HYPERBOLIC TRIGONOMETRY DERIVED FROM THE POINCARE MODEL

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HYPERBOLIC TRIGONOMETRY DERIVED FROM THE POINCARÉ MODEL

1. INTRODUCTION

The trigonometric formulas of hyperbolic geometry have been derived in a number of ingenious ways. As early as 1766 Lambert (9) noted that the geometry of the "third hypothesis" could be verified on a sphere of imaginary radius, and all the formulas of hyperbolic plane trigonometry could be obtained from those of ordinary spherical trigonometry by replacing the radius r by ir. The historical process, developed by both Bolyai and Lobachewsky, made use of the elegant fact that the geometry of horocycles on the horosphere is euclidean in nature (12, pp. 360-374). Sommerville (14, pp. 56 ff. and 84) has presented an excellent elementary treatment along these lines. Some early writers, however, regretted this appeal to solid geometry in order to derive formulas for a plane trigonometry. Clever methods were devised to remedy the seeming defect, one of the neatest being due to Liebmann (10, chapt. III), and subsequently reproduced by such writers as Carslaw (1, chapt. IV) and Wolfe (15, chapt. V). Other dodges were devised by Gérard (7), Young (16), and Fulton (6). A very careful treatment based upon the fact that hyperbolic geometry is euclidean in character in an infinitesimal domain was supplied by Coolidge (3, chapt. IV).

In order to establish the relative consistency of hyperbolic geometry and some second body of mathematics, it suffices to devise a model in the second body containing elements, with appropriate connecting relations, which, when substituted for the undefined elements and relations of a postulate set for hyperbolic geometry, will interpret those postulates as true theorems in the chosen body of mathematics. Since it is usually desired to establish the relative consistency of the hyperbolic and euclidean geometries, many euclidean models have been devised, the most famous ones being due to Beltrami, Cayley, Klein, and Poincaré (4, chapt. XIV). Once such a model has been devised it is conceivable that some theorems of hyperbolic geometry might be more readily established by demonstrating the analogues in the model rather than the originals directly from the accepted postulate basis (see appendix I). In particular, it may be that the trigonometry of the hyperbolic plane, which is usually established directly within the hyperbolic system only by means of more or less clever and complicated devices, can be rather easily established from one of the euclidean models. It is the purpose of this paper to so develop hyperbolic plane trigonometry, selecting for the model one that was exploited by Poincaré (11), and which is singularly elementary in nature. Carslaw (1, chapt. VIII, and 2) has already shown the utility of this particular model by very simply establishing from it several difficult theorems of hyperbolic geometry, and O. D. Smith, in his Oregon State College Master's Thesis (13), has, in a very elementary manner, developed a large portion of the hyperbolic geometry in this way.

2. DESCRIPTION OF THE POINCARÉ MODEL

As an acceptable postulate set for plane hyperbolic geometry let us select that of Hilbert (8) (see appendix II). The primitive terms for this postulate set are point, line, between (applied to three points on a line), congruent segments, and congruent angles. The postulates are statements concerning these primitive terms. A euclidean model of plane hyperbolic geometry must, then, be a system of geometrical elements and relations which, when substituted for the primitive terms, convert the Hilbert postulates into true theorems in euclidean geometry. The Poincaré model accomplishes this as follows. A fixed circle, Σ , is selected and called the fundamental circle. We then set up the following "dictionary":

1. a point of the hyperbolic plane	1. a point interior to Σ (hereafter called a nominal point)
2. a line of the hyperbolic plane	2. the arc interior to Σ of any circle orthogonal to Σ (hereafter called a nominal line)
3. point C lies between A and B	3. nominal point C lies between nominal points A and B on the nominal segment determined by A and B

We now define the (positive) nominal length of a nominal segment AB as

$$\overline{AB} = \log_a(AB,TS) = k \ln(AB,TS), \quad k = -\ln a,$$

where S and T are the points where the nominal line AB meets

Σ , A lying between S and B, and (AB,TS) denotes the anharmonic ratio (AT/BT)/(AS/BS) of the circular range A, B, T, S. Also, we define the nominal measure of the angle between two intersecting nominal lines as the ordinary radian measure of the angle between the two circles on which the nominal lines lie. Concluding our "dictionary" we then take

4. segment AB is congruent to segment A'B'	4. nominal segments AB and A'B' have equal nominal lengths (hereafter said to be nominally congruent)
5. angle ACB is congruent to angle A'C'B'	5. nominal angles ACB and A'C'B' have equal nominal measures (hereafter said to be nominally congruent)

It can be shown that with this "dictionary" the Hilbert postulates for plane hyperbolic geometry become true theorems in euclidean geometry. For every theorem in hyperbolic plane geometry there is the euclidean counterpart in the Poincaré model, and the establishment of the latter carries with it that of the former. We now proceed to establish hyperbolic plane trigonometry by obtaining the necessary counterparts in the Poincaré model.

3. PRELIMINARY THEOREMS

Consider any nominal right triangle O'P'Q', right angled at Q' (see fig. 1), and let the circles determined by the nominal lines O'P' and O'Q' intersect again in C. Invert the figure with respect to C as center and with a power that carries Σ into itself. Since inversion is a conformal transformation, the circles CP'O', CQ'O', being orthogonal to Σ and passing through the center of inversion C, invert into two diametral lines of Σ. Thus, by the inversion, the right triangle O'P'Q' is carried into the right triangle OPQ, where OP and OQ are radial lines of Σ. Since both angles and anharmonic ratios are preserved under inversion, it follows that nominal triangles O'P'Q' and OPQ are nominally congruent, and, to obtain the fundamental formulas of hyperbolic plane trigonometry, it suffices to study the relations connecting the nominal lengths of the sides and the nominal measures of the angles of the specially placed right triangle OPQ. We shall consistently distinguish euclidean lengths from nominal lengths by placing bars over the latter. Since angles have the same nominal and euclidean measures, no bars are here needed.

Let the circle Π determined by the nominal line QP cut Σ in S and T, Q lying between S and P (see fig. 2), and let IOQJ and MOPN be diameters of Σ, IJ cutting Π again in W.

We now establish a short chain of theorems connected with fig. 2.

THEOREM 3.1. <u>If</u> WS <u>and</u> WT <u>cut</u> Σ <u>again in</u> U <u>and</u> V, <u>then</u> UV <u>is the diameter of</u> Σ <u>perpendicular to diameter</u> IJ.

Select W as center of inversion, and choose a power such that Σ inverts into itself. Then S inverts into U, and T into V. Since Π is orthogonal to both Σ and IJ, it follows that UV is the diameter of Σ perpendicular to diameter IJ.

THEOREM 3.2. <u>Let</u> WP <u>cut</u> UV <u>in</u> R, <u>and designate the lengths of</u> OW <u>and</u> OR <u>by</u> m <u>and</u> n, <u>and the radius of</u> Σ <u>by</u> r. <u>Let</u> K <u>be the center of</u> Π <u>and let</u> M <u>and</u> N <u>be the feet of the perpendiculars dropped from</u> P <u>on</u> OW <u>and</u> OR <u>respectively</u>. <u>Then</u>

(a) $KP = (m^2 - r^2)/2m,$

(b) $OM = m(n^2 + r^2)/(m^2 + n^2),$

(c) $OP = (m^2n^2 + r^4)^{\frac{1}{2}}/(m^2 + n^2)^{\frac{1}{2}},$

(d) $OQ = r^2/m.$

Since $KP = OW - OK = m - (r^2 + KP^2)^{\frac{1}{2}}$, it follows that

$$KP = (m^2 - r^2)/2m.$$

Also, since tan PWO = n/m, and since angle PKO is twice angle PWO, it follows that

$$\tan PKO = 2mn/(m^2 - n^2), \quad \sin PKO = 2mn/(m^2 + n^2).$$

Therefore

$$MP = KP \sin PKO = n(m^2 - r^2)/(m^2 + n^2).$$

And, from similar triangles RNP and ROW,

$OM = NP = (OW)(NR)/ON = m(n - MP)/n = m(n^2 + r^2)/(m^2 + n^2).$

Then

$$OP^2 = MP^2 + OM^2 = (m^2n^2 + r^4)/(m^2 + n^2).$$

Finally,

$$OQ = OW - 2KP = m - 2(m^2 - r^2)/2m = r^2/m.$$

THEOREM 3.3. <u>The segments</u> OP, OQ, OR <u>are connected by the relation</u>

$$\frac{r^2 + OP^2}{r^2 - OP^2} = \frac{r^2 + OR^2}{r^2 - OR^2} \cdot \frac{r^2 + OQ^2}{r^2 - OQ^2} .$$

For, by theorem 3.2 (c),

$$\frac{r^2 + OP^2}{r^2 - OP^2} = \frac{r^2(m^2 + n^2) + m^2n^2 + r^4}{r^2(m^2 + n^2) - m^2n^2 - r^4} = \frac{(r^2 + n^2)(m^2 + r^2)}{(r^2 - n^2)(m^2 - r^2)}$$

$$= \frac{r^2 + n^2}{r^2 - n^2} \cdot \frac{r^2 + r^4/m^2}{r^2 - r^4/m^2} = \frac{r^2 + OR^2}{r^2 - OR^2} \cdot \frac{r^2 + OQ^2}{r^2 - OQ^2} ,$$

since OR = n and, by theorem 3.2 (d), $OQ = r^2/m$.

THEOREM 3.4. <u>If</u> OQ <u>is any radial segment of</u> Σ, <u>then</u>

(a) $\cosh(\overline{OQ}/k) = (r^2 + OQ^2)/(r^2 - OQ^2),$

(b) $\sinh(\overline{OQ}/k) = 2rOQ/(r^2 - OQ^2),$

(c) $\tanh(\overline{OQ}/k) = 2rOQ/(r^2 + OQ^2).$

For, since $\overline{OQ}/k = \ln(OQ, IJ)$, we have

$$\begin{aligned}
\cosh(\overline{OQ}/k) &= \left[\exp(\overline{OQ}/k) + \exp(-\overline{OQ}/k)\right]/2 \\
&= \left[(OQ, IJ) + (OQ, JI)\right]/2 \\
&= \left[(OI/QI)/(OJ/QJ) + (OJ/QJ)/(OI/QI)\right]/2 \\
&= \left[QJ/IQ + IQ/QJ\right]/2 \\
&= \left[(r - OQ)/(r + OQ) + (r + OQ)/(r - OQ)\right]/2 \\
&= (r^2 + OQ^2)/(r^2 - OQ^2),
\end{aligned}$$

and relation (a) is established. Relations (b) and (c) follow in a similar manner, or from the identities $\sinh^2 x = \cosh^2 x - 1$ and $\tanh x = (\sinh x)/(\cosh x)$.

4. HYPERBOLIC PLANE TRIGONOMETRY

We are now ready to derive the formulas of hyperbolic plane trigonometry. It is well known that the formulas for the general hyperbolic triangle, such as the law of sines, the law of cosines, etc., are readily derived by purely analytical procedures from the formulas for the hyperbolic right triangle. If we are given such a right triangle ABC, right angled at C, and if we designate the lengths of the sides opposite A, B, C by a, b, c, and let k be the parameter of hyperbolic geometry, the formulas for the hyperbolic right triangle are

(1) $$\cosh c/k = \cosh a/k \cosh b/k,$$

(2.1) $$\cos A = (\tanh b/k)/(\tanh c/k),$$

(2.2) $$\cos B = (\tanh a/k)/(\tanh c/k),$$

(3.1) $$\sin A = (\sinh a/k)/(\sinh c/k),$$

(3.2) $$\sin B = (\sinh b/k)/(\sinh c/k),$$

(4.1) $$\tan A = (\tanh a/k)/(\sinh b/k),$$

(4.2) $$\tan B = (\tanh b/k)/(\sinh a/k),$$

(5.1) $$\cosh a/k = \cos A / \sin B,$$

(5.2) $$\cosh b/k = \cos B / \sin A,$$

(6) $$\cot A \cot B = \cosh c/k.$$

We shall now establish the first two formulas, (1) and (2.1), and then show that all the other formulas of the list can be obtained from these two by purely analytical procedures.

THEOREM 4.1. *In figure* 2

$$\cosh(\overline{OP}/k) = \cosh(\overline{OQ}/k)\ \cosh(\overline{QP}/k).$$

As an immediate consequence of theorems 3.3 and 3.4 (a) we have

$$\cosh(\overline{OP}/k) = \cosh(\overline{OQ}/k)\ \cosh(\overline{OR}/k).$$

But, by theorem 3.1, $(QP,ST) = W(QP,ST) = (OR,UV)$, whence $\overline{OR} = \overline{QP}$, and the theorem is established.

THEOREM 4.2. *In figure* 2

$$\cos QOP = \tanh(\overline{OQ}/k)\ /\ \tanh(\overline{OP}/k).$$

For, by theorem 3.4 (c),

$$\tanh(\overline{OQ}/k)\ /\ \tanh(\overline{OP}/k) = OQ(r^2+OP^2)/OP(r^2+OQ^2).$$

Substituting the expressions for OP and OQ as given by theorem 3.2 (c) and (d), and simplifying, we find

$$\tanh(\overline{OQ}/k)\ /\ \tanh(\overline{OP}/k)$$
$$= m(r^2+n^2)/(m^2n^2+r^4)^{\frac{1}{2}}(m^2+n^2)^{\frac{1}{2}}$$
$$= \left[m(r^2+n^2)/(m^2+n^2)\right]/\left[(m^2n^2+r^4)^{\frac{1}{2}}/(m^2+n^2)^{\frac{1}{2}}\right]$$
$$= OM/OP \qquad \text{(theorem 3.2 (b) and (c))}$$
$$= \cos QOP,$$

and the theorem is established.

Relation (2.2) follows because A in (2.1) was arbitrary.

We now establish (3.1) from (1), (2.1), and elementary hyperbolic identities. We have

$$\sin^2 A = 1 - \cos^2 A = 1 - (\tanh^2 b/k)/(\tanh^2 c/k)$$
$$= (\tanh^2 c/k - \tanh^2 b/k)/(\tanh^2 c/k)$$
$$= (\operatorname{sech}^2 b/k - \operatorname{sech}^2 c/k)/(\tanh^2 c/k)$$
$$= [(\cosh^2 c/k)/(\cosh^2 b/k) - 1]/(\sinh^2 c/k)$$
$$= (\cosh^2 a/k - 1)/(\sinh^2 c/k)$$
$$= (\sinh^2 a/k)/(\sinh^2 c/k).$$

Thus, choosing the positive square root since A is an acute angle, relation (3.1) is established, and with it falls (3.2).

Relation (4.1) follows from (1), (2.1), and (3.1), for

$$\tan A = (\sinh a/k)(\tanh c/k)/(\sinh c/k)(\tanh b/k)$$
$$= (\sinh a/k)(\cosh b/k)/(\cosh c/k)(\sinh b/k)$$
$$= (\tanh a/k)/(\sinh b/k).$$

Relation (4.2) follows similarly from (1), (2.2), and (3.2).

Relation (5.1) follows from (1), (2.1), and (3.2), for

$$\cos A / \sin B = (\tanh b/k)(\sinh c/k)/(\tanh c/k)(\sinh b/k)$$
$$= (\cosh c/k)/(\cosh b/k)$$
$$= \cosh a/k.$$

Relation (5.2) follows similarly from (1), (2.2), and (3.1).

Finally, from (1), (5.1), and (5.2), we have

$$\cosh c/k = (\cosh a/k)(\cosh b/k)$$
$$= (\cos A / \sin B)(\cos B / \sin A)$$
$$= \cot A \cot B,$$

which is relation (6), and we conclude our derivations.

5. APPENDIX I

Eisenhart, in his text on coordinate geometry (5, appendix to chapt. I), gives an exposition of the relation between a set of postulates for euclidean geometry and the algebraic foundations of cartesian coordinate geometry. He uses a slight modification of a set of postulates by Hilbert. It is interesting to note that he answers in the affirmative the question: Do the methods of cartesian coordinate geometry enable one to solve any problem in euclidean plane geometry? This, of course, is dependent upon establishing a one-to-one correspondence between the primitive terms of euclidean geometry and appropriate algebraic elements and equalities.

So we see that the idea of developing euclidean geometry to a high degree of perfection by the use of a model is not new but is employed in analytical geometry and the calculus. These subjects develop geometry by means of a model based in the real number system. The power of developing a subject from a model is thus adequately illustrated.

6. APPENDIX II

The following, paraphrased from Eisenhart's coordinate geometry (5, appendix to chapt. I), is a simplified presentation of Hilbert's postulate set for plane hyperbolic geometry.

AXIOM 1. There is one and only one line passing through any two given (distinct) points.

AXIOM 2. Every line contains at least two points, and given any line there is at least one point not on it.

AXIOM 3. If a point B lies between the points A and C, then A, B, and C all lie on the same line, and B lies between C and A, and C does not lie between B and A, and A does not lie between B and C.

AXIOM 4. Given any two (distinct) points A and C, there can always be found a point B which lies between A and C, and a point D such that C lies between A and D.

AXIOM 5. If A, B, C are (distinct) points on the same line, one of the three points lies between the other two.

DEFINITION. The <u>segment</u> (or <u>closed</u> <u>interval</u>) AC consists of the points A and C and of all points which lie between A and C. A point B is said to be <u>on</u> the segment AC if it lies between A and C, or is A or C.

DEFINITION. Two lines, a line and a segment, or two segments, are said to <u>intersect</u> each other if there is a point which is on both of them.

DEFINITION. The <u>triangle</u> ABC consists of the three segments AB, BC, and CA (called the <u>sides</u> of the triangle), provided the points A, B, and C (called the <u>vertices</u> of the triangle) are not on the same line.

AXIOM 6. A line which intersects one side of a triangle and does not pass through any of the vertices must also intersect one other side of the triangle.

AXIOM 7. If A and B are (distinct) points and A' is a point on a line L, there exist two and only two (distinct) points B' and B" on L such that the pair of points A', B' is congruent to the pair A, B and the pair of points A', B" is congruent to the pair A, B; moreover A' lies between B' and B".

AXIOM 8. Two pairs of points congruent to the same pair of points are congruent to each other.

AXIOM 9. If B lies between A and C, and B' lies between A' and C', and A, B is congruent to A', B', and B, C is congruent to B', C', then A, C is congruent to A', C'.

DEFINITION. Two segments are <u>congruent</u> if their end points are congruent pairs of points.

DEFINITION. The <u>ray</u> AC consists of all points B which lie between A and C, the point C itself, and all points D such that C lies between A and D. (In consequence of preceding axioms it is readily proved that if C' is any point on the ray AC the rays AC' and AC are identical.) The ray AC is said to be <u>from</u> the point A.

DEFINITION. The <u>angle</u> BAC consists of the point A (the <u>vertex</u> of the angle) and the two rays AB and AC (the <u>sides</u> of the angle).

DEFINITION. If ABC is a triangle, the three angles BAC, ACB, CBA are called the <u>angles</u> of the triangle. Moreover the angle BAC is said to be <u>included</u> between the sides AB and AC of the triangle (and similarly for the other two angles of the triangle).

AXIOM 10. If BAC is an angle whose sides do not lie in the same line, and B' and A' are (distinct) points, there exist two and only two (distinct) rays, A'C' and A'C", from A' such that the angle B'A'C' is congruent to the angle BAC, and the angle B'A'C" is congruent to the angle BAC; moreover if E' is any point on the ray A'C' and E" is any point on the ray A'C", the segment E'E" intersects the line A'B'.

AXIOM 11. Every angle is congruent to itself.

AXIOM 12. If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the remaining angles of the first triangle are congruent each to the corresponding angle of the second triangle.

AXIOM 13. Through a given point A not on a given line L there passes more than one line which does not intersect L.

AXIOM 14. If A, B, C, D are (distinct) points, there exist on the ray AB a finite set of (distinct) points A_1, A_2, ..., A_n such that (1) each of the pairs A, A_1; A_1, A_2; A_2, A_3; ...; A_{n-1}, A_n is congruent to the pair C, D and (2) B lies between A and A_n.

AXIOM 15. The points of a line form a system of points such that no new points can be added to the space and assigned to the line without causing the line to violate one of the first eight axioms or Axiom 14.

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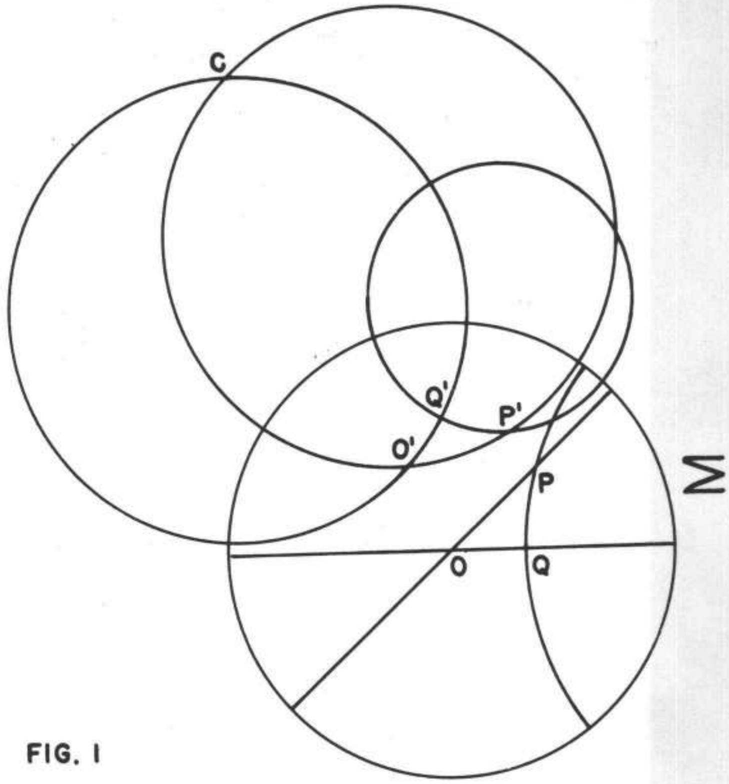

FIG. 1

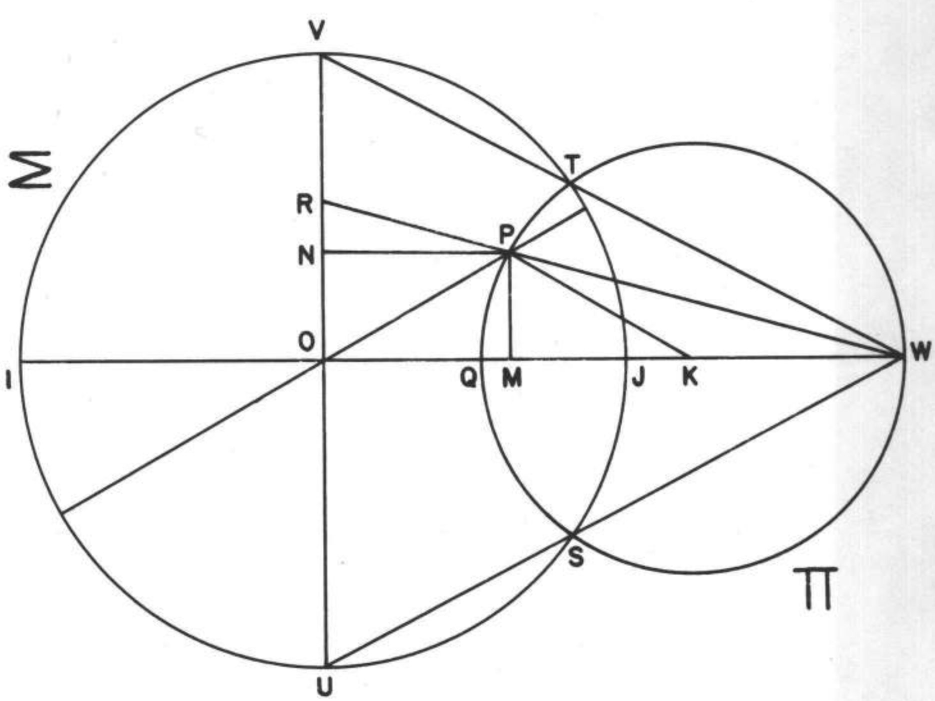

FIG. 2