Hypercomplex Signals—A Novel Extension of the Analytic Signal to the Multidimensional Case

Thomas Bülow and Gerald Sommer

***Abstract*—The construction of Gabor's complex signal—which is also known as the analytic signal—provides direct access to a real one–dimensional (1-D) signal's local amplitude and phase. The complex signal is built from a real signal by adding its Hilbert transform—which is a phase-shifted version of the signal—as an imaginary part to the signal. Since its introduction, the complex signal has become an important tool in signal processing, with applications, for example, in narrowband communication. Different approaches to an n-D analytic or complex signal have been proposed in the past. We review these approaches and propose the hypercomplex signal as a novel extension of the complex signal to n-D. This extension leads to a new definition of local phase, which reveals information on the intrinsic dimensionality of the signal. The different approaches are unified by expressing all of them as combinations of the signal and its partial and total Hilbert transforms. Examples that clarify how the approaches differ in their definitions of local phase and amplitude are shown. An example is provided for the two–dimensional (2-D) hypercomplex signal, which shows how the novel phase concept can be used in texture segmentation.**

***Index Terms*—Analytic signal, Clifford Fourier transform, complex signal, Hilbert transform, hypercomplex Fourier transform, local amplitude, local phase, quaternionic Fourier transform.**

I. INTRODUCTION

SINCE the introduction of the *complex signal* of a real one-dimensional (1-D) signal by Gabor in 1946 [1], this construction has become an important tool in 1-D signal processing. The complex signal is constructed by suppressing all negative frequency components of a real signal. This results in a complex signal that is the sum of the given real 1-D signal and a purely imaginary component that is the Hilbert transform of the original signal. The Hilbert transform performs a phase shift of the signal by $-\pi/2$. From the complex signal, the local amplitude (the *envelope*), and the local phase of the original signal can be derived as modulus and angular argument, respectively. Today, the term *analytic signal* is sometimes used instead of the term *complex signal*, which was coined by Gabor himself. We will use the term complex signal in the following. In applications, Gabor filters that yield an approximation of the complex signal of a bandpass filtered version of the input signal (see, e.g., [2] and [3]) are often used. Among others, the complex signal has found applications in narrowband communication [4], NMR-spectroscopy [5], geophysics [6], [7], and image processing. The latter application area indicates that there is the need of a definition of the complex signal for multidimensional signals. In this paper, we review known ways to define the complex signal of such signals. Furthermore, we introduce hypercomplex signals that are a novel extension of the complex signal to n-D.

In image processing, the most common definition of the complex signal is the line-wise calculation of the 1-D complex signal. This results from the definition of negative frequencies with respect to one half plane of the frequency domain. Line-wise evaluation is suitable for intrinsically 1-D signals, which vary merely along the preselected orientation. Generalizing a definition given by Krieger and Zetzsche [8] for 2-D signals, we call an n-D signal f intrinsically m-dimensional if it is constant with respect to $n-m$ orthogonal orientations (see Fig. 1).

Formally, a signal $f\colon \mathbb{R}^n \to \mathbb{R}$ is of intrinsic dimension m if it can be expressed as

$$f(\boldsymbol{x}) = g(\boldsymbol{A}\boldsymbol{x}), \qquad g\colon \mathbb{R}^m \to \mathbb{R}$$

for some real $m \times n$ matrix $\boldsymbol{A}$ and no other $l \times n$ matrix with $l < m$.

An approach to the complex signal that takes into account the fact that multidimensional signals are generally not intrinsically 1-D is the complex signal with single orthant spectrum introduced by Hahn [9]. Following this approach, 2^{n-1} complex signals of an n-D signal must be evaluated in order to be able to reconstruct the original signal. We introduce the hypercomplex signal, which is based on a combination of Hahn's complex signal with single orthant spectrum and hypercomplex Fourier transforms [10]. The original signal can be reconstructed from the hypercomplex signal by simply taking its real part.

As mentioned above, in 1-D, the polar representation of the complex signal yields access to the local amplitude and the local phase of the signal. In image processing, the intrinsically 1-D local phase can, e.g., be used to classify straight features into lines and edges. Oppenheim and Lim [11] showed that the main information content of an image lies in its phase components. Recently, Kovesi used phase congruence, i.e., the degree of congruence of the local phase over several scales for the detection of features in images [12].

We compare the consequences that the different definitions of the complex or hypercomplex signal have on the notions of local amplitude and local phase. It turns out that in 2-D, the hypercomplex signal leads to a novel type of local phase that is

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T. Bülow is with the GRASP Laboratory, University of Pennsylvania, Philadelphia, PA 19104 USA.

G. Sommer is with Christian Albrechts University Kiel, Kiel, Germany.

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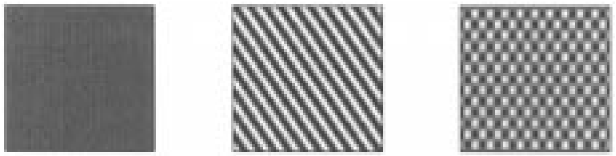

Fig. 1. From left to right: An intrinsically 0-D, 1-D, and 2-D signal.

related to the intrinsic dimensionality of the signal. We investigate this phase explicitly and show how it can be of use in image processing applications.

In Section II, we recap the main definitions and properties of the 1-D complex signal. In Section III, the known approaches toward an n-D complex signal are presented and compared with respect to the reconstructibility of the original signal from the complex signals. In Section IV, hypercomplex Fourier transforms are introduced with special emphasis on the 2-D transform, which is the quaternionic Fourier transform (QFT). The hypercomplex Fourier transforms are used in Section V in order to define the hypercomplex signal. Based on the 2-D hypercomplex signal (the *quaternionic signal*), the local quaternionic phase of a signal is introduced. In Section VI, the different complex signals are compared wiht the hypercomplex signal, and examples of the resulting definitions of the local amplitude and the local phase are given. It is shown that the phase of the quaternionic signal presents a new feature. An application of this phase to texture segmentation is presented.

II. One-Dimensional Complex Signal

Before delving into the n-D domain, we will recap the main definitions concerning the 1-D complex signal. First, the 1-D Hilbert transform is defined. The complex signal is defined as the sum of the signal and its Hilbert transform as the imaginary part.

Definition 1: The Hilbert transform f_{Hi} of a real 1-D signal f is given by

$$f_{Hi}(x) = f(x) * \frac{1}{\pi x} \tag{1}$$

where $*$ denotes convolution.

On the right-hand side of (1), Cauchy's principle value of the integral has to be evaluated:

$$f_{Hi}(x) = \frac{1}{\pi}\, V.p. \int_{\mathbb{R}} \frac{f(\xi)}{x-\xi}\, d\xi \tag{2}$$

$$= \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\int_{-\infty}^{x-\epsilon} \frac{f(\xi)}{x-\xi}\, d\xi + \int_{x+\epsilon}^{\infty} \frac{f(\xi)}{x-\xi}\, d\xi \right). \tag{3}$$

In the frequency domain, the Hilbert transform is given by

$$F_{Hi}(u) = -i\,\mathrm{sign}(u) F(u) \text{ with } \mathrm{sign}(u) = \begin{cases} 1, & \text{if } u > 0 \\ 0, & \text{if } u = 0 \\ -1, & \text{if } u < 0 \end{cases} \tag{4}$$

where F and F_{Hi} are the Fourier transforms of f and f_{Hi}, respectively. As mentioned above, the Hilbert transform is used as a tool for phase shifting the signal by $-\pi/2$. There is the following vivid explanation for this effect of the Hilbert transform: Every signal f can be represented as a linear combination of pure frequency components $\cos(2\pi ux + \varphi)$. The phase-shifted version of this is $\mathrm{sign}(u)\sin(2\pi ux + \varphi)$ and can be derived from the cosine function by applying the operator $-(1/2\pi|u|)\,(\partial/\partial x) ○\!\!-\!\!● -i(u/|u|) = -i\,\mathrm{sign}(u)$. The latter is identical to the transfer function of an ideal Hilbert transformer.[1]

The complex signal f_A of f is the sum of the original signal and the phase-shifted signal, where the shifted signal is added as an imaginary part[2]

$$f_A(x) = f(x) + i f_{Hi}(x) = f(x) * \left(\delta(x) + \frac{i}{\pi x} \right). \tag{5}$$

In the frequency domain, this reads

$$f_A(x) ○\!\!-\!\!● F_A(u) = F(u) + iF_{Hi}(u) = F(u)(1 + \mathrm{sign}(u)). \tag{6}$$

Thus, $F_A(u) = 0$ if $u < 0$, $F_A(u) = 2F(u)$ if $u > 0$, and $F_A(0) = F(0)$.

Writing f_A in exponential form $f_A(x) = |f_A(x)| \exp(i\phi(x))$ yields immediate access to the *local amplitude* $|f_A(x)|$ and the *local phase* $(\phi(x)\,\mathrm{smod}\,2\pi)$ of f. The local phase is uniquely defined in an interval of length 2π. In the following, we will always deal with intervals centered around 0: $\phi(x) \in [-\pi, \pi)$. Therefore, we define the symmetric modulo operator $(a\,\mathrm{smod}\,b) := ((a + b/2) \bmod b) - b/2$. As a simple example, we find the complex signal of the cosine function $f(x) = \cos(2\pi ux)$, $u > 0$ to be the complex exponential $f_A(x) = \exp(i2\pi ux)$.[3] That is, we obtain a constant local amplitude of $|f_A(x)| = 1$ and a linear local phase $\phi(x) = (2\pi ux\,\mathrm{smod}\,2\pi)$.[4]

III. n-D Complex Signal

As shown in the introduction, the complex signal can be used to separate the amplitude and the phase information of a given real 1-D signal. Since such a separation would be of much use for multidimensional signals as well—e.g., for feature extraction and classification in image processing, it is natural that there have been attempts to define the complex signal in n-D. There exist three different approaches toward this aim. We recapitulate these three definitions in the following.

A. Three Different Definitions of n-D Complex Signals

In the following, we will denote n-D variables by boldface characters: $\boldsymbol{x} = (x_1, \ldots, x_n)^T$. The first two definitions pre-

[1] We use the symbolic notation $f ○\!\!-\!\!● F$ or $f(x) ○\!\!-\!\!● F(u)$ in order to express that F is the Fourier transform of f.

[2] Sometimes, the Hilbert transform is defined with an opposite sign (e.g., in [13]). In these cases, the complex signal is defined as $f_A = f - if_{Hi}$, such that the complex signal is the same in both conventions.

[3] If negative u are admitted, we find $f_A(x) = \exp(i2\pi|u|x)$.

[4] It should be noted that the complex signal of the real signal $a(x)\cos(\phi(x))$ is, in general, not equal to $a(x)\exp(i\phi(x))$. This equality is only true if $a(x)$ and $\cos(\phi(x))$ have disjunct support in the frequency domain, where all frequencies contained in $a(x)$ are lower than all frequencies of $\cos(\phi(x))$. See [14] for more details. This condition is fulfilled in many practical applications such that the complex signal is still a useful practical tool.

sented here define the complex signal as a combination of the original signal and its Hilbert transform, as in 1-D. Using this approach, first, an n-D generalization of the Hilbert transform has to be defined. In the theory of the complex signal, two such extensions have been used: the *total Hilbert transform* [15] and the *partial Hilbert transform* [16].

Definition 2: The total Hilbert transform f_{Hi} of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is given by

$$f_{Hi}(\boldsymbol{x}) = V.p. \int_{\mathbb{R}^n} \frac{f(\boldsymbol{\xi})}{\pi^n \prod_{j=1}^{n} (x_j - \xi_j)} \, d^n \boldsymbol{\xi}. \tag{7}$$

In the frequency domain, the total Hilbert transform is constructed as follows:

$$f_{Hi}(\boldsymbol{x}) \circ\!\!-\!\!\bullet F_{Hi}(\boldsymbol{u}) = (-i)^n F(\boldsymbol{u}) \prod_{j=1}^{n} \operatorname{sign}(u_j). \tag{8}$$

Definition 3: The partial Hilbert transform $f_{Hi}^{\boldsymbol{n}}$ of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ with respect to the orientation $\boldsymbol{n}$ is given by

$$f_{Hi}^{\boldsymbol{n}}(\boldsymbol{x}) \circ\!\!-\!\!\bullet F_{Hi}^{\boldsymbol{n}}(\boldsymbol{u}) = -iF(\boldsymbol{u}) \operatorname{sign}(\boldsymbol{u}^T \cdot \boldsymbol{n}). \tag{9}$$

The partial Hilbert transform with respect to the coordinate directions $\boldsymbol{x}_j$ ($(\boldsymbol{x}_j)_k = 1$ if $j = k$ and $(\boldsymbol{x}_j)_k = 0$ else) is denoted as f_{Hi}^j. It is built in the spatial domain by

$$f_{Hi}^j(\boldsymbol{x}) = V.p. \int_{\mathbb{R}} \frac{f(\boldsymbol{\xi})}{\pi(x_j - \xi_j)} \, d\xi_j. \tag{10}$$

The total n-D Hilbert transform can be considered to be the successive application of partial Hilbert transforms with respect to all n coordinate directions.

Considering Definition 2, the *total complex signal* can be defined.

Definition 4: The total complex signal f_{tot} of a signal $f\colon \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f_{tot}(\boldsymbol{x}) = f(\boldsymbol{x}) + if_{Hi}(\boldsymbol{x}). \tag{11}$$

Considering Definition 4 in the frequency domain shows that the suppression of certain frequency components has no direct correspondence for n-D signals with even n. The Fourier transform F_{tot} of f_{tot} from Definition 4 is given by

$$F_{tot}(\boldsymbol{u}) = F(\boldsymbol{u}) \left[1 - (-i)^{n+1} \prod_{j=1}^{n} \operatorname{sign}(u_j) \right]. \tag{12}$$

The second possible way to introduce a complex signal in n-D is the combination of the original signal with the *partial* Hilbert transform as proposed by Peyrin *et al.* [17].

Definition 5: The *partial complex signal* $f_{part}^{\boldsymbol{n}}$ of a signal $f\colon \mathbb{R}^n \to \mathbb{R}$ with respect to $\boldsymbol{n}$ is defined by

$$f_{part}^{\boldsymbol{n}}(\boldsymbol{x}) = f(\boldsymbol{x}) + if_{Hi}^{\boldsymbol{n}}(\boldsymbol{x}). \tag{13}$$

In frequency domain, that reads

$$F_{part}^{\boldsymbol{n}}(\boldsymbol{u}) = \begin{cases} 2F(\boldsymbol{u}), & \text{for } (\boldsymbol{u}^T \cdot \boldsymbol{n}) > 0 \\ F(\boldsymbol{u}), & \text{for } (\boldsymbol{u}^T \cdot \boldsymbol{n}) = 0 \\ 0, & \text{for } (\boldsymbol{u}^T \cdot \boldsymbol{n}) < 0. \end{cases} \tag{14}$$

This corresponds to the 1-D case, where negative frequency components are suppressed while positive frequency components are multiplied by two. Here, a frequency $\boldsymbol{u}$ is called positive or negative (with respect to $\boldsymbol{n}$) if $(\boldsymbol{u}^T \cdot \boldsymbol{n}) > 0$ or $(\boldsymbol{u}^T \cdot \boldsymbol{n}) < 0$, respectively. The partial complex signal with respect to the coordinate direction $\boldsymbol{x}_j$ is denoted by f_{part}^j.

The third approach to the n-D complex signal has been proposed by Hahn [9]. In analogy to the 1-D complex signal Hahn defines a complex signal the spectrum of which is zero everywhere except from one orthant of the frequency domain. In 1-D, an *orthant* is a half-axis, in 2-D a quadrant, and so on.

Definition 6: Let f be an n-D signal and F its Fourier transform. The *complex signal with single orthant spectrum* f_{so_1} of f is then given by

$$f_{so_1}(\boldsymbol{x}) \circ\!\!-\!\!\bullet F_{so_1}(\boldsymbol{u}) = F(\boldsymbol{u}) \prod_{k=1}^{n} (1 + \operatorname{sign}(u_k)). \tag{15}$$

Recently, it could be shown that the n-D complex signal with single orthant spectrum is the boundary distribution of an analytic function [18], which was up to then only known for the 1-D complex signal. This justifies the name *analytic* signal instead of *complex signal*.

For each of the above definitions, we can define a "local amplitude" and "local phase" like in the 1-D case as the modulus and the angular phase of the complex signal. The properties of the different local phases and amplitudes will be investigated in Section VI. In the following section, we will compare the information content of the different definitions of the n-D complex signal presented so far.

B. Reconstructibility of a Signal From its Complex Signal

In this section, we consider the question of whether the original real signal f can be recovered from its complex signal or not. The 1-D complex signal is made up of the real signal f and a purely imaginary part if_{Hi}. In this case, f can be recovered from f_A trivially by taking its real part $f = \mathcal{R}\{f_A\}$. Thus, f_A contains the full information of f, although one half of the values in the frequency domain was set to zero. The reason for this is the Hermite symmetry of the Fourier transform F of any real signal f: $F(-\boldsymbol{u}) = F^*(\boldsymbol{u})$ [13]. Thus, F contains 50% redundant information, which can be canceled without loss of relevant information. The same is true for the complex signals f_{tot} and $f_{part}^{\boldsymbol{n}}$ (see Definitions 4 and 5). In both cases, f is the real part of its complex signal: $f(\boldsymbol{x}) = \mathcal{R}\{f_{tot}(\boldsymbol{x})\}$ and $f(\boldsymbol{x}) = \mathcal{R}\{f_{part}^{\boldsymbol{n}}(\boldsymbol{x})\}$. In the complex signal f_{so_1}, merely one orthant of the spectrum is maintained. Since in n-D there are 2^n orthants, that corresponds to a suppression of $(2^n - 1)/2^n$% of information. It is possible to define the complex signal with the single orthant spectrum with respect to any other orthant as well [16]. The totality of 2^{n-1} complex signals constructed from different orthants allows the reconstruction of the original signal. This also affects the representation of the signal via local amplitude and local phase. To each n-D signal there correspond 2^{n-1} amplitude and 2^{n-1} phase signals.

We will demonstrate the reconstructibility of f from two complex signals with single orthant spectrum for 2-D signals. According to Definition 6, the complex signal with single orthant (in the 2-D case "single quadrant") spectrum is given by $f_{so_1}(\boldsymbol{x}) \circ\!\!-\!\!\bullet F_{so_1}(\boldsymbol{u})$ with

$$F_{so_1}(\boldsymbol{u}) = F(\boldsymbol{u})(1+\operatorname{sign}(u_1))(1+\operatorname{sign}(u_2)) \tag{16}$$

$$= (F(\boldsymbol{u}) - F_{Hi}(\boldsymbol{u})) + i(F_{Hi}^1(\boldsymbol{u}) + F_{Hi}^2(\boldsymbol{u})). \tag{17}$$

Another complex signal for the second quadrant (upper left) is given by

$$F_{so_2}(\boldsymbol{u}) = F(\boldsymbol{u})(1-\operatorname{sign}(u_1))(1+\operatorname{sign}(u_2)) \tag{18}$$

$$= (F(\boldsymbol{u}) + F_{Hi}(\boldsymbol{u})) - i(F_{Hi}^1(\boldsymbol{u}) - F_{Hi}^2(\boldsymbol{u})). \tag{19}$$

From the two complex signals f_{so_1} and f_{so_2}, the original signal is recovered by

$$f(\boldsymbol{x}) = \tfrac{1}{2}\,\mathcal{R}\{f_{so_1}(\boldsymbol{x}) + f_{so_2}(\boldsymbol{x})\}. \tag{20}$$

In the following section, hypercomplex Fourier transforms will be presented. Later, they will be used in order to modify the complex signal with single orthant spectrum. This modification will lead to a signal with a single orthant hypercomplex spectrum containing the full signal information.

IV. Hypercomplex Fourier Transforms

Within the above framework, it is possible either to define a complex signal accounting for a 1-D even–odd symmetry of the n-dimensional signal or to define a set of complex signals accounting for the full symmetry of the input-signal. Our interest is to combine the two approaches in constructing *one hypercomplex signal* representing the full symmetry.

The 1-D Hilbert transform transforms signals of even symmetry to signals of odd symmetry and vice versa. The construction of the 1-D complex signal relies on the Hermite symmetry of the Fourier transform of a real input signal. Since we have the same property for the Fourier transform $F(\boldsymbol{u})$ of a real n-D signal f, i.e., $F^*(\boldsymbol{u}) = F(-\boldsymbol{u})$, one half plane of the frequency domain contains redundant information. Thus, it is possible to suppress this half plane, as is done in Definition 5, without losing spectral information.

A modification of Definition 6 that allows the suppression of all but one orthant of the spectrum while keeping enough spectral information for reconstructing the input signal requires a frequency domain representation with a redundancy of 75% for real 2-D signals and $1-2^{-n}$ for n-D signals. *Hypercomplex Fourier transforms* are transforms that yield frequency representation with the required redundancy.

The general definition of a hypercomplex Fourier transform of an n-D signal is

$$F^h(\boldsymbol{u}) = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \prod_{k=1}^{n} \exp(-\boldsymbol{e}_k 2\pi u_k x_k)\, d^n\boldsymbol{x}. \tag{21}$$

Here, the symbols $\boldsymbol{e}_k$ are imaginary units and, thus, obey the rules $\boldsymbol{e}_k^2 = -1$. Additionally, we define the product between two symbols with different indices by introducing a new symbol: $\boldsymbol{e}_k\boldsymbol{e}_l =: \boldsymbol{e}_{kl},\ k<l$. These elements are called elements of grade two, whereas $\boldsymbol{e}_k$ is an element of grade one. There are $\binom{n}{2}$ elements of grade two. Continuing in the same manner, we get $\binom{n}{3}$ elements of grade three ($\boldsymbol{e}_k\boldsymbol{e}_l\boldsymbol{e}_m =: \boldsymbol{e}_{klm}$, where $k<l<m$) and, finally, one element of grade n: $\boldsymbol{e}_1\boldsymbol{e}_2\cdots\boldsymbol{e}_n =: \boldsymbol{e}_{12\cdots n}$. The unification of these elements from grade one up to grade n constitutes a basis of an 2^n-dimensional $\mathbb{R}$-algebra. Multiplication in this algebra is associative. For a complete definition of this algebra, we have to define whether the multiplication of two elements of grade one is commutative or anticommutative.

In case of a commutative product $\boldsymbol{e}_k\boldsymbol{e}_l = \boldsymbol{e}_l\boldsymbol{e}_k$, the 2^n-D algebra is a commutative hypercomplex $\mathbb{R}$-algebra, and the corresponding transform is a commutative hypercomplex Fourier transform. Otherwise ($\boldsymbol{e}_k\boldsymbol{e}_l = -\boldsymbol{e}_l\boldsymbol{e}_k$), we have the Clifford algebra of the Euclidean space $\mathbb{R}^n$ [19]. The corresponding *Clifford Fourier transform* was introduced by Brackx *et al.* [10].

These transforms are invertible, and the inverse transforms are given by

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^n} F^h(\boldsymbol{u}) \prod_{k=1}^{n} \exp(\boldsymbol{e}_{n+1-k} 2\pi u_{n+1-k} x_{n+1-k})\, d^n\boldsymbol{u}. \tag{22}$$

In the current context, i.e., the construction of the complex signal, it makes no difference whether we use the commutative algebra or the Clifford algebra. However, the definition of the phase of an element of the algebra may strongly depend on the type of algebra. Thus far, the phase concept exists merely for the case $n=2$, where we use the 4-D Clifford algebra. This algebra is isomorphic to the quaternions.

In 2-D, we have the possibility to profit from a reordering of the factors of the hypercomplex Fourier transform. We present the corresponding transform [the quaternionic Fourier transform (QFT)] in the following. Because of the different order of the factors under the integral, the QFT is not identical with one of the n-D hypercomplex Fourier transforms for $n=2$, as defined above.

A. Quaternionic Fourier Transform

Before defining the quaternionic Fourier transform (QFT), we recap the relevant properties of Hamilton's quaternions. For a more thorough introduction, see, e.g., [20]. Quaternions are the set

$$\mathbb{H} = \{a + ib + jc + kd \mid a,\, b,\, c,\, d \in \mathbb{R}\} \tag{23}$$

together with the multiplication rules

$$ij = -ji = k \quad \text{and} \quad i^2 = j^2 = -1 \tag{24}$$

as well as component-wise addition and multiplications by real numbers. From (24), the missing multiplication rules follow: $k^2=-1$, $jk=-kj=i$, and $ki=ik=j$. Quaternions form an associative $\mathbb{R}$-algebra. Note that according to (24), the multiplication of quaternions is not commutative. The *conjugate* $\overline{q}$ of a quaternion $q = a+ib+jc+kd$ is defined by

$$\overline{q} = a - ib - jc - kd. \tag{25}$$

The *norm* of q is given by $|q| = \sqrt{q\overline{q}}$. It can be shown that $\mathbb{H}$ is a *normed algebra*, i.e., for $q_1, q_2 \in \mathbb{H}$, we have $|q_1||q_2| = |q_1 q_2|$. $\mathbb{H}$ forms a group under multiplication. The multiplicative inverse is given by $q^{-1} = \overline{q}/|q|^2$. For the components of a quaternion $q = a + ib + jc + kd$, we sometimes write

$$a = \mathcal{R}(q), \quad b = \mathcal{I}(q), \quad c = \mathcal{J}(q), \quad d = \mathcal{K}(q). \tag{26}$$

There are three nontrivial involutions defined on $\mathbb{H}$:

$$\begin{aligned} \alpha_i(q) &= -iqi = a + ib - jc - kd \\ \alpha_j(q) &= -jqj = a - ib + jc - kd \\ \alpha_k(q) &= -kqk = a - ib - jc + kd. \end{aligned} \tag{27}$$

These functions will be used in order to extend the notion of Hermite symmetry to quaternion-valued functions, as defined in [21].

Definition 7: A function $f: \mathbb{R}^2 \rightarrow \mathbb{H}$ is called quaternionic Hermitian if

$$f(-x_1, x_2) = \alpha_j(f(x_1, x_2))$$

and

$$f(x_1, -x_2) = \alpha_i(f(x_1, x_2)) \tag{28}$$

for each $(x_1, x_2) \in \mathbb{R}^2$.

Definition 8: The QFT F^q of a real 2-D signal f is defined as

$$F^q(\boldsymbol{u}) = \int_{\mathbb{R}^2} e^{-i2\pi u_1 x_1} f(\boldsymbol{x}) e^{-j2\pi u_2 x_2}\, d^2\boldsymbol{x}. \tag{29}$$

The QFT is invertible, and the inverse transform is given by

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2} e^{i2\pi u_1 x_1} F^q(\boldsymbol{u}) e^{j2\pi u_2 x_2}\, d^2\boldsymbol{u}. \tag{30}$$

Note that the QFT is not identical to the 2-D Clifford Fourier transform since the signal f is sandwiched between the two exponential functions rather than standing on their left side. The reason for this choice is that it allows the construction of shift and modulation theorems for the QFT that closely resemble the corresponding theorems of the complex Fourier transform [22]. For *real* 2-D signals, the QFT is identical to the 2-D Clifford Fourier transform.

A 2-D hypercomplex transform was first introduced by Ernst *et al.* [5], [23] without reference to a specific 4-D hypercomplex algebra. Ell [24], [25] introduced the QFT in the form (29) in the context of partial differential systems. Sangwine [26] used the QFT as a Fourier transform for color images. Chernov used the discrete QFT for the development of 2-D FFT algorithms [27].

The complex Fourier transform decomposes a real signal f into two parts of different symmetry: the even symmetric part $f_e(\boldsymbol{x}) \circ\!\!-\!\!\bullet F_e(\boldsymbol{u})$, which is real-valued, and the odd symmetric part $f_o(\boldsymbol{x}) \circ\!\!-\!\!\bullet F_o(\boldsymbol{u})$, which is purely imaginary.[5] Considering real 2-D signals, the QFT extends this property to the splitting of the signal into four different components: $f_{ee}(\boldsymbol{x}) \overset{q}{\circ\!\!-\!\!\bullet} F^q_{ee}(\boldsymbol{u})$, $f_{oe}(\boldsymbol{x}) \overset{q}{\circ\!\!-\!\!\bullet} iF^q_{oe}(\boldsymbol{u})$, $f_{eo}(\boldsymbol{x}) \overset{q}{\circ\!\!-\!\!\bullet} jF^q_{eo}(\boldsymbol{u})$, and $f_{oo}(\boldsymbol{x}) \overset{q}{\circ\!\!-\!\!\bullet} kF^q_{oo}(\boldsymbol{u})$. Here, the subscripts e and o denote *even* and *odd* symmetry components, e.g., $f_{eo}(x_1, x_2)$ is even with respect to the variable x_1 and odd with respect to x_2: $f_{eo}(x_1, x_2) = f_{eo}(-x_1, x_2) = -f_{eo}(x_1, -x_2)$. $f \overset{q}{\circ\!\!-\!\!\bullet} F^q$ symbolizes the fact that F^q is the QFT of f. Analogously, $\overset{h}{\circ\!\!-\!\!\bullet}$ is used for general hypercomplex Fourier transforms. In Section V, the QFT will be used in order to define the quaternion-valued signal of a real 2-D signal.

[5]When we say that a function is *even* or *even symmetric*, we mean that $f(\boldsymbol{x}) = f(-\boldsymbol{x})$, and *odd* means $f(\boldsymbol{x}) = -f(-\boldsymbol{x})$.

V. Hypercomplex Signals

Hypercomplex signals result from a combination of the single orthant approach outlined in Section III (Definition 6) and a hypercomplex Fourier transform. We postulated above that the appropriate spectral representation of a signal for the single orthant approach should contain all the relevant information of a real signal in one orthant. In 2-D, this is provided by the QFT, as shown by the following theorem and corollary.

Theorem 1: The QFT F^q of a real 2-D signal f is quaternionic Hermitian (see Definition 7).

Proof: It follows from Definition 8 that the QFT of a real 2-D signal has the form

$$F^q(\boldsymbol{u}) = F^q_{ee}(\boldsymbol{u}) + iF^q_{oe}(\boldsymbol{u}) + jF^q_{eo}(\boldsymbol{u}) + kF^q_{oo}(\boldsymbol{u}) \tag{31}$$

where the functions F^q_{xy}, $x, y \in \{e, o\}$ are real-valued. Here, e.g., $F^q_{eo}(x_1, x_2)$ is even with respect to x_1 and odd with respect to x_2. Applying the automorphisms α_i and α_j yields

$$\begin{aligned} \alpha_i(F^q(\boldsymbol{u})) &= F^q_{ee}(\boldsymbol{u}) + iF^q_{oe}(\boldsymbol{u}) - jF^q_{eo}(\boldsymbol{u}) - kF^q_{oo}(\boldsymbol{u}) \\ &= F^q(u_1, -u_2) \\ \alpha_j(F^q(\boldsymbol{u})) &= F^q_{ee}(\boldsymbol{u}) - iF^q_{oe}(\boldsymbol{u}) + jF^q_{eo}(\boldsymbol{u}) - kF^q_{oo}(\boldsymbol{u}) \\ &= F^q(-u_1, u_2). \end{aligned}$$ ■

From Theorem 1, the following corollary follows immediately.

Corollary 1: The QFT F^q of a real 2-D signal f can be reconstructed from the first quadrant of the frequency domain.

Proof: If F^q is known in the first quadrant ($u_1 \geq 0$ and $u_2 \geq 0$), the values in the other quadrants can be found as follows: The quadrant $u_1 \geq 0, u_2 < 0$ can be reconstructed from the first one by $\alpha_i(F^q(\boldsymbol{u})) = F^q(u_1, -u_2)$, the quadrant $u_1 < 0, u_2 \geq 0$ is obtained by $\alpha_j(F^q(\boldsymbol{u})) = F^q(-u_1, u_2)$, and finally, the quadrant $u_1 < 0, u_2 < 0$ is found by $\alpha_i(\alpha_j(F^q(u_1, u_2))) = \alpha_i(F^q(-u_1, u_2)) = F^q(-u_1, -u_2)$. ■

Equivalent results hold for n-D hypercomplex transforms, showing that a real n-D signal can be reconstructed from one orthant of its hypercomplex transform. Combining the definition of the complex signal with single orthant spectrum with the hypercomplex Fourier transforms yields the n-D hypercomplex signals.

Definition 9: Let f be an n-D signal, and let F^h be one of its hypercomplex Fourier transforms (either commutative of anticommutative). The corresponding hypercomplex signal f^h_{so} of f is then given by

$$f^h_{so}(\boldsymbol{x}) \overset{h}{\circ\!\!-\!\!\bullet} F^h(\boldsymbol{u}) \prod_{k=1}^{n} (1 + \operatorname{sign}(u_k)). \tag{32}$$

Depending on the chosen algebra (Clifford algebra or commutative hypercomplex algebra) f^h_{so} will take values in this specific algebra. Its *components*, however, will be the same, independent of the algebra chosen. We therefore omit another index specifying the used algebra. The original signal is contained in its hypercomplex signal as real part $\mathcal{R}(f_{so}^h(\boldsymbol{x})) = f(\boldsymbol{x})$.

Gabor's 1-D complex signal makes accessible the local amplitude and phase of a signal. As mentioned above, this is done by taking the modulus and the angular phase of the complex signal, respectively. Dealing with hypercomplex numbers, we have to find equivalents to the notions modulus and phase. We will restrict this analysis to 2-D signals and their hypercomplex signal based on the QFT as defined below.

Definition 10: Let f be a real 2-D signal, and let F^q be its quaternionic Fourier transform. The quaternionic signal f_{so}^q of f is then given by

$$f_{so}^q(\boldsymbol{x}) \circ\!\!\overset{q}{-}\!\!\bullet F^q(\boldsymbol{u})(1 + \text{sign}(u_1))(1 + \text{sign}(u_2)). \tag{33}$$

Based on the quaternionic signal, we define the local amplitude of a 2-D signal f as $|f_{so}^q| = \sqrt{f_{so}^q \overline{f_{so}^q}}$, where $|\cdot|$ is the norm of a quaternion, as given above. A possible definition of the phase of a quaternion will be given in the following section.

A. Phase of a Quaternion

Interpreting a quaternion q as a vector in $\mathbb{R}^4$, the norm of q is the length of this vector, whereas $q/|q|$ is a unit vector pointing to the hypersphere S^3. On first glance, it seems obvious to represent points on S^3 by polar coordinates and interpret these as the phase of a quaternion. However, it turns out that this representation is not adequate in the context of the QFT. For example, there exists no equivalent to the shift theorem of the complex Fourier transform using this representation [21]. There is another way of representing a quaternion. Each unit quaternion ($|q| = 1$) corresponds to a rotation in $\mathbb{R}^3$. The operational realization of a rotation by a unit quaternion q is given by $x' = qx\overline{q}$, where $x = ix_1 + jx_2 + kx_3$ is a called a *pure quaternion* representing the vector $\boldsymbol{x} = (x_1, x_2, x_3)^T$, and $q = \cos(\phi) + n\sin(\phi)$. Here, n is a pure quaternion representing a unit vector $\boldsymbol{n}$. The mapping $x' \mapsto x$ then performs a rotation by the angle 2ϕ about the axis defined by $\boldsymbol{n}$. Obviously, q and $-q$ represent the same rotation, such that there is a two-to-one correspondence between unit quaternions and rotations in $\mathbb{R}^3$. This is why $SO(3)$ and $SU(2)$ are not isomorphic but merely locally isomorphic.[6] The vector $\boldsymbol{n}$ and the angle ϕ could serve as a candidate of the phase of a quaternion. However, this parameterization does not lead to a phase definition with the desirable analogies to the phase of a 1-D signal. We use the correspondence between rotations and unit quaternions in order to find a reasonable definition of the phase of a quaternion. This definition will lead to three phase angles, two of which represent the 1-D phases in horizontal and vertical direction of a 2-D signal. The third phase component is related to the intrinsic local structure of the signal. These interpretations will be illustrated in the examples in Section VI.

Every rotation in 3-D can be expressed as a concatenation of three rotations about the coordinate axes. That is, each 3×3 rotation matrix R can be written as

$$R = R_{x_1}(\alpha)R_{x_3}(\beta)R_{x_2}(\gamma) \tag{34}$$

where $R_{x_i}(\alpha)$ is the matrix performing a rotation by α about the x_i axis. The angles α, β, and γ are called the *Euler-angles* of R [28].[7] Let q be a unit quaternion representing the same rotation as the matrix R. Then, the factorization of q equivalent to (34) is

$$q = e^{i\alpha/2}e^{k\beta/2}e^{j\gamma/2}. \tag{35}$$

Since we are interested in the quaternion representation and not in the matrix representation, we will replace α, β, and γ by $\phi = \alpha/2, \psi = \beta/2$, and $\theta = \gamma/2$ in order to eliminate the factors $1/2$ in (35). The above considerations lead to the following corollary and a definition of the phase of a quaternion.

Corollary 2: Any quaternion q can be represented as

$$q = |q|e^{i\phi}e^{k\psi}e^{j\theta} \tag{36}$$

where $|q|$ is the modulus of q, and (ϕ, θ, ψ) is called the phase of q. The phase is almost uniquely defined within the interval

$$(\phi, \theta, \psi) \in [-\pi, \pi[\times[-\pi/2, \pi/2[\times[-\pi/4, \pi/4]. \tag{37}$$

We have to say *almost uniquely* since there are the two singular cases $\psi = \pm\pi/4$. In this case, all values of ϕ and θ satisfying $\phi \mp \theta = C$ for some constant C fulfill (36).[8] We solve this situation by setting $\theta = 0$ whenever $\psi = \pm\pi/4$. A recipe for the computation of the phase of a quaternion can be summarized as follows: Normalize q: $\tilde{q} = q/|q| = a + ib + jc + kd$. Then, ψ is found to be

$$\psi = -\arcsin(2(bc - ad))/2. \tag{38}$$

If $\psi = \pm\pi/4$, we set $\theta = 0$, and find

$$\phi' = \tfrac{1}{2}\,\text{atan}\,2(2(-cd + ab),\, a^2 - b^2 - c^2 + d^2). \tag{39}$$

Else, we have

$$\phi' = \tfrac{1}{2}\,\text{atan}\,2(2(cd + ab),\, a^2 - b^2 + c^2 - d^2) \tag{40}$$
$$\theta = \tfrac{1}{2}\,\text{atan}\,2(2(bd + ac),\, a^2 + b^2 - c^2 - d^2). \tag{41}$$

Here, atan 2 is the four quadrant arctangent: $\text{atan}\,2(b, a) = \arctan(b/a)$, if $a > 0$; $\text{atan}\,2(b, 0) = \pm\pi$, if $\text{sign}(b) = \pm 1$; $\text{atan}\,2(b, a) = \arctan(b/a) - \pi$, if $a < 0$ and $b < 0$; $\text{atan}\,2(b, a) = \arctan(b/a) + \pi$, if $a < 0$ and $b \geq 0$. The expression $e^{i\phi'}e^{k\psi}e^{j\theta}$ can take one of the values $\tilde{q}$ or $-\tilde{q}$. If $e^{i\phi'}e^{k\psi}e^{j\theta} = -\tilde{q}$, set $\phi = (\phi' + \pi)\,\text{smod}\,2\pi$. Else, we set $\phi = \phi'$. We will also use the notation $\Phi(q) = \phi$, $\Theta(q) = \theta$, and $\Psi(q) = \psi$. For a complete derivation, see [21].

VI. Results

A. Analytic Results

All the above approaches toward n-D complex or hypercomplex signals are based on different combinations of the original

[6]In terms of Lie group theory, $SO(2)$ is the two-fold covering of $SO(3)$ [19].

[7]Every other permutation of rotations about the three coordinate axes is possible as well. Since the rotations do not commute, each permutation yields another set of angles for a given rotation R. It is even possible to represent a rotation as $R = R_{x_3}(\alpha)R_{x_1}(\beta)R_{x_3}(\gamma)$. Sometimes, the angles α, β, and γ according to *this* definition, where two rotations are about the x_3-axis with an intermediate rotation about the x_1-axis, are called *Euler angles* [29].

[8]This ambiguity is known as Gimbal lock when talking about rotations represented by Euler angles [28].

signal and its partial and total Hilbert transforms. For clarity, we summarize the different definitions for the 2-D case:

$$f_{tot} = f + i f_{Hi} \tag{42}$$
$$f_{part}^{j} = f + i f_{Hi}^{j} \tag{43}$$
$$f_{so_1} = (f - f_{Hi}) + i(f_{Hi}^1 + f_{Hi}^2) \tag{44}$$
$$f_{so_2} = (f + f_{Hi}) - i(f_{Hi}^1 - f_{Hi}^2) \tag{45}$$
$$f_{so}^{q} = f + i f_{Hi}^1 + j f_{Hi}^2 + k f_{Hi}. \tag{46}$$

All these constructions lead to different definitions of the local amplitude of a 2-D signal, where the amplitude is defined as the absolute value of the complex or hypercomplex signal. However, in the case of separable signals, some of the local amplitudes coincide.

Theorem 2: Let f be a separable signal $f(\boldsymbol{x}) = g(x_1)h(x_2)$. Then, the local amplitude with respect to the complex signals with single orthant spectra and the quaternionic signal are identical:

$$|f_{so}^{q}(\boldsymbol{x})| = |f_{so_1}(\boldsymbol{x})| = |f_{so_2}(\boldsymbol{x})|. \tag{47}$$

Proof: Expressing the local amplitudes in terms of the partial and total Hilbert transforms using (44)–(46) shows that (47) holds iff

$$f_{Hi}(\boldsymbol{x}) f(\boldsymbol{x}) = f_{Hi}^1(\boldsymbol{x}) f_{Hi}^2(\boldsymbol{x}). \tag{48}$$

Using the separability of f, we find $f_{Hi}^1 = g_{Hi}h$, $f_{Hi}^2 = g h_{Hi}$, and $f_{Hi} = g_{Hi}h_{Hi}$. Thus, (48) holds true, which proves the theorem. ■

For separable signals, the first two components of the quaternionic phase equal the 1-D phases of the separable components, as shown in the following theorem. The third component is zero in this case.

Theorem 3: Let f be a separable signal $f(\boldsymbol{x}) = g(x_1)h(x_2)$. Then, the following equalities hold true.

$$\Phi(f_{so}^{q}(\boldsymbol{x})) = \operatorname{atan2}(g_{Hi}(x_1),\, g(x_1)) \tag{49}$$
$$\Theta(f_{so}^{q}(\boldsymbol{x})) = \arctan(h_{Hi}(x_2)/h(x_2)) \tag{50}$$
$$\Psi(f_{so}^{q}(\boldsymbol{x})) \equiv 0. \tag{51}$$

The proof is based on the trigonometric identity

$$\operatorname{atan2}(2ab,\, a^2 - b^2)/2 = \arctan(b/a). \tag{52}$$

The ψ-component of the quaternionic phase, which is zero for separable signals, is a measure for the "degree of separability" of a signal. If we consider signals of the form

$$f(\boldsymbol{x};\, \lambda) = (1-\lambda)\cos(\omega_1 x_1 + \omega_2 x_2) + \lambda \cos(\omega_1 x_1 - \omega_2 x_2)$$

we find that ψ varies monotonically with the value of $\lambda \in [0,\, 1]$:

$$\Psi(f_{so}^{q}(\boldsymbol{x};\, \lambda)) \equiv -\frac{1}{2}\arcsin\left[\frac{2(1-2\lambda)}{1+(2\lambda-1)^2}\right]. \tag{53}$$

Thus, $\Psi(f_{so}^{q}(\boldsymbol{x};\lambda))$ equals zero for $\lambda = 1/2$, where $f(\boldsymbol{x};\, \lambda)$ is separable, and tends to $\pi/4$ or $-\pi/4$ if λ goes to 1 or 0, respectively. The ψ component of the quaternionic phase is the truly novel result here. It is sensitive to the local 2-D structure of the signal as shown in Fig. 2. This property seems to suggest the use of the quaternionic signal for texture analysis applications. One such example is presented in the following section.

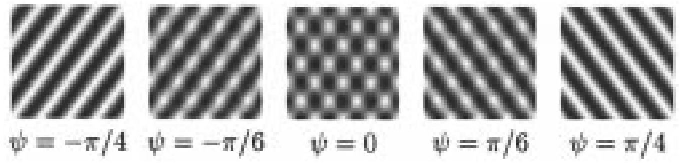

Fig. 2. Five patterns with different intrinsic dimensionality and the corresponding values of the ψ-component of the local phase.

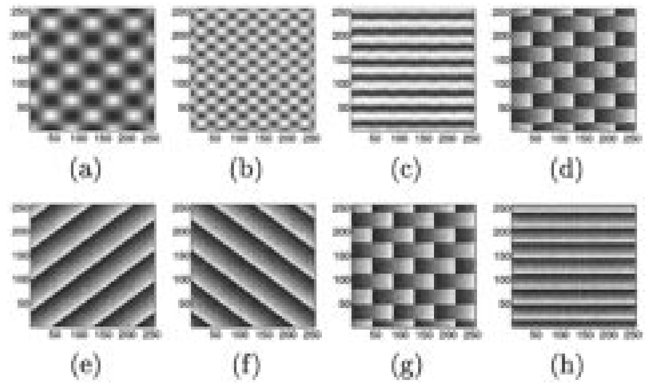

Fig. 3. Example 1. (a) Two-dimensional input signal f with $\omega_1 = 6\pi$, $\omega_2 = 8\pi$. (b) Local amplitude according to f_{tot}. (c) Local amplitude and (d) phase with respect to f_{part}^1. Local phase with respect to (e) f_{so_1} and (f) f_{so_2}. (g) ϕ and (h) θ resulting from f_{so}^q.

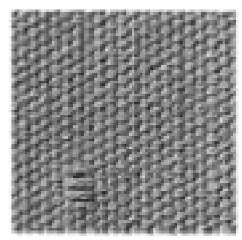

Fig. 4. Subregion of Brodatz' texture D77 taken from [30].

B. Examples

In order to compare the different approaches to an n-D complex signal, we present two examples of real 2-D signals and their complex and hypercomplex signals. In the first example, we investigate an intrinsically 2-D, separable signal, whereas the signal in the second example is the image of a textile texture.

Example 1: Let f be given by

$$f(\boldsymbol{x}) = \cos(\omega_1 x_1 + \varphi_1)\cos(\omega_2 x_2 + \varphi_2) \tag{54}$$

with $\omega_1, \omega_2 > 0$. The partial and total Hilbert transforms of f are given by

$$f_{Hi}^1(\boldsymbol{x}) = \sin(\omega_1 x_1 + \varphi_1)\cos(\omega_2 x_2 + \varphi_2) \tag{55}$$
$$f_{Hi}^2(\boldsymbol{x}) = \cos(\omega_1 x_1 + \varphi_1)\sin(\omega_2 x_2 + \varphi_2) \tag{56}$$
$$f_{Hi}(\boldsymbol{x}) = \sin(\omega_1 x_1 + \varphi_1)\sin(\omega_2 x_2 + \varphi_2). \tag{57}$$

Consequently, the quaternionic signal of f is

$$f_{so}^{q}(\boldsymbol{x}) = \exp(i(\omega_1 x_1 + \varphi_1))\exp(j(\omega_2 x_2 + \varphi_2)).$$

The local amplitude of f is $|f_{so}^{q}(\boldsymbol{x})| \equiv 1$. The local phase is given by $\theta(\boldsymbol{x}) = \Theta(f_{so}^{q}) = (\omega_2 x_2 + \varphi_2)\,\mathrm{smod}\,\pi$, $\phi(\boldsymbol{x}) = \Phi(f_{so}^{q}(\boldsymbol{x})) = (\omega_1 x_1 + \varphi_1)\,\mathrm{smod}\,2\pi$ if $\theta(\boldsymbol{x}) = \Theta(f_{so}^{q}(\boldsymbol{x})) =$

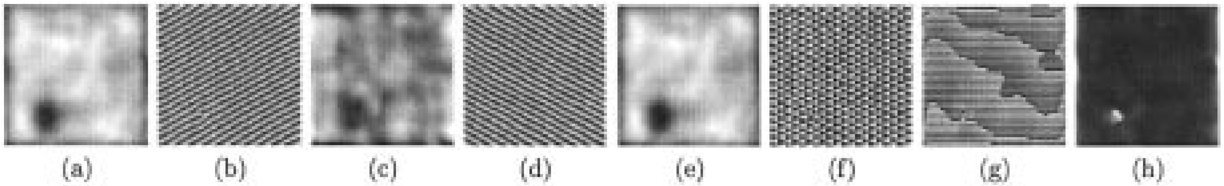

Fig. 5. Example 3. f_{so_1}. (a) Local amplitude. (b) Local phase. f_{so_2}. (c) Local amplitude and (d) local phase. f^q. (e) Local amplitude. (f) φ. (g) θ. (h) ψ.

$(\omega_2 x_2 + \varphi_2)\,\mathrm{smod}\,2\pi$, $\phi(\boldsymbol{x}) = \Phi(f_{so}^q(\boldsymbol{x})) = (\omega_1 x_1 + \varphi_1 + \pi)\,\mathrm{smod}\,2\pi$ else. $\psi(\boldsymbol{x}) = \Psi(f_{so}^q(\boldsymbol{x})) \equiv 0$.

Note that the local phase components ϕ and θ are identical to the local phases of the separable components of the signal modulo π. This is generally true for separable signals, as shown above (Theorem 3).

The two complex signals with single orthant spectra (see Definition 6) of f are

$$f_{so_1}(\boldsymbol{x}) = \exp(i(\omega_1 x_1 + \omega_2 x_2 + \varphi_1 + \varphi_2)) \tag{58}$$

$$f_{so_2}(\boldsymbol{x}) = \exp(i(-\omega_1 x_1 + \omega_2 x_2 - \varphi_1 + \varphi_2)). \tag{59}$$

Both local amplitudes are equal to 1 everywhere: $|f_{so_1}(\boldsymbol{x})| = |f_{so_2}(\boldsymbol{x})| \equiv 1$. The total complex signal is found to be

$$f_{tot}(\boldsymbol{x}) = \cos(\omega_1 x_1 + \varphi_1)\cos(\omega_2 x_2 + \varphi_2) + i\sin(\omega_1 x_1 + \varphi_1)\sin(\omega_2 x_2 + \varphi_2). \tag{60}$$

The partial complex signal is

$$f_{part}^1(\boldsymbol{x}) = \cos(\omega_1 x_1 + \varphi_1)\cos(\omega_2 x_2 + \varphi_2) + i\sin(\omega_1 x_1 + \varphi_1)\cos(\omega_2 x_2 + \varphi_2) \tag{61}$$

with a local amplitude $|f_{part}^1(\boldsymbol{x})| = |\cos(\omega_2 x_2 + \varphi_2)|$.

Some of the above results are shown in Fig. 3. Obviously, the total and the partial complex signals do not lead to reasonable definitions of the local amplitude in this case. The complex signals with single orthant spectra yield the correct local amplitude; the local phase is evaluated along the diagonals. The quaternionic signal yields the correct local amplitude too; the local phase components ϕ and θ correspond to the horizontal and vertical local phase.

Example 2: This example shows the complex signals with single orthant spectra and the quaternionic signal of the texture image shown in Fig. 4. As is often done in practical applications of the complex signal, we evaluate the complex/quaternionic signals of a bandpass filtered version of the original signal. The bandpass filter in this example was tuned to the dominant frequency of the presented texture.

It can be seen from Fig. 5(h) that the ψ-component of the quaternionic phase is sensitive to changes of a structure like the flaw in the textile shown in Fig. 4, which could be localized by thresholding the ψ-phase. Although this flaw can be clearly detected in the local amplitude image Fig. 5(e) as well, the amplitude is not stable under inhomogeneous lighting conditions, whereas the ψ-phase is almost not affected by that (see Fig. 6).

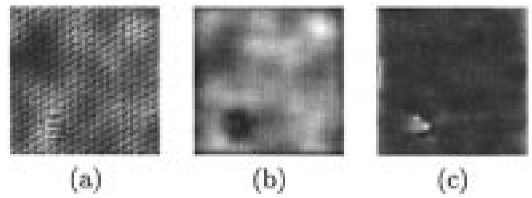

Fig. 6. Texture from Fig. 4 seen under (a) inhomogeneous lighting conditions. (b) Magnitude and (c) ψ-phase of its quaternionic signal.

VII. Conclusion

In this paper, we have reviewed different approaches toward n-D extensions of the concept of the analytic or complex signal first introduced by Gabor. Combining the complex signal with single orthant spectrum [9] and the hypercomplex Fourier transform, we proposed the hypercomplex signal as an n-D extension of the analytic signal. It could be shown that all these approaches are related since they all rely on different combinations of the input signal with its partial and total Hilbert transforms. The partial complex signal and the total complex signal contain the input signal as real part. The partial complex signal is an intrinsically 1-D concept. It can be obtained by evaluating the 1-D analytic signal along lines parallel to the reference orientation and, thus, is insensitive to intrinsically n-D signal structure. The total complex signal does not lead a useful definition of the local phase, as can be seen from the examples given in Section VI. The complex signal with single orthant spectrum proposed by Hahn yields an intrinsically n-D complex signal. In order to keep the full signal information, 2^{n-1} complex signals with a single orthant spectra have to be constructed. The hypercomplex signal proposed in this paper gives access to intrinsically n-D signal structure (for 2-D, this could be shown using the novel feature of the quaternionic phase) while being complete at the same time. The local phase of a hypercomplex signal has so far only been defined for 2-D signals.

Applications of the concepts introduced here can be found in [21]. In image processing, the concept of the quaternionic signal can be used, where local filters that yield a quaternionic signal as their filter response are applied. These filters are quaternionic extensions of the well-known complex Gabor filters. As was demonstrated in an example in this paper, the quaternionic phase, especially the ψ-component, can be a useful feature in texture analysis.

In addition to the extensions made here, there is yet another possible extension of the Hilbert transform to n-D, namely, Riesz transforms [31]. These transforms should be considered for the use in signal processing as well. First results of their rediscovery in the framework of geometric algebra as spherical Hilbert transform can be found in [32]. In [33], Riesz transforms are used in the context of image processing. Riesz transforms are convenient for the use with intrinsically 1-D signals of arbitrary orientation embedded in n-D space, whereas the present paper has dealt with intrinsically n-D signals.

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Thomas Bülow received the diploma in theoretical physics in 1995 and the Ph.D. degree in computer science in 1999, both summa cum laude from the Christian Albrechts University of Kiel, Kiel, Germany.

Currently, he is a research associate with the GRASP Laboratory, University of Pennsylvania, Philadelphia, and holds an Emmy Noether research grant from the German Research Association (DFG). His research interests include image- and multidimensional signal processing and low-level computer vision.

Dr. Bülow held a scholarship from the German National Merit Foundation (*Studienstiftung des Deutschen Volkes*) from 1995 to 1999.

Gerald Sommer received the diploma degree in physics from Friedrich Schiller Universität Jena, Jena, Germany, in 1969, the Ph.D. degree in physics from the same university in 1975, and a habilitation degree in engineering from Technical University Ilmenau, Ilmenau, Germany, in 1988.

From 1969 to 1991, he was with several departments of the Friedrich Schiller University Jena. From 1991 to 1993, he was the Head of the Division for Medical Image Processing, Research Center for Environment and Health (GSF-Medis), Munich–Neuherberg, Germany. Since 1993, he has been a Professor of computer science at the Christian Albrechts University Kiel, Kiel, Germany. He leads a research group in cognitive systems. His main interests are the design of behavior-based systems. His research covers signal theory and signal processing, neural computation for pattern recognition, and robot control.