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# Hyperbolic Systems of Partial Differential Inclusions

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## 0. - Introduction

Let $X, Y, Z$ denote finite dimensional vector-spaces, $f : X \times Y \mapsto X$ be a single-valued map, $G : X \times Y \rightsquigarrow Y$ be a set-valued map and $A \in \mathcal{L}(Y, Y)$ a linear operator. We set throughout this paper $\lambda = \min_{\|x\|=1} \langle Ax, x \rangle$.

We recall that the contingent cone $T_K(x)$ to a subset $K \subset X$ at $x \in K$ is defined by

$$T_K(x) := \left\{ v \in X \mid \liminf_{h \to 0+} \frac{d(x + hv, K)}{h} = 0 \right\}.$$

and that the *contingent derivative* $DR(x, y)$ of a set-valued map $R : X \rightsquigarrow Y$ at $(x, y) \in \text{Graph}(R)$ is defined by

$$\text{Graph}(DR(x, y)) := T_{\text{Graph}(R)}(x, y).$$

When $R = r$ is single-valued, we set $Dr(x) := Dr(x, r(x))$. Naturally, $Dr(x)(u) = r'(x)u$ whenever $r$ is differentiable at $x$.

Usually, a Lipschitz map $r$ is not differentiable, but *contingently differentiable* in the sense that its contingent derivative has nonempty values. In this case, it associates to every direction $u \in X$ the subset

$$Dr(x)(u) := \left\{ v \in Y \mid \liminf_{h \to 0+} \left\| v - \frac{r(x + hu) - r(x)}{h} \right\| = 0 \right\}.$$

See [8, Chapter 5] for more details on differential calculus of set-valued maps.

In this paper, we shall look for single-valued and set-valued *contingent solutions* to hyperbolic systems of partial differential inclusions, i.e., single-valued maps $r : X \mapsto Y$ *with closed graph* satisfying

$$\forall x \in X, \;\; Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x))$$

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and set-valued maps $R : X \rightsquigarrow Y$ with *closed graph* satisfying

$$\forall x \in X,\ \forall y \in R(x),\ Ay \in DR(x,y)(f(x,y)) - G(x,y).$$

We observe that when $r$ is differentiable, the contingent differential inclusion boils down to a *quasi-linear hyperbolic system of first-order partial differential equations*[1]

$$\forall j = 1,\dots,m,\ \sum_{k=1}^{m} a_j^k r_k(x) = \sum_{i=1}^{n} \frac{\partial r_j}{\partial x_i} f_i(x, r(x)) - g_j(x, r(x)).$$

**Motivations: Tracking Property** — Consider the *system of differential inclusions*

$$\text{(1)} \qquad \begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) \in Ay(t) + G(x(t), y(t)) \end{cases}$$

The solutions to the inclusion

$$\forall x \in X,\ Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x))$$

are the maps $r : X \mapsto Y$, regarded as *observation maps*, satisfying what is called the *tracking property*: for every $x_0 \in X$, there exists a solution $(x(\cdot), y(\cdot))$ to this system of differential inclusions (1) starting at $(x_0, y_0 = r(x_0))$ and satisfying

$$\forall t \geq 0,\ y(t) = r(x(t)).$$

One can also look for set-valued contingent solutions $R : X \rightsquigarrow Y$ to the inclusion

$$\text{(2)} \qquad \forall (x,y) \in \text{Graph}(R),\ Ay \in DR(x,y)(F(x,y)) - G(x,y)$$

characterizing the *tracking property*: for every $x_0 \in \text{Dom}(R)$ and every $y_0 \in R(x_0)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions

$$\begin{cases} x'(t) \in F(x(t), y(t)) \\ y'(t) \in Ay(t) + G(x(t), y(t)) \end{cases}$$

starting at $(x_0, y_0)$ and satisfying

$$\forall t \geq 0,\ y(t) \in R(x(t)).$$

[1] For several special types of systems of differential equations, the graph of such a map $r$ (satisfying some additional properties) is called a *center manifold*.

**Motivations: Inclusions governing feedback controls** — The partial differential inclusions governing the feedback controls $r : K \mapsto Y$ regulating solutions of a control system $(U, f)$:

$$(3) \quad \begin{cases} \text{i)} & x'(t) = f(x(t), u(t)) \text{ for almost all } t \geq 0 \\ \text{ii)} & u(t) \in U(x(t)) \end{cases}$$

belong to the class studied in this paper, as it was mentioned in [9,11,12]. Here, $U : X \rightsquigarrow Y$ is a closed set-valued map, $f : \text{Graph}(U) \mapsto X$ a continuous (single-valued) map with linear growth and $K = \text{Dom}(U)$. Let $\varphi : \text{Graph}(U) \mapsto \mathbb{R}_+$ be a nonnegative continuous function with linear growth (in the sense that $\varphi(x, u) \leq c(\|x\| + \|u\| + 1)$).

We look for feedback controls $r$ satisfying the following property: *for any $x_0 \in K$, there exists a solution to the differential equation*

$$x'(t) = f(x(t), r(x(t))) \ \& \ x(0) = x_0$$

*such that $u(t) := r(x(t)) \in U(x(t))$ is absolutely continuous and fulfils the growth condition*

$$\|u'(t) - Au(t)\| \leq \varphi(x(t), u(t))$$

*for almost all $t$.* Such feedback controls $r$ are *solutions to the following contingent differential inclusion*

$$\forall x \in K, \ Ar(x) \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraints

$$\forall x \in K, \ r(x) \in U(x).$$

**Outline** — We extend in the first section Hadamard's formula of solutions to linear hyperbolic differential equations to the set-valued case. Namely, we shall prove the existence of a set-valued contingent solutions $R_\star$ to the *decomposable system*

$$\forall (x, y) \in \text{Graph}(R_\star), \ Ay \in DR_\star(x, y)(\Phi(x)) - \Psi(x)$$

where $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ are two Marchaud maps[2], $K \subset X$ is closed and $A \in \mathcal{L}(Y, Y)$.

If we denote by $S_\Phi(x, \cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at $x$, then the set-valued map $R_\star : X \rightsquigarrow Y$ defined by

$$\forall x \in X, \ R_\star(x) := -\int_0^\infty e^{-At}\Psi(S_\Phi(x, t))dt$$

[2] A Marchaud map $\Phi : K \rightsquigarrow Y$ is an upper semicontinuous set-valued map with nonempty compact convex images and with linear growth.

is the *largest contingent solution with linear growth* to this partial differential inclusion when $\lambda := \min_{\|x\|=1}\langle Ax, x\rangle > 0$ is large enough. We also show that it is Lipschitz whenever $\Phi$ and $\Psi$ are Lipschitz and compare the solutions associated with maps $\Phi_i$ and $\Psi_i$ $(i = 1, 2)$.

We then turn our attention in the second section to partial differential inclusions of the form

$$\forall x \in X, \;\; Ar(x) \in Dh(x)(f(x, h(x))) - G(x, h(x))$$

when $\lambda > 0$ is large enough, $f : X \times Y \mapsto X$ is Lipschitz, $G : X \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and satisfies[3]

$$\forall x, y, \;\; \|G(x, y)\| \leq c(1 + \|y\|).$$

When $G$ is single-valued, we obtain a global *Center Manifold Theorem*, stating the existence and uniqueness of an invariant manifold for systems of differential equations with Lipschitz right-hand sides (existence and uniqueness of a contingent solution $r$ has been proved by viscosity methods in [6,7] when $A = \lambda\mathbf{1}$).

We end this paper with comparison theorems between single-valued and set-valued solutions to such partial differential inclusions, using both the extension of Hadamard's formula and some kind of maximum principle.

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**Notations** — If $r : X \mapsto Y$, we set

$$\|r\|_\infty := \sup_{x\in X}\|r(x)\| \in [0, \infty] \;\;\&\;\; \|r\|_\Lambda := \sup_{x\neq y}\frac{\|r(x) - r(y)\|}{\|x - y\|} \in [0, \infty]$$

and we denote by $\mathcal{C}_\Lambda(X, Y)$ the set of all Lipschitz maps from $X$ to $Y$.

When $G : X \rightsquigarrow Y$ is Lipschitz with nonempty closed images, we denote by $\|G\|_\Lambda$ its Lipschitz constant, the smallest of the constants $l$ satisfying

$$\forall z_1, z_2 \in X, \;\; G(z_1) \subset G(z_2) + l\|z_1 - z_2\|B$$

where $B$ is the closed unit ball in $Y$.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L, M) := \sup_{y\in L}\inf_{z\in M} d(y, z) = \sup_{y\in L} d(y, M)$$

3 We set $\|K\| := \sup_{x\in K}\|x\|$ when $K \subset X$. It is equal to $-\infty$ whenever $K$ is empty.

their *semi-Hausdorff distance*[4], and recall that $\Delta(L, M) = 0$ if and only if $L \subset M$. If $\Phi$ and $\Psi$ are two set-valued maps from $X$ to $Y$, we set

$$\Delta(\Phi, \Psi)_\infty = \sup_{x \in X} \Delta(\Phi(x), \Psi(x)) := \sup_{x \in X} \sup_{y \in \Phi(x)} d(y, \Psi(x)).$$

We recall that solutions are always understood as set-valued or single-valued maps *with closed graph.*

## 1. - Contingent Solutions to Decomposable Systems

We need first to establish some properties of contingent set-valued solutions to decomposable systems.

Let $K \subset X$ be a closed subset, $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ be two Marchaud maps and $A \in \mathcal{L}(Y, Y)$. We say that $K$ is a *viability domain* of $\Phi$ if

$$\forall x \in K, \;\; \Phi(x) \cap T_K(x) \neq \emptyset.$$

We set

$$\lambda := \inf_{\|x\|=1} \langle Ax, x \rangle$$

and we observe that

$$\forall y \in Y, \;\; \left\|e^{-At} y\right\| \leq e^{-\lambda t} \|y\|.$$

We look for a solution $R_\star : K \rightsquigarrow Y$ to the *decomposable system*

$$\text{(4)} \qquad \forall (x, y) \in \text{Graph}(R_\star), \;\; Ay \in DR_\star(x, y)(\Phi(x)) - \Psi(x).$$

Denote by $\mathcal{S}_\Phi(x, \cdot)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at $x$ viable in $K$ (in the sense that $x(t) \in K$ for all $t \geq 0$), which exists thanks to the Viability Theorem (see [2,3]).

4 The *Hausdorff distance* between $L$ and $M$ is $\max\left\{\Delta(L, M), \Delta(M, L)\right\}$, which may be equal to $\infty$.

We introduce the set-valued map $R_\star : K \rightsquigarrow Y$ defined[5] by

$$\text{(5)} \qquad \forall x \in K, \;\; R_\star(x) := -\int_0^\infty e^{-At}\Psi(S_\Phi(x,t))dt.$$

THEOREM 1.1. *Assume that $\Phi : K \rightsquigarrow X$ and $\Psi : K \rightsquigarrow Y$ are Marchaud maps and that $K$ is a closed viability domain of $\Phi$. If $\lambda$ is large enough, then $R_\star : K \rightsquigarrow Y$ defined by* (5) *is the* largest contingent solution to inclusion (4) with linear growth *and is bounded whenever $\Psi$ is bounded.*

*More precisely, if there exist positive constants $\alpha, \beta$ and $\gamma$ such that*

$$\forall x \in K, \;\; \|\Phi(x)\| \leq \alpha(\|x\| + 1) \;\; \& \;\; \|\Psi(x)\| \leq \beta + \gamma\|x\|$$

*and if $\lambda > \alpha$, then*

$$\text{(6)} \qquad \forall x \in K, \;\; \|R_\star(x)\| \leq \frac{\beta}{\lambda} + \frac{\gamma}{\lambda - \alpha}(\|x\| + 1).$$

*Furthermore, if $K := X$ and $\Phi$, $\Psi$ are Lipschitz, then $R_\star : X \rightsquigarrow Y$ is also Lipschitz (with nonempty values) whenever $\lambda$ is large enough:*

$$\text{If } \lambda > \|\Phi\|_\Lambda, \;\; R_\star(x_1) \subset R_\star(x_2) + \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda}\|x_1 - x_2\|B$$

*for every $x_1, x_2 \in X$.*

Formula (5) shows also that the graph of $R_\star$ is convex (respectively a convex cone) whenever the graphs of the set- valued maps $\Phi$ and $\Psi$ are convex (respectively are convex cones).

PROOF.

1. — We prove first that the graph of $R_\star$ satisfies contingent inclusion (4).

Indeed, choose an element $y$ in $R_\star(x)$. By definition of the integral of a set-valued map, this means that there exist a solution $x(\cdot) \in S_\Phi(x, \cdot)$ to the

[5] By definition of the integral of a set-valued map (see [8, Chapter 8] for instance), this means that for every $y \in R_\star(x)$, there exist a solution $x(\cdot) \in S_\Phi(x, \cdot)$ to the differential inclusion $x'(t) \in \Phi(x(t))$ starting at $x$ and $z(t) \in \Psi(x(t))$ such that

$$y := -\int_0^\infty e^{-At}z(t)dt.$$

differential inclusion $x'(t) \in \Phi(x(t))$ starting at $x$ which is viable in $K$ and $z(t) \in \Psi(x(t))$ such that

$$y := -\int_0^\infty e^{-At} z(t)dt \in R_\star(x).$$

We check that for every $\tau > 0$

$$-\int_0^\infty e^{-At} z(t+\tau)dt \in R_\star(x(\tau)) = R_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right).$$

By observing that

$$\begin{cases} \frac{1}{\tau}\int_0^\infty e^{-At}(z(t) - z(t+\tau))dt \\ = -\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At}z(t)dt + \frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At}z(t)dt \end{cases}$$

we deduce that

$$\begin{cases} y + \tau\left(-\frac{e^{A\tau}-1}{\tau}\int_0^\infty e^{-At}z(t)dt + \frac{e^{A\tau}}{\tau}\int_0^\tau e^{-At}z(t)dt\right) \\ \qquad \in R_\star\left(x + \tau\left(\frac{1}{\tau}\int_0^\tau x'(t)dt\right)\right). \end{cases}$$

Since $\Phi$ is upper semicontinuous, we know that for any $\varepsilon > 0$ and $t$ small enough, $\Phi(x(t)) \subset \Phi(x) + \varepsilon B$, so that $x'(t) \in \Phi(x) + \varepsilon B$ for almost all small $t$. Therefore, $\Phi(x)$ being closed and convex, we infer that for $\tau > 0$ small enough, $\frac{1}{\tau}\int_0^\tau x'(t)dt \in \Phi(x) + \varepsilon B$ thanks to the Mean-Value Theorem. This latter set being compact, there exists a sequence of $\tau_n > 0$ converging to 0 such that $\frac{1}{\tau_n}\int_0^{\tau_n} x'(t)dt$ converges to some $u \in \Phi(x)$.

In the same way, $\Psi$ being upper semicontinuous, $\Psi(x(t)) \subset \Psi(x) + \varepsilon B$ for any $\varepsilon > 0$ and $t$ small enough, so that $z(t) \in \Psi(x) + \varepsilon B$ for almost all small $t$. The Mean-Value Theorem implies that

$$\forall n > 0,\ \ z_n := \frac{1}{\tau_n}\int_0^{\tau_n} z(t)dt \in \Psi(x) + \varepsilon B$$

since this set is compact and convex. Furthermore, there exists a subsequence of $z_n$ converging to some $z_0 \in \Psi(x)$. Hence, since

$$\frac{1}{\tau_n}\int_0^{\tau_n} \left(e^{-At} - 1\right) z(t)dt \to 0$$

we infer that

$$Ay + z_0 \in DR_\star(x,y)(u)$$

so that $Ay \in DR_\star(x,y)(\Phi(x)) - \Psi(x)$.

2. — Let us prove now that the graph of $R_\star$ is closed when $\lambda$ is large enough. Consider for that purpose a sequence of elements $(x_n, y_n)$ of the graph of $R_\star$ converging to $(x,y)$. There exist solutions $x_n(\cdot) \in S_\Phi(x_n, \cdot)$ to the differential inclusion $x' \in \Phi(x)$ starting at $x_n$, viable in $K$ and measurable selections $z_n(t) \in \Psi(x_n(t))$ such that

$$y_n := -\int_0^\infty e^{-At} z_n(t)dt \in R_\star(x_n).$$

The growth of $\Phi$ being linear, there exists $\alpha > 0$ such that the solutions $x_n(\cdot)$ obey the estimate

$$\|x_n(t)\| \le (\|x_n\| + 1)e^{\alpha t} - 1 \qquad \& \qquad \|x_n'(t)\| \le \alpha(\|x_n\| + 1)e^{\alpha t}.$$

By [8, Theorem 10.1.9], we know that there exists a subsequence (again denoted by) $x_n(\cdot)$ converging uniformly on compact intervals to a solution $x(\cdot) \in S_\Phi(x, \cdot)$.

The growth of $\Psi$ being also linear, we deduce that, setting $u_n(t) := e^{-At} z_n(t)$,

$$\|z_n(t)\| \le \beta + \gamma(\|x_n\| + 1)e^{\alpha t}$$
$$\|u_n(t)\| \le \beta e^{-\lambda t} + \gamma(\|x_n\| + 1)e^{-(\lambda-\alpha)t}.$$

When $\lambda > \alpha$, Dunford-Pettis' Theorem implies that a subsequence (again denoted by) $u_n(\cdot)$ converges weakly to some $u(\cdot) \in L^1(0,\infty;Y)$. This implies that $z_n(\cdot)$ converges weakly to some $z(\cdot)$ in the space $L^1(0,\infty;Y;e^{-\lambda t}dt)$. The Convergence Theorem [8, Therem 7.2.2] states that $z(t) \in \Psi(x(t))$ for almost every $t$. Since the integrals $y_n$ converge to $-\int_0^\infty e^{-At}z(t)dt$, we have proved that

$$y = -\int_0^\infty e^{-At} z(t)dt \in R_\star(x).$$

3. — Estimate (6) is obvious since any solution $x(\cdot) \in S_\Phi(x, \cdot)$ satisfies

$$\forall t \geq 0, \;\; \|x(t)\| \leq (\|x\| + 1)e^{\alpha t}$$

so that, if $\lambda > \alpha$,

$$\|R_\star(x)\| \leq \int_0^\infty e^{-\lambda t} \left(\beta + \gamma(\|x\| + 1)e^{\alpha t}\right) dt = \frac{\beta}{\lambda} + \frac{\gamma}{\lambda - \alpha}(\|x\| + 1).$$

Assume now that $M : K \rightsquigarrow Y$ is any set-valued contingent solution to inclusion (4) with linear growth: there exists $\delta > 0$ such that for all $x \in K$, $\|M(x)\| \leq \delta(\|x\| + 1)$. Since Graph($M$) enjoys the viability property for the set-valued map $(x, y) \rightsquigarrow (\Phi(x), Ay + \Psi(x))$, we know that for any $(x, y) \in \mathrm{Graph}(M)$, there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions

$$(7) \qquad \begin{cases} \text{i)} & x'(t) \in \Phi(x(t)) \\ \text{ii)} & y'(t) - Ay(t) \in \Psi(x(t)) \end{cases}$$

starting at $(x, y)$ such that $y(t) \in M(x(t))$ for all $t \geq 0$. We also know that $\|x(t)\| \leq (\|x\| + 1)e^{\alpha t}$ so that $\|y(t)\| \leq \delta(1 + (\|x\| + 1)e^{\alpha t})$. The second differential inclusion of the above system implies that

$$t \mapsto z(t) := y'(t) - Ay(t)$$

is a measurable selection of $\Psi(x(\cdot))$ satisfying the growth condition

$$\|z(t)\| \leq \beta + \gamma(\|x\| + 1)e^{\alpha t}.$$

Therefore, if $\lambda > \alpha$, the function $e^{-At}z(t)$ is integrable. On the other hand, integrating by parts $e^{-At}z(t) := e^{-At}y'(t) - e^{-At}Ay(t)$, we obtain

$$e^{-AT}y(T) - y = \int_0^T e^{-At}z(t)dt$$

which implies that

$$y = -\int_0^\infty e^{-At}z(t)dt \in R_\star(x)$$

by letting $T \mapsto \infty$. Hence we have proved that[6] $M(x) \subset R_\star(x)$.

4. — Assume now that $K = X$ and that $\Phi$ and $\Psi$ are Lipschitz, take any

6 This proof actually implies that any set-valued contingent solution $M$ with polynomial

pair of elements $x_1$ and $x_2$ and $y_1 = -\int_0^\infty e^{-At} z_1(t)dt \in R_\star(x_1)$, where

$$\text{for some } x_1(\cdot) \in S_\Phi(x_1, \cdot) \ \& \ z_1(t) \in \Psi(x_1(t)) \text{ a.e. in } [0, +\infty[.$$

By the Filippov Theorem[7] there exists a solution $x_2(\cdot) \in S_\Phi(x_2, \cdot)$ such that

$$\forall t \geq 0, \ \|x_1(t) - x_2(t)\| \leq e^{\|\Phi\|_\Lambda t}\|x_1 - x_2\|.$$

We denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi(x_2(t))$, which is measurable thanks to [8, Corollary 8.2.13] and which satisfies

$$\forall t \geq 0, \ \|z_1(t) - z_2(t)\| \leq \|\Psi\|_\Lambda \|x_1(t) - x_2(t)\| \leq \|\Psi\|_\Lambda e^{\|\Phi\|_\Lambda t}\|x_1 - x_2\|.$$

growth in the sense that for some $\rho \geq 0$,

$$\forall x \in X, \ \|M(x)\| \leq \delta(\|x\|^\rho + 1)$$

is contained in $R_\star$ whenever $\lambda > \alpha\rho$, i.e., that there is no contingent solution with polynomial growth other than with linear growth (and bounded when $\gamma = 0$).

[7] Adapted to the case of solutions defined on $[0, \infty[$. Filippov's Theorem (see [5, Theorem 2.4.1] for instance), yields an estimate on any finite interval $[0, T]$: If $\Phi$ is $c$-Lipschitz with nonempty closed values, and if an absolutely continuous function $y(\cdot)$ and an initial state $x_0$ are given, then there exists a solution $x(\cdot)$ to the differential inclusion (7)i) defined on $[0, T]$ starting at $x_0$ and satisfying the estimate

$$\|x(t) - y(t)\| \leq e^{ct}\left(\|x_0 - y(0)\| + \int_0^t d(y'(s), \Phi(y(s)))e^{-cs}ds\right). \tag{8}$$

We can extend it to the interval $[0, +\infty[$. Indeed, there exists a solution $x(\cdot)$ to the differential inclusion defined on $[0, T]$ starting at $x_0$ satisfying estimate (8) and in particular

$$\|x(T) - y(T)\| \leq e^{cT}\left(\|x_0 - y(0)\| + \int_0^T d(y'(s), \Phi(y(s)))e^{-cs}ds\right).$$

There also exists a solution $z(\cdot)$ to the differential inclusion (7)i) starting at $x(T)$ estimating the function $t \mapsto y(t + T)$ and satisfying

$$\|z(t) - y(t + T)\| \leq e^{ct}\left(\|z(0) - y(T)\| + \int_0^t d(y'(s + T), \Phi(y(s + T)))e^{-cs}ds\right).$$

Hence we can extend $x(\cdot)$ on the interval $[0, 2T]$ by concatenating it with the function $t \mapsto x(t) := z(t - T)$ on the interval $[T, 2T]$, we check that the above estimates yield (8) for $t \in [0, 2T]$ and we reiterate this process. See the forthcoming monograph [23].

Therefore, if $\lambda > \|\Phi\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $R_\star(x_2)$ and satisfies

$$\|y_1 - y_2\| \le \int_0^\infty \|\Psi\|_\Lambda e^{-t(\lambda - \|\Phi\|_\Lambda)} \|x_1 - x_2\| dt \le \frac{\|\Psi\|_\Lambda}{\lambda - \|\Phi\|_\Lambda} \|x_1 - x_2\| \qquad \square$$

THEOREM 1.2. *Consider now two pairs* $(\Phi_1, \Psi_1)$ *and* $(\Phi_2, \Psi_2)$ *of Marchaud maps defined on* $X$ *and their associated solutions*

$$\forall x \in X, \ \ R_{\star i}(x) := -\int_0^\infty e^{-At} \Psi_i(S_{\Phi_i}(x,t))dt \qquad (i = 1, 2)$$

*to inclusion* (4). *If the set-valued maps* $\Phi_2$ *and* $\Psi_2$ *are Lipschitz, and if* $\lambda > \|\Phi_2\|_\Lambda$, *then*

$$\Delta(R_{\star_1}, R_{\star_2})_\infty \le \frac{1}{\lambda} \Delta(\Psi_1, \Psi_2)_\infty + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)} \Delta(\Phi_1, \Phi_2)_\infty.$$

PROOF. Choose $y_1 = -\int_0^\infty e^{-At} z_1(t)dt \in R_{\star 1}(x)$ where

$$x_1(\cdot) \in S_{\Phi_1}(x, \cdot) \ \ \& \ \ z_1(t) \in \Psi_1(x_1(t)).$$

In order to compare $x_1(\cdot)$ with the solution-set $S_{\Phi_2}(x, \cdot)$ via the Filippov Theorem, we use the estimate

$$d(x_1'(t), \Phi_2(x_1(t))) \le \sup_{z \in \Phi_1(x_1(t))} d(z, \Phi_2(x_1(t))) \le \Delta(\Phi_1, \Phi_2)_\infty.$$

Therefore, there exists a solution $x_2(\cdot) \in S_{\Phi_2}(x, \cdot)$ such that

$$\forall t \ge 0, \ \ \|x_1(t) - x_2(t)\| \le \Delta(\Phi_1, \Phi_2)_\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda}$$

by Filippov's Theorem. As before, we denote by $z_2(t)$ the projection of $z_1(t)$ onto the closed convex set $\Psi_2(x_2(t))$, which is measurable and satisfies

$$\begin{cases} \forall t \ge 0, \ \ \|z_1(t) - z_2(t)\| \le \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \|x_1(t) - x_2(t)\| \\ \le \Delta(\Psi_1, \Psi_2)_\infty + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \left(e^{t\|\Phi_2\|_\Lambda} - 1\right) / \|\Phi_2\|_\Lambda. \end{cases}$$

Therefore, if $\lambda > \|\Phi_2\|_\Lambda$, $y_2 = -\int_0^\infty e^{-At} z_2(t)dt$ belongs to $R_{\star 2}(x)$ and satisfies

$$\begin{cases} \|y_1 - y_2\| \\ \leq \int_0^\infty e^{-\lambda t} \Delta(\Psi_1, \Psi_2)_\infty dt + \|\Psi_2\|_\Lambda \Delta(\Phi_1, \Phi_2)_\infty \int_0^\infty \frac{e^{t\|\Phi_2\|_\Lambda} - 1}{\|\Phi_2\|_\Lambda} e^{-\lambda t} dt \\ \leq \frac{\Delta(\Psi_1, \Psi_2)_\infty}{\lambda} + \frac{\|\Psi_2\|_\Lambda}{\lambda(\lambda - \|\Phi_2\|_\Lambda)} \Delta(\Phi_1, \Phi_2)_\infty. \quad \square \end{cases}$$

When $\Phi := \varphi$, $\Psi := \psi$ are single-valued, we obtain:

PROPOSITION 1.3. *Assume that $\varphi : X \mapsto X$ and $\psi : X \mapsto Y$ are Lipschitz and that $\psi$ is bounded. Then when $\lambda > 0$, the map $r := \Gamma(\varphi, \psi)$ defined by*

$$r(x) = -\int_0^\infty e^{-At} \psi(S_\varphi(x,t)) dt$$

*is the unique bounded single-valued solution to the contingent inclusion*

$$(9) \qquad Ar(x) \in Dr(x)(\varphi(x)) - \psi(x)$$

*and satisfies*

$$(10) \qquad \|r\|_\infty \leq \frac{\|\psi\|_\infty}{\lambda} \ \& \ \text{ if } \lambda > \|\varphi\|_\Lambda, \ \|r\|_\Lambda \leq \frac{\|\psi\|_\Lambda}{\lambda - \|\varphi\|_\Lambda}.$$

*Furthermore, for all Lipschitz single-valued maps $\varphi_i : X \mapsto X$, $\psi_i : X \mapsto Y$, $i = 1, 2$ such that $\psi_1$, $\psi_2$ are bounded and all $\lambda > \|\varphi_2\|_\Lambda$*

$$(11) \qquad \|\Gamma(\varphi_1, \psi_1) - \Gamma(\varphi_2, \psi_2)\|_\infty \leq \frac{\|\psi_1 - \psi_2\|_\infty}{\lambda} + \frac{\|\psi_2\|_\Lambda}{\lambda(\lambda - \|\varphi_2\|_\Lambda)} \|\varphi_1 - \varphi_2\|_\infty.$$

The proof can be derived from Theorems 1.1 and 1.2 or directly from the properties of linear systems of hyperbolic equations established in [7].

## 2. - Existence of a Lipschitz contingent solution

We shall now prove the existence of a contingent single-valued solution to inclusion

$$(12) \qquad \forall x \in X, \ Ar(x) \in Dr(x)(f(x, r(x))) - G(x, r(x)).$$

THEOREM 2.1. *Assume that the map $f : X \times Y \mapsto X$ is Lipschitz, that $G : X \rightsquigarrow Y$ is Lipschitz with nonempty convex compact values and that*

$$\forall x, y, \ \ \|G(x, y)\| \leq c(1 + \|y\|)$$

*for some $c > 0$.*

*Then if $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ (where $\nu$ is the dimension of $X$), there exists a bounded Lipschitz contingent solution to the partial differential inclusion* (12).

PROOF. Since for every Lipschitz single-valued map $s(\cdot)$, the set-valued map $x \rightsquigarrow G(x, s(x))$ is Lipschitz (with constant $\|G\|_\Lambda(1 + \|s\|_\Lambda)$ and has convex compact values, [8, Theorem 9.4.1] implies that the subset $G_s$ of Lipschitz selections $\psi$ of the set-valued map $x \rightsquigarrow G(x, s(x))$ with Lipschitz constant not larger than $\nu\|G\|_\Lambda(1 + \|s\|_\Lambda)$ is not empty (where $\nu$ denotes the dimension of $X$). We denote by $\varphi_s$ the Lipschitz map defined by $\varphi_s(x) := f(x, s(x))$, with Lipschitz constant equal to $\|f\|_\Lambda(1 + \|s\|_\Lambda)$.

The solutions $r$ to inclusion (12) are the fixed points to the set-valued map $\mathcal{H} : \mathcal{C}_\Lambda(X, Y) \rightsquigarrow \mathcal{C}(X, Y)$ defined by

$$\mathcal{H}(s) := \{\Gamma(\varphi_s, \psi)\}_{\psi \in G_s}. \tag{13}$$

Indeed, if $r \in \mathcal{H}(r)$, there exists a selection $\psi \in G_r$ such that

$$Ar(x) \in Dr(x)(f(x, r(x))) - \psi(x) \subset Dr(x)(f(x, r(x)))G(x, r(x)).$$

Since $\|G(x, y)\| \leq c(1 + \|y\|)$, we deduce that any selection $\psi \in G_s$ satisfies

$$\|\psi\|_\infty \leq c(1 + \|s\|_\infty).$$

Therefore, Proposition 1.3 implies that if $\lambda$ is large enough,

$$\forall r \in \mathcal{H}(s), \ \ \|r\|_\infty \leq \frac{c}{\lambda}(1 + \|s\|_\infty) \ \& \ \|r\|_\Lambda \leq \frac{\nu\|G\|_\Lambda(1 + \|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1 + \|s\|_\Lambda)}.$$

We first observe that when $\lambda > c$,

$$\forall s \in \mathcal{C}_\Lambda(X, Y) \text{ such that } \|s\|_\infty \leq \frac{c}{\lambda - c}, \ \forall r \in \mathcal{H}(s), \ \ \|r\|_\infty \leq \frac{c}{\lambda - c}.$$

When $\lambda > 4\nu\|f\|_\Lambda\|G\|_\Lambda$, we denote by

$$\rho(\lambda) := \frac{\lambda - \|f\|_\Lambda - \nu\|G\|_\Lambda\sqrt{\lambda^2 - 2\lambda(\|f\|_\Lambda + \nu\|G\|_\Lambda) + (\|f\|_\Lambda - \nu\|G\|_\Lambda)^2}}{2\|f\|_\Lambda}$$

the smallest root of the equation

$$\lambda\rho = \|f\|_\Lambda\rho^2 + (\|f\|_\Lambda + \nu\|G\|_\Lambda)\rho + \nu\|G\|_\Lambda$$

which is positive. We observe that

$$\lim_{\lambda \to +\infty} \lambda \rho(\lambda) = \nu \|G\|_\Lambda$$

and infer that

$$\forall s \in \mathcal{C}_\Lambda(X,Y) \text{ such that } \|s\|_\Lambda \leq \rho(\lambda), \ \forall r \in \mathcal{H}(s), \ \|r\|_\Lambda \leq \rho(\lambda)$$

because $r$ being of the form $\Gamma(\varphi_s, \psi_s)$, satisfies by Proposition 1.3:

$$\|r\|_\Lambda \leq \frac{\|\psi_s\|_\Lambda}{\lambda - \|\varphi_s\|_\Lambda} \leq \frac{\nu\|G\|_\Lambda(1+\|s\|_\Lambda)}{\lambda - \|f\|_\Lambda(1+\|s\|_\Lambda)} \leq \frac{\nu\|G\|_\Lambda(1+\rho(\lambda))}{\lambda - \|f\|_\Lambda(1+\rho(\lambda))} = \rho(\lambda).$$

Let us denote by $B^1_\infty(\lambda)$ the subset of $\mathcal{C}_\Lambda(X,Y)$ defined by

$$B^1_\infty(\lambda) := \left\{ r \in \mathcal{C}_\Lambda(X,Y) \mid \|r\|_\infty \leq \frac{c}{\lambda - c} \ \& \ \|r\|_\Lambda \leq \rho(\lambda) \right\}$$

which is compact (for the compact convergence topology) thanks to Ascoli's Theorem.

We have therefore proved that if $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$, the set-valued map $\mathcal{H}$ sends the compact subset $B^1_\infty(\lambda)$ to itself.

It is obvious that the values of $\mathcal{H}$ are convex. Kakutani's Fixed-Point Theorem implies the existence of a fixed point $r \in \mathcal{H}(r)$ if we prove that the graph of $\mathcal{H}$ is closed.

Actually, the graph of the restriction of $\mathcal{H}$ to $B^1_\infty(\lambda)$ is compact. Indeed, let us consider any sequence $(s_n, r_n) \in \text{Graph}(\mathcal{H})$ such that $s_n \in B^1_\infty(\lambda)$. Since $B^1_\infty(\lambda)$ is compact, a subsequence (again denoted by) $(s_n, r_n)$ converges to some function

$$(s, r) \in B^1_\infty(\lambda) \times B^1_\infty(\lambda).$$

But there exist bounded Lipschitz selections $\psi_n \in G_{s_n}$ with Lipschitz constant $\nu\|G\|_\Lambda(1+\rho(\lambda))$ such that

$$\forall n \geq 0, \ \ r_n = \Gamma(\varphi_{s_n}, \psi_n).$$

Therefore a subsequence (again denoted by) $\psi_n$ converges to some function $\psi \in G_s$. Since $\varphi_{s_n}$ converges obviously to $\varphi_s$, we infer that $r_n$ converges to $\Gamma(\varphi_s, \psi)$, i.e., that $r \in \mathcal{H}(s)$, since $\Gamma$ is continuous by formula (11) of Proposition 1.3.

## 3. - Comparison Results

The point of this section is to compare two solutions to inclusion (12), or even, a single-valued solution and a contingent set-valued solution $M : X \rightsquigarrow Y$.

We first deduce from Theorem 1.2 the following "localization property":

THEOREM 3.1. *We posit the assumptions of Theorem* 2.1 *with* $A \in \mathcal{L}(Y,Y)$ *such that* $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$ *(where* $\nu$ *is the dimension of* $X$*). Let* $\Phi : X \rightsquigarrow X$ *and* $\Psi : X \rightsquigarrow Y$ *be two Lipschitz and Marchaud maps with which we associate the set-valued map* $R_\star$ *defined by*

$$\forall x \in X,\ \ R_\star(x) := -\int_0^\infty e^{-At}\Psi(S_\Phi(x,t))dt.$$

*Then any single-valued contingent solution* $r(\cdot)$ *to inclusion* (12) *having linear growth satisfies the following estimate*

$$\begin{cases} \forall x \in X,\ \ d(r(x), R_\star(x)) \le \\ \dfrac{1}{\lambda}\sup_{x\in X}\Delta(G(x,r(x)),\Psi(x)) + \dfrac{\|\Psi\|_\Lambda}{\lambda(\lambda - \|\Phi\|_\Lambda)}\sup_{x\in X} d(f(x,r(x)),\Phi(x)). \end{cases}$$

*In particular, if we assume that*

$$\forall y \in Y,\ \ f(x,y) \in \Phi(x)\ \&\ G(x,y) \subset \Psi(x)$$

*then all single-valued contingent solutions* $r(\cdot)$ *to inclusion* (12) *with linear growth are selections of* $R_\star$.

PROOF. Let $r$ be any single-valued contingent solution to inclusion (12) with linear growth. One can show that $r$ can be written in the form

$$r(x) = -\int_0^\infty e^{-At}z(t)dt \ \text{ where } z(t) \in G(x(t), r(x(t)))$$

by using the same arguments as in the third part of the proof of Theorem 1.1.

We also adapt the proof of Theorem 1.2 with $\Phi_1 := f(x, r(x))$, $z_1(t) := z(t)$, $\Phi_2 := \Phi$ and $\Psi_2 := \Psi$, to show that the estimates stated in the theorem hold true. □

The next comparison results are consequences of the following kind of *maximum principle*.

We recall that when $M$ is Lipschitz around $x$ and $y \in M(x)$, its *adjacent derivative* $D^\flat M(x,y) \subset DM(x,y)$ is defined by

$$v \in D^\flat M(x,y)(u) \text{ if and only if } \lim_{h\to 0+} d\left(v, \frac{M(x+hu)-y}{h}\right) = 0.$$

A set-valued map $M$ is said to be *derivable* at $(x,y) \in \text{Graph}(M)$ if the contingent and adjacent derivatives coincide at $(x,y)$ and derivable if it is derivable at every point of its graph. See [8, Chapter 5] for more details.

LEMMA 3.2. (MAXIMUM PRINCIPLE) *We posit the assumptions of Theorem 2.1 with $A \in \mathcal{L}(Y,Y)$ such that $\lambda > \max(c, 4\nu\|f\|_\Lambda\|G\|_\Lambda)$. Let $M$ be a Lipschitz set-valued map such that $D^\flat M(x,y)(f(x,y))$ is nonempty for every $(x,y) \in$ Graph$(M)$. Let $r$ be any Lipschitz single-valued solution to (12) and set*

$$\Gamma(x) := G(x, r(x)) \cap (Dr(x)(f(x, r(x))) - Ar(x)).$$

*If the supremum*

$$\delta := \sup_{(x,y)\in \text{Graph}(M)} \|r(x) - y\|$$

*is finite, then*

$$\delta \leq \frac{1}{\lambda} \sup_{(x,y)\in\text{Graph}(M)} d\left(\Gamma(x), \overline{co}(D^\flat M(x,y)(f(x,r(x)))) - Ay\right).$$

*The same conclusion holds true if we assume that the solution $r$ is derivable and when we replace the adjacent derivative of $M$ by its contingent derivative.*

PROOF. It is sufficient to consider the case when the supremum

$$\delta := \sup_{(x,y)\in\text{Graph}(M)} \|r(x) - y\| = \|r(\bar{x}) - \bar{y}\|$$

is achieved[8] at some $(\bar{x}, \bar{y})$ of the graph of $M$ and when $\delta > 0$.

Let us take $\psi := v - Ar(\bar{x})$ in the set

$$G(\bar{x}, r(\bar{x})) \cap (Dr(\bar{x})(f(\bar{x}, r(\bar{x}))) - Ar(\bar{x})).$$

Set $u := f(\bar{x}, r(\bar{x}))$. Since $r$ is Lipschitz, there exists a sequence $h_n > 0$ converging to 0 such that

$$\frac{r(\bar{x} + h_n u) - r(\bar{x})}{h_n} \text{ converges to } v.$$

Since $M$ is Lipschitz, we deduce that for any $w \in D^\flat M(\bar{x}, \bar{y})(u)$, there exists a sequence $w_n$ converging to $w$ such that $\bar{y} + h_n w_n \in M(\bar{x} + h_n u)$. Thus

$$\|r(\bar{x}) - \bar{y}\| \geq \left\| r(\bar{x}) - \bar{y} + h_n\left(\frac{r(\bar{x} + h_n u) - r(\bar{x})}{h_n} - w_n\right)\right\|.$$

Therefore,

$$\forall w \in D^\flat M(\bar{x}, \bar{y})(u), \langle r(\bar{x}) - \bar{y}, v - w\rangle \leq 0$$

8 If the nonnegative bounded function $\chi(x,y) := \|r(x) - y\|$ does not achieve its maximum, we use a standard argument which can be found in [17,26] for instance. One can find approximate maxima $(x_n, y_n)$ such that $\chi(x_n, y_n)$ converges to $\sup_{(x,y)\in\text{Graph}(M)} \chi(x,y)$ and $\chi'(x_n, y_n)$ converges to 0.

and we infer that

$$\forall w \in \overline{co}(D^\flat M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x})))),\ \langle r(\bar{x})-\bar{y}, A(r(\bar{x})-\bar{y})+A\bar{y}+\psi-w\rangle \le 0$$

from which we obtain the estimate

$$\begin{cases} \lambda\|r(\bar{x})-\bar{y}\| \\ \le \inf_{\psi\in\Gamma(\bar{x}),\, w\in\overline{co}(D^\flat M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x}))))} \|A\bar{y}+\psi-w\| \\ = d\Big(\Gamma(\bar{x}), \overline{co}(D^\flat M(\bar{x},\bar{y})(f(\bar{x},r(\bar{x})))) - A\bar{y}\Big). \end{cases} \qquad \square$$

We use this Lemma to compare two solutions to inclusion (12):

THEOREM 3.3. *We posit the assumptions of Theorem* 2.1. *Let* $r_1$ *and* $r_2$ *be two Lipschitz contingent solutions to* (12). *If* $r_2$ *is differentiable and if* $\lambda > \|r_2\|_\Lambda \|f\|_\Lambda$, *then*

$$\|r_1-r_2\|_\infty \le \sup_{x\in X} \frac{\|G(x,r_1(x))-G(x,r_2(x))\|}{\lambda - \|r_2\|_\Lambda\|f\|_\Lambda}.$$

*When* $f$ *does not depend on* $y$, *we can take* $\|f\|_\Lambda = 0$ *in the above estimate.*

*In particular, when* $G$ *does not depend on* $y$, *we deduce that*

$$\|r_1-r_2\|_\infty \le \sup_{x\in X} \frac{\mathrm{Diam}(G(x))}{\lambda - \|r_2\|_\Lambda\|f\|_\Lambda}.$$

More generally, let us consider a set-valued contingent solution $M : X \rightsquigarrow Y$ to the inclusion

$$(14) \qquad \forall (x,y) \in \mathrm{Graph}(M),\ Ay \in DM(x,y)(f(x,y)) - G(x,y).$$

THEOREM 3.4. *We posit the assumptions of Theorem* 2.1. *Let* $r$ *be a Lipschitz contingent solution to* (12) *and* $M$ *be a Lipschitz set-valued contingent solution to inclusion* (14) *in the stronger sense that for every* $(x,y) \in \mathrm{Graph}(M)$, *there exists a Lipschitz closed convex process* $E(x,y) \subset \overline{co}(D^\flat M(x,y))$ *satisfying*

$$\forall (x,y) \in \mathrm{Graph}(M),\ Ay \in E(x,y)(f(x,y)) - G(x,y)$$

*and*

$$\|E\|_\Lambda := \sup_{(x,y)\in\mathrm{Graph}(M)} \|E(x,y)\|_\Lambda < +\infty.$$

*Assume also that the supremum*

$$\delta := \sup_{(x,y)\in\mathrm{Graph}(M)} \|r(x)-y\|$$

*is finite and that* $\lambda > \|E\|_\Lambda \|f\|_\Lambda$. *Then*

$$\sup_{(x,y)\in \mathrm{Graph}(M)} \|r(x) - y\| \leq \sup_{(x,y)\in \mathrm{Graph}(M)} \frac{\|G(x, r(x)) - G(x, y)\|}{\lambda - \|E\|_\Lambda \|f\|_\Lambda}$$

*or, equivalently,*

$$\forall (x, y) \in \mathrm{Graph}(M),\ \ M(x) \subset r(x) + \sup_{(x,y)\in \mathrm{Graph}(M)} \frac{\|G(x, r(x)) - G(x, y)\|}{\lambda - \|E\|_\Lambda \|f\|_\Lambda} B.$$

*When* $f$ *does not depend on* $y$, *we can take* $\|f\|_\Lambda = 0$ *in the above estimates.*

*In particular, when* $G$ *does not depend on* $y$, *we deduce that*

$$\forall (x, y) \in \mathrm{Graph}(M),\ \ M(x) \subset r(x) + \sup_{x\in \mathrm{Dom}(M)} \frac{\mathrm{Diam}(G(x))}{\lambda - \|E\|_\Lambda \|f\|_\Lambda} B.$$

PROOF. By Lemma 3.2, it is enough to show that for every $(x, y) \in \mathrm{Graph}(M)$ and

$$\psi \in G(x, r(x)) \cap \Big(Dr(x)(f(x, r(x))) - Ar(x)\Big)$$

there exists

$$w \in \overline{co}\Big(D^\flat M(x, y)(f(x, r(x)))\Big)$$

such that

$$\|\psi - (w - Ay)\| \leq \|G(x, r(x)) - G(x, y)\| + \|E\|_\Lambda \|f\|_\Lambda \delta.$$

Take any such $\psi$. By assumption, we know that the norms of the closed convex processes $E(x, y)$ are bounded by $\|E\|_\Lambda$ and that

$$\begin{cases} Ay \in E(x, y)(f(x, y)) - G(x, y) \\ \subset E(x, y)(f(x, r(x))) + E(x, y)(f(x, y) - f(x, r(x))) - G(x, y). \end{cases}$$

Then there exist

$$w \in E(x, y)(f(x, r(x))) \subset \overline{co}\Big(D^\flat M(x, y)(f(x, r(x)))\Big)$$

and $\psi' \in G(x, y)$ satisfying

$$\|Ay - w + \psi'\| \leq \|E\|_\Lambda \|f\|_\Lambda \|r(x) - y\| \leq \|E\|_\Lambda \|f\|_\Lambda \delta.$$

Hence

$$\begin{cases} \|\psi - (w - Ay)\| \leq \|Ay - w + \psi'\| + \|\psi - \psi'\| \\ \leq \|E\|_\Lambda \|f\|_\Lambda \delta + \|G(x, r(x)) - G(x, y)\| \\ \leq \|E\|_\Lambda \|f\|_\Lambda \delta + \sup_{(x,y) \in \mathrm{Graph}(M)} \|G(x, r(x)) - G(x, y)\| \end{cases}$$

from which the conclusion of Theorem 3.4 follows. □

Uniqueness follows when $\lambda$ is large enough and when we assume the existence of a set-valued map $M$ the graph of which is an *invariance domain* of the set-valued map $(x,y) \rightsquigarrow f(x,y) \times (Ay + G(x,y))$, in the sense that[9]

$$\forall (x,y) \in \mathrm{Graph}(M), \; G(x,y) + Ay \subset DM(x,y)(f(x,y)).$$

We need to use the *circatangent derivative* $CM(x,y)$ of $M$ at $(x,y)$ defined by

$$v \in CM(x,y)(u) \text{ if and only if } \lim_{\substack{(x',y') \to_G (x,y) \\ h \to 0+}} d\left(v, \frac{M(x' + hu) - y'}{h}\right) = 0$$

where $\to_G$ denotes the convergence in the graph of $G$. See [8, Chapter 4] for more details.

THEOREM 3.5. *We posit the assumptions of Theorem* 2.1. *Assume that the graph of the Lipschitz set-valued map* $M$ *is an invariance domain of* $(x,y) \rightsquigarrow f(x,y) \times (Ay + G(x,y))$ *and that there exists Lipschitz closed convex process* $E$ *satisfying*

$$\forall (x,y) \in \mathrm{Graph}(M), \; CM(x,y) \subset E(x,y) \subset \overline{co}(D^\flat M(x,y))$$

*and that*

$$\|E\|_\Lambda := \sup_{(x,y) \in \mathrm{Graph}(M)} \|E(x,y)\|_\Lambda < +\infty.$$

9 One can prove that when $F$ is Lipschitz with closed values and the graph of $M$ is closed, then Graph($M$) is an *invariance domain* if and only if it is invariant in the sense that for any $(x_0, y_0) \in$ Graph($M$), every solution to the system of differential inclusions

$$\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) \in Ay(t) + G(x(t), y(t)) \end{cases}$$

starting at $(x_0, y_0)$ satisfies

$$\forall t \geq 0, \; y(t) \in M(x(t)).$$

*If $\lambda$ is large enough, then $M(x) = \{r(x)\}$ for any (single-valued) contingent solution $r$ to inclusion* (12) *such that the supremum*

$$\delta := \sup_{(x,y)\in\mathrm{Graph}(M)} \|r(x) - y\|$$

*is finite.*

PROOF. Since $f$ and $G$ are lower semicontinuous, we know from [8, Theorem 4.1.9] that inclusion

$$\forall (x,y) \in \mathrm{Graph}(M),\ \ G(x,y) + Ay \subset DM(x,y)(f(x,y))$$

holds true with the circatangent derivative $CM(x,y)$ (which is a closed convex process), so that

$$\forall (x,y) \in \mathrm{Graph}(M),\ \ G(x,y) + Ay \subset CM(x,y)(f(x,y)) \subset E(x,y)(f(x,y)).$$

Observe that it is sufficient to prove that

$$\lambda\delta \leq \|G\|_\Lambda \delta + \|E\|_\Lambda \|f\|_\Lambda \delta$$

which implies that $\delta = 0$ whenever $\lambda > \|G\|_\Lambda + \|E\|_\Lambda\|f\|_\Lambda$.

By Lemma 3.2, it is enough to show that for every $(x,y) \in \mathrm{Graph}(M)$ and

$$\psi \in G(x, r(x)) \cap \Big(Dr(x)(f(x, r(x))) - Ar(x)\Big)$$

there exists

$$w \in \overline{co}\Big(D^\flat M(x,y)(f(x, r(x)))\Big)$$

such that

$$\|\psi - (w - Ay)\| \leq \|G\|_\Lambda \delta + \|E\|_\Lambda \|f\|_\Lambda \delta.$$

Take any such $\psi$. Since $G$ is Lipschitz, we infer that

$$\psi \in G(x, r(x)) \subset G(x,y) + \|G\|_\Lambda \|r(x) - y\| B \subset G(x,y) + \|G\|_\Lambda \delta B.$$

Therefore,

$$Ay + \psi \in Ay + G(x,y) + \|G\|_\Lambda \delta B \subset CM(x,y)(f(x,y)) + \|G\|_\Lambda \delta B$$

and, $E(x,y)$ being a closed convex process with a norm less than or equal to $\|E\|_\Lambda$,

$$\begin{cases} E(x,y)(f(x,y)) \subset E(x,y)(f(x,r(x))) + E(x,y)(f(x,y) - f(x,r(x))) \\ \subset E(x,y)(f(x,r(x))) + \|E\|_\Lambda \|f\|_\Lambda \delta. \end{cases}$$

Hence there exists

$$w \in E(x,y)(f(x,r(x))) \subset \overline{co}\Big(D^{\flat}M(x,y)(f(x,r(x)))\Big)$$

such that

$$\|Ay + \psi - w\| \leq \|G\|_{\Lambda}\delta + \|E\|_{\Lambda}\|f\|_{\Lambda}\delta. \qquad \square$$

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