Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 347–358. doi:10.2298/AADM0902347J

HYPERGRAPH DOMINATION AND STRONG INDEPENDENCE

Bibin K. Jose, Zsolt Tuza[1]

We solve several conjectures and open problems from a recent paper by Acharya [**2**]. Some of our results are relatives of the Nordhaus–Gaddum theorem, concerning the sum of domination parameters in hypergraphs and their complements. (A dominating set in $\mathcal{H}$ is a vertex set $D \subseteq X$ such that, for every vertex $x \in X \setminus D$ there exists an edge $E \in \mathcal{E}$ with $x \in E$ and $E \cap D \neq \emptyset$.) As an example, it is shown that the tight bound $\gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}}) \leq n+2$ holds in hypergraphs $\mathcal{H} = (X, \mathcal{E})$ of order $n \geq 6$, where $\overline{\mathcal{H}}$ is defined as $\overline{\mathcal{H}} = (X, \overline{\mathcal{E}})$ with $\overline{\mathcal{E}} = \{X \setminus E \mid E \in \mathcal{E}\}$, and $\gamma\gamma$ is the minimum total cardinality of two disjoint dominating sets. We also present some simple constructions of balanced hypergraphs, disproving conjectures of the aforementioned paper concerning strongly independent sets. (Hypergraph $\mathcal{H}$ is balanced if every odd cycle in $\mathcal{H}$ has an edge containing three vertices of the cycle; and a set $S \subseteq X$ is strongly independent if $|S \cap E| \leq 1$ for all $E \in \mathcal{E}$.)

1. INTRODUCTION

In graphs, the theory of dominating sets is extensively studied, with well over 1000 publications, see e.g. the book [**6**] and the recent papers [**3**, **7**]. On the other hand, hypergraph domination is a very recent issue, introduced in [**1**] and further studied in [**2**, **5**]. The goal of our present note is to solve several open problems and conjectures posed in [**2**].

Hypergraphs. Unless otherwise stated, we use the terminology of Berge [**4**]. Given a set X, a *hypergraph* $\mathcal{H}$ is a pair $\mathcal{H} = (X, \mathcal{E})$ where $\mathcal{E}$ is a collection of subsets of X. The elements of X and of $\mathcal{E}$ are called *vertices* and *edges*, respectively.

2000 Mathematics Subject Classification. 05C65, 05C69.
Keywords and Phrases. Balanced hypergraph, strongly independent set, inverse domination number, independent domination number, disjoint domination number.
[1]Research supported in part by the Hungarian Scientific Research Fund, OTKA grant no. T-049613.

Traditionally, $E \neq \emptyset$ is required for all $E \in \mathcal{E}$, and we shall also assume that X itself is not an edge; that is, $1 \leq |E| \leq |X| - 1$ for every edge E.

Having fixed $\mathcal{H} = (X, \mathcal{E})$, two vertices v and w are *adjacent* if there exists an edge $E \in \mathcal{E}$ that contains both v and w, and non-adjacent otherwise. In graphs, it is equivalent to assume that a set does not contain any edge, or any two of its vertices are non-adjacent. In hypergraphs, however, the former condition is weaker than the latter. Here we shall be interested in *strongly independent sets*. By that we mean sets $S \subseteq X$ in which no two vertices in S are adjacent; or, equivalently, $|S \cap E| \leq 1$ for all $E \in \mathcal{E}$.

A *pendent vertex* is a vertex incident with exactly one edge of $\mathcal{H}$. Given an integer $k > 0$, the *k-section* of $\mathcal{H}$ is defined as the hypergraph $\mathcal{H}_{(k)} = (X, \mathcal{E}_{(k)})$ with edge set

$$\mathcal{E}_{(k)} = \{F \mid F \subset X,\ 1 \leq |F| \leq k,\ F \subset E\ for\ some\ E \in \mathcal{E}\}$$

So, the 2-section $\mathcal{H}_{(2)}$ of $\mathcal{H}$ is a graph with the same vertices as $\mathcal{H}$, and with a loop attached to each vertex. We denote by $[\mathcal{H}]_2$ the graph obtained from this 2-section by *omitting loops* (loop = 1-element edge).

For a set $Y \subseteq X$, we say that $(Y, \mathcal{F})$ is an *induced subhypergraph* of $\mathcal{H}$, or the subhypergraph *induced by* Y in $\mathcal{H}$, if $\mathcal{F} = \{E \in \mathcal{E} \mid E \subseteq Y\}$. We shall use the notation $\langle Y \rangle$ for the induced subhypergraph $(Y, \mathcal{F})$ if $\mathcal{H}$ is understood.

Dominating sets. Let $\mathcal{H} = (X, \mathcal{E})$ be a hypergraph. A set $D \subseteq X$ is a *dominating set* if, for every $x \in X \setminus D$, there exists a $y \in D$ such that x and y are adjacent; or, equivalently, if for every vertex $x \notin D$ there exists an edge $E \in \mathcal{E}$ such that $x \in E$ and $E \cap D \neq \emptyset$. The minimum cardinality of a dominating set of $\mathcal{H}$ is called the *domination number* of $\mathcal{H}$, denoted $\gamma(\mathcal{H})$. We denote by $D^o(\mathcal{H})$ and $D^m(\mathcal{H})$ the set of all minimum dominating sets (of cardinality $\gamma(\mathcal{H})$) and set of all (inclusion-wise) minimal dominating sets, respectively.

Let $D \in D^o(\mathcal{H})$. An *inverse dominating set* with respect to D is any dominating set D' of $\mathcal{H}$ such that $D' \subseteq X \setminus D$. The *inverse domination number* of $\mathcal{H}$ is defined as

$$\gamma^{-1}(\mathcal{H}) = \min\{|D'| \,\big|\, D \in D^o(\mathcal{H}), D' \text{ is an inverse dominating set with respect to } D\}.$$

Furthermore,

$$\gamma\gamma(\mathcal{H}) = \min\{|S_1| + |S_2| \,\big|\, S_1, S_2 \in D^m(\mathcal{H}),\ S_1 \cap S_2 = \phi\}$$

is called the *disjoint domination number* of $\mathcal{H}$. Finally, the least cardinality of a strongly independent dominating set is called the *independent domination number* and is denoted by $\gamma_i(\mathcal{H})$.

A slight restriction. We assume throughout this paper that

(1) every vertex of $\mathcal{H}$ is incident with some edge of cardinality at least 2.

Equivalently, $\mathcal{H}$ has no isolated vertices and every loop, if present in $\mathcal{H}$, is contained in a non-loop edge of $\mathcal{H}$.

The *complement* of $\mathcal{H}$ is defined as $\overline{\mathcal{H}} = (X, \overline{\mathcal{E}})$, where $\overline{\mathcal{E}} = \{X \setminus E \mid E \in \mathcal{E}\}$. Condition (1), when required for $\overline{\mathcal{H}}$, implies the following consequence for $\mathcal{H}$:

(1′) for every vertex x of $\mathcal{H}$ there is an edge of cardinality at most $|X|-2$ avoiding x.

Throughout this paper we restrict our attention to hypergraphs satisfying both conditions (1) and (1′), and also $|X| \geq 4$, in order to avoid the need to discuss trivial anomalies. For instance, if a vertex violates (1), then all dominating sets contain it, therefore inverse domination number and disjoint domination number cannot be defined in this case.

Results on domination. Acharya raised the problem of finding attainable lower and upper bounds for $\gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}})$ [2, Problem 3(iv)]. We solve this problem in Section 2, and for the upper bound we prove even a stronger statement. Namely, we shall prove that $\gamma(\mathcal{H})+\gamma^{-1}(\mathcal{H})+\gamma(\overline{\mathcal{H}})+\gamma^{-1}(\overline{\mathcal{H}}) \leq \max\{8, n+2\}$ holds, which is tight for all $n \geq 4$ also for $\gamma\gamma(\mathcal{H})+\gamma\gamma(\overline{\mathcal{H}})$. (It follows by definition that $\gamma\gamma(\mathcal{H}) \leq \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$, cf. [**2**].) Moreover, $\min\{\gamma\gamma(\mathcal{H}), \gamma\gamma(\overline{\mathcal{H}})\} = \min\{\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}), \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}})\} \leq 4$.

On the other hand, we disprove Conjecture 3.8 of [**2**] (which stated that $\gamma(\mathcal{H}) = \gamma_i(\mathcal{H})$ implies $\gamma\gamma(\mathcal{H}) = \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$ for every connected $\mathcal{H}$) by giving an infinite family of counterexamples.

Balanced hypergraphs. A *cycle of length* q in $\mathcal{H}$ is a sequence $(x_1, E_1, x_2, E_2, \ldots, x_q, E_q, x_{q+1})$ such that

- $x_1, x_2, \ldots, x_q$ are all distinct vertices of $\mathcal{H}$.
- $E_1, E_2, \ldots, E_q$ are all distinct edges of $\mathcal{H}$.
- $x_k, x_{k+1} \in E_k$ for $k = 1, 2, \ldots, q$.
- $q > 1$ and $x_{q+1} = x_1$.

If q is odd, then the cycle is called an *odd cycle.* A hypergraph is said to be *balanced* if every odd cycle in $\mathcal{H}$ has an edge containing three vertices of the cycle.

In Section 3 we give counterexamples to Conjecture 3.17 of [**2**], which stated that every balanced hypergraph has two disjoint maximal strongly independent sets. Moreover, we observe that Problem 2 of [**2**], about the characterization of hypergraphs having two disjoint maximal strongly independent sets, is reducible to the same problem on graphs.

2. DISJOINT AND INVERSE DOMINATION NUMBERS

In this section we prove results concerning disjoint domination and inverse domination, the first theorem solving (and extending) Problem 3(iv) posed by Acharya [**2**] concerning $\gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}})$, in the spirit of the famous Nordhaus–Gaddum theorem.

In proving upper bounds, the following simple assertion will be very useful.

Lemma 2.1. *Every non-adjacent pair of $\mathcal{H}$ dominates $\overline{\mathcal{H}}$.*

Proof. Let x, x' be non-adjacent in $\mathcal{H}$, and let $z \notin \{x, x'\}$ be any vertex. By $(1')$ there exists $E \in \mathcal{E}$ such that $z \notin E$. Since x and x' are non-adjacent, E contains at most one of x and x'. Hence, the complementary edge $\overline{E} = X \setminus E \in \overline{\mathcal{E}}$ contains z and at least one of x and x'. Consequently, every z is dominated by $\{x, x'\}$. □

Theorem 2.2. *For every integer $n \geq 4$,*

$$4 \leq \gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}}) \leq \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) + \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}}) \leq \max\{8, n+2\}$$

and the bounds are tight.

Proof. First we show that the bounds are tight.

For the lower bound, construct the hypergraph $\mathcal{H} = (X, \mathcal{E})$ with n vertices as follows. Partition the vertex set into four nonempty sets, $X = X_1 \cup \cdots \cup X_4$. Set $\mathcal{E} = \{X_i \cup X_j \mid 1 \leq i < j \leq 4\}$. It is clear that $\mathcal{H}$ and $\overline{\mathcal{H}}$ are isomorphic hypergraphs, and all the vertices are adjacent to each other in both $\mathcal{H}$ and $\overline{\mathcal{H}}$. Hence, $\gamma\gamma(\mathcal{H}) = \gamma\gamma(\overline{\mathcal{H}}) = 2$, therefore the lower bound is attainable for all $n \geq 4$.

To see tightness of the upper bound, if $4 \leq n \leq 6$, we can again take a 4-partition $X = X_1 \cup \cdots \cup X_4$ and set $\mathcal{E} = \{X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4\}$. Then the edge set of $\overline{\mathcal{H}}$ is $\{X_1 \cup X_2, X_1 \cup X_4, X_3 \cup X_4\}$, and we have $\gamma\gamma(\mathcal{H}) = \gamma\gamma(\overline{\mathcal{H}}) = 4$. For n larger, we can make $\mathcal{H}$ a tree graph as follows. Take two adjacent vertices x, x' and join half of $X \setminus \{x, x'\}$ — say, the set Y — to x, the other half — say $Y' = X \setminus (Y \cup \{x, x'\}$ — to x'. There are exactly four minimal dominating sets, forming two disjoint pairs and yielding $\gamma\gamma(\mathcal{H}) = n$. This completes the proof of tightness.

Next, we prove the validity of the inequalities. The lower bound is clear by definition, since every dominating set is nonempty, therefore $\gamma\gamma(\mathcal{H}) \geq 2$, and $\gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}}) \geq 4$ always holds. Also, the middle inequality is a direct consequence of the definitions, as observed in [**2**].

Since $\max\{8, n+2\} = n+2$ for all $n \geq 6$, the value '8' is relevant for $n = 4, 5$ only, and the case $n = 4$ is trivial because any two disjoint (dominating) sets together can have at most n vertices. So, the assertion for '8' boils down to the claim that if a hypergraph $\mathcal{H}$ of order 5 satisfying the restrictions (1), $(1')$ has $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) = 5$, then $\gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}}) \leq 3$. This is not hard to show, but needs a little argument.

Let $n = 5$, $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) = 5$, $\gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}}) \geq 4$. Note that the domination number is at most $\lfloor n/2 \rfloor = 2$, because of (1). To simplify explanation, below we do not distinguish between cases which can be obtained from each other by re-naming vertices or other objects; e.g., the role of $\mathcal{H}$ and $\overline{\mathcal{H}}$ is symmetric.

Suppose first that there is a dominating vertex x in $\mathcal{H}$; i.e., $\gamma(\mathcal{H}) = 1$. We should then have $\gamma^{-1}(\mathcal{H}) = 4$, what would imply in particular that no edge of $\mathcal{H}$ is disjoint from x, leading to the contradiction that $(1')$ is violated.

Suppose next that $\gamma(\mathcal{H}) = \gamma(\overline{\mathcal{H}}) = 2$. We need to prove $\gamma^{-1}(\mathcal{H}) = 2$. Let $D = \{x, x'\} \in D^o(\mathcal{H})$ be a minimum dominating set in $\mathcal{H}$, and let $Y = \{y, y', y''\} = X \setminus D$. Consider the graph G_Y obtained from the 2-section graph of hypergraph $(Y, \{E \cap Y \mid E \in \mathcal{E}\})$ by removing the loops. If G_Y has no isolated vertices, then any two vertices of Y dominating $\{x, x'\}$ form an inverse dominating set in $\mathcal{H}$. (Select one neighbor of x, and one of x' non-adjacent to x). On the other hand, if y is an isolated vertex of G_Y, we may suppose that x and y are adjacent, and also that $\{x', y'\}$ are adjacent, since x is not a dominating vertex. Then $\{y, y'\}$ is an inverse dominating set, because both pairs $\{y, y''\}$, $\{y', y''\}$ cannot be non-adjacent (for otherwise $\{x, y\}$ and $\{x', y'\}$ would be disjoint 2-element edges, and y'' would dominate $\overline{\mathcal{H}}$). This completes the proof for $n \leq 5$.

From now on, suppose $n \geq 6$. Let us denote by G_N and $G_{\overline{N}}$ the graph of non-adjacent pairs in $\mathcal{H}$ and in $\overline{\mathcal{H}}$, respectively. (These are the complementary graphs of $[\mathcal{H}]_2$ and $[\overline{\mathcal{H}}]_2$.) Due to (1), the vertex degrees are at most $n-2$ in both of them. By direct inspection and applying Lemma 2.1, we further have:

- If G_N has at least two isolated vertices, then $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) = 2$.
- If G_N has two disjoint edges, then $\gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}}) \leq 4$.

Analogous consequences are derived from $G_{\overline{N}}$ by switching $\mathcal{H}$ with $\overline{\mathcal{H}}$.

Hence, if two isolated vertices occur in *at least one* of G_N and $G_{\overline{N}}$, then we have nothing to do; moreover, two disjoint edges in *both* G_N and $G_{\overline{N}}$ yield the upper bound '8', and the proof is done also in this case.

Hence, from now on we assume that any two edges of G_N share a vertex, and G_N has at most one isolated vertex. This reduces to the unique possibility

- $G_N \cong K_{1,n-2}$

Indeed, intersecting edges mean triangle or star, but a star with $n-1$ edges would violate (1), whereas a star with fewer than $n-2$ edges would yield two isolated vertices in G_N. Moreover, a triangle would imply $n-3 > 1$ isolated vertices.

Let x be the center of the star, and y be the isolated vertex of G_N. We are going to prove that

$$\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) \leq 3 \qquad \text{and} \qquad \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}}) \leq 4$$

hold in this case, much better than the desired total upper bound $n + 2$.

Since y is not incident with any non-adjacent pair, we have $\gamma(\mathcal{H}) = 1$, with $\{y\}$ as the unique minimum dominating set. Further, $\gamma^{-1}(\mathcal{H}) = 2$ and the minimal inverse dominating sets are precisely the edges of G_N, because x has to be in each of them, and any other vertex dominates the entire set $X \setminus \{x\}$.

Turning to $\overline{\mathcal{H}}$, every edge of G_N dominates $\overline{\mathcal{H}}$ by Lemma 2.1, hence $\gamma(\overline{\mathcal{H}}) \leq 2$. We will also find a dominating set D with $|D| \leq 2$ and $x \notin D$. Restriction (1′) implies that there is an edge $E \in \mathcal{E}$ with $|E| \leq n-2$ and $x \notin E$. Moreover, the adjacent pair $E' = \{x, y\}$ is the unique edge incident with x in $\mathcal{H}$. Consider the edges $\overline{E} = X \setminus E$ and $\overline{E'} = X \setminus \{x, y\}$ in $\overline{\mathcal{H}}$. If $\overline{E} \cap \overline{E'} \neq \emptyset$, then $F = \{y, z\}$ dominates $\overline{\mathcal{H}}$ for any $z \in \overline{E} \cap \overline{E'}$. And if $\overline{E} \cap \overline{E'} = \emptyset$, then $F = \{y, z\}$ dominates $\overline{\mathcal{H}}$ for any $z \in \overline{E}$.

Now, three situations can occur. If x dominates $\overline{\mathcal{H}}$, then $\gamma(\overline{\mathcal{H}}) = 1$ and F is an inverse dominating set, hence $\gamma^{-1}(\overline{\mathcal{H}}) \leq 2$. If some $w \in X \setminus \{x\}$ dominates $\overline{\mathcal{H}}$, then again $\gamma(\overline{\mathcal{H}}) = 1$ and any edge of G_N not incident with w is an inverse dominating set (there are $n - 3 > 0$ of them), so $\gamma^{-1}(\overline{\mathcal{H}}) \leq 2$. Finally, if none of these cases occur, then $\gamma(\overline{\mathcal{H}}) = \gamma^{-1}(\overline{\mathcal{H}}) = 2$ because F dominates $\overline{\mathcal{H}}$ and any edge of G_N disjoint from F is an inverse dominating set. □

Remark 2.3. Tightness of the bounds has already been verified in the proof above, but many further examples of hypergraphs $\mathcal{H}$ attaining the equality $\gamma\gamma(\mathcal{H}) + \gamma\gamma(\overline{\mathcal{H}}) = n + 2$ can be given. For instance, for n even, consider the graph mP_2 with $m = n/2$. Now, form hypergraph $\mathcal{H}$ by adding one edge E such that E contains exactly one vertex from each P_2. It is easy to verify that $\gamma\gamma(\mathcal{H}) = n$. The general principle for such constructions is to specify a minimum dominating set (in the present case it is E) and create at least one pendent vertex attached at each of its vertices.

With the method in the proof of Theorem 2.2, a further upper bound can be obtained.

Theorem 2.4. *For every $n \geq 4$ and every hypergraph $\mathcal{H}$ of order n,*

$$\min\{\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}), \gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}})\} \leq 4$$

and the bound is tight.

Proof. Tightness follows from the construction attaining '8' in the previous theorem, hence we only have to prove the upper bound for $n \geq 5$. We refer to the main part of the previous proof, denoting by G_N and $G_{\overline{N}}$ the graph of non-adjacent pairs in $\mathcal{H}$ and in $\overline{\mathcal{H}}$, respectively. The proof is done by Lemma 2.1 if any of G_N and $G_{\overline{N}}$ has two disjoint edges or two isolated vertices. Otherwise, since $n \geq 5$, we have $G_N \cong G_{\overline{N}} \cong K_{1,n-2}$, and the argument above yields that $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$ or $\gamma(\overline{\mathcal{H}}) + \gamma^{-1}(\overline{\mathcal{H}})$ is at most 3. □

The following result disproves Conjecture 3.8 of [**2**] by displaying infinitely many counterexamples to "$\gamma\gamma(\mathcal{H}) = \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$".

Proposition 2.5. *For every integer $n \geq 7$, there exists a hypergraph $\mathcal{H}$ of order n with $\gamma(\mathcal{H}) = \gamma_i(\mathcal{H})$ but $\gamma\gamma(\mathcal{H}) \neq \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$.*

Proof. Let $P_5 = (a, b, c, d, e)$ be a path graph on five vertices. Construct a new hypergraph $\mathcal{H} = (X, \mathcal{E})$ from P_5 by adding vertices $b_1, b_2, \ldots, b_s, d_1, d_2, \ldots, d_t$ ($s \geq 1$, $t \geq 1$, $s + t = n - 5$) and two new hyperedges E_1, E_2 in the following way, shown in Figure 1: $X = V(P_5) \cup \{b_1, b_2, \ldots, b_s, d_1, d_2, \ldots, d_t\}$ and $\mathcal{E} = E(P_5) \cup \{E_1, E_2\}$, where $E_1 = \{b, b_1, b_2, \ldots, b_s\}$ and $E_2 = \{d, d_1, d_2, \ldots, d_t\}$.

First, we prove that $\gamma(\mathcal{H}) = \gamma_i(\mathcal{H})$. Since no vertex of $\mathcal{H}$ is adjacent to all vertices, we have $\gamma(\mathcal{H}) \neq 1$. On the other hand, the set $D = \{b, d\}$ dominates $\mathcal{H}$ and hence $\gamma(\mathcal{H}) = 2$. Moreover, since the vertices b and d are not adjacent in $\mathcal{H}$, the independent domination number $\gamma_i(\mathcal{H})$ is also 2. Hence, $\gamma(\mathcal{H}) = \gamma_i(\mathcal{H}) = 2$.

Now we prove that $\gamma\gamma(\mathcal{H}) \neq \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$. Since $\gamma(\mathcal{H}) = 2$ and $D = \{b, d\}$ is the *unique* minimum dominating set, we need to find an inverse dominating set D' of $\mathcal{H}$ inside $X - D$. Since b and d are the only vertices dominating the vertices a, c and e, we must select a, c, e into D'. Also, since $\{a, c, e\}$ does not dominate the vertices $b_1, b_2, \ldots, b_s, d_1, d_2, \ldots, d_t$, two vertices b_i, d_j for some i and j must be in D'. Hence, all minimal inverse dominating sets are of the form $D' = \{a, c, e, b_i, d_j\}$.

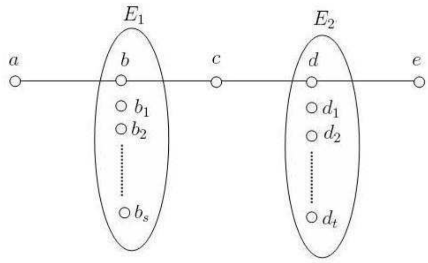

Figure 1. Inverse vs. disjoint domination

Consequently, $\gamma^{-1}(\mathcal{H}) = 5$ and $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) = 7$. On the other hand, consider the sets $S_1 = \{a, b_i, d\}$ and $S_2 = \{b, d_j, e\}$ for some i and j. Clearly, $S_1 \cap S_2 = \emptyset$, and both S_1 and S_2 are dominating sets of $\mathcal{H}$. Hence, $\gamma\gamma(\mathcal{H}) \leq |S_1| + |S_2| = 6 < 7 = \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$. (In fact, $\gamma\gamma(\mathcal{H}) = 6$.) Thus, $\gamma\gamma(\mathcal{H}) \neq \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$, as claimed. □

In a disconnected hypergraph $\mathcal{H}$, the difference between $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$ and $\gamma\gamma(\mathcal{H})$ is equal to the sum of differences in its connected components, so that it can be arbitrarily large. But this unboundedness remains valid even if we restrict ourselves to *connected* hypergraphs.

Theorem 2.6. *For every integer $k \geq 1$ there exists a connected hypergraph $\mathcal{H}$ such that $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) - \gamma\gamma(\mathcal{H}) = k$.*

Proof. The construction is a generalization of that in the proof of Proposition 2.5. We start from a path with vertices $v_1, v_2, \ldots, v_{5k}$ and edges $\{v_i, v_{i+1}\}$ ($i = 1, \ldots, 5k - 1$). Moreover, we attach mutually disjoint edges E_j to the vertices v_j for $j \equiv 2, 4 \pmod 5$. The case $k = 1$ exactly means the hypergraph exhibited in Figure 1.

We first prove that $\gamma(\mathcal{H}) = 2k$ holds, and that $D_0 := \{v_i \mid i \equiv 2, 4 \pmod 5\}$ is the unique dominating set of minimum size in $\mathcal{H}$. The lower bound $\gamma(\mathcal{H}) \geq 2k$ is easily seen since the vertices in any $E_j \setminus \{v_j\}$ are dominated by the vertices of E_j only. To prove uniqueness, let D be any minimal dominating set of $\mathcal{H}$. It suffices to show that if $D \neq D_0$ then D contains at least one vertex outside $\bigcup_{j \equiv 2,4 \pmod 5} E_j$. Assuming $D \neq D_0$ there must occur a vertex $y \in E_i \setminus \{v_i\}$ for some subscript i. It means $v_i \notin S$, therefore D has to contain vertices from both pairs $\{v_{i-2}, v_{i-1}\}$ and $\{v_{i+1}, v_{i+2}\}$ (to dominate v_{i-1} and v_{i+1}, respectively; for $i = 2$ or $i = 5k = 1$ we view v_0 or v_{5k+1} as a dummy vertex). Since one of those pairs is disjoint from all E_j (independently of the actual value of i), the assertion follows.

Next, let D' be an inverse dominating set. Since D_0 is the unique dominating set of minimum cardinality, $D' \cap D_0 = \emptyset$ holds. Consequently, D' has to contain a vertex from each of the following sets:

- $\{v_i\}$, $i \equiv 3 \pmod 5$ — k vertices in D',
- $E_j \setminus \{v_j\}$, $j \equiv 2, 4 \pmod 5$ — $2k$ vertices in D',
- $\{v_i, v_{i+1}\}$, $i \equiv 0 \pmod 5$ — $k+1$ vertices in D'.

Thus, $\gamma^{-1}(\mathcal{H}) = 4k+1$ and $\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) = 6k+1$.

The proof of the theorem will be completed if we show that $\gamma\gamma(\mathcal{H}) = 5k+1$. To prove $\gamma\gamma(\mathcal{H}) \le 5k+1$, we specify one vertex $y_j \in E_j \setminus \{v_j\}$ for each $j \equiv 2, 4 \pmod 5$, and consider the following two sets:

$$S_1 = \{v_i \mid i \equiv 4, 7, 10 \pmod{10}\} \cup \{y_j \mid j \equiv 2, 9 \pmod{10}\} \cup \{v_1\},$$
$$S_2 = \{v_i \mid i \equiv 2, 5, 9 \pmod{10}\} \cup \{y_j \mid j \equiv 4, 7 \pmod{10}\}.$$

It can be checked that the sets S_1 and S_2 are disjoint, and each of them dominates $\mathcal{H}$. Moreover, if k is even, then $|S_1| = 5k/2 + 1$ and $|S_2| = 5k/2$; and if k is odd, then $|S_1| = |S_2| = (5k+1)/2$. Thus, in either case, $\gamma\gamma(\mathcal{H}) \le |S_1| + |S_2| = 5k+1$.

Conversely, to prove $\gamma\gamma(\mathcal{H}) \ge 5k+1$, let S_1 and S_2 be two disjoint dominating sets of $\mathcal{H}$, such that $|S_1| + |S_2| = \gamma\gamma(\mathcal{H})$. Denoting $S = S_1 \cup S_2$, we observe:

- $|S \cap E_j| \ge 2$ for all $j \equiv 2, 4 \pmod 5$, because both S_1 and S_2 dominate E_j,
- $S \cap \{v_{5i}, v_{5i+1}\} \ne \emptyset$ for all $1 \le i \le k-1$, because if $v_\ell \notin S$ ($\ell = 5i, 5i+1$) then the two neighbors of v_ℓ must occur in S, and hence at most one vertex of each pair can be missing,
- $v_1 \in S$ and $v_{5k} \in S$, because each of v_1 and v_{5k} has just one neighbor.

Summing up, we obtain $\gamma\gamma(\mathcal{H}) = |S| \ge 4k + (k-1) + 2 = 5k+1$. □

3. DISJOINT STRONGLY INDEPENDENT SETS

To simplify some statements in the sequel, let us say that two sets $S_1, S_2 \subseteq X$ in hypergraph $\mathcal{H} = (X, \mathcal{E})$ form a *strongly independent disjoint pair* — called an *SID-pair*, for short — if $S_1 \cap S_2 = \emptyset$, moreover both of the S_i are strongly independent and *maximal under inclusion*; i.e., each $x \in X \setminus S_i$ is adjacent to some vertex of S_i. Due to the condition of maximality, not every $\mathcal{H}$ has an SID-pair. The study of hypergraphs having at least one SID-pair was initiated by Acharya [**2**], where a couple of conjectures were proposed. In this section we put some related remarks.

Problem 2 of [**2**] asks for a characterization of hypergraphs having SID-pairs. The next observation shows that this question is in fact a problem in graph theory, since we can reduce it from hypergraphs to simple graphs.

Proposition 3.1. *A hypergraph $\mathcal{H}$ has an SID-pair if and only if so does its loopless 2-section $[\mathcal{H}]_2$.*

Proof. Two vertices are adjacent in $\mathcal{H}$ if and only if they are adjacent in $[\mathcal{H}]_2$. Hence, the strongly independent sets maximal under inclusion, and also the SID-pairs, are exactly the same in $\mathcal{H}$ and in $[\mathcal{H}]_2$. □

Hence, the problem reduces to characterizing *graphs* in which there exist two disjoint independent sets maximal under inclusion.

On the other hand, despite that SID-pairs occur in either both or none of $\mathcal{H}$ and $[\mathcal{H}]_2$, in some classes of hypergraphs there can be more structure than in their 2-sections. In this direction it was proposed in Conjecture 3.17 of [**2**] that perhaps all balanced hypergraphs have SID-pairs. We give counterexamples to this.

As a preliminary observation towards counterexamples, it is clear by definition that every vertex of a cycle in a hypergraph is contained in at least two edges. Hence, the following property is immediate.

($*$) Pendent vertices do not belong to any cycle in any hypergraph.

Proposition 3.2. *For every integer $n \geq 6$, there exists a balanced hypergraph of order n, which does not have any SID-pair.*

Proof. We first describe an example on six vertices. Let $\mathcal{H} = (X, \mathcal{E})$ with $X = \{a, b, c, d, e, f\}$ and $\mathcal{E} = \{E_1, E_2, \ldots, E_6\}$, where $E_1 = \{a, b\}$, $E_2 = \{b, c\}$, $E_3 = \{a, b, c\}$, $E_4 = \{a, d\}$, $E_5 = \{b, e\}$, $E_6 = \{c, f\}$ (see Figure 2).

It is clear that the only cycle in $\mathcal{H}$ is $C = (a, E_1, b, E_2, c, E_3, a)$. Although it is of odd length, edge E_3 contains three vertices of C, hence $\mathcal{H}$ is a balanced hypergraph.

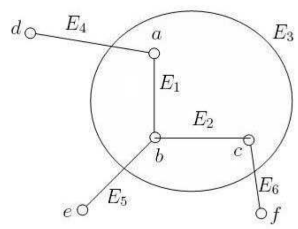

Figure 2. Balanced hypergraph

On the other hand, strongly independent sets S of $\mathcal{H}$ can have at most one vertex in E_3, and by the assumption of maximality, specifying $S \cap E_3$ there is a unique choice of S for each of the four possibilities. Namely, the maximal strongly independent sets are

$\{a, e, f\}$, $\{b, d, f\}$, $\{c, d, e\}$, $\{d, e, f\}$.

Each of them has at least two vertices in $\{d, e, f\}$, therefore no two of them are disjoint.

Attaching more than one pendent vertex to a, b, and/or c, constructions on any number $n \geq 6$ of vertices can be obtained. □

Remark 3.3. By a slight modification of the construction above, counterexamples to Proposition 3.14, Corollary 3.15 and Proposition 3.21 of the paper [**2**] can be given. Those assertions stated (or, in 3.21, assumed) that hypergraphs without cycles — and, more generally, those without odd cycles — would contain SID-pairs. To disprove this, we observe that removing the edges E_1 and E_2 from the hypergraph exhibited in Figure 2 all cycles are eliminated and still no SID-pair occurs. Actually, the edge E_3 can be made of arbitrary size, and we only need to maintain at least three pendent vertices whose neighbors are mutually distinct. Moreover, as a combination of the previous ideas, a path of 2-element edges can be inserted inside the enlarged edge. All these transformations result in balanced hypergraphs without SID-pairs.

If the goal is just to avoid containment between edges without changing their number, one can modify $\mathcal{H}$ of Figure 2 to a new hypergraph $\mathcal{H}_1 = (X_1, \mathcal{E}_1)$ by inserting two further pendant vertices u and v in such a way that E_1 and E_2 are extended to $E_1' = E_1 \cup \{u\}$ and to $E_2' = E_2 \cup \{v\}$, respectively. That is, the new vertex set is $X_1 = X \cup \{u, v\}$, the new edge set is $\mathcal{E}_1 = \mathcal{E} \setminus \{E_1, E_2\} \cup \{E_1', E_2'\}$, and no SID-pairs occur.

The following assertion shows that excluding cut vertices (assuming that the hypergraph is 2-connected) is not sufficient for an SID-pair.

Proposition 3.4. *For every integer $n \geq 15$, there exists a 2-connected balanced hypergraph of order n, which does not have any SID-pair.*

Proof. Let $n = 3k$ ($k \geq 5$) first. Consider the hypergraph $\mathcal{H} = (X, \mathcal{E})$ with vertex set $X = X_1 \cup X_2 \cup X_3$, where $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,k}\}$ for $i = 1, 2, 3$, and with edge set

$$\mathcal{E} = \{X_1, X_3\} \cup \{\{x_{1,j}, x_{2,j}\} \mid 1 \leq j \leq k\} \cup \{\{x_{2,j} x_{3,j}\} \mid 1 \leq j \leq k\}$$

This $\mathcal{H}$ remains connected after the removal of any one vertex, and all cycles have length 6 in it. Hence, $\mathcal{H}$ is 2-connected and balanced.

Suppose that $S \subset X$ is a maximal independent set. Then, since $|S \cap X_1| \leq 1$ and $|S \cap X_3| \leq 1$, maximality implies $|S \cap X_2| \geq k - 2 > k/2$ for $k \geq 5$. Thus, any two maximal independent sets share a vertex inside X_2. This settles the question for orders n which are multiples of 3.

To prove the assertion for $n = 3k + 1$ and $n = 3k + 2$, we select one or two indices j, and for the selected $x_{2,j}$ we take a 'false twin' $x_{2,j}'$ adjacent to both $x_{1,j}$ and $x_{3,j}$, but not to $x_{2,j}$. Then the hypergraph remains balanced, because $x_{2,j}$ and $x_{2,j}'$ occur together in just one new 4-cycle. Moreover, 2-connectivity is preserved, too. Finally, no SID-pair can occur, because any maximal independent set contains either both or none of $x_{2,j}$ and $x_{2,j}'$. □

4. CONCLUDING REMARKS

So far there are very few papers on hypergraph domination, therefore lots of new questions arise. Below we list some problems which are closely related to the results presented here.

Problem 1. Characterize the hypergraphs $\mathcal{H}$ satisfying

(a) $\gamma\gamma(\mathcal{H}) = \gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})$,

(b) $\gamma\gamma(\mathcal{H}') = \gamma(\mathcal{H}') + \gamma^{-1}(\mathcal{H}')$ for all induced subhypergraphs $\mathcal{H}' \subseteq \mathcal{H}$,

(c) $\gamma\gamma(\mathcal{H}') = \gamma(\mathcal{H}') + \gamma^{-1}(\mathcal{H}')$ for all subhypergraphs $\mathcal{H}' \subseteq \mathcal{H}$,

where condition (1) is assumed throughout.

Problem 2. For which classes $\mathfrak{H}$ of hypergraphs is it true that

$$\sup \{\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H}) - \gamma\gamma(\mathcal{H}) \mid \mathcal{H} \in \mathfrak{H}\}$$

is finite?

Problem 3. Does there exist a universal upper bound on $\dfrac{\gamma(\mathcal{H}) + \gamma^{-1}(\mathcal{H})}{\gamma\gamma(\mathcal{H})}$ for all hypergraphs $\mathcal{H}$?

Also, concerning Proposition 3.4 it remains an open problem to investigate, what kind of additional properties make balanced hypergraphs necessarily contain SID-pairs.

Problem 4. Does there exist an integer k such that every k-connected, balanced hypergraph $\mathcal{H}$ has an SID-pair?

Acknowledgment. The authors are thankful to the organizers of ICDM'08 for providing an opportunity for fruitful discussions of which this paper is an outcome. Thanks are also due to a referee for constructive remarks.

REFERENCES

1. B. D. Acharya: *Domination in hypergraphs.* AKCE J. Combin., 4:2 (2007), 117–126.
2. B. D. Acharya: *Domination in hypergraphs* II. New directions, Proc. Int. Conf. – ICDM 2008, Mysore, India, pp. 1–16.
3. G. Bacsó: *Complete description of forbidden subgraphs in the structural domination problem.* Discrete Math., 309:8 (2009), 2466–2472.
4. C. Berge: *Hypergraphs*: *Combinatorics of Finite Sets*. North-Holland, Amsterdam, 1989.
5. Bibin K. Jose, K. A. Germina, Kumar Abhishek: *On Some open Problems of Stable Sets and Domination in hypergraphs.* Submitted to AKCE J. Combin.
6. T. W. Haynes, S. T. Hedetniemi, P. J. Slater: *Fundamentals of Domination in Graphs.* Marcel Dekker, Inc., New York, N.Y., 1998.
7. Zs. Tuza: *Hereditary domination in graphs*: *Characterization with forbidden induced subgraphs.* SIAM J. Discrete Math., 22:3 (2008), 849–853.

Post Graduate Department of Mathematics,
St. Dominic's College Kanjirappally,
Kottayam - 686512, Kerala,
India
E–mail: bibinkjose2002@yahoo.com

(Received May 3, 2009)
(Revised August 9, 2009)

Computer and Automation Institute,
Hungarian Academy of Sciences,
H–1111 Budapest, Kende u. 13–17,
Hungary

Department of Computer Science
and Systems Technology,
University of Pannonia,
H–8200 Veszprém, Egyetem u. 10,
Hungary
E–mail: tuza@sztaki.hu