Hyperelliptic linear systems on a K3 surface

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0 Introduction

Let X be a K3 surface[1] and $L \in \operatorname{Pic} X$ a line bundle such that the linear system $|L|$ has no fixed components, and such that $L^2 > 0$.

Theorem 0.1 (see [1], [2]) *$|L|$ has no base points, and hence defines a morphism*

$$\phi_L \colon X \to \mathbb{P}^g,$$

(where $L^2 = 2g - 2$). Furthermore,

(i) if $|L|$ contains a nonhyperelliptic curve (of genus g), ϕ_L is birational onto a surface $\overline{X}$ of degree $2g-2$, having only isolated rational double points (Du Val singularities);

(ii) if $|L|$ contains a hyperelliptic curve, then ϕ_L is a generically 2-to-1 mapping of X onto a surface F of degree $g-1$.

In the second case, $|L|$ will be called a *hyperelliptic linear system* and X a hyperelliptic K3. The object of this article is to exploit the well known classification of surfaces of degree $g-1$ in $\mathbb{P}^g$ (recalled in Section 1) to give a complete classification of hyperelliptic K3s. The principal result is as follows:

(i) any hyperelliptic K3 X is a double cover of one of the surface $\mathbb{P}^2$ or $\mathbb{F}_n$ (with $n = 0, 1, 2, 3$ or 4), with ramification curve $C \in |-2K_F|$;

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[1] Definition: $K_X = \mathcal{O}_X$, and $q = h^1(\mathcal{O}_X) = 0$. All varieties, line bundles, etc., are defined over a fixed algebraically closed ground field of characteristic $\neq 2$.

(ii) conversely, if $C \in |-2K_{\mathbb{F}_n}|$ is any curve having only "reasonable" singularities (see Theorem 2.2), then there is a K3 double covering $X \to F$ ramified in C, and this X has hyperelliptic linear systems $|L_r| = |L_0 + rE|$ (with L_0 and $E \in \operatorname{Pic} X$) of degrees $2n + 4r$ (for $r \geq 0$).

To interpret these results, let ${}^0\mathcal{M}_g$ denote the moduli space of K3 surfaces X, together with a line bundle $L \in \operatorname{Pic} X$ which is primitive (not a multiple in $\operatorname{Pic} X$), and such that $|L|$ is without fixed components and $L^2 = 2g-2$; ${}^0\mathcal{M}_g$ can be constructed as a quasiprojective variety by the methods of [5]; alternatively, it can be discussed in the local analytic context by the methods of Tyurina [1, Chapter IX].

The pairs (X, L_r), with $X \to \mathbb{F}_n$ as above, define a subvariety $\mathcal{F}_{n;r} \subset {}^0\mathcal{M}_g$ with $g-1 = n+2r$; and the union of these is just the subvariety of ${}^0\mathcal{M}_g$ consisting of pairs (X, L) which fall into case (ii) of Theorem 0.1.

It turns out that in a certain sense the double covers of $\mathbb{F}_2$ and $\mathbb{F}_3$ are degenerate cases of double covers of $\mathbb{F}_0$ and $\mathbb{F}_1$ (see Theorem 3.5 for a precise statement). This is equivalent to the inclusions

$$\mathcal{F}_{2;r} \subset \overline{\mathcal{F}_{0;r+1}} \quad \text{and} \quad \mathcal{F}_{3;r} \subset \overline{\mathcal{F}_{1;r+1}}$$

for the $\mathcal{F}_{n;r} \subset {}^0\mathcal{M}_g$.

Thus if g is even, the hyperelliptic subvariety of ${}^0\mathcal{M}_g$ consists of the single component $\overline{\mathcal{F}_{1;r}}$ (with $r = g/2 - 1$), whereas if g is odd it has two components $\overline{\mathcal{F}_{0;r+2}}$ and $\mathcal{F}_{4;r}$ (with $r = (g-5)/2$); these components are in all cases of codimension 1 in ${}^0\mathcal{M}_g$, that is, they are 18-dimensional.

Note added in proof The results and methods of this article overlap substantially with those of Dolgachev [9].

1 The surfaces $\mathbb{F}_n$ and a theorem of del Pezzo

In this section I recall what is needed about rational scrolls. Proofs of del Pezzo's theorem may be found in [3] and [4]. On $\mathbb{P}^1$ consider the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}$. The surface $\mathbb{F}_n$ is by definition the associated projective bundle:

$$\mathbb{F}_n = \operatorname{Proj}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}\bigr)$$

(defined for $n \geq 0$; some of the statements to follow require minor modification for the case $n = 0$). $\pi \colon \mathbb{F}_n \to \mathbb{P}^1$ is the projection map.

There are two obvious line bundles on $\mathbb{F}_n$, namely $L = \pi^*\mathcal{O}(1)$ – so that the linear system $|L|$ is just the ruling $|A|$ of $\mathbb{F}_n$, and the "tautological" Grothendieck bundle M; $\pi_* M = \mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}$ (the bundle we started from), so that $|M|$ is the fixed section B of the ruling $\mathbb{F}_n \to \mathbb{P}^1$. The following assertions are easy:

Proposition 1.1 *(i) The intersection pairings relating A and B are*

$$A^2 = 0, \quad A \cdot B = 1, \quad B^2 = -n;$$

(ii) the classes of A and B generate $\operatorname{Pic} \mathbb{F}_n$*;*

(iii) the canonical class $K_{\mathbb{F}_n}$ is given by

$$-K_{\mathbb{F}_n} = (n+2)A + 2B;$$

(iv) B is a fixed component of $|aA+bB|$ if and only if $a < nb$; B is the only curve on $\mathbb{F}_n$ to be fixed in any linear system, and is the only irreducible curve with negative self-intersection.

Proposition 1.2 *For any $r \geq 0$ the linear system $|B + (n+r)A|$ is without base points and defines a morphism*

$$\phi_{n;r} \colon \mathbb{F}_n \to \mathbb{P}^{n+2r+1};$$

except for the case $n = r = 0$, the image $\mathbb{F}_{n;r}$ of $\phi_{n;r}$ is a surface of degree $n + 2r$.

(i) If $r > 0$, $\phi_{n;r} \colon \mathbb{F}_n \to \mathbb{F}_{n;r}$ is an isomorphism.

(ii) $\mathbb{F}_{n;0}$ is a cone with vertex $O = \phi_{n;0}(B)$; $\phi_{n;0}$ induces an isomorphism of $\mathbb{F}_n \setminus B$ with $\mathbb{F}_{n;0} \setminus O$, taking the ruling $|A|$ of $\mathbb{F}_n$ into the generators of $\mathbb{F}_{n;0}$ (passing through O).

Thus the surface $\mathbb{F}_n$ may be regarded as the natural desingularisation of the cone on the twisted $n$ic rational curve in $\mathbb{P}^n$.

Theorem 1.3 (del Pezzo) *Let $F \subset \mathbb{P}^g$ be an irreducible surface of degree $g - 1$ and not contained in any hyperplane of $\mathbb{P}^g$. Then F is projectively equivalent to either $\mathbb{F}_{n;r}$ for some n, r with $n + 2r = g - 1$, or the Veronese surface $V_4 \subset \mathbb{P}^5$.*

Remark 1.4 $\mathbb{F}_{n;r}$ is distinguished among the surfaces of degree $n + 2r$ by the fact that it has a ruling by straight lines of $\mathbb{P}^g$, and a hyperplane section can be found containing $n + r$ lines of the ruling, but no more; that is, if H is the hyperplane section, and A a line of the ruling, then

$$H^0(\mathcal{O}_F(H - (n+r)A) \neq 0, \quad H^0(\mathcal{O}_F(H - (n+r+1)A) = 0.$$

2 Double coverings of the surfaces $\mathbb{F}_n$

Let X be a K3 surface, and D a hyperelliptic divisor on X as in Theorem 0.1 (ii). $D^2 > 0$, so that $\phi_D\colon X \to F \subset \mathbb{P}^g$, with $g \geq 2$; ϕ_D has degree 2, so that F has degree $g-1$, and since F does not lie in any hyperplane of $\mathbb{P}^g$ it is either $\mathbb{P}^2$, $\mathbb{P}^2$ in its Veronese embedding or one of the $\mathbb{F}_{n;r}$. In this section we consider only the last possibility. If F is $\mathbb{F}_{n;r}$ and $r > 0$, then F is isomorphic to $\mathbb{F}_n$, so that ϕ_D defines a double cover $X \to \mathbb{F}_n$; in the case $r = 0$ the same also holds:

Lemma 2.1 *Let $\phi_D\colon X \to \mathbb{F}_{g-1;0}$. Then ϕ_D factorises through $\phi_{g-1;0}$;*

$$\begin{tikzcd} X \arrow[rr, "\phi_D"] \arrow[dr] & & \mathbb{F}_{g-1;0} \\ & \mathbb{F}_{g-1} \arrow[ur, "\phi_{g-1;0}"'] & \end{tikzcd}$$

Proof $\mathbb{F}_{g-1;0}$ is a cone with vertex O. The proper transform $\phi_D^{-1}(|A|)$ of the ruling $|A|$ of $\mathbb{F}_{g-1;0}$ is obviously a pencil $|E|$ of irreducible curves on X. I am going to show that $|E|$ is in fact an elliptic pencil (that is, $E^2 = 0$), so that then $|D + E|$ defines a morphism $\phi_{D+E}\colon X \to \mathbb{F}_{g-1;1} \cong \mathbb{F}_{g-1}$ to complete the above triangle.

Taking a hyperplane section of $\mathbb{F}_{g-1;0}$ through the vertex O gives

$$D \sim (g-1)E + \sum n_i E_i,$$

the E_i being components of $\phi_D^{-1}(0)$, and $n_i \geq 0$; E and the E_i have no components in common, so $E \cdot E_i \geq 0$. But now

$$D \cdot E = 2A \cdot (\text{hyperplane}) = 2 = (g-1)E^2 + \sum n_i E \cdot E_i;$$

$E^2 = 0$ then follows since $g \geq 3$ and E^2 is even.

$D \cdot \sum n_i E_i = 0$, and so there is no positive divisor in $|D - gE| = |\sum n_i E_i - E|$. The linear system $|D + E|$ is still hyperelliptic (since $|E|$ cuts out a g^1_2 on any curve in $|D + E|$), and ϕ_{D+E} takes X onto $\mathbb{F}_{g-1;1}$, since it certainly goes into some ruled surface $\mathbb{F}_{n;r}$ with $n + 2r = g + 1$, and precisely g of the lines of the ruling can be contained in a hyperplane section. The commutativity of the above triangle is obvious, and the lemma is proved.

Hence we know that there is a morphism $\phi\colon X \to \mathbb{F}_n$ such that the linear system $|\phi^*((n+r)A + B)|$ is hyperelliptic (for some $r > 0$). It is easy to see by arguing on the genus of $E = \phi^*(A)$ and of $\phi^*((n+r)A+B)$ that the ramification curve C of ϕ is linearly equivalent to $(2n+4)A+4B \sim -2K_F$. This proves that any such K3 is of the type constructed in the following theorem.

Theorem 2.2 *Let C be a curve on $\mathbb{F}_n$ in the linear system $|-2K_{\mathbb{F}_n}|$. Suppose that C is either nonsingular or has only isolated singularities with (analytic) local equations of the type (i) $y^2 = z^{n+1}$; (ii) $y^2z = z^{n-1}$; (iii) $y^3 = z^4, z^3y$ or z^5. Then there is a K3 surface X and a double covering $X \to \mathbb{F}_n$ whose ramification locus is precisely the curve C.*

Remark 2.3 If C has isolated singularities not of the above analytic type, then the double cover of $\mathbb{F}_n$ ramified in C is not birationally equivalent to a K3 surface, since it has $p_g = 0$. We shall also subsequently need the obvious fact that if C contains B as a repeated component then the double cover ramified in C is rational, since it has a pencil of curves of genus 0.

Proof of Theorem 2.2 The double covering $\pi\colon \overline{X} \to \mathbb{F}_n$ given locally by the equation $z^2 =$ (equation of C) has only isolated Du Val singularities, and $\overline{X}$ has a canonical desingularisation $f\colon X \to \overline{X}$ with trivial canonical class (see Appendix).

Let us prove that $H^1(X, \mathcal{O}_X) = 0$; first, if $f\colon X \to \overline{X}$ is the desingularisation map then $R^i f_*\mathcal{O}_X = 0$ for $i > 0$ since $\overline{X}$ has only rational singularities. Hence $H^1(X, \mathcal{O}_X) = H^1(\overline{X}, \mathcal{O}_{\overline{X}})$ by the Leray spectral sequence. Since $\pi\colon \overline{X} \to F$ is finite, $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = H^1(F, \pi_*\mathcal{O}_{\overline{X}})$; but by definition of $\overline{X}$ (see Appendix), $\pi_*\mathcal{O}_{\overline{X}} = \mathcal{O}_{\mathbb{F}} \oplus K_{\mathbb{F}}$, and hence the result.

Corollary 2.4 *Let X be a K3 surface, $|D|$ a hyperelliptic linear system, and suppose that $\phi_D\colon X \to \mathbb{F}_n$ is the corresponding double cover; then $n = 0, 1, 2, 3$ or 4.*

Proof $-2K_{\mathbb{F}_n} \sim (2n+4)A + 4B$; but by Proposition 1.1 (iv), $2B$ will be fixed in $|-2K_{\mathbb{F}_n}|$ if $2n+4 < 3n$, that is, if $n \geq 5$. Then the ramification curve will have B as double component, which contradicts the hypothesis that X is a K3 surface, by Remark 2.3 above.

Appendix to Section 2

I summarise some well known facts on double coverings.

Definition Let S be a scheme; a *finite double covering* of S is a finite S-scheme $\pi\colon Z \to S$ whose corresponding $\mathcal{O}_S$-algebra $\pi_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_S$-module of rank 2.

Proposition 2.5 *Let B be a locally free $\mathcal{O}_S$-algebra of rank 2; let L denote the cokernel of the structure map $\mathcal{O}_S \to B$. Then L is a line bundle (locally free $\mathcal{O}_S$-module of rank 1).*

Let 2 be invertible in $\mathcal{O}_S$. Then the exact sequence

$$0 \to \mathcal{O}_S \to B \to L \to 0$$

has a splitting $\alpha\colon L \to B$ which is uniquely determined by the requirement that α takes any local section of L into a local section of B whose square is in $\mathcal{O}_S$.

In other words, we have an isomorphism of coherent $\mathcal{O}_S$-algebras $B = \mathcal{O}_S \oplus L$, the multiplication in $\mathcal{O}_S \oplus L$ being given by an $\mathcal{O}_S$-linear map $L \otimes L \to \mathcal{O}_S$, that is, a section of $L^{\otimes -2}$; since two different sections of $L^{\otimes -2}$ define isomorphic algebras B if and only if they differ by multiplication by the square of an invertible element of $H^0(\mathcal{O}_S)$, we have the following result.

Corollary 2.6 *There is a bijection between the following two sets:*

$$\left\{\begin{matrix}\textit{finite double}\\ \textit{coverings of } S\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{pairs } (L,s) \textit{ with } L \in \operatorname{Pic} S, \textit{ and}\\ s \in H^0(S, L^{\otimes -2})/(H^0(\mathcal{O}_S)^*)^2\end{matrix}\right\}.$$

We shall see that this correspondence makes sense of the equation $z^2 = s$ in the affine space $\mathbf{L}$ corresponding to L.

The proposition is proved in three steps: (i) by localisation one can assume that S is affine, and that B is a free $\mathcal{O}_S$-module of rank 2; (ii) B can be given a basis $(1, t)$ with $t \in B$; (iii) completing the square, B can be given a basis $(1, s)$ with $s^2 \in \mathcal{O}_S$.

Since Z is defined as $\operatorname{Spec}_S(B)$, and B is a quotient of the symmetric algebra $\bigoplus L^{\otimes n}$, we have the inclusion

$$\begin{array}{ccc} Z & \subset & \mathbf{L} = \operatorname{Spec}_S(\bigoplus L^{\otimes n}) \\ & \searrow \quad \swarrow_f & \\ & S & \end{array}$$

and indeed, as a divisor in $\mathbf{L}$, Z is defined by the vanishing of the single section $(z^2 - s)$ of $f^*(L^{\otimes -2}) = \mathcal{O}_{\mathbf{L}}(Z)$. From this we get the following adjunction formula:

Proposition 2.7 *Suppose that S is a nonsingular variety*[2] *with canonical line bundle K_S, and let $s \in H^0(S, L^{\otimes -2})$ be such that the divisor $D \in |-2L|$ (ramification locus) is nonsingular; then Z is nonsingular, and the canonical class of Z is given by*

$$K_Z = \pi^*(K_S \otimes L^{-1}).$$

Proof Since the tangent space to $\mathbf{L}$ is $f^*(T_S \oplus L^{-1})$, we have

$$\Omega^{n+1}_{\mathbf{L}} = f^*(\Omega^n_S \otimes L)$$

for the top exterior powers of the cotangent bundles ($n = \dim S$); the formula for K_Z then follows from the usual adjunction formula.

Proposition 2.8 *Suppose that S is a nonsingular surface, and that $s \in H^0(L^{\otimes -2})$ defines a curve C which has only isolated singularities with the local analytic equation of Theorem 2.2.*

Then the finite double cover $\overline{X} = Z$ provided by Corollary 2.6 has only Du Val singularities, viz.

$$\begin{array}{rll}
\text{(i)} & x^2 + y^2 + z^{n+1} = 0 & A_n \\
\text{(ii)} & x^2 + y^2 z + z^{n-1} = 0 & D_n \\
\text{(iii)} & x^2 + y^3 + z^4 = 0 & E_6 \\
 & x^2 + y^3 + yz^3 = 0 & E_7 \\
 & x^2 + y^3 + z^5 = 0 & E_8
\end{array}$$

and $\overline{X}$ admits a desingularisation $f\colon X \to \overline{X}$ for which the adjunction formula of Proposition 2.7 holds, that is

$$K_X = (f\pi)^*(K_S \otimes L^{-1}).$$

Proof It is well known (indeed, this is Du Val's characterisation) that these singularities have the property that they are isolated double points, and that the surface obtained by blowing them up has also only isolated double points; $\overline{X}$ thus has a desingularisation

$$X_N \to X_{N-1} \to \cdots \to X_1 \to \overline{X},$$

the nth step of which consists of blowing up an isolated double point of X_{n-1}. $\overline{X}$ is given as a divisor on a nonsingular 3-fold F, say, and so we can

[2]More generally, if S is Gorenstein, and s is arbitrary, then Z will also be Gorenstein, with K_Z given by the same formula.

set $K_{\overline{X}} = (K_F \otimes \mathcal{O}_F(\overline{X})) \otimes \mathcal{O}_{\overline{X}}$, and call this the "canonical line bundle"[3] of the singular surface $\overline{X}$. Proposition 2.8 follows at once from the easy lemma:

Lemma 2.9 *Let $P \in \overline{X}$ be an isolated double point, and let $f\colon X_1 \to \overline{X}$ be the blowing up of P in $\overline{X}$. Then $K_{X_1} = f^* K_{\overline{X}}$.*

Proof $\overline{X}$ is embedded in a nonsingular 3-fold, and the blowing up of $\overline{X}$ can also be embedded in a diagram of the form

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & \overline{X} \\ \cap & & \cap \\ F_1 & \xrightarrow{g} & F, \end{array}$$

with g the blowing up of P in F; letting $e = g^{-1}P$ be the exceptional locus of the blowing up, we get

$$K_{F_1} = g^* K_F \otimes \mathcal{O}_{F_1}(2e); \quad \mathcal{O}_{F_1}(X_1) = g^* \mathcal{O}_F(X) \otimes \mathcal{O}_{F_1}(-2e),$$

and the lemma follows. This argument actually only uses the fact that $\overline{X}$ can be *locally* embedded in a nonsingular 3-fold, and is identical to Du Val's original argument [6].

3 A description of the surfaces arising

Let X be a K3 surface, and let $|L|$ be a linear system on X without fixed components and with $L^2 > 0$. The following easy result can be found in [2]:

Proposition 3.1 $|L|$ *is hyperelliptic if and only if one of the following holds:*

(i) either $L^2 = 2$ or $L' = \frac{1}{2}L \in \operatorname{Pic} X$, and $L'^2 = 2$;

(ii) there is an elliptic pence $|E|$ on X with $E \cdot L = 2$.

[3]Note that this is nothing by Grothendieck's dualising sheaf $\omega_{\overline{X}}$.

Proof If (ii) holds then it is obvious that every nonsingular curve $D \in |L|$ is hyperelliptic. If $L^2 = 2$ then $\phi_L \colon X \to \mathbb{P}^2$ is a double covering, and if $L = 2L'$ with $L'^2 = 2$ then ϕ_L is the composite of $\phi_{L'}$ with the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$.

Conversely, if $|L|$ is hyperelliptic then ϕ_L factors through a double covering $X \to F$ (with $F = \mathbb{P}^2$ or one of the $\mathbb{F}_n$), and a morphism $F \to \mathbb{P}^g$ which can be (i) the identity $\mathbb{P}^2 \to \mathbb{P}^2$ or the Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$, or (ii) one of the $\phi_{n;r}$. In the second case the ruling $|A|$ on F gives rise to an elliptic pencil $|E|$ on X with $E \cdot L = 2$.

Denote $\mathcal{F}_n$ (resp. $\mathcal{P}^2$) the quotient of the open set of $|-2K_{\mathbb{F}_n}|$ consisting of curves with only the "reasonable" singularities of Theorem 2.2 by the natural action of $\operatorname{Aut} F$. This quotient is a coarse moduli space [5] for K3 surfaces double covering $\mathbb{F}_n$ (resp. $\mathbb{P}^2$). Its points are denoted by the same letter X as the corresponding surface.

Theorem 3.2 *A K3 surface $X \in \mathcal{P}^2$ has an irreducible linear system $|M|$ with $M^2 = 2$; conversely, given a K3 surface X and $|M|$ irreducible with $M^2 = 2$, then $|M|$ and $|2M|$ are hyperelliptic.*

A K3 surface $X \in \mathbb{F}_n$ (for $n = 0, 1, 2, 3$ or 4) has the following divisors: an irreducible elliptic pencil $|E|$, together with a positive divisor D such that

(i) $H^0(\mathcal{O}_X(D - E)) = 0$;

(ii) $D \cdot E = 2$;

(iii) $D^2 = -2n$;

and every component θ of D moves in $|D + nE|$ so that in particular

(iv) $\theta \cdot (D + nE) \geq 0$.

Conversely, given a K3 surface X, and elliptic pencil $|E|$ on X, and a divisor D satisfying (i–iv), then define

$$D_r = D + (n + r)E$$

(for $r \geq 0$); then $|D_r|$ is without fixed components and hyperelliptic, and

$$\phi_{D_r} \colon X \to \mathbb{F}_{n;r} \subset \mathbb{P}^{n+2r+1}.$$

Remark 3.3 For the various values of n, D must be of the following form (where there is more than one possibility, the generic case is given first):

$n = 0$**:** $|D|$ is an irreducible elliptic pencil;

$n = 1$**:** $|D|$ is irreducible, or $D = C + C_1 + \cdots + C_n + C'$ (see Figure 1) with $n \geq 0$, or $D = 2C + \sum n_i C_i$ (Figure $1'$);

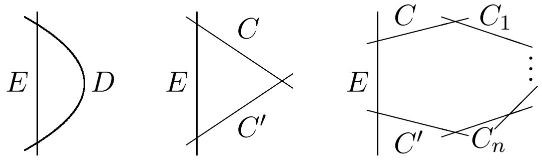


Figure 1: $n = 1$ (general case)

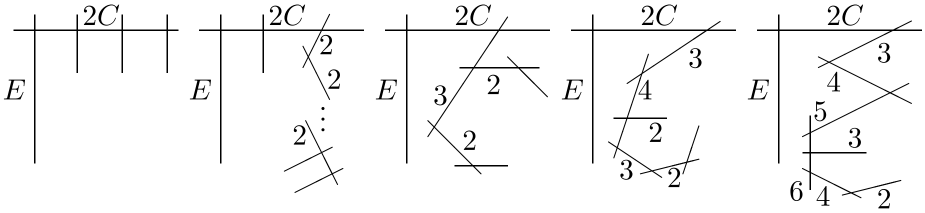
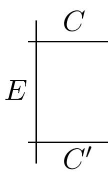
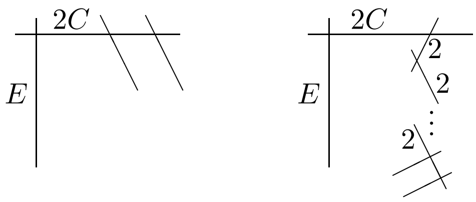


Figure $1'$: $n = 1$ (special cases)

$n = 2$**:** $D = C + C'$ (Figure 2), or $D = 2C + \sum n_i C_i$ (Figure $2'$);





$n = 2$: Figure 2 Figure $2'$: (special cases)

$n = 3$**:** $D = 2C + C_1$ (see Figure 3);

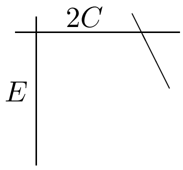


Figure 3: $n = 3$

$n = 4$**:** $D = 2C$, with C irreducible, $C^2 = -2$ and $E \cdot C = 1$.

In the above figures, C, C' and the C_i are irreducible rational curves with $C^2 = -2$; the C_i have $C_i \cdot E = 0$, and so are components of a reducible fibre of $|E|$; the (transversal) intersections of the various curves, and the multiplicities (when other than 1) are as indicated.

Proof of Theorem 3.2 If $f\colon X \to \mathbb{F}_n$ is the double covering, then $E = f^*A$ and $D = f^*B$ clearly give us an irreducible $|E|$ and a positive D satisfying (i–iv). Let $|E|$ and D be given on X satisfying (i–iii), and assume for the moment that $|D + nE|$, and hence $|D_r|$ for $r \geq 0$, is without fixed components. Then $|D_r|$ is hyperelliptic, and to prove that $\phi_{D_r}\colon X \to \mathbb{F}_{n;r} \subset \mathbb{P}^{n+2r+1}$ it suffices to note that $|D_r - (n+r)E| = |D|$ is nonempty, but that $|D_r - (n+r+1)E| = |D - E|$ is empty. The fact that (iv) implies that D_0 is without fixed components follows from the following proposition, which is a rearrangement of some results of [2]:

Proposition 3.4 *Let D be a divisor on a K3 surface with $D^2 > 0$. Then the following three assertions are equivalent:*

(i) D is 2-connected;

(ii) every component θ of D satisfies $D \cdot \theta \geq 0$, and $D \neq C + nE$ with $|E|$ an elliptic pencil, $E \cdot C = 1$, and $C^2 = -2$;

(iii) $|D|$ is without fixed components.

The table of possibilities for D is obtained in either of the two following ways: firstly, $D = f^*B$, where $f\colon X \to \mathbb{F}_n$ is the double covering, so the possibilities for D are determined by the various ways in which B meets the ramification curve. For example, if $n = 1$, then we can have

(a) B is not contained in the ramification divisor, and has intersection number 2 with it; in this case, according as B meets the ramification curve in 2 distinct points, touches it at a nonsingular point, or passes transversally through a "generalised cusp" (with local equation $y^2 = z^{n+1}$), we get the possibilities of Figure 1.

(b) B is contained in the ramification locus, and has three points of intersection with the rest of it – two of which are necessarily distinct, since otherwise we do not have a "reasonable" singularity; this gives the possibilities of Figure 1′ for D.

Secondly, one can enumerate the possibilities for the components using the numerical conditions (ii–iv): thus if $n = 2$, $D \cdot E = 2$ implies that the components of D having positive intersection with E are

(a) C and C' with $E \cdot C = E \cdot C' = 1$;

(b) $2C$, with $E \cdot C = 1$;

(c) C with $E \cdot C = 2$.

In fact (c) is impossible, since then the remaining components C_i of D satisfy $E \cdot C_i = 0$, and hence (by (iv)) $C_i \cdot D \geq 0$; but then $D^2 = -4$ gives $C \cdot D \leq -4$, which is absurd.

A similar argument shows that in (a) we necessarily have $C \cdot C' = 0$, and it easily follows that $D = C + C'$.

For (B), the same argument shows that the remaining components must have $E \cdot C_i = D \cdot C_i = 0$, and it is easy to see that Figure $2'$ gives all the possibilities.

Theorem 3.5 *Let X be a K3 surface, and let D be a divisor on X such that $\phi_D \colon X \to \mathbb{P}^g$ is a double cover of X onto $\mathbb{F}_{2;r}$ (resp. $\mathbb{F}_{3,r}$).*

Then the pair (X, D) is a specialisation of a pair (X', D'), with X' a K3 surface, and D' a divisor such that $\phi_{D'} \colon X' \to \mathbb{P}^g$ is a double cover of X' onto $\mathbb{F}_{0;r+1}$ (resp. $\mathbb{F}_{1;r+1}$).

First proof (in the case of the complex ground field only) By the methods of Tyurina [1, Chapter IX], it is clear that for any primitive subgroup $M \subset \operatorname{Pic} X$ there exist arbitrarily small deformations X' of X such that the algebraic classes in $H^2(X', \mathbb{Z}) = H^2(X, \mathbb{Z})$ are precisely the classes of elements of M. Let D be the given divisor on X, and $|E|$ the elliptic pencil as in Theorem 3.2. I claim that $M = \langle D, E \rangle$ is primitive – since

$$\det \begin{vmatrix} 2n & 2 \\ 2 & 0 \end{vmatrix} = -4,$$

we need only check that $\frac{1}{2}D, \frac{1}{2}E$ and $\frac{1}{2}(D + E)$ are not in $\operatorname{Pic} X$, which is easy. Hence X has a deformation X' in which divisors D' and E' span $\operatorname{Pic} X'$. It is clear that $H^0(\mathcal{O}_X(D')) = 0$, but $H^0(\mathcal{O}_X(D' + E')) > 0$, so that (X', D') does belong to the case $\mathbb{F}_0$ or $\mathbb{F}_1$.

Second proof (for any ground field) By Theorem 3.2, $|D-rE|$ is a linear system without fixed components and ϕ_{D-rE} maps X 2-to-1 onto $\mathbb{F}_{2;0} \subset \mathbb{P}^3$ (resp. $\mathbb{F}_{3;0} \subset \mathbb{P}^4$). In the first case, $\mathbb{F}_{2;0}$ is an ordinary quadric cone, and the ramification curve is cut out on $\mathbb{F}_{2;0}$ by a quartic surface of $\mathbb{P}^3$. Deforming $\mathbb{F}_{2;0}$ into a nonsingular quadric $\mathbb{F}_{0;1}$ and keeping the intersection with the same quartic as ramification curve, we obtain the desired deformation X', and linear system $(D-rE)', E'$ and hence D'. The same construction works in the second case if we take $\mathbb{F}_{3;0} \subset \mathbb{P}^4$ to be a hyperplane section of a suitable 3-fold scroll in $\mathbb{P}^5$, and the ramification curve in $\mathbb{F}_{3;0}$ to be cut out by a quartic of $\mathbb{P}^5$ containing two planes of the scroll. The theorem is proved.

The assertion that $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}_4$ have dimension 18 follows from the methods of [1, Chapter IX], or by direct computation of the dimensions of $|-2K_{\mathbb{F}_n}|$ and $\operatorname{Aut}\mathbb{F}_n$.

4 Some remarks on "trigonal" linear systems

Apart from the exceptional cases with $g = 2$ or 5, this paper has been concerned mainly with linear systems $|D|$ on a K3 surface X such that

(i) $|D|$ is without fixed components;

(ii) $D \cdot E = 2$ for some elliptic pencil $|E|$ of X.

An analogous study can be made of "trigonal" linear systems, for which $D \cdot E = 3$ for some elliptic pencil $|E|$ of X; according to [2], these are just the exceptions to $\phi_D(X) = \overline{X}$ being an intersection of quadrics (for $g \geq 5$, with one exceptional case with $g = 6$). In this case, we get a model of X as a divisor on a 3-fold scroll:

$$\phi_{D+rE} \colon X \to \overline{X} \subset \mathbb{F} \subset \mathbb{P}^N,$$

with $\mathbb{F} = \operatorname{Proj}_{\mathbb{P}^1}\big(\mathcal{O} \oplus \mathcal{O}(m) \oplus \mathcal{O}(n)\big)$ with $0 \leq m \leq n$; and $\overline{X} \in |-K_{\mathbb{F}}|$. The condition that $\overline{X}$ has only isolated singularities implies that $m \leq 2$ and $n - m \leq 2 + m$. We thus get finitely many families of "trigonal" K3 surfaces; these can be discussed in the spirit of Section 3, and in particular the analogue of Theorem 3.2 can be proved.

In both the hyperelliptic and the trigonal case, one of the families is the family of K3 surfaces with are Jacobians of elliptic fibrations; if the elliptic pencil $|E|$ has a section $C \subset X$ then $|D| = |2C + rE|$ (for $r \geq 4$) is hyperelliptic, and $|D| = |3C+rE|$ (for $r \geq 6$) is trigonal. In both cases these family occupy the extreme position in the classification – corresponding to $n = 4$ in the hyperelliptic case, and to $(m, n) = (2, 6)$ in the trigonal case.

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