Open access • Journal Article • DOI:10.1007/S00208-010-0560-0

Hyperbolicity, CAT(-1)-spaces and the Ptolemy inequality — Source link

Thomas Foertsch, Viktor Schroeder

Institutions: University of Bonn, University of Zurich

Published on: 01 Jun 2011 - Mathematische Annalen (Springer-Verlag)

Topics: Boundary (topology)

Hyperbolicity, CAT(−1)-spaces and the Ptolemy inequality

Foertsch, T; Schroeder, V

http://dx.doi.org/10.1007/s00208-010-0560-0.

Originally published at:
Foertsch, T; Schroeder, V (2011). Hyperbolicity, CAT(−1)-spaces and the Ptolemy inequality. Mathematische Annalen, 350(2):339-356.

Hyperbolicity, CAT(-1)-spaces and the Ptolemy inequality

Abstract

Using a four points inequality for the boundary of CAT(-1)-spaces we study the relation between Gromov hyperbolic spaces and CAT(-1)-spaces.

Hyperbolicity, CAT(−1)-spaces and the Ptolemy Inequality

Thomas Foertsch & Viktor Schroeder*

Abstract

Using a four points inequality for the boundary of CAT(−1)-spaces we study the relation between Gromov hyperbolic spaces and CAT(−1)-spaces.

1 Introduction

From various hyperbolic cone constructions it is known that every bounded, complete metric space can appear as the visual boundary of a Gromov hyperbolic space. Here visual boundary means the boundary of a Gromov hyperbolic space endowed with a visual metric. In order to study the relation of (rough geodesic) Gromov hyperbolic spaces and CAT(−1)-spaces in terms of asymptotic methods, it is of major importance to understand which metric spaces can appear as visual boundaries of CAT(−1)-spaces. Surprisingly enough, due to our knowledge there does not appear any necessary condition for this in the literature. One of the main purposes of this paper is to provide a first such condition, namely a four point relation, which we will call the *Ptolemy Inequality*:

Theorem 1.1. *Let Y be the boundary of a* CAT(−1)*-space endowed with a Bourdon or a Hamenstädt metric $|\ |$. Let $y_1, y_2, y_3, y_4 \in Y$, then*

$$|y_1y_3||y_2y_4| \leq |y_1y_2||y_3y_4| + |y_2y_3||y_4y_1|.$$

Equality holds if and only if the convex hull of the four points is isometric to an ideal quadrilateral in the hyperbolic plane $\mathbb{H}^2$ such that the geodesics y_1y_3 and y_2y_4 are the diagonals.

Note that the formulation of this four point inequality is Möbius invariant. Thus, if the metric $|\ |$ is replaced by a Möbius equivalent metric, the inequality is invariant. Therefore it holds for all Bourdon metrics and also for all Hamenstädt metrics on Y (for a discussion of these metrics compare

*Supported by Swiss National Science Foundation

Section 2.3). As a consequence the inequality is well adapted to the geometry of the boundary of a CAT(-1)-space. It is a classical theorem attributed to Ptolemy (85-165), that if $y_1, \ldots, y_4$ are points in this order on a circle in the Euclidean plane, then we have equality in this formula. The classical Ptolemy theorem is equivalent to the *if direction* of the equality discussion in Theorem 1.1. We will obtain the inequality from a detailed study of the proof of a result of Bourdon [B2].

We use Theorem 1.1 to study the relation between Gromov hyperbolic spaces and CAT(-1)-spaces. Clearly every CAT(-1)-space is Gromov hyperbolic. Since Gromov hyperbolicity is not a local curvature condition, the opposite is not true in general. Given a Gromov hyperbolic space X one can ask the following question: Is X rough isometric to some CAT(-1)-space W? Here a map $f : X \to W$ between metric spaces is called a *rough isometric embedding*, if there exists a constant $R \geq 0$ such that

$$|xx'| - R \leq |f(x)f(x')| \leq |xx'| + R.$$

If in addition the image $f(X) \subset W$ is R-dense, then f is called a *rough isometry*.

We look for answers to this question for Gromov hyperbolic spaces which are *visual* (see Section 2.2). The visual condition can be viewed as a quasiisometry-invariant version of the condition of extendable geodesics. For simplicity of the exposition the reader may think that X is a geodesic space with a basepoint $o \in X$ such that for every point $x \in X$ there exists $y \in \partial_\infty X$ such that x lies on a geodesic oy.

Coming back to our question we remark that Gromov hyperbolicity is invariant under arbitrary scaling of the metric while the CAT(-1)-condition is only invariant under scaling with factors $\lambda \leq 1$. Therefore the formulation of the problem is not yet good enough for our purposes.

If (X, d) is a metric space, we can consider the whole family of scaled metric spaces $(X, \lambda\, d)$ with $\lambda > 0$ and look for a distinguished normalization. We use the asymptotic upper curvature bound $K_u(X)$ defined in [BF] as a normalization (see Section 2.2).

This normalization is only possible if $K_u(X)$ is finite. In the case that $K_u(X) = -\infty$, the space X looks very much like a tree. We call a visual Gromov hyperbolic space *treelike* if $K_u(X) = -\infty$. This definition is justified by the result in [BF], that a visual Gromov hyperbolic space with $K_u(X) = -\infty$ is rough isometric to a tree provided that in addition $\partial_\infty X$ is doubling. Since trees are CAT(-1), it is not a substantial restriction to consider only nontreelike spaces.

Thus in the sequel we will consider only nontreelike visual Gromov hyperbolic spaces. Let X be such a space, then we can normalize X such that $K_u(X) = -1$. We call the normalized metric on X the critical metric and use the symbol d_0 for it.

Main Question: Let (X, d_0) be a nontreelike visual Gromov hyperbolic space endowed with its critical metric d_0. Does there exist a CAT(-1) space W, such that (X, d_0) is rough isometric to W?

In this setting one can reformulate the Bonk-Schramm embedding result (compare [BoS]). It says that under the doubling condition on $\partial_\infty X$ the answer to this question is *almost yes* in the sense that one only needs an arbitrarily small scaling of the critical metric to obtain the desired rough isometry. More precisely the Bonk-Schramm result (which relies on the Assouad embedding theorem) implies the following:

Theorem 1.2. *Let (X, d_0) be a nontreelike visual Gromov hyperbolic space endowed with its critical metric. Assume in addition that the boundary $\partial_\infty X$ is doubling. Then for every positive $\lambda < 1$ there exists a rough isometry of $(X, \lambda\, d_0)$ to a* CAT(-1)*-space W.*

Remark 1.3. Actually in [BoS] it was proven more explicitly that for every positive $\lambda < 1$ there exists a number N, such that $(X, \lambda\, d_0)$ is rough isometric to a convex subset W of the standard hyperbolic space $\mathbb{H}^N$.

We want to remark that it follows from the definition of the critical metric, that $(X, \lambda d_0)$ cannot be rough isometric to a CAT(-1) space for any $\lambda > 1$ (compare Remark 2.3).

A related embedding result can be obtained by combining results of Lang-Schlichenmaier (see [LS]) and Alexander-Bishop (see [AB]).

Theorem 1.4. *Let (X, d_0) be as above and assume now that the boundary $\partial_\infty X$ has finite Assouad Nagata dimension. Then there exists some $\lambda < 1$ such that $(X, \lambda\, d_0)$ is rough isometric to a* CAT(-1)*-space.*

Details for a proof of this theorem will be given elsewhere. Just note that by a theorem of Lang-Schlichenmeier every metric space of finite Assouad Nagata dimension admits a snowflake embedding into a product of a finite number of metric trees. This product certainly is a CAT(1)-space. Now Alexander-Bishop construct CAT(-1)-spaces as certain metric warped products with fibers that are CAT(1)-spaces. In order to establish the validity of Theorem 1.4, it only remains to verify that the fiber's CAT(1)-metric actually yields a visual metric on the boundary at infinity of such a CAT(-1) warped product. The embedding statement of Theorem 1.4 then just follows exactly as the one of Theorem 1.2.

Although the proof of Theorem 1.4 needs some constant λ bounded away from 1 (more precisely: the proof of Theorem 1.3 in [LS] needs this constant), it is unknown if the result is true for any positive $\lambda < 1$ (similar as in the Bonk-Schramm-Assouad result).

One main result of our paper is the existence of an example of a Gromov hyperbolic space (X, d_0) such that the optimal λ for which $(X, \lambda\, d_0)$ is rough isometric to a CAT(-1)-space is bounded away from 1. This implies in particular that the main question as stated above has a negative answer.

Theorem 1.5. *There exists a visual Gromov hyperbolic space (X, d_0) with the following property. If $\frac{1}{2} < \lambda$, then there does not exist a* CAT(-1)-*space W which is rough isometric to $(X, \lambda\, d_0)$. However, $(X, \frac{1}{2}\, d_0)$ is rough isometric to a* CAT(-1)-*space.*

Our result allows now to reformulate the question more quantitatively and to introduce a new invariant λ_0 for visual nontreelike Gromov hyperbolic spaces:

Questions: Let (X, d_0) be a visual Gromov hyperbolic space with its critical metric. Does there exist some $0 < \lambda \leq 1$ such that $(X, \lambda\, d_0)$ is rough isometric to a CAT(-1)-space? If such λ exists, what is the supremum λ_0 of these λ? Is $(X, \lambda_0\, d_0)$ rough isometric to some CAT(-1)-space?

Remark 1.6. Note that the set λ, such that $(X, \lambda\, d_0)$ is rough isometric to some CAT(-1)-space is either empty or an interval of the form $(0, a)$ or $(0, a]$ with $a \leq 1$.

We give an outline of the paper. Let X be a Gromov hyperbolic space. We denote by $Z = \partial_\infty X$ the boundary at infinity of X. Given a basepoint $o \in X$, the expression $e^{-(.|.)_o}$ defines a quasi-metric on Z, here $(.|.)_o$ denotes the Gromov product.

If o and o' are different basepoints then the quasi-metrics $e^{-(.|.)_o}$ and $e^{-(.|.)_{o'}}$ are bi-Lipschitz. Thus the bi-Lipschitz class $[\rho]$ of the quasi-metric $\rho = e^{-(.|.)_o}$ is well defined and does not depend on the basepoint.

If we scale the metric on X by a factor λ, then the Gromov product $(.|.)_o$ is transformed into $\lambda(.|.)_o$ and the corresponding quasi-metric on Z is taken to the power λ. Thus it is reasonable to consider the whole family ρ^λ of quasi-metrics and not only the particular quasi-metric $\rho = e^{-(.|.)_o}$.

Given a general quasi-metric space (Z, ρ) we can associate to ρ a critical exponent $s_0 \in (0, \infty]$ (see Section 2.1). If $s_0 \neq \infty$, we say that ρ^{s_0} is the critical quasi-metric on Z. In the case that X is a visual Gromov hyperbolic space, consider $Z = \partial_\infty X$ endowed with the quasi-metric $\rho = e^{-(.|.)_o}$. Then there is a relation of the critical exponent s_0 of ρ and the asymptotic upper curvature bound $K_u(X)$ defined in [BF]. Indeed it holds $K_u(X) = -s_0^2$.

If X is nontreelike (i.e. $s_0 \neq \infty$), then one can scale the metric on X in a unique way, such that $e^{-(.|.)_o}$ (where now the Gromov product is taken with respect to the scaled metric) is in the critical class. This corresponds to the scaling $K_u(X) = -1$. In this way we find a distinguished metric d_0 on X.

We are interested in the question, if one can embed X rough isometrically into some CAT(-1)-space W. The existing embedding theorems work in the following way. First find an embedding of the boundary $\partial_\infty X = Z$ into the boundary of some CAT(-1)-space, i.e. a map $f : Z \to Y$, where Y is the boundary of some W. Then one extends this embedding to an embedding $F : X \to W$. The idea of the extension is easily explained in the case that X is a geodesic Gromov hyperbolic space with extendable geodesics. Given a basepoint $o \in X$ and an arbitrary point $x \in X$, there exists a point $z \in Z$ and a geodesic oz, such that $x \in oz$. The extension F is now defined as follows. Choose some basepoint $o' \in W$. Now define $F(x)$ to be the point on the geodesic $o'f(z)$ such that $|ox| = |o'F(x)|$.

Bonk and Schramm proved that F is a rough isometric embedding if and only if f is a bi-Lipschitz map. In this case F is a rough isometry onto the convex hull of $F(X) \subset Y$, which turns out to be CAT(-1) itself.

Using this extension construction, the embedding problem can be reduced to an embedding problem $f : Z \to Y$, where Z is some complete bounded quasi-metric space, and Y is the boundary of a CAT(-1)-space (endowed with a Bourdon metric).

To discuss this embedding problem, we recall here the definition of a snowflake map. A map $f : Z \to Y$ between quasi metric spaces is called a *q-snowflake map*, if there exists $c > 1$ such that for all $z, z' \in Z$

$$\frac{1}{c}|zz'|^q \leq |f(z)f(z')| \leq c|zz'|^q.$$

This means that the quasi-metric ρ^q embeds bi-Lipschitz into the metric space Y. In the case that ρ is critical, we conclude in particular that $q \leq 1$ and $q = 1$ can only occur, if the critical quasi-metric ρ is actually bi-Lipschitz to a metric.

Thus we have the following: Let X be a nontreelike visual Gromov hyperbolic space, then $(X, \lambda\, d_0)$ can be rough isometrically embedded into a CAT(-1)-space, if there exists a λ-snowflake map from $\partial_\infty X$ to the boundary Y of a CAT(-1)-space.

To obtain our example we denote with Z the unit ball in ℓ^1, i.e. a point $z \in Z$ is a sequence $(z_1, z_2, \dots)$ with $\sum |z_i| \leq 1$. We prove

Theorem 1.7. *For $q > \frac{1}{2}$ there does not exist a q-snowflake embedding $f : Z \to Y$ where Y is a space satisfying the Ptolemy inequality.*

However we show there exists a $\frac{1}{2}$-snowflake map of Z into some Hilbert space, which is the boundary of the infinite dimensional hyperbolic space (see Section 5).

Finally note that Theorem 1.7 is sharp with respect to our methods of proof in the following sense: What we actually prove is that Z does not admit a q-snowflake embedding into a metric space satisfying the Ptolemy inequality for $q > \frac{1}{2}$. However, in Section 3 we prove

Proposition 1.8. *Let (X, d) be an arbitrary metric space. Then $(X, d^{1/2})$ satisfies the Ptolemy inequality.*

Thus, if we want to obtain a non embedding theorem similar in spirit to Theorem 1.7 with snowflake parameters $q \leq \frac{1}{2}$, then we need other necessary conditions for a metric space to appear as Bourdon or Hamenstädt metrics on the boundary of CAT(-1)-spaces.
We would like to emphazise that the most natural candidate allowing such a non embedding result seems to be the unit ball in l^∞.

It is a pleasure to thank Sergei Buyalo, Mario Bonk, Alexander Lytchak and Urs Lang for many discussions about hyperbolic spaces.

2 Preliminaries

2.1 Quasi-metrics and metrics

A *quasi-metric space* is a set Z with a function $\rho : Z \times Z \to [0, \infty)$ which satisfies the conditions:
(1) $\rho(z, z') \geq 0$ for every z, $z' \in Z$ and $\rho(z, z') = 0$ if and only if $z = z'$;
(2) $\rho(z, z') = \rho(z', z)$ for every z, $z' \in Z$;
(3) $\rho(z, z'') \leq K \max\{\rho(z, z'), \rho(z', z'')\}$ for every z, z', $z'' \in Z$ and some fixed $K \geq 1$.

Let (Z, ρ) be a quasi-metric space. By $[\rho]$ we denote the bi-Lipschitz class of ρ, i.e. for a map $\rho' : Z \times Z \to [0, \infty)$ we have $\rho' \in [\rho]$ if and only if there exists $c \geq 1$ such that for all $z, z' \in Z$

$$\frac{1}{c}\ \rho(z, z') \leq \rho'(z, z') \leq c\ \rho(z, z').$$

We are interested in obtaining a metric on Z. Since the only problem is the triangle inequality, the following approach is very natural. Define a map $d : Z \times Z \to [0, \infty]$, $d(z, z') = \inf \sum_i \rho(z_i, z_{i+1})$, where the infimum is taken over all sequences $z = z_0, \ldots, z_{n+1} = z'$ in Z. By definition d satisfies the triangle inequality. We call this approach to the triangle inequality the *chain approach*.

For a quasi-metric ρ we denote with $d = \mathrm{ca}(\rho)$ the pseudometric which we obtain when applying the chain approach to ρ.

The problem with the chain approach is that $d(z, z')$ could be 0 for different points z, z' and axiom (1) is not longer satisfied for (Z, d).

Frink [Fr] realized that the chain approach works for 2-quasi-metric spaces.

Proposition 2.1. *Let ρ be a 2-quasi-metric on a set Z and let for $z, z' \in Z$, $d(z, z') = \inf \sum_i \rho(z_i, z_{i+1})$, where the infimum is taken over all sequences $z = z_0, \ldots, z_{n+1} = z'$ in Z. Then d is a metric on Z with $\frac{1}{4} d \leq \rho \leq d$.*

If (Z, ρ) is a quasi-metric space, then ρ^s is a 2-quasi-metric if $s > 0$ is sufficiently small.

Definition 2.2. A quasi-metric space (Z, ρ) is called *LM-space* (Lipschitz metrizable), if $\mathrm{ca}(\rho) \in [\rho]$.

Hence a quasi-metric space is LM if and only if the following two conditions hold:
(1) the chain approach gives a metric.
(2) the metric from the chain approach is bi-Lipschitz to ρ.
Clearly the LM property is a bi-Lipschitz invariant.

One easily proves the following: If ρ is LM, then ρ^s also is LM for every $0 < s \leq 1$.

Note that ρ^s is a 2-quasi-metric for s small enough. Thus to every quasi-metric space (Z, ρ) which is not bi-Lipschitz to an ultrametric one can associate in a unique way a *critical exponent* $s_0 \in (0, \infty]$ with the property: ρ^s is LM for all $s < s_0$ and ρ^s is not LM for all $s > s_0$.

Remark 2.3. It follows from the definition of the critical exponent, that for every $s > s_0$ there cannot exist a bi-Lipschitz map $(Z, \rho^s) \to Y$, where Y is a metric space.

We shortly discuss the situation $s_0 = \infty$. It follows from [BF]:

Theorem 2.4. *Let (Z, ρ) be a doubling quasi-metric space. Then $s_0 = \infty$ if and only if ρ is bi-Lipschitz to an ultrametric .*

Without the doubling assumption Theorem 2.4 fails in general. This follows from an example due to Leonid Kovalev: *Consider* the set of integers $\mathbb{N}$ endowed with the metric d, where $d(m, n) := \log(1 + |m - n|)$. For this metric the critical exponent s_0 is ∞, but $(\mathbb{N}, d)$ is not bi-Lipshitz equivalent to an ultrametric. Nevertheless, we call (ad hoc) a quasi-metric ρ *ultrametriclike* if $s_0 = \infty$.

2.2 Gromov hyperbolic spaces

Let X be a metric space. For $o, x, x' \in X$ let

$$(x|x')_o := \frac{1}{2}(|ox| + |ox'| - |xx'|).$$

The space X is called *δ-hyperbolic* if for $o, x, x', x'' \in X$

$$((x|x')_o, (x|x'')_o, (x'|x'')_o) \text{ is a } \delta\text{-triple} \tag{1}$$

in the sense that the two smallest of the three numbers differ by at most δ.

X is called *hyperbolic*, if it is δ-hyperbolic for some $\delta \geq 0$. The relation (1) is called the *δ-inequality* with respect to the point $o \in X$.

If X satisfies the δ-inequality for one individual basepoint $o \in X$, then it satisfies the 2δ-inequality for any other basepoint $o' \in X$ (see for example [G]). Thus, to check hyperbolicity, one has to check this inequality only for one basepoint.

Let X be a hyperbolic space and $o \in X$ be a base point. A sequence $\{x_i\}$ of points $x_i \in X$ *converges to infinity,* if

$$\lim_{i,j\to\infty}(x_i|x_j)_o = \infty.$$

Two sequences $\{x_i\}$, $\{x_i'\}$ that converge to infinity are *equivalent* if

$$\lim_{i\to\infty}(x_i|x_i')_o = \infty.$$

Using the δ-inequality, one easily sees that this defines an equivalence relation for sequences in X converging to infinity. The *boundary at infinity* $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging to infinity.

For points y, $y' \in \partial_\infty X$ we define their Gromov product by

$$(y|y')_o = \inf \liminf_{i\to\infty}(x_i|x_i')_o,$$

where the infimum is taken over all sequences $\{x_i\} \in y$, $\{x_i'\} \in y'$. Note that $(y|y')_o$ takes values in $[0, \infty]$ and that $(y|y')_o = \infty$ if and only if $y = y'$. In a similar way we define for $\xi \in \partial_\infty X$, $x \in X$

$$(y|x)_o = \inf \liminf_{i\to\infty}(x_i|x)_o.$$

If X is δ-hyperbolic and if $y, y', y'' \in \partial_\infty X$, then $((y|y')_o, (y|y'')_o, (y'|y''))$ is a δ-triple. This implies that the expression $\rho(y, y') = e^{-(y|y')_o}$ defines a K-quasi-metric on $\partial_\infty X$ where $K = e^\delta$.

A Gromov hyperbolic space is called *visual*, if there exists a point $o \in X$ and a constant $D \geq 0$ such that for every $x \in X$ there exists $y \in \partial_\infty X$ with $|ox| - (x|y)_o \leq D$.

Roughly speaking, in a visual Gromov hyperbolic space the position of a point x is (up to a universal constant), given by some point $y \in \partial_\infty X$ and the distance $|ox|$ from the basepoint. It turns out that for these spaces almost all information is encoded in the properties of $\partial_\infty X$.

On the other hand, if some bounded metric space Y is given, then it is possible to construct a Gromov hyperbolic space X such that $\partial_\infty X$ as a set coincides with Y and the quasi-metric $e^{-(\cdot|\cdot)_o}$ is bi-Lipschitz to the given metric on Y (see for example [BoS]).

Also the following holds. Let X be a visual Gromov hyperbolic space. Then $\partial_\infty X$, endowed with the quasi-metric $\rho = e^{-(\cdot|\cdot)_o}$, is bi-Lipschitz to an ultrametric if and only if X is rough isometric to a tree.

In [BF] the notion $K_u(X)$ of an upper asymptotic curvature bound is introduced. In the case of visual Gromov hyperbolic spaces this notion is strongly related to the critical exponent of the quasi-metric $e^{-(\cdot|\cdot)_o}$ on $\partial_\infty X$. The following relation holds (see Theorem 1.5 in [BF]): $K_u(X) = -s_0^2$.

A visual Gromov hyperbolic space is called *treelike*, if $K_u(X) = -\infty$.

This definition is motivated by the following result (see [BF]):

Theorem 2.5. *Let X be a visual Gromov hyperbolic space. Assume in addition that $\partial_\infty X$ is doubling. Then $K_u(X) = -\infty$ if and only if X is rough isometric to a tree.*

2.3 CAT(-1) spaces

Let now X be a CAT(-1) space, i.e. X is a complete geodesic metric space, such that triangles are thinner than comparison triangles in the hyperbolic plane $\mathbb{H}^2$. In particular X is also Gromov hyperbolic. Let $Y = \partial_\infty X$. Given $x \in X$ and $w \in X \cup \partial_\infty X$ there exists a unique geodesic segment xw from x to w. If $y_1, y_2 \in \partial_\infty X$ are different points, there is also a unique geodesic line y_1y_2 joining these points.

Given a point $o \in X$ and points $y_1, y_2 \in \partial_\infty X$ we denote by $\angle_o(y_1, y_2)$ the local angle at x, i.e. the angle between the initial directions of the geodesics from oy_1 and oy_2. By $\theta_o(y_1, y_2)$ we denote the asymptotic comparison angle. I.e. let $y_i(t)$ be the point on the ray oy_i with distance t to o. Let $\overline{o}, \overline{y_1(t)}, \overline{y_2(t)}$ be the comparison triangle in $\mathbb{H}^2$, and let $\overline{\gamma_t}$ be the angle of this triangle at $\overline{o}$. Then $\theta_o(y_1, y_2) = \lim_{t\to\infty} \overline{\gamma_t}$. Let $\rho_o(y_1, y_2) = \sin(\frac{1}{2}\theta_o(y_1, y_2))$ be the Bourdon metric (with basepoint o). Indeed Bourdon proved [B1] that this expression is a metric and satisfies the triangle inequality. One can also express ρ_o in terms of the Gromov product and obtains the formula $\rho_o(y_1, y_2) = e^{-(y_1|y_2)_o}$, i.e. in the CAT$(-1)$ situation the Bourdon metric corresponds to the quasi-metric considered earlier. For the convenience of the reader we give a proof of this formula. The computation is in the hyperbolic plane $\mathbb{H}^2$. The triangle $\overline{o}, \overline{y_1(t)}, \overline{y_2(t)}$ has a limit ideal triangle given by geodesic rays $\gamma_i : [0, \infty) \to \mathbb{H}^2$ starting from $\overline{o}$ with angle θ_o. We then have

$$e^{-(y_1|y_2)_o} = \lim_{t\to\infty} (e^{h_t} e^{-2t})^{1/2},$$

where $h_t = d(\gamma_1(t), \gamma_2(t))$ is the distance in $\mathbb{H}^2$. From the hyperbolic law of cosine

$$\cosh(h_t) = \cosh^2(t) - \sinh^2(t) \cos\theta_o$$

and the trigonometric formula $1 - \cos\theta_o = 2\sin^2(\theta_o/2)$, we easily obtain

$$e^{h_t} \sim e^{2t} \sin^2(\theta_o/2)$$

as $t \to \infty$. Hence, the claim.

Bourdon metrics with respect to different basepoints are Möbius equivalent. Thus given a fixed Bourdon metric $|\,|$ on Y, we have for an arbitrary $o \in X$ that

$$\frac{\rho_o(y_1, y_2)\rho_o(y_3, y_4)}{\rho_o(y_1, y_3)\rho_o(y_2, y_4)} = \frac{|y_1y_2||y_3y_4|}{|y_1y_3||y_2y_4|}.$$

The Möbius invariance can be seen as follows. Let $o' \in X$ be a different basepoint, then a trivial computation shows for $x_1, x_2 \in X$ that

$$(x_1|x_2)_{o'} = |oo'| + (x_1|x_2)_o - (x_1|o')_o - (x_2|o')_o,$$

a formula which extends to points $y_1, y_2 \in \partial_\infty X$. Thus

$$\rho_{o'}(y_1, y_2) = \mu \frac{\rho_o(y_1, y_2)}{\lambda(y_1)\lambda(y_2)},$$

where $\mu = e^{-|oo'|}$ and $\lambda(y) = e^{-(y|o')_o}$. This clearly implies that $\rho_{o'}$ is Möbius equivalent to ρ_o.

We should mention here that Hamenstädt [H] introduced (even earlier) a metric on $\partial_\infty X \setminus \{\omega\}$, where X is a Hadamard manifold with curvature ≤ -1 and $\omega \in \partial_\infty X$ is a distinguished point. We do not describe her construction verbatim but modify her construction such that it works also for general CAT(-1) spaces: fix a point $\omega \in \partial_\infty X$ and consider a Busemann function b for the point ω. Define the Gromov product with respect to this Busemann function, i.e.

$$(x|x')_b = \frac{1}{2}(b(x) + b(x') - |xx'|),$$

which also extends to points at infinity. The corresponding *Hamenstädt metric* ρ_b is then defined as $e^{-(\cdot|\cdot)_b}$. If b is the Busemann function at ω such that $b(o) = 0$ for some point $o \in X$, then one easily computes $b(x) = |ox| - 2(\omega|x)_o$ and by straightforward calculation one obtains the formula

$$(x|x')_b = (x|x')_o - (\omega|x)_o - (\omega|x')_o,$$

which also extends to infinity. Hence

$$\rho_b(y, y') = \frac{\rho_o(y, y')}{\rho_o(y, \omega)\rho_o(y', \omega)}.$$

Thus the Hamenstädt metric can be obtained by involution at the point ω from the Bourdon metric and in particular these metrics are Möbius equivalent.

Note that by definition ρ_b is only a quasi-metric on $\partial_\infty X \setminus \{\omega\}$, since the involution of an arbitrary metric does not necessarily satisfy the triangle inequality. However in our situation the triangle inequality

$$\rho_b(y, y'') \leq \rho_b(y, y') + \rho_b(y', y'')$$

is equivalent to the Ptolemy inequality

$$\rho_o(y, y'')\rho_o(y', \omega) \leq \rho_o(y, y')\rho_o(y'', \omega) + \rho_o(y', y'')\rho_o(y, \omega),$$

which is proved in the next section. Thus ρ_b is actually a metric. This observation can be considered as the first application of the Ptolemy inequality. As a sideremark of this observation we formulate this in larger generality:

Remark 2.6. Let (Z, d) be an arbitray metric space. For $z \in Z$ consider the involution $d_z : Z \setminus \{z\} \times Z \setminus \{z\} \to [0, \infty)$, $d_z(a, b) = d(a, b)/(d(a, z)d(b, z))$. Then d_z is a metric for all $z \in Z$ iff d satisfies the Ptolemy inequality.

Observe that the CAT(-1) condition implies that $\angle_o(y_1, y_2) \leq \theta_o(y_1, y_2)$. However the following holds: if $\theta_o(y_1, y_2) = \pi$ then $\angle_o(y_1, y_2) = \pi$. ($\theta_o(y_1, y_2) = \pi$ implies that $d(y_1(t), y_2(t)) = 2t$ and hence $\angle_o(y_1, y_2) = \pi$.)

Thus we conclude that $\rho_o(y_1, y_2) \leq 1$ with equality if and only if o lies on the geodesic y_1y_2.

We will use the following

Lemma 2.7. *Let X be a* CAT(-1) *space, $y_1, \ldots, y_4 \in \partial_\infty X$ be different points and $o \in X$, then*

$$\sin \frac{1}{2}\theta_o(y_1, y_2) \sin \frac{1}{2}\theta_o(y_3, y_4) \leq \frac{|y_1y_2||y_3y_4|}{|y_1y_3||y_2y_4|}.$$

Equality holds if and only if the geodesics from y_1y_3 and y_2y_4 intersect in the point o.

Proof. We have

$$\begin{aligned} \sin \frac{1}{2}\theta_o(y_1, y_2) \sin \frac{1}{2}\theta_o(y_3, y_4) &= \rho_o(y_1, y_2)\rho_o(y_3, y_4) \\ &\leq \frac{\rho_o(y_1, y_2)\rho_o(y_3, y_4)}{\rho_o(y_1, y_3)\rho_o(y_2, y_4)} = \frac{|y_1y_2||y_3y_4|}{|y_1y_3||y_2y_4|}. \end{aligned}$$

Equality holds if and only if $\rho_o(y_1, y_3) = \rho_o(y_2, y_4) = 1$, which is equivalent to o lying on $y_1y_3 \cap y_2y_4$.

□

3 The Ptolemy Inequality

We start this section with a proof of Proposition 1.8, which states that the square root of any metric space satisfies the Ptolemy inequality. The following proof is due to Alexander Lytchak:

Proof of Proposition 1.8: Let $\{x_1, y_1, x_2, y_2\}$ be an ordered quadrupel in (X, d) and set $p_1 := d(x_1, x_2)$, $p_2 := d(y_1, y_2)$, $q_1 := d(x_1, y_1)$, $q_2 := d(x_2, y_1)$, $q_3 := d(x_2y_2)$ and $q_4 := d(x_1, y_2)$. Without loss of generality we may assume that

$$\begin{aligned} p_1 &\leq q_1 + q_2 \leq q_3 + q_4 \quad \text{and} \\ p_2 &\leq q_2 + q_3 \leq q_1 + q_4. \end{aligned}$$

Our claim follows, once we verify that $\sqrt{p_1p_2} \leq \sqrt{q_1q_3}\sqrt{q_2q_4}$. We prove this inequality by showing that for suitable $p_1' \geq p_1$, $p_2' \geq p_2$ and $q_4' \leq q_4$ one obtains $\sqrt{p_1'p_2'} \leq \sqrt{q_1q_3}\sqrt{q_2q_4'}$.

The numbers p_1', p_2' and q_4' are obtained as follows. First set $p_1' := q_1 + q_2$ and $p_2' := q_2 + q_3$. Then choose q_4' such that

$$\begin{aligned} &q_3 + q_4' = q_1 + q_2 \quad \text{and} \quad q_2 + q_3 \leq q_1 + q_4' \qquad \text{or} \qquad (2) \\ &q_3 + q_4' \leq q_1 + q_2 \quad \text{and} \quad q_2 + q_3 = q_1 + q_4'. \end{aligned}$$

Without loss of generality we may assume that the relations (2) are satisfied. It follows that $q_1 - q_3 = q_4 - q_2$ and $-(q_1 - q_3) \leq q_4 - q_2$ from which we deduce $q_4 \geq q_2 \geq 0$ and $\epsilon := q_1 - q_3 \geq 0$. Now the inequality

$$2q_2q_3 \;\leq\; 2\sqrt{q_3(q_3+\epsilon)q_2(q_2+\epsilon)}$$

implies

$$\begin{aligned} \sqrt{(q_2+q_3+\epsilon)(q_2+q_3)} \;&\leq\; \sqrt{q_3(q_3+\epsilon)} \;+\; \sqrt{q_2(q_2+\epsilon)} \\ \Longleftrightarrow \qquad \sqrt{p_1'p_2'} \;&\leq\; \sqrt{q_1q_3}\,\sqrt{q_2q_4'}, \end{aligned}$$

from which the claim follows.

□

Next we prove

Theorem 3.1. *Let Y be the boundary of a* CAT(-1)*-space with a Bourdon or Hamenstädt metric $|.|$. Let $y_1, y_2, y_3, y_4 \in Y$, then*

$$|y_1y_3||y_2y_4| \leq |y_1y_2||y_3y_4| + |y_2y_3||y_4y_1|.$$

Equality holds if and only if the convex hull of the y_i is isometric to an ideal quadrilateral in $\mathbb{H}^2$, such that the geodesics y_1y_3 and y_2y_4 intersect.

Remark 3.2. We note that this is a Möbius invariant comparison statement. I.e. if the equality is true for some metric $|.|$, then it also true for every Möbius equivalent metric. Thus this kind of comparison result is suitable for the boundary of a CAT(-1)-space.

Proof of Theorem 3.1: We use an idea from the proof of Lemma 3.1 in Bourdon's paper [B2].

Consider the geodesic y_2y_4 in our CAT(-1)-space. By continuity there exists a point $x \in y_2y_4$ with $\theta_x(y_1, y_2) = \theta_x(y_3, y_4)$. Denote this angle by β. Thus, by Lemma 2.7 we obtain the following estimate

$$\sin^2 \frac{1}{2}\beta \leq \frac{|y_1y_2||y_3y_4|}{|y_1y_3||y_2y_4|}.$$

Let $\gamma = \theta_x(y_2, y_3)$ and $\delta = \theta_x(y_4, y_1)$, then the same argument shows

$$\sin \frac{1}{2}\gamma \sin \frac{1}{2}\delta \leq \frac{|y_2y_3||y_4y_1|}{|y_1y_3||y_2y_4|}.$$

Since $x \in y_2y_4$ we see $\angle_x(y_2, y_3) + \angle_x(y_3, y_4) \geq \pi$ and $\angle_x(y_4, y_1) + \angle_x(y_1, y_2) \geq \pi$. Since $\angle_x \leq \theta_x$ we obtain $\beta + \gamma \geq \pi$ and $\beta + \delta \geq \pi$. Consequently $\sin \frac{1}{2}\gamma \geq \sin \frac{1}{2}(\pi - \beta) = \cos \frac{1}{2}\beta$ and also $\sin \frac{1}{2}\delta \geq \cos \frac{1}{2}\beta$. It follows

$$\sin \frac{1}{2}\gamma \sin \frac{1}{2}\delta \geq \cos^2 \frac{1}{2}\beta,$$

and hence

$$\cos^2 \frac{1}{2}\beta \leq \frac{|y_2y_3||y_4y_1|}{|y_1y_3||y_2y_4|}.$$

Since $\cos^2 \frac{1}{2}\beta + \sin^2 \frac{1}{2}\beta = 1$, we obtain the Ptolemy inequality.

If we have equality, then we have actually equality in all estimates. This implies now the rigidity statement by the following arguments. By Lemma 2.7 the diagonals y_1y_3 and y_2y_4 intersect at the point x. Furthermore we have equality for the angles $\angle_x(y_1, y_2) = \theta_x(y_1, y_2)$, $\angle_x(y_2, y_3) = \theta_x(y_2, y_3)$, $\angle_x(y_3, y_4) = \theta_x(y_3, y_4)$, $\angle_x(y_4, y_1) = \theta_x(y_4, y_1)$. This implies by standard rigidity results that the ideal triangles xy_iy_{i+1} are isometric to triangles in $\mathbb{H}^2$. Since, moreover, the angles at x add up to 2π we finally see that the span of $y_1, \dots, y_4$ is isometric to an ideal quadrilateral in the hyperbolic plane. □

Examples: At the end of this section we give two examples. We show that any four point metric space which satisfies the Ptolemy inequality can be Möbius embedded into the boundary of some CAT(-1)-space. Then we give an example of a metric space (of six points) satisfying the Ptolemy inequality but which cannot be Möbius embedded into the boundary of a CAT(-1)-space.

Let us first consider a four point metric space $W = \{w_1, \dots, w_4\}$ with metric d which satisfies the Ptolemy inequality. Let $a_1 = d(w_1, w_2)$, $a_2 = d(w_3, w_4)$, $b_1 = d(w_2, w_3)$, $b_2 = d(w_4, w_1)$, $c_1 = d(w_1, w_3)$, $c_2 = d(w_2, w_4)$. We define a new metric d' on W by setting $a'_1 = a'_2 = a' = \sqrt{a_1a_2}$, $b'_1 = b'_2 = b' = \sqrt{b_1b_2}$, $c'_1 = c'_2 = c' = \sqrt{c_1c_2}$.

Any nonsingular triangle in (W, d') has the distances a', b', c' and since the numbers a'^2, b'^2, c'^2 satisfy the triangle inequality by the Ptolemy inequality, also a', b', c' satisfies the triangle inequality. We can (after renumbering the points) assume that $c' \geq \max\{a', b'\}$. Finally we scale the metric and let $c = 1$, $a = a'/c'$, $b = b'/c'$.

This new metric is Möbius equivalent to d and thus we can start with this metric. The Ptolemy inequality now says $a^2 + b^2 \geq 1$. Define the angles α and β such that $\sin \frac{1}{2}\alpha = a$ and $\sin \frac{1}{2}\beta = b$. Then $a^2 + b^2 \geq 1$ implies $\alpha + \beta \geq \pi$. Let Δ_α be the ideal triangle oyy' in $\mathbb{H}^2$, such that $\angle_o(y,y') = \alpha$. Now glue four triangles $\Delta_\alpha\Delta_\beta\Delta_\alpha\Delta_\beta$ cyclically together to obtain an ideal quadrilateral formed from hyperbolic pieces with cone angle $2(\alpha + \beta) \geq 2\pi$. Thus the space is CAT($-1$), and by construction the Bourdon metric at the cone point coincides with our given metric.

To obtain the six point example we start with some general remark. Let X be a CAT(-1)-space and let $Y = \partial_\infty X$. Let $|.|$ be some fixed Bourdon or Hamenstädt metric on Y.

Assume that there are points $y_1, \dots, y_4 \in Y$ such that we have equality in the Ptolemy inequality. Then by the equality case above the geodesics y_1y_3 and y_2y_4 intersect in some point $x \in X$. Let a, b be the positive numbers such that

$$a^2 = \frac{\rho_x(y_1,y_2)\rho_x(y_3,y_4)}{\rho_x(y_1,y_3)\rho_x(y_2,y_4)} = \frac{|y_1y_2||y_3y_4|}{|y_1y_3||y_2y_4|},$$
$$b^2 = \frac{\rho_x(y_2,y_3)\rho_x(y_4,y_1)}{\rho_x(y_1,y_3)\rho_x(y_2,y_4)} = \frac{|y_2y_3||y_4y_1|}{|y_1y_3||y_2y_4|}.$$

Note that $\rho_x(y_1,y_3) = \rho_x(y_2,y_4) = 1$, since x lies on the corresponding geodesics. Since we have equality in the Ptolemy inequality, we see that $a^2+b^2=1$. Let $\alpha \in (0,\pi)$ such that

$$\sin^2\frac{1}{2}\alpha = \rho_x(y_1,y_2)\rho_x(y_3,y_4) = a^2,$$

$$\cos^2\frac{1}{2}\alpha = \rho_x(y_2,y_3)\rho_x(y_4,y_1) = b^2.$$

Then by the above calculations α and $(\pi-\alpha)$ is the intersection angle of the geodesics y_1y_3 and y_2y_4. An immediate consequence is:

Lemma 3.3. *Let $y_1,\ldots,y_4$ be points in Y with equality in the Ptolemy inequality. Assume in addition that $|y_1y_2||y_3y_4| = |y_2y_3||y_4y_1|$. Then the four points span a hyperbolic quadrilateral and the geodesics y_1y_3 and y_2y_4 intersect orthogonaly.*

To construct our example consider first the standard hyperbolic 3-space $\mathbb{H}^3$. Let $o \in \mathbb{H}^3$ and consider three geodesics through o which are pairwise orthogonal. We denote by $e_i^\pm \in \partial_\infty\mathbb{H}^3$, $i=1,2,3$, the endpoints of these geodesics. Then the Bourdon metric ρ_o satisfies $\rho_o(e_i^+,e_i^-)=1$ for $i=1,2,3$ and all other distances are equal to $\frac{1}{\sqrt{2}}$.

We now define on this 6 point space $\{e_i^\pm | i=1,2,3\}$ another metric $|.|$. Therefore choose $a,b,c \in \mathbb{R}$ numbers close to 1. Define $|e_i^+e_i^-|=1$ for $i=1,2,3$.

$$|e_1^+e_2^+| = |e_1^+e_2^-| = \frac{a}{\sqrt{2}}, \quad |e_1^-e_2^+| = |e_1^-e_2^-| = \frac{1}{a\sqrt{2}},$$

$$|e_2^+e_3^+| = |e_2^+e_3^-| = \frac{b}{\sqrt{2}}, \quad |e_2^-e_3^+| = |e_2^-e_3^-| = \frac{1}{b\sqrt{2}},$$

$$|e_3^+e_1^+| = |e_3^+e_1^-| = \frac{c}{\sqrt{2}}, \quad |e_3^-e_1^+| = |e_3^-e_1^-| = \frac{1}{c\sqrt{2}}.$$

If the numbers a,b,c are close to 1, then the triangle inequality is still satisfied. Note that for the three four-point-configurations $\{e_1^\pm,e_2^\pm\}$, $\{e_1^\pm,e_3^\pm\}$, $\{e_2^\pm,e_3^\pm\}$ we have still equality in the corresponding Ptolemy inequality. One easily checks that all other four-point-configurations still satisfy the Ptolemy inequality if a,b,c are close to 1.

We claim that the metric space $(\{e_i^\pm\},|.|)$ cannot be Möbius embedded into the boundary of a CAT(-1)-space. Assume to the contrary that it embeds into $\partial_\infty X$. Then by the above Lemma the three geodesics $e_i^+ e_i^-$ intersect pairwise orthogonaly. Let us first assume that the three geodesics intersect in one common point $x \in X$. Then the above Lemma together with the fact, that each pair of geodesics spans a hyperbolic quadrilateral implies that the metric ρ_x coincides with the metric ρ_o in $\mathbb{H}^3$. But note that (for a, b, c different from 1), the metic ρ_o is not Möbius equivalent to the metric $|.|$: e.g.

$$\frac{|e_1^+ e_2^-||e_2^+ e_3^+|}{|e_1^+ e_2^+||e_2^- e_3^+|} = b^2 \neq 1 = \frac{\rho_o(e_1^+, e_2^-)\rho_o(e_2^+, e_3^+)}{\rho_o(e_1^+, e_2^+)\rho_o(e_2^-, e_3^+)}.$$

Thus the three geodesics do not intersect in a common point. Hence we obtain three intersection points x_{12}, x_{23}, x_{31} of the corresponding geodesics which form a triangle with three right angles. Since the space is CAT(-1), such a triangle cannot exist.

Finally note that every CAT(0)-space satisfies the Ptolemy inequality. This follows from the fact that the euclidean plane satisfies the Ptolemy inequality (which is classical), and the fact that every four point configuration in a CAT(0)-space admits a subembedding into the euclidean plane (compare [BH], p. 164).

4 Proof of Theorem 1.7

The proof is inspired by results of Enflo (compare [E1] and [E2]). Let $Y = \partial_\infty X$, where X is a CAT(-1) space. On Y we consider a Bourdon metric.

4.1 Cubes in spaces satisfying the Ptolemy inequality

An m cube in a metric space is a subset of 2^m points which are indexed by the set $\{0,1\}^m$. Thus a 2-cube in Y are four points $y_{(0,0)}$, $y_{(0,1)}$, $y_{(1,1)}$, $y_{(1,0)}$. On the set of indices we consider the Hamming metric d_H, i.e. the distance between two indices is the number of different entries. The points y_I, $I \in \{0,1\}^m$ are called vertices. A pair of points y_I, y_J is called a d-diagonal, if $d_H(I,J) = d$. The 1-diagonals are also called *sides*. The distance $|y_I y_J|$ is called the length of the diagonal.

We denote by $S_{n,m} = \{I \in \{0,1\}^n | d_H(I,0) = m\}$. Thus $S_{n,m}$ is the set of $\{0,1\}$-sequences of length n containing exactly m entries 1. Note that $d_H(I,J)$ is even for $I, J \in S_{n,m}$.

We will first consider certain homothetic embeddings of the cube $\{0,1\}^m$ into $S_{n,m}$ where $n \geq 2m$. For $1 \leq i < j \leq n$ consider the map $\varphi_{i,j} : \{0,1\} \to \{0,1\}^n$ defined by $\varphi_{i,j}(0) = e_i$, $\varphi_{i,j}(1) = e_j$. For a sequence $1 \leq k_1 < k_2 < \ldots < k_{2m} \leq n$ we define $\varphi_{k_1 \cdots k_{2m}} : \{0,1\}^m \to S_{n,m}$ by $\varphi_{k_1 \cdots k_{2m}}(i_1, \ldots, i_m) = \varphi_{k_1 k_2}(i_1) + \ldots + \varphi_{k_{2m-1} k_{2m}}(i_m)$. For example consider $\varphi_{1245} : \{0,1\}^2 \to S_{5,2}$, which maps

$(0,0) \mapsto (1,0,0,1,0)$; $(0,1) \mapsto (1,0,0,0,1)$; $(1,1) \mapsto (0,1,0,0,1)$; $(1,0) \mapsto (0,1,0,1,0)$.

These maps are homotheties with factor 2, i.e. for every multiindex $K = k_1 \cdots k_{2m}$ and for all $I, J \in \{0,1\}^m$ we have $d_H(\varphi_K(I), \varphi_K(J) = 2d_H(I,J)$.

Theorem 4.1. *For every $m \in \mathbb{N}$ there exists some $n = n_m \geq 2m$ with the following property: Let Y be a metric space satisfying the Ptolemy inequality and let $\Phi : S_{n,m} \to Y$ be a map into Y, such that $|\Phi(I)\Phi(J)| \leq b$ if $d_H(I,J) = 2$ for some $b > 0$ (recall that 2 is the minimal nontrivial distance). Then there exists a multiindex K with $1 \leq k_1 < k_2 < \ldots < k_{2m} \leq n$, such that the map $\Phi \circ \varphi_K : \{0,1\}^m \to Y$ is a map of an m-cube into Y such that there exists a diagonal in $\{0,1\}^m$ the image of which under $\Phi \circ \varphi_K$ has length $\leq \sqrt{m}\, b$.*

Proof. The proof is by induction on m, where the case $m = 1$ is trivial. Let us assume that the result is true for the value $m-1$, with n_{m-1} being the corresponding n-value. Let

$$p = 2^{m-1}\binom{n_{m-1}}{2m-2} + 1.$$

We define $n = n_{m-1} + p$ and will show now that the result is true for $n = n_m$. Let therefore $\Phi : S_{n,m} \to Y$ be a map as in the statement of the theorem. For $1 \leq i \leq p$ we define canonical embeddings $\rho_i : S_{n_{m-1},m-1} \to S_{n,m}$ by $\rho_i(I) = (I, 0, \ldots, 0, 1, 0 \ldots, 0)$, where we put the entry 1 at the i'th additional place, i.e. at the $(n_{m-1} + i)$'th place of the sequence. We now apply the induction hypothesis to the $\Phi_i = \Phi \circ \rho_i : S_{n_{m-1},m-1} \to Y$. Thus for every i there exists a multiindex K_i such that the maps $\tau_i = \Phi \circ \rho_i \circ \varphi_{K_i}$ satisfy the requirement of the statement. There are only $\binom{n_{m-1}}{2m-2}$ different multiindices. By the choice of p we see that there exists a common multiindex K' with $1 \leq k_1 < k_2 < \ldots < k_{2m-2} \leq n_{m-1}$ such that at least $2^{m-1} + 1$ of the maps $\tau_i = \Phi \circ \rho_i \circ \varphi_{K'}$ satisfy the requirement of the statement. Since the cube $\{0,1\}^{m-1}$ has 2^{m-1} diagonals, there are at least two of the i's (lets call them $i_1 < i_2$), where the same diagonal in the image has length $\leq \sqrt{m-1}\, b$. Let $a, \overline{a} \in \{0,1\}^{m-1}$ be the endpoints of the diagonal. Define $k_{2m-1} = n_{m-1} + i_1$ and $k_{2m} = n_{m-1} + i_2$ and consider the multiindex K which is obtained from K' by extending it through $k_{2m-1}k_{2m}$. We show that the map $\Phi \circ \varphi_K : S_{n,m} \to Y$ contains a diagonal of length $\leq \sqrt{m}$.

Note that restricted to $\{0,1\}^{m-1} \times \{0\}$ this map coincides with τ_{i_1} and restricted to $\{0,1\}^{m-1} \times \{1\}$ with τ_{i_2}. We consider the images $y_{(a,0)}$, $y_{(a,1)}$, $y_{(\overline{a},0)}$, $y_{(\overline{a},1)}$ of the corresponding four points in $\{0,1\}^m$. We have $|y_{(a,0)}y_{(\overline{a},0)}|, |y_{(a,1)}y_{(\overline{a},1)}| \leq \sqrt{m-1}\, b$ by induction hypothesis. Furthermore we have $|y_{(a,0)}y_{(a,1)}|, |y_{(\overline{a},0)}y_{(\overline{a},1)}| \leq b$ by assumption on Φ. By Theorem 3.1 this implies that the product

$$|y_{(a,0)}y_{(\overline{a},1)}|\ |y_{(\overline{a},0)}y_{(a,1)}| \leq (m-1)b^2 + b^2 = mb^2,$$

which implies that at least one of the diagonals $y_{(a,0)}y_{(\overline{a},1)}$ or $y_{(a,1)}y_{(\overline{a},0)}$ has length $\leq \sqrt{m}\ b$. $\square$

We now prove Theorem 1.7.

Assume that there exists a (q,c)-snowflake map $\Psi : Z \to Y$, where Z is the unit ball in ℓ^1. Let $m \in \mathbb{N}$ be given and let $n = n_m$ as above. Consider the map $s_m : S_{n,m} \to Z$, $I \mapsto \frac{1}{m}I$. Here we consider $\frac{1}{m}I$ as a sequence in ℓ^1 (with m entries of value $\frac{1}{m}$ and all other values 0). Thus s_m is just a scaling map the image of which lies on the unit sphere of Z. We consider $\Phi = \Psi \circ s_m$ and apply Theorem 4.1 to it. Note that for $I, J \in S_{m,n}$ with $d_H(I,J) = 2$ we have

$$|\Phi(I)\Phi(J)| \leq c2^q \frac{1}{m^q},$$

since $d_Z(s_m(I), s_m(J)) = 2\frac{1}{m}$ and the snowflake property of Ψ. If I, J have distance $d_H(I,J) = 2m$, then $d_Z(s_m(I), s_m(J)) = 2$ and hence

$$|\Phi(I)\Phi(J)| \geq \frac{2^q}{c}$$

by the snowflake property. By Theorem 4.1 there exists a diagonal, i.e. points I, J with $d_H(I,J) = 2m$, such that

$$|\Phi(I)\Phi(J)| \leq \sqrt{m}c2^q \frac{1}{m^q}.$$

Thus $\sqrt{m}\frac{1}{m^q} \geq \frac{1}{c^2}$ which implies (since we can choose m arbitrarily large and independent from c) that $q \leq 1/2$.

5 An example

Let Z be as above the unit ball in ℓ^1. We construct in this section a $\frac{1}{2}$-snowflake map $f : Z \to Y$, where Y is the boundary of some CAT(-1)-space X. Therefore we first give a $\frac{1}{2}$-snowflake embedding of Z into the Hilbert space ℓ^2.

By the Assouad embedding theorem there exists $N \in \mathbb{N}$ and a bi-Lipschitz embedding $(\mathbb{R}, d^{1/2}) \to \mathbb{R}^N$, i.e. a $\frac{1}{2}$ snowflake map $h : \mathbb{R} \to \mathbb{R}^n$. Thus there is some constant c, such that

$$\frac{1}{c}|t-s| \leq |h(t)h(s)|^2 \leq c|t-s|,$$

where we consider the Euclidean metric on $\mathbb{R}^N$. Now the map $g : Z \to \ell^2$, $(z_1, z_2, \ldots) \mapsto (h(z_1), h(z_2), \ldots)$ satisfies also

$$\frac{1}{c}|zz'| \leq |g(z)g(z')|^2 \leq c|zz'|,$$

for all $z, z' \in Z$. Thus by g the space Z is mapped into a bounded ball of a Hilbert space and g is a $\frac{1}{2}$-snowflake map.

To map it to the boundary of a CAT(-1) space, we consider as X the infinite dimensional version of the hyperbolic space in the unit ball model. The boundary Y is the unit sphere in a Hilbert space. The relation is as in the classical Euclidean situation with the classical stereographic projection $\varphi : S^n \to \widehat{\mathbb{R}}^n$,

$$\varphi(x) = \frac{1}{1 - x_0}(x_1, \dots, x_n) \text{ for } x = (x_0, \dots, x_n).$$

Here we consider $S^n \subset \mathbb{R}^{n+1}$, where on $\mathbb{R}^{n+1}$ we have coordinates $x = (x_0, \dots, x_n)$.

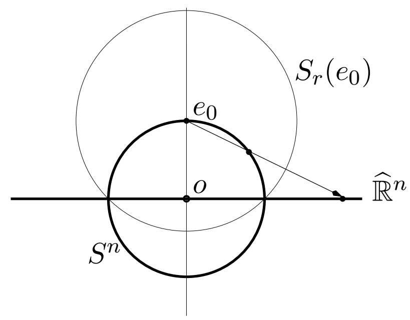

Figure 1: The stereographic projection as an inversion

The inversion $\widehat{\varphi} : \widehat{\mathbb{R}}^{n+1} \to \widehat{\mathbb{R}}^{n+1}$ of the extended $\widehat{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \infty$ with respect to the sphere $S_r(e_0) \subset \mathbb{R}^{n+1}$, $e_0 = (1, \dots, 0)$, $r = \sqrt{2}$, restricted to the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$, coincides with the stereographic projection, $\widehat{\varphi}|S^n = \varphi$. Thus φ as well as its inverse $\pi : \widehat{\mathbb{R}}^n \to S^n$ are Möbius maps.

We put $o = (0, \dots, 0) \in \mathbb{R}^{n+1}$ and denote by ρ the standard metric on $\mathbb{R}^{n+1}$, $\rho(x, y) = |x - y|$, canonically extended to $\widehat{\mathbb{R}}^{n+1}$. We use the same notation ρ for the induced metric on $S^n \subset \mathbb{R}^{n+1}$, and for the induced metric on $\widehat{\mathbb{R}}^n = \{x_{n+1} = 0\} \cup \{\infty\} \subset \widehat{\mathbb{R}}^{n+1}$.

We can generalize this classical situation to the infinite dimensional case, by replacing $\mathbb{R}^n$ by ℓ^2, and $\mathbb{R}^{n+1}$ by $\mathbb{R} \times \ell^2$ with an additional 0-coordinate. Then the unit sphere S^∞ in $\mathbb{R} \times \ell^2$ is the boundary of the infinite dimensional hyperbolic space, where the Bourdon metric (with respect to the origin) is the metric $\frac{1}{2}\rho$. The map $\pi : \widehat{\ell^2} \to S^\infty$ restricted to the bounded subset $g(Z) \subset \ell^2$ is bi-Lipschitz. Thus $f : Z \to S^\infty$, $f = \pi \circ g$, is a snowflake map.

References

[AB] S. Alexander & R. Bishop, *Curvature bounds for warped products of metric spaces*, Geom. Funct. Anal. 14, 1143-1181 (2004)

[BF] M. Bonk & Th. Foertsch, *Asymptotic upper curvature bounds in coarse geometry*, to appear in Mathematische Zeitschrift

[BoS] M. Bonk & O. Schramm, *Embeddings of Gromov hyperbolic spaces* Geom. Funct. Anal. 10, no. 2, 266–306 (2000)

[B1] M. Bourdon, *Structure conforme au bord et flot géodésique d'un* CAT(-1)*-espace* Enseign. Math. (2) no. 41 (1995), 63–102.

[B2] M. Bourdon, *Sur le birapport au bord des* CAT(-1)*-espaces* Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 95–104.

[BH] M. Bridson& A. Haefliger, *Metric spaces of non-positive curvature* Grundlehren der Mathematischen Wissenschaften 319 Springer-Verlag, Berlin, 1999. xxii+643 pp. ISBN: 3-540-64324-9

[E1] P. Enflo, *On the nonexistence of uniform homeomorphisms between L_p-spaces* Ark. Mat. 8 (1969), 103–105.

[E2] P. Enflo, *On a problem of Smirnov* Ark. Mat. 8 (1969), 107–109.

[Fr] A. H. Frink, *Distance functions and the metrization problem*, Bull. AMS 43 (1937), 133–142.

[G] M. Gromov, *Hyperbolic Groups*, in "Essays in Group Theory" (G. Gersten ed.), Math. Sc. Res. Inst. Publ. Springer (1987), 75–263

[H] U. Hamenstädt *A new description of the Bowen-Margulis measure*, Ergodic Theory Dynam. Systems 9 (1989), no. 3, 455–464.

[LS] U. Lang & T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, Int. Math. Res. Not. 58 (2005), 3625–3655

Thomas Foertsch,
Institut für Mathematik
Universität Bonn
Beringstr. 1
D-53111 Bonn
foertsch@math.uni-bonn.de

Viktor Schroeder,
Institut für Mathematik
Universität Zürich
Winterthurer Strasse 190
CH-8057 Zürich, Switzerland
vschroed@math.unizh.ch