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Hyperinvariant subspaces and extended eigenvalues

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Abstract. An extended eigenvalue for an operator A is a scalar λ for which the operator equation $AX = \lambda XA$ has a nonzero solution. Several scenarios are investigated where the existence of non-unimodular extended eigenvalues leads to invariant or hyperinvariant subspaces.

For a bounded operator A on a complex Hilbert space $\mathcal{H}$, the set $\mathrm{EE}(A)$ of extended eigenvalues for A is defined to be the set of those complex numbers λ for which there is an operator $T \neq 0$ satisfying $AT = \lambda TA$. T is then referred to as a λ eigen-operator for A. The eigenvalue terminology, although not perfectly accurate, seems useful on two levels. The first was described in [2]; briefly, if A has dense range, then the equation

$$AX = \phi(X)A; \quad \phi(X) \in \mathcal{L}(\mathcal{H})$$

has a unital algebra as its solution set, and ϕ is a unital homomorphism. Our extended eigenvalues are precisely the eigenvalues for ϕ. The second point of view is that one can easily show that for an operator on a finite dimensional space, the set of extended eigenvalues for that operator is the set of ratios of eigenvalues, with the obvious restriction on the use of 0. This is shown explicitly in [3]. In other works this concept of extended eigenvalue has appeared as α commuting or λ commuting, but we choose to use a term which is parameter free.

For $A \in \mathcal{L}(\mathcal{H})$ (the set of bounded operators on $\mathcal{H}$), a (closed, linear) subspace of $\mathcal{H}$ is a nontrivial invariant subspace (n.i.s.) for A if it is neither $\mathcal{H}$ nor $\{0\}$ and is invariant under A. This space is hyperinvariant for A if it is invariant for every operator in $(A)'$, the commutant of A. More generally, a subspace is defined to be invariant for a set of operators if it is invariant for each member of that set.

Extended eigenvalues and invariant subspaces. For a given $\lambda \in \mathrm{EE}(A)$ we define $\mathcal{E} = \mathcal{E}(A, \lambda)$ as the set of all λ eigen-operators for A. This is a (weakly) closed linear space of operators, and $\mathcal{E}(A, 1)$ is $(A)'$, the commutant of A; that is, the set of all operators commuting with A. Direct multiplication leads to the next result:

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Lemma 1. *$\mathcal{E}(A,\alpha)\mathcal{E}(A,\beta) \subset \mathcal{E}(A,\alpha\beta)$. In particular, for each α,*

$$(A)'\mathcal{E}(A,\alpha)(A)' = \mathcal{E}(A,\alpha).$$

Proposition 2. *Let A be an invertible operator. Then:*

(a) *There are positive numbers a and b, $a < b$, such that every extended eigenvalue for A is contained within the annulus $R_{a,b}$ with inner radius a and outer radius b.*
(b) *If λ is an extended eigenvalue for A with $|\lambda| \neq 1$ then there is a positive integer N such that every product of N members of $\mathcal{E}$ is zero. In particular, every operator in $\mathcal{E}$ is nilpotent of order no greater than N.*
(c) *If A has an extended eigenvalue of modulus other than 1, then it has an extended eigenvalue for which the N in part* (b) *above may be taken as 2.*

Proof. Choose $c > 0$ such that for every vector x, $\|Ax\| \geq c\|x\|$. Let λ be an extended eigenvalue for A and let T be a corresponding eigen-operator. It follows that for every positive integer n,

$$AT^n = \lambda^n T^n A.$$

Then for each x and n,

$$c\|T^n x\| \leq \|AT^n x\| = |\lambda|^n \|T^n AX\| \leq |\lambda|^n \|T^n\| \|Ax\|.$$

It then follows that

$$c\|T^n\| \leq |\lambda|^n \|T^n\| \|A\|.$$

From the case $n = 1$ we have $\frac{c}{\|A\|} \leq |\lambda|$. We may then choose $a = \frac{c}{\|A\|}$ to establish the inner radius as required for part (a). Since A is invertible and $T \neq 0$, λ cannot be zero. Appropriate multiplications then show that

$$A^{-1}T = (1/\lambda)TA^{-1},$$

and the analysis above for A and T applies equally well for A^{-1} and T. This guarantees the existence of a finite outer radius for our annulus R_{ab}.

As for part (b), suppose λ is an extended eigenvalue for A with modulus other than 1. Then there is a smallest N such that $\lambda^N \notin R_{ab}$. Let $\{T_1, \cdots, T_N\} \subset \mathcal{E}(A,\lambda)$. Then $AT_1 \cdots T_N = \lambda^N T_1 \cdots T_N A$. Since λ^N is not an extended eigenvalue for A, we must have $T_1 \cdots T_N = 0$.

Again, suppose that λ is an extended eigenvalue for A with modulus not equal to 1, and let N be the smallest positive integer such that $\lambda^N \notin \mathrm{EE}(A)$. Since N is at least 2, we may choose a positive integer $k < N$ for which $2k \geq N$. It follows that there is a nonzero operator S satisfying $AS = \lambda^k SA$. But then we have $AS^2 = \lambda^{2k} S^2 A$, forcing the conclusion that $S^2 = 0$. □

Remark. The preceding result holds in any Banach algebra setting. The only change would be to replace operator lower bounds as follows: If a is invertible in a Banach algebra, then there is a constant $\alpha > 0$ such that for all b, $\|ab\| \geq \alpha\|b\|$.

Our goal is to use $\mathcal{E}(A,\lambda)$ in the establishment of invariant and perhaps hyperinvariant subspaces (under specific hypotheses) for A. Our motivation for this comes from the independent and virtually contemporaneous works of S. Brown ([4]), and of H. W. Kim, R. Moore, and C. M. Pearcy ([7]). These works provided extensions of V. Lomonosov's classic result [8] (see also [5, pp. 181–182]). Specifically, it was shown that if A has a compact extended eigen- operator then A has a proper hyperinvariant subspace. Kim *et al* proved a somewhat more general result involving the functional calculus of a compact operator.

Note that if λ is an extended eigenvalue for A and if x is any nonzero vector, then $(A)'\mathcal{E}(A,\lambda)x \subset \mathcal{E}(A,\lambda)x$. We eliminate one roadblock to hyperinvarinace as follows:

Lemma 3. *Let* $\lambda \in \mathrm{EE}(A)$ *and let* $\mathcal{E} = \mathcal{E}(A,\lambda)$*. Define* $\mathcal{S} = \{x : \mathcal{E}x = \{0\}\}$*. Then* $\mathcal{S}$ *is hyperinvariant for* A*.*

Proof. Note that

$$\mathcal{S} = \bigcap_{T\in\mathcal{E}} \ker T$$

so that $\mathcal{S}$ is a closed linear subspace of $\mathcal{H}$. For $x \in \mathcal{S}, T \in \mathcal{E}$, and $B \in (A)'$, we have $TB \in \mathcal{E}$; thus $TBx = 0$. Consequently $Bx \in \mathcal{S}$, and so $(A)'\mathcal{S} \subset \mathcal{S}$. □

Note that $\mathcal{S}$ cannot be $\mathcal{H}$ since by assumption $\mathcal{E}$ contains at least one nonzero operator. However it could possibly be the zero subspace.

If $AT = \tau TA$ and $AS = \sigma SA$, then $ATS = \tau\cdot\sigma TSA$. This seems to indicate that $\mathrm{EE}(A)$ is a (unital) semigroup under multiplication. However, there is no general reason why $TS \neq 0$ in the equation above. Later we will present an absolutely continuous multiplication operator illustrating this failure of multiplicative closure. However, failure of the semigroup property gives us valuable information about the operator.

Theorem 4. *Suppose that* $\mathrm{EE}(A)$ *is not closed under multiplication. Then* A *has a proper hyperinvariant subspace.*

Proof. Let $S \in \mathcal{E}(A,\sigma)$ and $T \in \mathcal{E}(A,\tau)$, and suppose that $\sigma\tau \notin \mathrm{EE}(A)$. Since

$$AST = \sigma SAT = \sigma\tau STA$$

and $\sigma\tau \notin \mathrm{EE}(A)$, we are forced to conclude that $ST = 0$. The symmetric argument shows that $TS = 0$. Thus for each vector $x, \mathcal{E}(A,\tau)x \subset \ker S$. In particular, for any choice of x, $\mathcal{E}(A,\tau)x$ is not dense in $\mathcal{H}$. On the other hand, since $T \neq 0$, we may choose x so that $\mathcal{E}(A,\tau)x \neq \{0\}$. But $(A)'\mathcal{E}(A,\tau) = \mathcal{E}(A,\tau)$; ensuring a proper hyperinvariant subspace for A. □

Corollary 5. *Suppose that* A *is an invertible operator and* $\lambda \in \mathrm{EE}(A)$ *with* $|\lambda| \neq 1$*. Then* A *has a proper hyperinvariant subspace.*

Proof. We have seen that $\mathrm{EE}(A)$ is contained in a true annulus, and so for any extended eigenvalue of modulus not equal to 1, a positive integer power of it is not an extended eigenvlaue. This means that Theorem 10 is applicable.

□

Corollary 6. *Suppose that* A *has an extended eigenvalue* λ *of modulus other than one, and* A *is either surjective or bounded below. Then* A *has a nontrivial hyperinvariant subspace.*

Proof. If A is bounded below, then it has closed range, and this range is hyperinvariant. Either it is proper or A is invertible. In either case A has a proper hyperinvariant subspace.

If A is surjective, then it is either invertible or $\ker A$ is a proper hyperinvariant subspace. □

Corollary 7. *Suppose that A is a semi-Fredholm operator with an extended eigenvalue of modulus other than 1. Then A has a proper hyperinvariant subspace.*

Proof. Being a semi-Fredholm operator, either A is invertible or its range or kernel is proper. In any case, A must have a proper hyperinvariant subspace. □

For the next corollary we let π denote the canonical projection of $\mathcal{L}(\mathcal{H})$ into the Calkin algebra. We write $\mathrm{EE}_\pi(a)$ for the set of extended eigenvalues of a member a of the Calkin algebra.

Corollary 8. *Suppose there exists $\lambda \in \mathrm{EE}(A) - \mathrm{EE}_\pi(\pi(A))$. Then A has a proper hyperinvariant subspace.*

Proof. Let T be a nonzero operator for which $AT = \lambda TA$. Then $\pi(A)\pi(T) = \lambda\pi(T)\pi(A)$. But $\lambda \notin \mathrm{EE}_\pi(\pi(A))$, so $\pi(T) = 0$; i.e., T is compact. Since $T \neq 0$ and $AT = \lambda TA$, [4] and [7] show that A has a proper hyperinvariant subspace. □

We are able to make some general remarks about the nature of extended eigenoperators for invertible operators:

Lemma 9. *Suppose that A is invertible and $T \in \mathcal{E}(A, \lambda)$, and λ is not a root of unity. Then the spectrum of T must be circularly symmetric.*

Proof. If $|\lambda| \neq 1$ then T is nilpotent and so its spectrum is $\{0\}$. If $|\lambda| = 1$ but no positive power of λ is 1, then $\{\lambda^n : n = 1, 2, \dots\}$ is dense in the unit circle.

Now for each positive integer n, $A^nT = \lambda^n TA^n$, so that

$$T = A^{-n}\lambda^n TA^n; \quad n = 1, 2, \dots .$$

But similar operators have identical spectra. Thus if σ is any member of the spectrum of T, then this spectrum contains the set $\{\sigma \cdot \lambda^n : n = 1, 2, \dots\}$. Since $\{\lambda^n : n = 1, 2, \dots\}$ is dense in the unit circle,and spectra are closed, this shows that the spectrum of T is circularly symmetric about the origin. □

Examples. (i) Let $\mathcal{H}= L^2(1, 2)$ (with respect to Lebesgue measure). For notational convenience, we consider members of $\mathcal{H}$ as being defined on the entire real line, but vanishing off $(1, 2)$. Let A be the operator on $\mathcal{H}$ of multiplication by the variable x. Let λ be a member of the interval $(\frac{1}{2}, 2)$. Let T be defined on $\mathcal{H}$ by $Tf(x) = f(x/\lambda)$. We then have

$$ATf(x) = xf(x/\lambda) \text{ and } TAf(x) = \chi_{(1,2)}(x/\lambda)f(x/\lambda).$$

Thus $AT = \lambda TA$. The choice of λ insures that $T \neq 0$; showing that $(\frac{1}{2}, 2) \subset \mathrm{EE}(A)$. In fact, it follows from a considerably more general result in [3] that these sets are equal, but in this specific context we can exhibit the operator T explicitly. Note that if $\lambda \neq 1$ then the corresponding operator T is indeed nilpotent.

(ii) This example illustrates the fact that $\mathrm{EE}(A) - \mathrm{EE}_\pi(\pi(A))$ can be nonempty. We first note that in any C^* algebra, if $ua = \lambda au$ for a unitary element u and nonzero element a, then $|\lambda| = 1$. This is not the case for isometries. Indeed, if U is the unilateral shift and $|\lambda| \geq 1$, then the diagonal operator T with n^{th}

diagonal entry λ^{-n} satisfies $UT = \lambda TU$. Since $\pi(U)$ is a unitary member of the Calkin algebra, we have our desired example.

Unfortunately, invertibility does not guarantee the existence of extended eigenvalues. The following proposition seems particularly curious as it was shown in [2] that the set of extended eigenvalues for the Volterra operator is the open half line $(0, \infty)$. In this case and in what follows, we are dealing with the classical Volterra operator

$$Vf(x) = \int_0^x f(t)dt \ ; \quad f \in L^2(0,1).$$

The following result uses a generalized form of a technique employed in [2].

Theorem 10. *Let γ be a nonzero scalar. Then* $\mathrm{EE}(\gamma + V) = \{1\}$.

Proof. Suppose $(\gamma + V)T = \lambda T(\gamma + V)$. Then

$$VT = T((\lambda - 1)\gamma I + \lambda V).$$

Since $(\lambda - 1)\gamma \neq 0$, and V is quasinilpotent, the operator $(\lambda - 1)\gamma I + \lambda V$ is invertible. Call this operator W^{-1}. Letting $D = V^{-1}$ be the (unbounded) differentiation operator, we see that $TW = DT$. Now, the spectrum of W^{-1} consists of the singleton $\{(\lambda - 1)\gamma\}$ and the spectrum of W is $\left\{\frac{1}{(\lambda-1)\gamma}\right\}$. Thus for all z of sufficiently large real part (say, $\Re\, z > r$) , we have

$$(z - D)^{-1}T = T(z - W)^{-1}.$$

Indeed the left side of this equation is an operator valued entire function (see, e.g., [1, Chapter IV, Section 3]). Let x and y be arbitrary members of $\mathcal{H}$, and define the continuous functions f and g on $(0, \infty)$ by

$$f(t) = \langle S_t T x, y\rangle \quad \text{and} \quad g(t) = \langle T e^{tW} x, y\rangle$$

where $\{S_t\}$ is the translation semigroup

$$S_t h(x) = h(x - t); h \in L^2(0,1).$$

Note that for $t \geq 1, S_t = 0$. Using the Laplace transform formula for resolvents (see [6]), for $\Re\, z > r$,

$$\int_0^1 e^{-zt} f(t)dt = \int_0^\infty e^{-zt} g(t)dt.$$

But the Laplace transform is injective, so $f = g$. As these functions are continuous, this is a true pointwise equality. In particular, $g(1) = f(1) = 0$. Thus for all vectors x and $y, Te^W x \perp y$. But e^W is invertible, and so $T = 0$. □

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