left(1 + \|u\|_{C^0(\Gamma)} + \|v\|_{C^0(\Gamma)}\right)\|z\|_{C^0(\Gamma)}\|v_t\|_{L^2(\Gamma)}\|\bar{v}_t\|_{L^2(\Gamma)} \\ &\leq C(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}^2 + \frac{1}{4}\|\bar{v}_t\|_{L^2(\Gamma)}^2, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} 2|\langle f'(u)z_t, \bar{v}_t\rangle_{L^2(\Gamma)}| &\leq C\left(1 + \|u\|_{C^0(\Gamma)}^2\right)\|z_t\|_{L^2(\Gamma)}\|\bar{v}_t\|_{L^2(\Gamma)} \\ &\leq C(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}^2 + \frac{1}{4}\|\bar{v}_t\|_{L^2(\Gamma)}^2. \end{aligned} \tag{4.17}$$

We emphasize again that by Lemma 4.6 the constants $C = C_\varepsilon(t^*)$ in estimates (4.13) and (4.14) depend on $\varepsilon > 0$. After combining (4.13)–(4.17) with the identity (4.12), we arrive at the differential inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\varepsilon\|\bar{v}_t\|_1^2 + \|\bar{v}_t\|_{L^2(\Gamma)}^2 + \|\Delta\bar{v}\|^2 + 2\langle f(u) - f(v), \bar{v}_t\rangle_{L^2(\Gamma)}\right\} \leq C_\varepsilon(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}^2. \tag{4.18}$$

(Recall that by the definition of a strong solution in Definition 3.5, $\bar{v}_{tt} \in L^2(0,\infty;L^2(\Gamma))$.) Now by integrating (4.18) over $(0,t^*)$ and once again applying the estimate (4.15), we are left with the bound

$$\varepsilon\|\bar{v}_t(t^*)\|_1^2 + \|\Delta\bar{v}(t^*)\|^2 \le C_\varepsilon(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}^2.$$

By standard H^2-elliptic regularity estimates (see (3.70) and (3.71) above), we obtain

$$\|\bar{\theta}(t^*)\|_{\mathcal{D}_\varepsilon} \le C_\varepsilon(t^*)\|\bar{\zeta}_0\|_{\mathcal{H}_\varepsilon}. \tag{4.19}$$

Inequality (4.2) now follows with $R_\varepsilon = \bar{\theta}(t^*)$ and $\Lambda^* = C_\varepsilon(t^*) \ge 0$. This finishes the proof. □

LEMMA 4.9. Condition (C3) holds.

Proof. We proceed exactly as in the proof of Lemma 4.6, differentiating (1.1)–(1.3) with respect to t and letting $h = u_t$. This time we obtain the bound

$$\|\varphi_t(t)\|_{\mathcal{H}_\varepsilon} \le Q_\varepsilon(R)$$

for $\varphi_t = (u_t, u_{tt})$ and some function Q_ε depending on $\varepsilon > 0$, where the size of the initial data now depends on the norm of $\mathcal{B}_\varepsilon^1$. Hence, on the compact interval $[t^*, 2t^*]$, the map $t \mapsto S_\varepsilon(t)\varphi_0$ is Lipschitz continuous for each fixed $\varphi_0 \in \mathcal{B}_\varepsilon^1$; i.e., there is a constant $L_\varepsilon = L_\varepsilon(t^*) > 0$ (which depends on $\varepsilon > 0$) such that

$$\|S_\varepsilon(t_1)\varphi_0 - S_\varepsilon(t_2)\varphi_0\|_{\mathcal{H}_\varepsilon} \le L_\varepsilon(t^*)|t_1 - t_2|.$$

Together with the continuous dependence estimate (3.8), (C3) follows. □

REMARK 4.10. According to Proposition 4.5, the semiflow $S_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_\varepsilon$ possesses an exponential attractor $\mathcal{M}_\varepsilon \subset \mathcal{B}_\varepsilon^1$, which attracts bounded subsets of $\mathcal{B}_\varepsilon^1$ exponentially fast (in the topology of $\mathcal{H}_\varepsilon$). In order to show that the attraction property in Theorem 4.1(iii) also holds, we can appeal once more to the transitivity of the exponential attraction [22, Theorem 5.1] and the result of Theorem 3.17 (also see Remark 4.7).

In contrast to the standard case of Dirichlet boundary conditions, where we have a complete treatment, due to [22] and [41], the situation with boundary condition (1.2) remains essentially less clear. Important questions remain open about the following:

- higher-order dissipative estimates which are uniform with respect to $\varepsilon > 0$,
- finite dimensionality of the exponential attractor $\mathcal{M}_\varepsilon$ (and global attractor $\mathcal{A}_\varepsilon$) which is uniform in $\varepsilon > 0$,
- existence of a robust (Holder continuous in $\varepsilon \in [0,1]$) family of exponential attractors $\{\mathcal{M}_\varepsilon\}$.

5. Appendix.

To make the paper reasonably self-contained, we include the statement of a frequently used Grönwall-type inequality [44, Lemma 5].

PROPOSITION 5.1. Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be an absolutely continuous function satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + 2\eta\Lambda(t) \le h(t)\Lambda(t) + k,$$

where $\eta > 0$, $k \geq 0$, and $\int_s^t h(\tau)\mathrm{d}\tau \leq \eta(t-s) + m$, for all $t \geq s \geq 0$ and some $m \geq 0$. Then, for all $t \geq 0$,

$$\Lambda(t) \leq \Lambda(0)e^m e^{-\eta t} + \frac{ke^m}{\eta}.$$

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