# Hypergraph connectivity augmentation

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**Abstract**

The hypergraph augmentation problem is to augment a hypergraph by hyperedges to meet prescribed local connectivity requirements. We provide here a minimax theorem for this problem. The result is derived from the degree constrained version of the problem by a standard method. We shall construct the required hypergraph for the latter problem by a greedy type algorithm. A similar minimax result will be given for the problem of augmenting a hypergraph by weighted edges (hyperedges of size two with weights) to meet prescribed local connectivity requirements. Moreover, a special case of an earlier result of Schrijver on supermodular colourings shall be derived from our theorem.

## 1. Introduction

Let us be given a hypergraph $\mathcal{G}$ on the finite set $V$ and for any pair $s, t$ of vertices a connectivity requirement $\lambda(s,t)$ between $s$ and $t$, where $\lambda(s,t)$ is a non-negative integer. We are looking for a hypergraph $\mathcal{H}$ so that adding the hyperedges of $\mathcal{H}$ to $\mathcal{G}$ the new hypergraph satisfies the connectivity requirements.

For a hypergraph $\mathcal{H}$ the degree of a set $X$ denoted by $d_{\mathcal{H}}(X)$ is the number of hyperedges $H$ of $\mathcal{H}$ for which none of $H \cap X$ and $H \cap (V-X)$ is empty.

Using this notation our problem is the following. Find a hypergraph $\mathcal{H}$ so that for all $X \subseteq V$

$$d_{\mathcal{G}+\mathcal{H}}(X) \geq \lambda(s,t) \text{ for all } s \in X, t \notin X.$$

Introducing the set function $R(X) := \max\{\lambda(s,t) : s \in X, t \notin X\}$, the inequalities above can be transformed into

$$d_{\mathcal{H}}(X) \geq R(X) - d_{\mathcal{G}}(X) \text{ for all } X \subseteq V.$$

Finally, let us define $p(X) := R(X) - d_{\mathcal{G}}(X)$. So the hypergraph $\mathcal{H}$ must satisfy

$$d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V.$$

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The set function $p(X)$ has the following properties: it is integer valued, symmetric and skew-supermodular (for definitions see Section 2). We remark here that $p(X)$ is not necessarily non-negative.

The question is what is the minimum value of a hypergraph $\mathcal{H}$ which satisfies a symmetric, skew–supermodular set function $p$, that is

$$d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V.$$

Here the value of the hypergraph $\mathcal{H}$ is not the number of the hyperedges in $\mathcal{H}$ because in that case the problem is trivial. (Let $\mathcal{H}$ contain the set $V$ $k$ times where $k = \max\{p(X) : X \subseteq V\}$.) Thus it is natural to define val($\mathcal{H}$) to be $\sum_{H \in \mathcal{H}} |H|$.

The following theorem solves the problem.

**Theorem 1.** *Let $p$ be a symmetric, skew–supermodular, integer valued function on the ground set $V$. Then*

$$\min\{\text{val}(\mathcal{H}) : d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V\} = \max\{\textstyle\sum p(V_i)\}$$

*where the maximum is taken over all subpartitions $\{V_1, ..., V_l\}$ of $V$.*

The edge–connectivity augmentation problem for graphs was solved by A.Frank [1]. He derived the minimax theorem from the degree constrained version of the problem. We shall follow this line.

**Theorem 2.** *Let $p$ be a skew–supermodular integer valued function on the ground set $V$. Furthermore, let $m(v)$ be a non–negative integer valued function on $V$. Then there exists a hypergraph $\mathcal{H}$ such that*

$$d_{\mathcal{H}}(v) = m(v) \text{ for all } v \in V, \tag{1}$$

$$d_{\mathcal{H}}(X) \geq p(X) \text{ for all } X \subseteq V, \tag{2}$$

*if and only if for all $X \subseteq V$*

$$p(X) \leq \min\{m(X), m(V - X)\}. \tag{3}$$

*Furthermore, $\mathcal{H}$ can be chosen so that $|\mathcal{H}| = k := \max\{p(X) : X \subseteq V\}$.*

We shall prove that the hyperedges of the required hypergraph can be constructed step by step by a greedy type algorithm.

In Section 5 we shall derive from Theorem 2 a result due to Schrijver [4] on supermodular colourings.

In the remaining part of the Introduction we show a simple application of Theorem 1. Let us consider the following problem. We want to augment a hypergraph $\mathcal{G}$ by a set $F$ of edges (hyperedges of size two) with suitable rational weights on the edges so that the resulting hypergraph satisfies prescribed local edge-connectivity requirements and the total weight of the new edges is minimum. As above, we can formulate this problem as follows.

Minimize $1c_F$ so that $d_{c_F}(X) \geq p(X)$ for all $X \subset V$, where $c_F$ is the weighting of $F$, $d_{c_F}(X)$ denotes the sum of the weights of the edges of $F$ leaving $X$ and $p$ is the above defined set function.

**Claim** $\min\{1c_F : d_{c_F}(X) \geq p(X) \text{ for all } X \subseteq V\} = \max\{1/2 \sum p(V_i)\}$,

*where the maximum is taken over all subpartitions* $\{V_1, ..., V_l\}$ *of* $V$.

**Proof.** min≥max: $1c_F \geq 1/2 \sum d_{c_F}(V_i) \geq 1/2 \sum p(V_i)$.

min≤max: Let $\mathcal{H}$ be a hypergraph and $\{V_1, ..., V_l\}$ be a subpartition of $V$ satisfying the minimax formula in Theorem 1. Using $\mathcal{H}$ we define the required edge set and weighting. Let us replace each hyperedge $H$ of $\mathcal{H}$ by a circuit $C_H$ on the vertex set of $H$ with weight $1/2$ on each edge of $C_H$. Then $d_{c_F}(X) \geq d_{\mathcal{H}}(X) \geq p(X)$ and $\sum p(V_i) = \sum_{H \in \mathcal{H}} |H| = 2(1c_F)$, and we are done. □

**Remarks**

1. Clearly, this claim is true for any integer valued, symmetric, skew–supermodular set function $p$.
2. Note that the above constructed weighting is always half integral.
3. In the special case, when the starting hypergraph is empty, the same result was obtained by Gomory and Hu [3], and when the starting hypergraph is a graph, this was proved later by A. Frank in [1].

## 2. Definitions, Preliminaries

In this paper all (set) functions are integer valued. A set function $p$ is called **skew–supermodular** if at least one of the following two inequalities holds

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \tag{$p1$}$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X), \tag{$p2$}$$

whenever $X$ and $Y$ are two subsets of $V$. We call $p$ **symmetric** if $p(X) = p(V - X)$ for all $X \subseteq V$. Let $\mathcal{H}$ be a hypergraph. Then the **value** of $\mathcal{H}$ is defined to be $\text{val}(\mathcal{H}) = \sum\{|H| : H \in \mathcal{H}\}$. $d_{\mathcal{H}}(X)$ denotes the number of hyperedges $H$ of $\mathcal{H}$ for which none of $H \cap X$ and $H \cap (V - X)$ is empty.

A.Frank showed in [2] that $R(X)$ defined in the Introduction is skew-supermodular. It is well-known that the degree function $d_{\mathcal{G}}(X)$ satisfies both $(p1)$ and $(p2)$ so $p(X)$ defined in the Introduction is skew-supermodular, indeed. In this section we present some simple propositions we shall need later. It is easy to see that the following holds.

**Proposition 3.** *Let* $p$ *be a skew–supermodular function on* $V$*. Then*

$$p'(X) := \max\{p(X), p(V - X)\}$$

*is a symmetric, skew–supermodular function.* □

**Proposition 4.** *Let $p$ be a skew–supermodular function on $V$. Let $Z$ be a subset of $V$. Then $p'(X) := \max\{p(X \cup X') : X' \subseteq Z\}$ is skew–supermodular on $V - Z$.*

**Proof.** Let $X, Y \subseteq V - Z$. Let $X'$ (respectively, $Y'$) be the subset of $Z$ which defines $p'(X)$ (resp. $p'(Y)$). $p$ is skew–supermodular, thus either $(p1)$ (Case I.) or $(p2)$ (Case II.) holds for $X \cup X'$ and $Y \cup Y'$.

**Case I.**

$$\begin{aligned} p'(X) + p'(Y) &= p(X \cup X') + p(Y \cup Y') \\ &\leq p((X \cup X') \cap (Y \cup Y')) + p((X \cup X') \cup (Y \cup Y')) \\ &= p((X \cap Y) \cup (X' \cap Y')) + p((X \cup Y) \cup (X' \cup Y')) \\ &\leq p'(X \cap Y) + p'(X \cup Y). \end{aligned}$$

**Case II.**

$$\begin{aligned} p'(X) + p'(Y) &= p(X \cup X') + p(Y \cup Y') \\ &\leq p((X \cup X') - (Y \cup Y')) + p((Y \cup Y') - (X \cup X')) \\ &= p((X - Y) \cup (X' - Y')) + p((Y - X) \cup (Y' - X')) \\ &\leq p'(X - Y) + p'(Y - X). \end{aligned}$$

□

**Proposition 5.** *Let $p$ be a skew–supermodular function on $V$. Let $V_1$ be a subset of $V$. Let*

$$p'(X) := \begin{cases} p(X) - 1 & \text{if } X \cap V_1 \neq \emptyset, \\ p(X) & \text{otherwise.} \end{cases}$$

*Then $p'$ is skew–supermodular on $V$.*

**Proof.** It is straightforward. □

Let $m : V \longrightarrow Z_+$ be a non–negative integer valued function on $V$. Let $p$ be a skew–supermodular set function on $V$. Assume that $p(X) \leq m(X)$ for all $X \subseteq V$. We call a set $X \subseteq V$ **tight** if $m(X) = p(X)$. Then we have the following.

**Proposition 6.** *Let $X$ and $Y$ be two tight sets. Then either $X \cap Y$ and $X \cup Y$ or $X - Y$, $Y - X$ are tight sets. Furthermore, in the latter case $m(X \cap Y) = 0$.*

**Proof.** Since $p$ is skew–supermodular, either $(p1)$ or $(p2)$ holds for $X$ and $Y$.

**Case I.** Assume first that $(p1)$ holds. Then

$$\begin{aligned} m(X) + m(Y) &= p(X) + p(Y) \\ &\leq p(X \cap Y) + p(X \cup Y) \\ &\leq m(X \cap Y) + m(X \cup Y) \\ &= m(X) + m(Y). \end{aligned}$$

Thus equality holds everywhere, implying that $X \cap Y$ and $X \cup Y$ are tight.

**Case II.** Assume that $(p2)$ holds. Then

$$\begin{aligned} m(X) + m(Y) &= p(X) + p(Y) \\ &\le p(X - Y) + p(Y - X) \\ &\le m(X - Y) + m(Y - X) \\ &= m(X) + m(Y) - 2m(X \cap Y). \end{aligned}$$

This implies that $X - Y$ and $Y - X$ are tight. Furthermore, $m(X \cap Y) = 0$. □

## 3. The proof of Theorem 2

In this section we prove Theorem 2. Moreover, we shall characterize the hyperedges which can be contained in a hypergraph satisfying the requirements of Theorem 2.

**Proof.** The only if part is trivial, so we prove the other direction. Let us consider a minimal counter–example, minimal with respect to $|V| + k$.

By Proposition 3, we may assume that $p$ is symmetric.

**Lemma 7.** *Let $H \subseteq V$. Then there exists a hypergraph $\mathcal{H}$ with*

*i.)* $H \in \mathcal{H}$,
*ii.)* $\mathcal{H}$ *satisfies (1) and (2),*
*iii.)* $\mathcal{H}$ *contains exactly* $k := \max\{p(X) : X \subseteq V\}$ *hyperedges*

*if and only if*

*a.)* $p$ *and* $m$ *satisfy (3),*
*b.)* $X \cap H \neq \emptyset$ *for all set* $X \subseteq V$ *with* $p(X) = k$,
*c.)* $|X \cap H| \le m(X) - p(X) + 1$ *for all* $X \subseteq V$,
$|H| \le m(X) - p(X)$ *for all* $V \supseteq X \supseteq H$,
*d.)* $m(v) \ge 1$ *for all* $v \in H$.

**Remark.** Note that in fact b.) and c.) imply a.).

**Proof.** First we show the necessity of the conditions. Assume the hypergraph $\mathcal{H}$ satisfies the requirement of the lemma. It is easy to see that a.), b.) and d.) hold. To see c.) assume that $H \in \mathcal{H}$. Let $\mathcal{H}' := \mathcal{H} - H$,

$$p'(X) := \begin{cases} p(X) - 1 & \text{if } X \cap H \neq \emptyset, \\ p(X) & \text{otherwise.} \end{cases}$$

$$m'(v) := \begin{cases} m(v) - 1 & \text{if } v \in H, \\ m(v) & \text{otherwise.} \end{cases}$$

Then $p'$ is skew–supermodular by Proposition 5, $m'$ is non-negative by d.) and clearly $\mathcal{H}'$ satisfies (1) and (2) for $p', m'$. Thus (3) holds for $p'$ and $m'$.

Let $X \subseteq V$. Then $p(X) - 1 \leq p'(X) \leq m'(X) = m(X) - |X \cap H|$.
Finally, let $H \subseteq X \subseteq V$. Then $p(X) = p(V - X) = p'(V - X) \leq m'(X) = m(X) - |H|$.

Secondly, we show the sufficiency of the conditions. Let $H$ be a subset of $V$ satisfying a.), b.), c.) and d.).
Let $p'$ and $m'$ be defined as above.

**Claim 8.** *$p'$ and $m'$ satisfy the conditions of Theorem 2.*

**Proof.** By Proposition 5, $p'$ is skew–supermodular and by d.) $m'$ is non-negative.
To prove (3), let $X$ be a subset of $V$.

If $X \cap H = \emptyset$, then $p'(X) = p(X) \leq m(X) = m'(X)$ by a.).
If $X \cap H \neq \emptyset$, then by c.), $p'(X) = p(X) - 1 \leq m(X) - |X \cap H| = m'(X)$.
Thus $p'(X) \leq m'(X)$ for all $X \subseteq V$. $(*)$

If $X \cap H = \emptyset$, then by c.), $p'(X) = p(X) = p(V - X) \leq m(V - X) - |H| = m'(V - X)$.
If $X \cap H \neq \emptyset$, then using the previous inequality $(*)$ and the symmetry of $p$, $p'(X) = p(X) - 1 = p(V - X) - 1 \leq p'(V - X) \leq m'(V - X)$.
Thus $p'(X) \leq m'(V - X)$ for all $X \subseteq V$. □

By b.) $\max\{p'(X) : X \subseteq V\} = k - 1$. By the minimality of the counter–example, there exists a hypergraph $\mathcal{H}'$ satisfying (1) and (2) for $p'$ and $m'$ and containing $k - 1$ hyperedges.

**Claim 9.** *$\mathcal{H} := \mathcal{H}' \cup \{H\}$ satisfies (1) and (2) for $p$ and $m$.*

**Proof.** First we prove that (1) holds.
If $v \notin H$ then $d_{\mathcal{H}}(v) = d_{\mathcal{H}'}(v) = m'(v) = m(v)$.
If $v \in H$ then $d_{\mathcal{H}}(v) = d_{\mathcal{H}'}(v) + 1 = m'(v) + 1 = m(v)$.

Secondly we prove that (2) holds. Let $X \subseteq V$.
If $X \cap H = \emptyset$, then $d_{\mathcal{H}}(X) = d_{\mathcal{H}'}(X) \geq p'(X) = p(X)$.
If $X \supseteq H$, then $d_{\mathcal{H}}(X) = d_{\mathcal{H}'}(X) = d_{\mathcal{H}'}(V - X) \geq p'(V - X) = p(V - X) = p(X)$.
If $X \cap H \neq \emptyset$ and $H - X \neq \emptyset$, then $d_{\mathcal{H}}(X) = d_{\mathcal{H}'}(X) + 1 \geq p'(X) + 1 = p(X)$. □

By Claim 9 the proof of Lemma 7 is complete. □

To prove Theorem 2 we have to show that there exists a subset $H$ of $V$ satisfying b.), c.) and d.) in Lemma 7. We shall need the following lemma.

**Lemma 10.** *$m(v) \geq 1$ for all $v \in V$.*

**Proof.** Let $Z := \{v \in V : m(v) = 0\}$ and suppose $Z$ is not empty. Let us define a set function $p'$ on $V - Z$ as follows.

$$p'(X) := \max\{p(X \cup X') : X' \subseteq Z\}.$$

Then by Proposition 4, $p'$ is skew–supermodular and it is symmetric since

$$p'(X) = \max\{p(X \cup X') : X' \subseteq Z\} = \max\{p(V - X - X') : X' \subseteq Z\} = p'((V - Z) - X).$$

Let $m'(v) := m(v)$ for $v \in V - Z$. Since $m(X') = 0$ for all $X' \subseteq Z$, we have for all $X \subseteq V - Z$

$$p'(X) = \max\{p(X \cup X') : X' \subseteq Z\} \leq \max\{m(X \cup X') : X' \subseteq Z\} = m(X) = m'(X).$$

Thus $p'$ and $m'$ satisfy the conditions of Theorem 2. Now the ground set is smaller, thus there exists a hypergraph $\mathcal{H}$ which satisfies (1) and (2) for $p'$ and $m'$. Since $p'$ is defined by as a maximum, this hypergraph is good for $p$ and $m$. This contradiction proves the assertion. □

Let $V_1$ be a minimal (for inclusion) set intersecting each set $Y$ for which $p(Y) = k$. We show that $V_1$ satisfies c.) and thus by Lemma 7 there exists a hypergraph satisfying (1) and (2) and containing this hyperedge.

**Lemma 11.** *$p(X) \leq m(X) + 1 - |V_1 \cap X|$ for all $X \subseteq V$. Furthermore, if $X \supseteq V_1$ then $p(X) \leq m(X) - |V_1|$.*

**Proof.** We prove the lemma by induction on $|X|$. For $X = \emptyset$ it is true by (3). Let $X$ be an arbitrary subset of $V$ and assume that the lemma holds for each smaller set. We may assume that there exists a vertex $y \in V_1 \cap X$, for otherwise the lemma holds by (3). By the minimality of $V_1$, there exists a set $Y \subseteq (V - V_1) \cup \{y\}$ for which $p(Y) = k$. Then

$$|V_1 \cap (X - Y)| = |V_1 \cap X| - 1. \tag{4}$$

Note that by Lemma 10 and (4) we have the following inequalities.

$$m(X - Y) \geq m((X - Y) \cap V_1) \geq |(X - Y) \cap V_1| = |V_1 \cap X| - 1. \tag{5}$$

$$m(X \cap Y) \geq 1. \tag{6}$$

Since $y \in X \cap Y$, $|X - Y| < |X|$ and the lemma is true for $X - Y$, that is

$$p(X - Y) \leq m(X - Y) + 1 - |V_1 \cap (X - Y)|. \tag{7}$$

By the skew–supermodularity of $p$, $(p1)$ (Case I) or $(p2)$ (Case II.) holds for $X$ and $Y$.

**Case I.**

$$\begin{aligned} p(X) + p(Y) &\leq p(X \cap Y) + p(X \cup Y) && \text{by (p1)} \\ &\leq m(X \cap Y) + p(X \cup Y) && \text{by (3)} \\ &= m(X) - m(X - Y) + p(X \cup Y) \\ &\leq m(X) + 1 - |V_1 \cap X| + p(X \cup Y). && \text{by (5)} \end{aligned}$$

This implies the desired inequalities, since $p(Y) = k, p(X \cup Y) \leq k$ and if $X \supseteq V_1$ then $V - (X \cup Y)$ is disjoint from $V_1$ thus $p(X \cup Y) = p(V - (X \cup Y)) \leq k - 1$.

**Case II.**

$$\begin{aligned} p(X) + p(Y) &\leq p(X - Y) + p(Y - X) && \text{by (p2)} \\ &\leq m(X - Y) + 1 - |V_1 \cap (X - Y)| + p(Y - X) && \text{by (7)} \\ &\leq m(X) - m(X \cap Y) + 1 - |V_1 \cap X| + 1 + p(Y - X) && \text{by (4)} \\ &\leq m(X) - |V_1 \cap X| + 1 + p(Y - X) && \text{by (6)} \\ &\leq m(X) - |V_1 \cap X| + k, && \text{since } (Y - X) \cap V_1 = \emptyset. \end{aligned}$$

□

Observe that $V_1$ can be constructed easily since the minimal sets $Y$ with $p(Y) = \max\{p(X) : X \subset V\}$ are pairwise disjoint by the skew-supermodularity of $p$.

## 4. Hypergraph connectivity augmentation

Now we are in a position to prove Theorem 1.

**Proof.** The min $\geq$ max being trivial we prove only the other direction. We shall prove it with a standard method using Theorem 2. Let $m : V \longrightarrow Z_+$ be a degree constraint on $V$ so that there exists a hypergraph $\mathcal{H}$ satisfying (1) and (2) in Theorem 2 for $p$ and $m$ and $m(V)$ is minimal. (Such an $m$ exists since $m(v) = \max\{p(X) : X \subseteq V\}$ for all $v \in V$ satisfies (3) in Theorem 2.)

Recall that a set $X$ is tight if $p(X) = m(X)$. Let $Z := \{v \in V : m(v) = 0\}$ and $V'; = V - Z$. By the minimality of $m(V)$ and the symmetry of $p$ each $v \in V'$ is contained in a tight set. Thus there exists a set system $\{X_1, ...X_l\}$ satisfying the following.

i.) $X_i$ is tight for all $i = 1, ..., l$,
ii.) $\bigcup_1^l X_i \supseteq V'$,
iii.) $\sum |X_i|$ is minimal.

**Claim.** $X_i \cap X_j = \emptyset$.

**Proof.** Assume that $X_i \cap X_j \neq \emptyset$. By Proposition 6, either $X_i \cup X_j$ is tight or $X_i - X_j$, $X_j - X_i$ are tight sets and in the latter case $m(X \cap Y) = 0$. Observe that this implies that $X \cap Y \subseteq Z$. $\{X_i, X_j\}$ will be replaced by $X_i \cup X_j$ in the former case, and by $X_i - X_j$, $X_j - X_i$ in the latter case. In both cases we have a contradiction with iii.) □

This subpartition shows the other direction.

min $\leq \mathrm{val}(\mathcal{H}) = \sum_{H \in \mathcal{H}} |H| = \sum_{v \in V} d_{\mathcal{H}}(v) = \sum_{v \in V} m(v) = m(V) = \sum_1^l m(X_i) = \sum_1^l p(X_i) \leq$ max. □□□

## 5. Supermodular colourings

In this section we show that Theorem 2 implies a special case of an earlier result of Schrijver [4] on supermodular colourings. Let $p$ be a supermodular function. (A set function $p$ is **supermodular** if ($p$1) holds for all sets $X$ and $Y$.) A colouring of $V$ with $k$ colours is called **good $k$–colouring with respect to** $p$ if for all subsets $X$ of $V$ the number of different colour classes intersecting $X$ is at least $p(X)$. Schrijver proved in [4] that the obvious necessary condition for the existence of a good $k$–colouring is also sufficient, that is the following holds.

**Theorem 12.** *Let $p$ be an integer valued supermodular function and let $k := \max\{p(X) : X \subseteq V\}$ be non-negative. Then there exists a good $k$–colouring if and only if $p(X) \leq |X|$ for all $X \subseteq V$.*

**Proof.** Let

$$V' := V \cup v,$$

$$p'(X) := \begin{cases} p(X) & \text{if } v \notin X \\ p(V-X) & \text{if } v \in X. \end{cases}$$

$$m'(u) := \begin{cases} 1 & \text{if } u \in V, \\ \max\{p(X) : X \subseteq V\} & \text{if } u = v. \end{cases}$$

It is easy to see that $p'$ is skew–supermodular.

**Lemma 13.** *$p'$ and $m'$ satisfy* (3) *in Theorem 2.*

**Proof.** First of all observe that $p'$ is symmetric. Let $X \subseteq V'$.
If $v \notin X$, then $p'(X) = p(X) \leq |X| = m'(X)$.
If $v \in X$, then $p'(X) = p(V - X) \leq m'(v) + |X \cap V| = m'(X)$.
Thus $p'(X) \leq m'(X)$ for all $X \subseteq V'$. This was to be proved since $p'$ is symmetric. □

By Theorem 2, there exists a hypergraph $\mathcal{H}$ so that

$$d_{\mathcal{H}}(u) = m'(u) \text{ for all } u \in V', \tag{8}$$

$$d_{\mathcal{H}}(X) \geq p'(X) \text{ for all } X \subseteq V', \tag{9}$$

The hypergraph $\mathcal{H}$ defines a partition of $V$ by (8) and this partition is a good $k$–colouring by (9) and by the definition of $m'(v)$. □

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